3. Prove Proposition 1.2.11.
4. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. Prove that $\triangle(X)$ is an example of a contractible space which is not an ANR.
5. Let $A \subseteq \mathbb{S}^{n-1}$. Observe that $B^n \setminus A$ is a convex subset of $\mathbb{R}^n$ and hence is an AR. Present a direct proof of this.
6. Prove that each AR is path-connected. Prove that every ANR is locally path-connected. Observe that, in particular, every ANR is locally connected.
7. Give an example of a subspace X of $\mathbb{R}$, such that X is an ANR, but $X \cup \{p\}$ is not an ANR for some $p \in \mathbb{R}$.
8. Let X be the $\sin(1/x)$-continuum in the plane. Prove that X is not an ANR.
9. ►Let A_1 and A_2 be closed subsets of the locally convex linear spaces L_1 and L_2, respectively. Prove that for each homeomorphism $h\colon A_1 \to A_2$ there exists a homeomorphism $H\colon L_1 \times L_2 \to L_1 \times L_2$ such that

$$H(a, \underline{0}) = \underline{0}, h(a)$$

for every $a \in A$.
10. Let L be a linear space. In addition, let X be a space, $A \subseteq X$ be closed and $f\colon A \to L$ be continuous. Finally, let $\mathcal{U}$ and $\{a_U : U \in \mathcal{U}\}$ be a Dugundji system for X and A. For each $x \in X \setminus A$ let

$$\mathcal{E}(x) = \{U \in \mathcal{U} : x \in U\}.$$

Define a function $F \Rightarrow L$ by

$$F(x) = \begin{cases} \{f(x)\} & (x \in A), \\ \operatorname{conv}(\{f(a_U) : U \in \mathcal{E}(x)\}) & (x \in X \setminus A). \end{cases}$$

Prove that F is LSC.
11. Use Exercise 1.2.10 to prove that a normed linear space satisfies the conclusion of the Dugundji Theorem 1.2.2.
12. Let L be a linear space. Assume that for every space X, every LSC function $F\colon X \Rightarrow L$ such that for every $x \in X$ the set $F(x)$ is compact and convex, has a continuous selection. Prove that every linear subspace of L is an AR.

1.3. Function spaces

The space $C^*(X)$ from Example 1.1.5 is a special case of a more general construction. Let X and Y be spaces and let ϱ be an admissible metric on Y. Define

$$C_\varrho(X, Y) = \{f \in C(X, Y) : \operatorname{diam}_\varrho(f[X]) < \infty\}.$$

It is sometimes convenient to refer to the elements of $C_\varrho(X, Y)$ as *bounded* functions. We shall endow $C_\varrho(X, Y)$ with a useful topology.

In §A.3 we observed that $\hat{\varrho}$ need not be a metric on $C(X,Y)$. Fortunately, on $C_\varrho(X,Y)$ it is a metric.

Lemma 1.3.1. *Let X and Y be spaces and let ϱ be an admissible metric on Y. Then*

(1) *for all $f,g \in C_\varrho(X,Y)$ we have $\hat{\varrho}(f,g) < \infty$,*
(2) *the function $\hat{\varrho}\colon C_\varrho(X,Y) \times C_\varrho(X,Y) \to [0,\infty)$ is a metric.*

Proof. For (1), take an arbitrary point $z \in X$ and observe that for all functions $f,g \in C_\varrho(X,Y)$ the following holds:

$$\hat{\varrho}(f,g) \le \operatorname{diam}_\varrho(f[X]) + \varrho\big(f(z),g(z)\big) + \operatorname{diam}_\varrho\big(g(X)\big) < \infty.$$

The proof of (2) is routine and is left as an exercise to the reader. □

From now on we shall endow $C_\varrho(X,Y)$ with the topology induced by $\hat{\varrho}$. Let us emphasize that the set $C_\varrho(X,Y)$ as well as its topology depend on the choice of the metric ϱ. It is a natural question to ask when the choice of ϱ is irrelevant.

Lemma 1.3.2. *Let X and Y be spaces with X compact. In addition, let ϱ_1 and ϱ_2 be admissible metrics for Y. Then*

(1) $C_{\varrho_1}(X,Y) = C_{\varrho_2}(X,Y) = C(X,Y)$,
(2) *the topologies on $C(X,Y)$ induced by $\hat{\varrho}_1$ and $\hat{\varrho}_2$ are the same.*

Proof. (1) is trivial.

For each $\varepsilon > 0, y \in Y$ and $i \in \{1,2\}$ we put

$$B_i(y,\varepsilon) = \{z \in Y : \varrho_i(y,z) < \varepsilon\}.$$

For (2), take $f \in C_{\varrho_1}(X,Y)$ and $\varepsilon > 0$, arbitrarily. Our aim is to prove that there exists $\delta > 0$ such that

$$\{g \in C_{\varrho_2}(X,Y) : \hat{\varrho}_2(f,g) < \delta\} \subseteq \{g \in C_{\varrho_1}(X,Y) : \hat{\varrho}_1(f,g) < \varepsilon\},$$

i.e., that the $\hat{\varrho}_2$-ball about f with radius δ is contained in the $\hat{\varrho}_1$-ball about f with radius ε.

To this end, observe that since $f[X]$ is compact, the open cover

$$\mathcal{U} = \{B_1(f(x), \tfrac{1}{2}\varepsilon) : x \in X\}$$

has a ϱ_2-Lebesgue number, say δ (Lemma A.5.3). We claim that this δ is as required. To see that this is indeed the case, take an arbitrary $g \in C_{\varrho_2}(X,Y)$ such that $\hat{\varrho}_2(f,g) < \delta$. For each $x \in X$ we have $\varrho_2\big(f(x),g(x)\big) < \delta$, so there exists $p_x \in X$ such that $\{f(x),g(x)\} \subseteq B_1(f(p_x), \varepsilon/2)$. Consequently, for each element $x \in X$ we have

$$\varrho_1\big(f(x),g(x)\big) < \varepsilon,$$

from which it follows by Exercise A.5.4 that $\hat{\varrho}_1(f,g) < \varepsilon$.

So we conclude that the topology on $C(X,Y)$ induced by $\hat{\varrho}_2$ is finer than the topology on $C(X,Y)$ induced by $\hat{\varrho}_1$. By interchanging the roles of ϱ_1 and ϱ_2 in the above argument we find that the induced topologies are indeed the same. $\square$

Let X and Y be spaces with X compact. From the above lemma we conclude that all the topologies we defined on the set $C(X,Y)$ coincide. So for compact X and any (Y,ϱ) we shall denote the space $C_\varrho(X,Y)$ simply by $C(X,Y)$. The topology on $C(X,Y)$ is called *the topology of uniform convergence.*

Observe that the norm topology on $C(X)$ for compact X defined in Example 1.1.5 coincides with the just defined topology on $C(X,\mathbb{R})$.

On Page 19 we defined the so-called compact-open topology on $C(X)$. This is again a special case of a more general construction. Indeed, let X and Y be spaces and for an arbitrary compact subset K in X and an arbitrary open subset U in Y define

$$[K,U] = \{f \in C(X,Y) : f[K] \subseteq U\}.$$

Topologize $C(X,Y)$ by taking the collection

$$\{[K,U] : K \subseteq X \text{ compact and } U \subseteq Y \text{ open}\}$$

as an open subbase. This topology is called the *compact-open topology* on the set $C(X,Y)$.

We will now show that for compact X and arbitrary Y the compact-open topology on $C(X,Y)$ coincides with the topology of uniform convergence on $C(X,Y)$. This allows us to prove quite easily that $C(X,Y)$ is separable.

Proposition 1.3.3. *Let X and Y be spaces with X compact. The topology of uniform convergence on $C(X,Y)$ coincides with the compact-open topology on $C(X,Y)$. As a consequence, $C(X,Y)$ is separable.*

Proof. Let ϱ be an admissible metric on Y. In addition, let $K \subseteq X$ be compact and $U \subseteq Y$ open. If $f \in [K,U]$ then $f[K]$ is a compact subset of U. By Corollary A.5.4 there exists $\delta > 0$ such that

$$B(f[K],\delta) \subseteq U.$$

This clearly implies that if $g \in C(X,Y)$ is such that $\hat{\varrho}(f,g) < \delta$ then $g[K]$ is contained in U, i.e.,

$$\{g \in C(X,Y) : \hat{\varrho}(f,g) < \delta\} \subseteq [K,U].$$

From this we conclude that $[K,U]$ is open in the topology of uniform convergence on $C(X,Y)$ and hence that the topology of uniform convergence is finer than the compact-open topology.

For the converse, let $\mathcal{B}$ and $\mathcal{E}$ be countable open bases for X and Y, respectively, which are both closed under finite unions. For $B \in \mathcal{B}$ and $E \in \mathcal{E}$ put

$$A(B, E) = [\overline{B}, E].$$

By the above, each $A(B, E)$ is open in the topology of uniform convergence on $C(X, Y)$. Let $\mathcal{A}$ be the (countable) collection of all $A(B, E)$'s. We claim that the family $\mathcal{A}^*$ of all finite intersections of elements of $\mathcal{A}$ is an open base for $C(X, Y)$ endowed with the topology of uniform convergence. This proves on the one hand that the compact-open topology is finer than the topology of uniform convergence and on the other hand that $C(X, Y)$ has a countable base.

Let $f \in C(X, Y)$ and $\varepsilon > 0$. We shall prove that there exists an element $F \in \mathcal{A}^*$ such that $f \in F \subseteq \{g \in C(X, Y) : \hat{\varrho}(f, g) < \varepsilon\}$. By compactness of the set $f[X]$, there are finitely many elements of $\mathcal{E}$, say $E_1, E_2, \ldots, E_n$, such that

(1) $f[X] \subseteq \bigcup_{i=1}^{n} E_i$,
(2) for every $i \leq n$, $\operatorname{diam}(E_i) < \varepsilon$.

Let $\mathcal{U} = \{f^{-1}[E_i] : i \leq n\}$. Since $\mathcal{B}$ is a base, there clearly is a cover $\mathcal{V}$ of X consisting of elements of $\mathcal{B}$ such that $\overline{\mathcal{V}} < \mathcal{U}$. By compactness of X, we may assume that $\mathcal{V}$ is finite. For each $i \leq n$ let W_i be the union of the elements of $\mathcal{V}$ the closures of which are contained in $f^{-1}[E_i]$. Since $\mathcal{B}$ is closed under finite unions, $\mathcal{W} = \{W_i : i \leq n\}$ is a subcollection of $\mathcal{B}$, $\mathcal{W}$ covers X, and $\mathcal{W}$ has the property that the closure of each W_i is contained in $f^{-1}[E_i]$. (This is so since W_i is a *finite* union of sets the closures of which are contained in $f^{-1}[E_i]$.) Now put

$$F = \bigcap_{i=1}^{n} A(W_i, E_i).$$

It is clear that $f \in F$. In addition, F is open in the topology of uniform convergence. We claim that $F \subseteq \{g \in C(X, Y) : \hat{\varrho}(f, g) < \varepsilon\}$. To this end, take $g \in F$ and $x \in X$. There exists $i \leq n$ with $x \in W_i$. Since $f, g \in F$, it follows that $f[W_i] \cup g[W_i] \subseteq E_i$. Consequently, both $f(x)$ and $g(x)$ belong to E_i from which we get by (2) that $\varrho\big(f(x), g(x)\big) < \varepsilon$. So we are done. □

This result can be used to estimate the number of continuous functions.

Corollary 1.3.4. *Let X and Y be spaces with X compact. Then $C(X, Y)$ has cardinality at most $\mathfrak{c}$.*

Proof. This follows from Proposition 1.3.3 and Exercise A.2.16. □

We now turn to completeness properties of $C(X, Y)$ and $C_\varrho(X, Y)$.

Proposition 1.3.5. *Let X and (Y,ϱ) be spaces. Let $(f_n)_n$ be a $\hat{\varrho}$-Cauchy sequence in $C(X,Y)$ such that for every $x \in X$, $\lim_{n\to\infty} f_n(x)$ exists. Then the function $f\colon X \to Y$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous. In addition, if $f_n \in C_\varrho(X,Y)$ for every n then $f \in C_\varrho(X,Y)$ and $f = \lim_{n\to\infty} f_n$ (in $C_\varrho(X,Y)$).*

Proof. For the first part of the proposition, see Lemma A.3.1.

For the remaining part, suppose that $f_n \in C_\varrho(X,Y)$ for every n. We shall prove that $\operatorname{diam}(f[X]) < \infty$. By Claim 1 in the proof of Lemma A.3.1 there exists an $M \in \mathbb{N}$ such that for every $x \in X$, $\varrho\big(f(x), f_M(x)\big) < 1$. Take arbitrary $x, z \in X$. Then

$$\begin{aligned}\varrho\big(f(x), f(z)\big) &\leq \varrho\big(f(x), f_M(x)\big) + \varrho\big(f_M(x), f_M(z)\big) + \varrho\big(f_M(z), f(z)\big)\\ &< 2 + \operatorname{diam}(f_M[X]),\end{aligned}$$

so $\operatorname{diam}(f[X]) < \infty$.

It remains to prove that $f = \lim_{n\to\infty} f_n$ (in $C_\varrho(X,Y)$). But this follows again easily from Claim 1 in the proof of Lemma A.3.1. □

Corollary 1.3.6. *Let X and (Y,ϱ) be spaces. Then ϱ is complete if and only if the metric $\hat{\varrho}$ on $C_\varrho(X,Y)$ is complete.*

Proof. Suppose that (Y,ϱ) is complete and let $(f_n)_n$ be a $\hat{\varrho}$-Cauchy sequence in $C_\varrho(X,Y)$. Fix $z \in X$ arbitrarily and let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\hat{\varrho}(f_n, f_m) < \varepsilon$ for all $n, m \geq N$. Since for all n and m,

$$\varrho\big(f_n(z), f_m(z)\big) \leq \hat{\varrho}(f_n, f_m),$$

we conclude that $\big(f_n(z)\big)_n$ is Cauchy in (Y,ϱ). The completeness of (Y,ϱ) and Proposition 1.3.5 now yield that the sequence $(f_n)_n$ converges.

Assume that $\hat{\varrho}$ is complete. Let $(y_n)_n$ be a ϱ-Cauchy sequence in Y. For each n let $f_n\colon X \to Y$ be the constant function with value y_n. It is easy to see that $(f_n)_n$ is a $\hat{\varrho}$-Cauchy sequence in $C_\varrho(X,Y)$. So $f = \lim_{n\to\infty} f_n$ exists and belongs to $C_\varrho(X,Y)$. It is trivial to prove that f is constant and that the sequence $(y_n)_n$ converges to the unique point in the range of f. □

Corollary 1.3.7. *Let X be compact and Y topologically complete. Then the space $C(X,Y)$ is topologically complete.*

Groups of homeomorphisms. We will now restrict our attention to spaces of homeomorphisms and will derive some basic properties of them.

Let X and Y be spaces and define

$$\mathcal{S}(X,Y) = \{f \in C(X,Y) : f \text{ is surjective}\}.$$

There are spaces X and Y for which $\mathcal{S}(X,Y)$ is empty, for example this is the case if X has smaller cardinality than Y.

Proposition 1.3.8. *Let X and Y be spaces with X compact. Then $\mathcal{S}(X,Y)$ is closed in $C(X,Y)$.*

Proof. Take an arbitrary $f \notin \mathcal{S}(X,Y)$. There exists a point $y \in Y \setminus f[X]$. So

$$f \in [X, Y \setminus \{y\}] \subseteq C(X,Y) \setminus \mathcal{S}(X,Y).$$

Since $[X, Y \setminus \{y\}]$ is open in $C(X,Y)$ (Proposition 1.3.3), this shows that

$$C(X,Y) \setminus \mathcal{S}(X,Y)$$

is a neighborhood of f. Hence $C(X,Y) \setminus \mathcal{S}(X,Y)$ is open in $C(X,Y)$. □

Let X and Y be spaces with X compact, and let $\varepsilon > 0$. A continuous function $f\colon X \to Y$ is called an ε-map if for every $y \in Y$,

$$\operatorname{diam}\left(f^{-1}(y)\right) < \varepsilon.$$

Let $\varepsilon > 0$ and put

$$C_\varepsilon(X,Y) = \{f \in C(X,Y) : f \text{ is an } \varepsilon\text{-map}\}$$

and

$$\mathcal{S}_\varepsilon(X,Y) = C_\varepsilon(X,Y) \cap \mathcal{S}(X,Y),$$

respectively. In addition, put

$$\mathcal{G}_\varepsilon(X,Y) = \mathcal{S}(X,Y) \setminus C_\varepsilon(X,Y).$$

Lemma 1.3.9. *Let X and Y be spaces with X compact. Then $C_\varepsilon(X,Y)$ is open in $C(X,Y)$ for every $\varepsilon > 0$. Consequently, $\mathcal{G}_\varepsilon(X,Y)$ is a closed subset of $C(X,Y)$.*

Proof. Take $f \in C_\varepsilon(X,Y)$. Since X is compact, $f\colon X \to f[X]$ is a closed map (Exercise A.5.5). We claim that for every $y \in f[X]$ there exists an open neighborhood U_y (in $f[X]$) such that

$$\operatorname{diam}\left(f^{-1}[U_y]\right) < \varepsilon.$$

This will be achieved in two steps. Take an arbitrary $y \in f[X]$. Then

$$\operatorname{diam}\left(f^{-1}(y)\right) < \varepsilon$$

since f is an ε-map. There clearly exists an open neighborhood U of $f^{-1}(y)$ such that $\operatorname{diam}(U) < \varepsilon$. Now by using that f is a closed map, Exercise A.1.15 gives us the required neighborhood U_y.

Let $\delta > 0$ be a Lebesgue number for the open covering

$$\{U_y : y \in f[X]\}$$

of $f[X]$ (Lemma A.5.3). Let $g \in C(X,Y)$ be such that $\hat{\varrho}(g,f) < \delta/2$. We claim that $g \in C_\varepsilon(X,Y)$. This clearly suffices. To this end, take an arbitrary $y \in Y$. Since we have $\hat{\varrho}(f,g) < \delta/2$ it follows easily that $\operatorname{diam}\left(fg^{-1}(y)\right) < \delta$. There consequently exists a point $z \in f[X]$ such that $fg^{-1}(y) \subseteq U_z$. This implies that

$$\operatorname{diam}\left(f^{-1}fg^{-1}(y)\right) < \varepsilon.$$

Since $g^{-1}(y) \subseteq f^{-1}fg^{-1}(y)$, we conclude that $\operatorname{diam}\left(g^{-1}(y)\right) < \varepsilon$, i.e., g is an ε-map. That $\mathcal{G}_\varepsilon(X,Y)$ is closed now follows by Proposition 1.3.8. □

Let X and Y be spaces. We introduce a few more interesting subsets of $C(X,Y)$. Let $\mathcal{I}(X,Y)$ denote the subset of $C(X,Y)$ consisting of all imbeddings of X into Y, and let $\mathcal{H}(X,Y)$ denote the set of all homeomorphisms from X onto Y. If $X = Y$ then for $\mathcal{H}(X,X)$ we shall simply write $\mathcal{H}(X)$. As usual, $\mathcal{H}(X)$ is called the *autohomeomorphism group* of X.

Lemma 1.3.10. *Let X and Y be spaces with X compact. Then*

$$\mathcal{I}(X,Y) = \bigcap_{n=1}^{\infty} C_{1/n}(X,Y).$$

As a consequence, $\mathcal{I}(X,Y)$ is a G_δ-subset of $C(X,Y)$, and $\mathcal{H}(X,Y)$ is a G_δ-subset of both $C(X,Y)$ and $\mathcal{S}(X,Y)$.

Proof. That $\mathcal{I}(X,Y) \subseteq \bigcap_{n=1}^{\infty} C_{1/n}(X,Y)$ is a triviality. Pick an arbitrary

$$f \in \bigcap_{n=1}^{\infty} C_{1/n}(X,Y).$$

Then f is a $1/n$-map for every n, hence f is one-to-one. So the compactness of X implies that f is an imbedding (Exercise A.5.9). The remaining statements are obvious. □

Corollary 1.3.11. *Let X and Y be spaces with X compact and Y topologically complete. Then both $\mathcal{I}(X,Y)$ and $\mathcal{H}(X,Y)$ are topologically complete.*

Proof. Since $C(X,Y)$ is completely metrizable by Corollary 1.3.6, this follows immediately from Lemma 1.3.10 and Theorem A.6.3. □

So $\mathcal{H}(X)$ is complete if X is compact. It will be convenient to describe an explicit complete metric for $\mathcal{H}(X)$ that generates its topology.

Proposition 1.3.12. *Let X be a compact space. For $f, g \in \mathcal{H}(X)$ define*

$$\sigma(f,g) = \hat{\varrho}(f,g) + \hat{\varrho}(f^{-1}, g^{-1}).$$

Then σ is a complete metric on $\mathcal{H}(X)$ that generates its topology.

Proof. That σ is a metric is left as an exercise to the reader. We shall first prove that $\hat{\varrho}$ and σ generate the same topology on $\mathcal{H}(X)$. Since for all $f, g \in \mathcal{H}(X)$ we have $\hat{\varrho}(f, g) \leq \sigma(f, g)$, the only thing to verify is that for every $\varepsilon > 0$ and every $f \in \mathcal{H}(X)$ there exists $\delta > 0$ such that

$$\text{if } g \in \mathcal{H}(X) \text{ and } \hat{\varrho}(f, g) < \delta \text{ then } \sigma(f, g) < \varepsilon.$$

Choose arbitrary $\varepsilon > 0$ and $f \in \mathcal{H}(X)$. By compactness, f^{-1} is uniformly continuous (Exercise A.5.18) and consequently there exists $\gamma > 0$ such that for all $x, y \in X$ with $\varrho(x, y) < \gamma$ we have $\varrho\big(f^{-1}(x), f^{-1}(y)\big) < \varepsilon/2$. Let

$$\delta = \min\{\gamma, \varepsilon/2\}.$$

Take $g \in \mathcal{H}(X)$ such that $\hat{\varrho}(f, g) < \delta$. Pick an arbitrary $x \in X$ and put

$$z = g^{-1}(x).$$

Since $\hat{\varrho}(f, g) < \delta$, it follows that

$$\varrho\big(f(z), g(z)\big) = \varrho(fg^{-1}(x), x) < \delta \leq \gamma.$$

As a consequence,

$$\varrho\big(g^{-1}(x), f^{-1}(x)\big) = \varrho\big(f^{-1}fg^{-1}(x), f^{-1}(x)\big) < \varepsilon/2.$$

We conclude that $\hat{\varrho}(g^{-1}, f^{-1}) < \varepsilon/2$ by Exercise A.5.4. Therefore,

$$\sigma(f, g) = \hat{\varrho}(f, g) + \hat{\varrho}(f^{-1}, g^{-1}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now let $(f_n)_n$ be a σ-Cauchy sequence in $\mathcal{H}(X)$. Then $(f_n)_n$ is a $\hat{\varrho}$-Cauchy sequence in $C(X, X)$ and therefore the limit $f = \lim_{n\to\infty} f_n$ exists and belongs to $C(X, X)$ (Corollary 1.3.6). Similarly, by the definition of σ, the limit $g = \lim_{n\to\infty} f_n^{-1}$ exists and belongs to $C(X, X)$. It is easily seen that $f \circ g = 1_X = g \circ f$ from which it follows that $f \in \mathcal{H}(X)$. □

Lemma 1.3.10 and Proposition 1.3.12 both imply by Theorem A.6.6 the following:

Corollary 1.3.13. *If X is compact then $\mathcal{H}(X)$ is a Baire space.*

Exercises for §1.3.

1. Let $\mathbb{N}$ denote the discrete space of natural numbers. Prove that $C_{|\cdot|}(\mathbb{N}, \mathbb{R})$ is not separable.
2. Let X, Y and Z be compact spaces. For $f \in C(Z, X)$ and $g, h \in C(X, Y)$ prove that
$$\hat{\varrho}(g \circ f, h \circ f) \leq \hat{\varrho}(g, h).$$
In addition, show that if f is surjective then
$$\hat{\varrho}(g \circ f, h \circ f) = \hat{\varrho}(g, h).$$
3. Let X be compact, let $f, g \in C(X, X)$ such that g is a homeomorphism. Prove that
$$\hat{\varrho}(1_X, f \circ g^{-1}) = \hat{\varrho}(f, g).$$

4. Let X be a compact space. Prove that the function
$$\xi\colon \mathcal{H}(X) \times \mathcal{H}(X) \to \mathcal{H}(X)$$
defined by
$$\xi(f, g) = f \circ g^{-1}$$
is continuous (i.e., $\mathcal{H}(X)$ is a topological group).

5. Prove that the function $f\colon \mathbb{I} \to \mathbb{I}$ defined by
$$f(x) = \begin{cases} 2x & (0 \leq x \leq 1/4), \\ 1/2 & (1/4 \leq x \leq 3/4), \\ 2x - 1 & (3/4 \leq x \leq 1). \end{cases}$$
belongs to the closure of $\mathcal{H}(\mathbb{I})$ in $C(\mathbb{I}, \mathbb{I})$. (Hence $\mathcal{H}(X)$ is even for compact X not necessarily a closed subspace of $C(X, X)$.)

6. Prove that $\mathcal{H}(\mathbb{I})$ has exactly two components.

7. Give an example of two nontrivial continua X and Y such that $\mathcal{S}(X, Y)$ is empty.

1.4. The Borsuk homotopy extension theorem

The aim of the present section is to prove that a continuous function f from a closed subspace A of a space X into an ANR Z is extendable over X if and only if f is homotopic to an extendable function $g\colon A \to Z$.

We shall need the following simple lemma:

Lemma 1.4.1. *Let A be a closed subset of a space X. Then for every neighborhood V of $B = (X \times \{0\}) \cup (A \times \mathbb{I})$ in $X \times \mathbb{I}$ there is a continuous function $\alpha\colon X \times \mathbb{I} \to V$ which is the identity on B.*

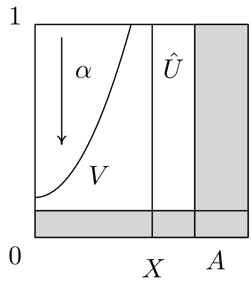

Figure 1.

Proof. Let $\pi\colon X \times \mathbb{I} \to X$ be the projection. By compactness of $\mathbb{I}$ and Exercise A.5.8, π is closed. We may assume without loss of generality that V is open. So $F = (X \times \mathbb{I}) \setminus V$ is a closed set which misses $A \times \mathbb{I}$. Hence
$$U = X \setminus \pi[F]$$

is an open neighborhood of A in X such that $\hat{U} = U \times \mathbb{I} \subseteq V$. Now let

$$\beta \colon X \to \mathbb{I}$$

be a Urysohn function (Lemma A.4.1) such that

$$\beta \restriction A \equiv 1, \quad \beta \restriction X \setminus U \equiv 0.$$

Define $\alpha \colon X \times \mathbb{I} \to X \times \mathbb{I}$ by

$$\alpha(x,t) = (x, \beta(x) \cdot t).$$

Then α is clearly continuous. It is easily seen that it restricts to the identity on B. We shall prove that $\alpha[X \times \mathbb{I}] \subseteq V$, thereby showing that α is as required. To this end, take $(x,t) \in X \times \mathbb{I}$ arbitrarily. If $x \notin U$ then $\beta(x) = 0$ and consequently, $\alpha(x,t) = (x,0) \in B \subseteq V$. On the other hand, if $x \in U$ then $\alpha(x,t) = (x, \beta(x) \cdot t) \in U \times \mathbb{I} \subseteq V$. ☐

We now come to the main result in this section.

The Borsuk Homotopy Extension Theorem 1.4.2. *Let A be a closed subspace of a space X, let Z be an* ANR *and let $H \colon A \times \mathbb{I} \to Z$ be a homotopy such that H_0 is extendable to a continuous function $f \colon X \to Z$. Then there is a homotopy $F \colon X \times \mathbb{I} \to Z$ such that*

(1) *$F_0 = f$,*
(2) *for every $t \in \mathbb{I}$, $F_t \restriction A = H_t$.*

Proof. Using the notation as in Lemma 1.4.1, define a function $\xi \colon B \to Z$ by

$$\xi(x,t) = \begin{cases} f(x) & (x \in X, t = 0), \\ H(x,t) & (x \in A, t \in \mathbb{I}). \end{cases}$$

Since $X \times \{0\}$ and $A \times \mathbb{I}$ are closed in B, it follows easily that ξ is continuous. As Z is an ANR and as B is closed in $X \times \mathbb{I}$, we can find a neighborhood V of B in $X \times \mathbb{I}$ such that ξ can be extended to a continuous function $\xi' \colon V \to Z$. Let $\alpha \colon X \times \mathbb{I} \to V$ be as in Lemma 1.4.1.

Define $F \colon X \times \mathbb{I} \to Z$ by the formula

$$F(x,t) = \xi'\big(\alpha(x,t)\big).$$

It is easily seen that F is as required. ☐

Theorem 1.4.2 has some interesting corollaries.

Corollary 1.4.3. *Let A be a closed subset of a space X. Let Z be an* ANR *and let $f \colon A \to Z$ be continuous. The following statements are equivalent:*

(1) *f can be extended over X,*
(2) *f is homotopic to an extendable function $g \colon A \to Z$.*

Proof. That (1) $\Rightarrow$ (2) is clear since $f \simeq f$ by Lemma A.12.1(1).

For (2) $\Rightarrow$ (1), let $H: A \times \mathbb{I} \to Z$ be a homotopy with $H_0 = g$ and $H_1 = f$. Since g is extendable, f is extendable as well (Theorem 1.4.2). $\square$

Our next goal is to investigate contractibility within the class of ANR's.

Theorem 1.4.4. *Let X be an* AR*. Then X is contractible.*

Proof. By Corollary 1.1.8, X can be thought of as a closed subset of a normed linear space L. Fix an arbitrary point $p \in L$. Since the function $F: L \times \mathbb{I} \to L$ defined by the formula

$$(x, t) \mapsto t \cdot p + (1 - t) \cdot x$$

is a contraction, it follows that X is contractible, being a retract of L (Theorem A.12.4). $\square$

These results yield the following interesting and useful characterization of AR's.

Corollary 1.4.5. *Let X be a space. The following statements are equivalent:*

(1) *X is an* AR,
(2) *X is a contractible* ANR.

Proof. The implication (1) $\Rightarrow$ (2) is trivial by Theorem 1.4.4.

For (2) $\Rightarrow$ (1), let

$$H: X \times \mathbb{I} \to X$$

be a homotopy such that $H_0 = 1$ and H_1 is constant, say with constant value c. In addition, let Y be a space, $A \subseteq Y$ be closed and $f: A \to X$ be continuous. Define $F: A \times \mathbb{I} \to X$ by

$$F(a, t) = H(f(a), t).$$

Then F is clearly a homotopy with $F_0 = f$ and F_1 a constant function. It is trivial that F_1 can be extended to a continuous function from Y to X. As X is an ANR, by Corollary 1.4.3 it follows that $F_0 = f$ can be extended over Y. We conclude that X is an AR. $\square$

For a nontrivial generalization of Corollary 1.4.5, see Theorem 4.2.20.

Cones. Let X be a space and consider its cone $\triangle(X)$. We observed in Exercise 1.2.2 that if $\triangle(X)$ is an ANR, then so is X. It is an interesting question whether the converse holds.

Theorem 1.4.6. *For a space X the following statements are equivalent:*

(1) *X is an* ANR,
(2) *$\triangle(X)$ is an* ANR,

1.5. Topological characterization of some familiar spaces

The aim of this section is to present topological characterizations of two familiar spaces, namely, the Cantor middle-third set C and the closed unit interval $\mathbb{I}$. (See §1.9 for topological characterizations of $\mathbb{Q}$ and $\mathbb{P}$.)

Topological characterization of the Cantor set. A space X is called *zero-dimensional* if it is nonempty and has a base consisting of clopen sets, i.e., if for every point $x \in X$ and for every neighborhood U of x there exists a clopen subset $C \subseteq X$ such that $x \in C \subseteq U$. It is clear that a nonempty subspace of a zero-dimensional space is again zero-dimensional and that products of zero-dimensional spaces are zero-dimensional. Observe that by Exercise A.2.12 it follows that a space is zero-dimensional if and only if it has a *countable* base consisting of clopen sets.

It is clear that no nontrivial connected space is zero-dimensional. As a consequence, if a space contains a nontrivial connected subspace then it is not zero-dimensional.

Proposition 1.5.1. *Let X be zero-dimensional. Then for every open cover $\mathcal{U}$ of X there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of pairwise disjoint clopen sets.*

Proof. Let $\mathcal{U}$ be an open cover of X. Since X is zero-dimensional, it can be refined by a clopen cover $\mathcal{E}$. We may assume without loss of generality that $\mathcal{E}$ is countable, say $\mathcal{E} = \{E_n : n \in \mathbb{N}\}$ (Corollary A.2.3). Now put $V_1 = E_1$ and

$$V_n = E_n \setminus \bigcup_{i<n} E_i$$

for every $n \geq 2$. Then $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is as required. □

Corollary 1.5.2. *Let X be a space. Then X is zero-dimensional if and only if for all disjoint closed sets A and B there exists a clopen subset $U \subseteq X$ such that $A \subseteq U$ and $B \cap U = \emptyset$.*

Proof. First assume that X is zero-dimensional. For every $x \in X$ let U_x be a neighborhood of x such that $U_x \cap A = \emptyset$ or $U_x \cap B = \emptyset$ (here we use that A and B are disjoint closed sets). By Proposition 1.5.1 there is a refinement $\mathcal{V}$ of $\{U_x : x \in X\}$ consisting of pairwise disjoint clopen sets. Now put

$$U = \bigcup \{V \in \mathcal{V} : V \cap A \neq \emptyset\}.$$

It is easy to see that U is as required.

If for all disjoint closed subsets A and B of X there exists a clopen set C with $A \subseteq C \subseteq X \setminus B$, then the clopen subsets of X clearly form a base. So then X is zero-dimensional. □

CHAPTER 2

Basic combinatorial topology

In this chapter we present some elementary combinatorial results and apply these to get nontrivial information about the topology of the Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$. The main result is the Brouwer Fixed-Point Theorem.

2.1. Affine notions

In this section we will introduce simplicial complexes and present some basic results on them. So-called subdivisions of simplicial complexes are important in the proof of Brouwer's Fixed-Point Theorem as well as in ANR-theory.

For completeness sake, we begin by reviewing some elementary Linear Algebra. The proofs of these results are left to the reader.

Let V denote a fixed vector space.

Let F be a finite subset of V, say $F = \{v_1, \dots, v_n\}$. An *affine combination* of $v_1, \dots, v_n$ is a vector v that can be written in the form $\sum_{i=1}^n t_i v_i$ with $t_1, \dots, t_n \in \mathbb{R}$ and $\sum_{i=1}^n t_i = 1$. Such a v is called *geometrically dependent* on F.

A *linear (affine) subspace* of V is a subset of V closed under the formation of linear (affine) combinations.

Theorem 2.1.1. *Let A be a subset of V and let $a \in A$. Then A is an affine subspace if and only if $A - a$ is a linear subspace.*

If $S \subseteq V$, then the intersection of all affine (linear) subspaces of V containing S is the smallest affine (linear) subspace of V containing S. This subset is called *affine hull (linear hull)* of S and is denoted by $\operatorname{aff}(S)$ $\big(\operatorname{lin}(S)\big)$.

Theorem 2.1.2. $\operatorname{aff}(S)$ *is the set of all affine combinations of elements of S. Moreover, for every $a \in S$ the following equality holds:*

$$\operatorname{aff}(S) - a = \operatorname{lin}(S - a).$$

We see that if $S \subseteq V$ then $\operatorname{aff}(S)$ is a translated linear subspace of V, since for every $a \in S$ the equality

$$\operatorname{aff}(S) = a + \operatorname{lin}(S - a)$$

holds. We say that S *spans* the affine subspace $\mathrm{aff}(S)$.

Let $v_1, \ldots, v_n \in V$. Then

(1) the vectors $v_1, \ldots, v_n$ are said to be *geometrically independent* if for all elements $t_1, \ldots, t_n \in \mathbb{R}$ with $\sum_{i=1}^n t_i = 0$ and $\sum_{i=1}^n t_i v_i = 0$ we have $t_1 = \cdots = t_n = 0$,
(2) a subset $S \subseteq V$ is *linearly (geometrically) independent* if and only if every finite subset is.

Theorem 2.1.3. *Let $S \subseteq V$. The following statements are equivalent:*

(1) *S is geometrically independent,*
(2) *no $x \in S$ is geometrically dependent on a finite subset $F \subseteq S \setminus \{x\}$,*
(3) *for every $s \in S$, $\{x - s : x \in S, x \neq s\}$ is linearly independent,*
(4) *for some $s \in S$, $\{x - s : x \in S, x \neq s\}$ is linearly independent.*

Corollary 2.1.4. *Let $S \subseteq \mathbb{R}^n$ be geometrically independent. Then S contains at most $n + 1$ points.*

Consider the affine subspace $\mathrm{aff}(S)$ and fix an arbitrary $a \in S$. We saw that $\mathrm{aff}(S) = a + \mathrm{lin}(S - a)$. So there exists a subset $T \subseteq S$ such that $\mathrm{lin}(S - a) = \mathrm{lin}(T - a)$ while moreover $T - a$ is linearly independent. By Theorem 2.1.3 it follows that $T' = \{a\} \cup T$ is geometrically independent. Then

$$\mathrm{aff}(T') = a + \mathrm{lin}(T' - a) = a + \mathrm{lin}(T - a) = a + \mathrm{lin}(S - a) = \mathrm{aff}(S).$$

So we conclude that S and its geometrically independent subset T' span the same affine subspace.

An affine subspace of a linear space is called *m-dimensional* if it is spanned by $m + 1$ geometrically independent vectors.

Proposition 2.1.5. *Let $S \subseteq V$ be geometrically independent. If $A, B \subseteq S$ then*

$$\mathrm{aff}(A) \cap \mathrm{aff}(B) = \mathrm{aff}(A \cap B).$$

The following result is a nice test for proving that subsets of V are geometrically independent.

Theorem 2.1.6. *Let $a_0, a_1, \ldots, a_n$ be elements of V such that*

$$a_{i+1} \notin \mathrm{aff}(\{a_0, \ldots, a_i\})$$

for every $i < n$. Then $\{a_0, \ldots, a_n\}$ is geometrically independent.

A function between affine subspaces of linear spaces is called *affine* if it preserves affine combinations. Images and preimages of affine sets under affine functions are again affine.

Theorem 2.1.7. *Let V_1 and V_2 be linear spaces, let $A_1 \subseteq V_1$ and $A_2 \subseteq V_2$ be affine subspaces and let $f\colon A_1 \to A_2$ be a function. Then the following statements are equivalent:*

(1) *f is affine,*
(2) *the composition*

$$A_1 - a_1 \xrightarrow{+a_1} A_1 \xrightarrow{f} A_2 \xrightarrow{-a_2} A_2 - a_2 \quad \big(a_1 \in A_1, a_2 = f(a_1)\big),$$

i.e., the function $\xi\colon A_1 - a_1 \to A_2 - a_2$ defined by

$$\xi(x) = f(x + a_1) - a_2,$$

is linear.

From Theorem 2.1.7 it follows that an affine function, the domain and range of which are both contained in a finite dimensional normed linear space, is continuous. This can be seen as follows.

We first claim that for arbitrary $n, m \in \mathbb{N}$, linear functions

$$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

are continuous. First observe that each linear function $f\colon \mathbb{R} \to \mathbb{R}$ is a multiplication and hence is continuous. Assume that every linear function $f\colon \mathbb{R}^i \to \mathbb{R}$ is continuous for $i \le n$, and let $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be linear. If $f_1 = F \restriction \mathbb{R}^n \times \{0\}$ and

$$f_2 = F \restriction \underbrace{\{0\} \times \{0\} \times \cdots \times \{0\}}_{n\times} \times \mathbb{R}.$$

then they are continuous by assumption. But then F is continuous by linearity, being the sum of the continuous functions

$$(x_1, \ldots, x_{n+1}) \mapsto f_1(x_1, \ldots, x_n, 0)$$

and

$$(x_1, \ldots, x_{n+1}) \mapsto f_2(0, \ldots, 0, x_{n+1}).$$

Consequently, each linear function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous since $\mathbb{R}^m$ is endowed with the product topology. By Exercise 1.1.18 it follows that each finite dimensional normed linear space is topologically isomorphic to some $\mathbb{R}^n$. So we conclude that a linear function between finite dimensional normed linear spaces is indeed continuous.

This gives us what we want since by Theorem 2.1.7 each affine function f, the domain and range of which are both contained in a finite dimensional normed linear space, is of the form

$$f = \xi \circ F \circ \eta,$$

where both ξ and η are translations and hence homeomorphisms (Exercise 1.1.3), and F is linear.

Simplices. Let V be a linear space. An *n-simplex* in V is a geometrically independent subset of V having precisely $n+1$ points. Simplices are denoted by Greek letters. If σ and τ are simplices and $\sigma \subseteq \tau$ then σ is called a *face* of τ; to indicate that σ is a face of τ we shall also use the notation: $\sigma \preccurlyeq \tau$. An n-simplex in V is sometimes also called an *n-dimensional simplex*.

Theorem 2.1.8. *Let σ be a simplex in V and let $A = \operatorname{aff}(\sigma)$. Then every element $x \in A$ can be written uniquely as an affine combination $\sum_{v\in\sigma} t_v \cdot v$ of σ. In addition, the functions $\alpha_v \colon A \to \mathbb{R}$ defined by $\alpha_v(x) = t_v$ are affine.*

Corollary 2.1.9. *Let σ and τ be simplices in V such that $\tau \preccurlyeq \sigma$ and let*

$$A = \operatorname{aff}(\sigma), \quad B = \operatorname{aff}(\tau).$$

If $x \in B$ then $\alpha_v(x) = 0$ for every $v \in \sigma \setminus \tau$.

The real numbers $\alpha_v(x)$ for $v \in \sigma$ are called the *affine coordinates* of x with respect to σ. We call the α_v the *coordinate functions* of $\operatorname{aff}(\sigma)$. This notation will remain in force throughout the book.

Observe that by the above remarks and Theorem 2.1.8 it follows that the coordinate functions of $\operatorname{aff}(\sigma)$ are continuous on $\operatorname{aff}(\sigma)$.

A *geometric simplex* is the convex hull of a simplex. We use $|\sigma|$ as an abbreviation for $\operatorname{conv}(\sigma)$ and sometimes say that $|\sigma|$ is the geometric simplex spanned by σ. If $x \in |\sigma|$ then it can be written as a convex combination of the elements of σ (Lemma 1.1.1). But a convex combination is a special case of an affine combination, and affine coordinates are unique by Theorem 2.1.8. So we conclude that the affine coordinates of $x \in |\sigma|$ are non-negative.

The elements of σ are called the vertices of $|\sigma|$. If τ is a face of σ then $|\tau|$ is also called a face of $|\sigma|$. The union of all proper faces of $|\sigma|$ is called the *(geometric) boundary* $\partial|\sigma|$ of $|\sigma|$ and its complement is the (geometric) interior $|\sigma|^\circ$ of $|\sigma|$.

It is clear that the geometric interior of a simplex is non-empty. Just observe that a point in $|\sigma|$ all whose affine coordinates are strictly positive belongs to the geometric interior of $|\sigma|$.

Since a geometric simplex $|\sigma|$ is the convex hull of the finite set σ it follows that $|\sigma|$ is compact $\big($Lemma 1.1.1(2)$\big)$; this remark will be used without explicit reference in the remaining part of this book.

Theorem 2.1.10. *If σ is a simplex and $\sigma_1, \sigma_2 \subseteq \sigma$ then $|\sigma_1| \cap |\sigma_2| = |\sigma_1 \cap \sigma_2|$.*

The diameter of a geometric simplex is attained at its vertices, as the next result shows.

Theorem 2.1.11. *Let $(V, \|\cdot\|)$ be a normed linear space, and let $\sigma \subseteq V$ be a simplex. Then $\operatorname{diam}(\sigma) = \operatorname{diam}(|\sigma|)$.*

Application 1: The No-Retraction Theorem. It is clear that the function $r\colon \mathbb{J}^n \to B^n$ defined by

$$r(x) = \begin{cases} \frac{x}{\|x\|} & (\|x\| \geq 1), \\ x & (\|x\| \leq 1). \end{cases}$$

is a retraction. Consequently, B^n has the fixed-point property by Theorem 2.4.5 and Exercise A.12.10(2) (observe that in fact $\mathbb{J}^n$ and B^n are homeomorphic, cf. Exercise 1.1.24).

Theorem 2.4.10 (No-Retraction Theorem). *For every $n \in \mathbb{N}$, $\mathbb{S}^{n-1}$ is not a retract of B^n.*

Proof. To the contrary, suppose that $\mathbb{S}^{n-1}$ is a retract of B^n. As was remarked above, B^n has the fixed-point property. Consequently, $\mathbb{S}^{n-1}$ has the fixed-point property by Exercise A.12.10(2). However, the antipodal mapping on $\mathbb{S}^{n-1}$ clearly demonstrates that $\mathbb{S}^{n-1}$ does not have the fixed-point property. Contradiction. □

Corollary 2.4.11. *No $\mathbb{S}^n$ is contractible.*

Proof. Suppose that $\mathbb{S}^n$ is contractible. Then it is an AR by Corollaries 1.2.13 and 1.4.5. Consequently, there exists a retraction $r\colon B^n \to \mathbb{S}^{n-1}$, which contradicts Theorem 2.4.10. □

Application 2: The Theorem on Partitions. The following result is the basis for dimension theory (see Chapter 3).

The Theorem on Partitions 2.4.12. *Consider $\mathbb{J}^n$ and for $i \leq n$ its opposite faces*

$$A_i = \{x \in \mathbb{J}^n : x_i = -1\}, \quad B_i = \{x \in \mathbb{J}^n : x_i = 1\}.$$

If C_i is a partition between A_i and B_i for $i \leq n$ then $\bigcap_{i=1}^n C_i \neq \emptyset$.

Proof. To the contrary, assume that for every $i \leq n$, C_i is a partition between A_i and B_i such that $\bigcap_{i=1}^n C_i = \emptyset$. There are closed subsets E_i and F_i in $\mathbb{J}^n$ for $i \leq n$ such that

$$A_i \subseteq E_i, \quad B_i \subseteq F_i, \quad E_i \cup F_i = \mathbb{J}^n, \quad E_i \cap F_i = C_i.$$

By Exercise A.4.9 there exist continuous functions $\xi_i\colon \mathbb{J}^n \to \mathbb{J}$ such that

$$\xi_i[A_i] = \{1\}, \quad \xi_i[B_i] = \{-1\}, \quad \xi_i^{-1}(0) = C_i.$$

Define $f\colon \mathbb{J}^n \to \mathbb{J}^n$ by $f(x) = \big(\xi_1(x), \ldots, \xi_n(x)\big)$. Then f is continuous and the point $\underline{0}$ does not belong to its range. For every $x \in \mathbb{J}^n \setminus \{\underline{0}\}$ the ray from $\underline{0}$ through x intersects the 'boundary' $B = \bigcup_{i=1}^n A_i \cup \bigcup_{i=1}^n B_i$ of $\mathbb{J}^n$ in precisely one point, say $r(x)$. The function $r\colon \mathbb{J}^n \setminus \{\underline{0}\} \to B$ is easily seen to be continuous. The function $g = r \circ f\colon \mathbb{J}^n \to B$ has the following properties:

$$g\big[(-1,1)^n\big] \cap (-1,1)^n = \emptyset,$$

and for every $i \leq n$,

$$g[A_i] \subseteq B_i, \quad g[B_i] \subseteq A_i.$$

Therefore, g has no fixed-point, which contradicts Theorem 2.4.5. □

Corollary 2.4.13. *Consider the Hilbert cube Q and its opposite faces*

$$W_i^{-1} = \{x \in Q : x_i = -1\}, \quad W_i^1 = \{x \in Q : x_i = 1\}.$$

If C_i is a partition between W_i^{-1} and W_i^1 for every i then $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$.

Proof. For every m, define $f_m \colon \mathbb{J}^m \to Q$ by

$$f_m(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, 0, \dots).$$

Then f_m is clearly an imbedding. It is easily seen that $f_m^{-1}[C_i]$ is a partition between A_i and B_i for every $i \leq m$. Consequently, by Theorem 2.4.12,

$$\bigcap_{i=1}^{m} f_m^{-1}[C_i] \neq \emptyset.$$

By compactness of Q we therefore obtain that

$$\bigcap_{i=1}^{\infty} C_i \neq \emptyset,$$

which is as desired. □

Application 3: The Non Homogeneity Theorem.

Theorem 2.4.14. *Let $n \in \mathbb{N}$. Then*

(1) *if $A \subseteq \mathbb{S}^{n-1}$ then $B^n \setminus A$ is contractible.*
(2) *if $A \subseteq B^n \setminus \mathbb{S}^{n-1}$ is nonempty then $B^n \setminus A$ is not contractible.*

Proof. (1) Define $H \colon (B^n \setminus A) \times \mathbb{I} \to B^n \setminus A$ by $H(x, t) = (1 - t)x$. It is easily seen that H is a contraction. (Alternatively, apply Exercise 1.2.5 and Theorem 1.4.4.)

(2) Without loss of generality, $\underline{0} \in A$. Assume to the contrary that

$$H \colon (B^n \setminus A) \times \mathbb{I} \to B^n \setminus A$$

is a contraction. Define $F \colon \mathbb{S}^{n-1} \times \mathbb{I} \to \mathbb{S}^{n-1}$ by

$$F(x, t) = \frac{H(x, t)}{|H(x, t)|}.$$

Then F contracts $\mathbb{S}^{n-1}$ to a point which contradicts Corollary 2.4.11. □

Since $\mathbb{J}^n$ and B^n are homeomorphic, cf. Exercise 1.1.24, this yields:

Corollary 2.4.15. *Let $n \in \mathbb{N}$. Then $\mathbb{I}^n$ is not homogeneous.*

CHAPTER 3

Basic dimension theory

Dimension theory enables us to assign to every topological space X a number, $\dim X$, in

$$\{-1, 0, 1, \dots\} \cup \{\infty\}$$

having, among other things, the following properties:

(1) if X and Y are homeomorphic spaces then $\dim X = \dim Y$,
(2) $\dim \mathbb{R}^n = n$ for every $n \in \mathbb{N}$.

So $\dim X$ is a topological invariant of X, and by (2) it distinguishes between the euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$. A space X for which $\dim X < \infty$ is called *finite dimensional*, and a space is *infinite-dimensional* if it is not finite dimensional.

The aim of this chapter is to present some basic results from dimension theory. Some of the presented results are classical: they mostly concern finite dimensional spaces. However, during the last decades significant contributions were made concerning the topology of infinite-dimensional spaces and the topology of hereditarily indecomposable continua. We shall also present some of these results in detail.

3.1. The covering dimension

Let X be a space and let Γ be an index set. The Theorem on Partitions 2.4.12 motivates the following definition: a family of pairs of disjoint closed sets $\tau = \{(A_i, B_i) : i \in \Gamma\}$ of X is called *essential* if for every family

$$\{L_i : i \in \Gamma\},$$

where L_i is an arbitrary partition between A_i and B_i for every i, we have

$$\bigcap_{i \in \Gamma} L_i \neq \emptyset;$$

if τ is not essential then it is called *inessential*. So Theorem 2.4.12 and Corollary 2.4.13 show that $\mathbb{I}^n$ has an essential family of n pairs of disjoint closed sets and Q has essential families of size n for every $n \leq \aleph_0$. It follows easily that every subfamily of an essential family is again essential (Exercise 3.1.1).

Theorem 3.1.1. *$\mathbb{I}^n$ has an essential family consisting of n pairs of disjoint closed sets, but every family consisting of at least $n+1$ pairs of disjoint closed sets is inessential.*

Proof. By the above remarks we need only to consider the second part of the theorem.

Let A and B be disjoint closed subsets of $\mathbb{I}^n$ and let $E \subseteq \mathbb{I}$ be dense.

Claim 1. There is a partition D between A and B such that

$$D \subseteq \{x \in \mathbb{I}^n : (\exists\, i \leq n)(x_i \in E)\}.$$

Proof. This is easy. Every point $x \in A$ has a neighborhood of the form

$$\prod_{i=1}^{n} [a_i, b_i],$$

with $a_i, b_i \in E \cup \{0, 1\}$ for every $i \leq n$, which misses B. There is a finite family $\mathcal{F}$ of these neighborhoods whose union covers A, and the boundary D of this union is contained in the union of the boundaries of the elements of $\mathcal{F}$. We conclude that D is the required partition between A and B. ♢

Now let $\tau = \{(A_i, B_i) : i \leq n+1\}$ be an arbitrary family consisting of $n+1$ pairs of disjoint closed subsets of $\mathbb{I}^n$. We will prove that τ is inessential. Indeed, there exist $n+1$ pairwise disjoint dense subsets of $\mathbb{I}$. One can take for example

$$E_1 = (\sqrt{2} + \mathbb{Q}) \cap \mathbb{I}, \quad E_2 = (\sqrt{3} + \mathbb{Q}) \cap \mathbb{I}, \quad E_3 = (\sqrt{5} + \mathbb{Q}) \cap \mathbb{I}, \quad \ldots,$$

respectively. By the above there exist partitions D_i between A_i and B_i such that

$$D_i \subseteq \{x \in \mathbb{I}^n : (\exists\, j \leq n)(x_j \in E_i)\} \qquad (i \leq n+1).$$

Since the E_i are pairwise disjoint, clearly $\bigcap_{i=1}^{n+1} D_i = \emptyset$. □

Observe that in Theorem 3.1.1 we formulated a topological property of $\mathbb{I}^n$ shared by no $\mathbb{I}^m$ for $m \neq n$. In particular we obtain:

Corollary 3.1.2. *Let $n, m \in \mathbb{N}$. If $n \neq m$ then $\mathbb{I}^n \not\approx \mathbb{I}^m$.*

These remarks suggest the following definition: for a space X define its *covering dimension*, $\dim X \in \{-1, 0, 1, \ldots\} \cup \{\infty\}$, by

$$\begin{aligned}
\dim X = -1 &\iff X = \emptyset,\\
\dim X \leq n &\iff \text{every family of } n{+}1 \text{ pairs of disjoint closed subsets of } X \text{ is inessential},\\
\dim X = n &\iff \dim X \leq n \text{ and } \dim X \not\leq n-1,\\
\dim X = \infty &\iff \dim X \neq n \text{ for every } n \geq -1.
\end{aligned}$$

A space X with $\dim X < \infty$ is called *finite dimensional*; if it is not finite dimensional then it is called *infinite-dimensional*.

Observe that $\dim X \geq n$ if and only if there is an essential family consisting of n pairs of disjoint closed subsets of X. So in a sense, there are n different 'directions' in X. Also observe that it is not clear at all why we call $\dim X$ the covering dimension of X: the term *partition degree* seems to be more appropriate. We will explain our terminology later.

It is easy to see that if X and Y are homeomorphic spaces then X and Y have the same covering dimension.

In our new terminology, Theorem 3.1.1 reads as follows:

Theorem 3.1.3. $\dim \mathbb{I}^n = n$ *for every* n.

In §1.5 we called a space zero-dimensional if it is nonempty and has a base consisting of clopen sets. In Corollary 1.5.2 it was shown that a space X is zero-dimensional if and only if $\emptyset$ is a partition between any pair of disjoint closed subsets of X. So, in our new terminology, a space X is zero-dimensional in the sense of §1.5 if and only if $\dim X = 0$. It will turn out that zero-dimensional spaces are, in a sense, the 'building blocks' of all the other finite dimensional spaces.

Without too much trouble, the proof of Theorem 3.1.1 can be adapted to show that $\dim \mathbb{R}^n = n$ for every n. This equality however will turn out to follow trivially from Theorem 3.1.1 and results to be derived in §3.2. For that reason we will not verify it here.

For later use we shall now study some elementary properties of essential families of pairs of disjoint closed sets.

The following simple result is fundamental for dimension theory since it allows to extend partitions in subspaces to partitions in the whole space.

Lemma 3.1.4. *Let* Y *be a subspace of a space* X. *In addition, let* A *and* B *be disjoint closed subsets of* X. *If* U *and* V *are open neighborhoods of* A *and* B, *respectively, having disjoint closures, and if* S *is a partition in* Y *between* $Y \cap \overline{U}$ *and* $Y \cap \overline{V}$, *then there is a partition* T *in* X *between* A *and* B *such that* $T \cap Y \subseteq S$.

Proof. Write $Y \setminus S$ as the disjoint union of two open (in Y) sets E and F such that

$$Y \cap \overline{U} \subseteq E, \quad Y \cap \overline{V} \subseteq F.$$

Since $E \cap V = \emptyset$ we obtain $\overline{E} \cap B = \emptyset$, and similarly $\overline{F} \cap A = \emptyset$. From this it follows easily that $A \cup E$ and $B \cup F$ are separated. By Corollary A.8.2 there exist disjoint open neighborhoods U' and V' of $A \cup E$ and $B \cup F$, respectively. Clearly, $T = X \setminus (U' \cup V')$ is as required. □

For a closed subspace we can do a little better.

Corollary 3.1.5. *Let Y be a closed subspace of a space X. In addition, let A and B be disjoint closed subsets of X. If S is a partition in Y between the sets $Y \cap A$ and $Y \cap B$, then there is a partition T in X between A and B such that $T \cap Y \subseteq S$.*

Proof. Write $Y \setminus S$ as the disjoint union of two open (in Y) sets E and F such that

$$Y \cap A \subseteq E, \quad Y \cap B \subseteq F.$$

Observe that $S \cup F \cup B$ is closed in X and that $A \cap (S \cup F \cup B) = \emptyset$. By Corollary A.4.3 there is an open neighborhood U of A in X such that

$$\overline{U} \cap (S \cup F \cup B) = \emptyset.$$

By a similar argument it is possible to find an open neighborhood V of B in X such that

$$\overline{V} \cap \big(S \cup E \cup \overline{U}\big) = \emptyset.$$

By construction, S is a partition between $\overline{U} \cap Y$ and $\overline{V} \cap Y$. Now apply Lemma 3.1.4. □

Corollary 3.1.6. *Let Y be a zero-dimensional subspace of X. Then for all disjoint closed subsets A and B of X there exists a partition S between A and B in X such that $S \cap Y = \emptyset$.*

Proof. Let A and B be disjoint closed subsets of X and let U and V be disjoint neighborhoods of A and B, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$ (Corollary A.4.3). Since Y is zero-dimensional, $\emptyset$ is in Y a partition between the sets $\overline{U} \cap Y$ and $\overline{V} \cap Y$. Now apply Lemma 3.1.4. □

Corollary 3.1.7. *If a space X is the union of at most $n+1$ zero-dimensional subspaces then* $\dim X \leq n$.

Remark 3.1.8. We will prove later in Corollary 3.3.9 that the converse of this result is also true; so a space is at most n-dimensional if and only if it is the union of a family of $n + 1$ (not necessarily pairwise distinct) zero-dimensional subspaces.

Proof. Let $X = \bigcup_{i=1}^{n+1} X_i$ with $\dim X_i \leq 0$ for $i \leq n + 1$. Let

$$\tau = \{(A_i, B_i) : i \leq n + 1\}$$

be an arbitrary family of $n + 1$ pairs of disjoint closed subsets of X. By Corollary 3.1.6, for each $i \leq n + 1$ there exists a partition L_i between A_i and B_i in X such that $L_i \cap X_i = \emptyset$. Then $\bigcap_{i=1}^{n+1} L_i = \emptyset$ so that τ is inessential. We conclude that $\dim X \leq n$. □

These results enable us to prove the following:

Theorem 3.1.9. *Let X be a space, let $\{(A_i, B_i) : i \in \Gamma\}$ be essential in X, and let Γ_0 be a proper subset of Γ. If for every $i \in \Gamma_0$, L_i is a partition between A_i and B_i and*

$$L = \bigcap_{i \in \Gamma_0} L_i$$

then

$$\{(L \cap A_i, L \cap B_i) : i \in \Gamma \setminus \Gamma_0\}$$

is essential in L.

Proof. For $i \notin \Gamma_0$ let E_i be a partition in L between $L \cap A_i$ and $L \cap B_i$. By Corollary 3.1.5, for $i \notin \Gamma_0$ we can find a partition F_i in X between A_i and B_i such that $F_i \cap L \subseteq E_i$. Then

$$\emptyset \neq \bigcap_{i \in \Gamma_0} L_i \cap \bigcap_{i \in \Gamma \setminus \Gamma_0} F_i \subseteq L \cap \bigcap_{i \in \Gamma \setminus \Gamma_0} E_i = \bigcap_{i \in \Gamma \setminus \Gamma_0} E_i,$$

which is as required. □

Corollary 3.1.10. *Let X be a space and let $n \geq 0$. If $\dim X \geq n$ then there exist disjoint closed subsets A and B of X such that if L is a partition between A and B then $\dim L \geq n - 1$.*

Proof. If $n = 0$ then there is nothing to prove. So assume that $n \geq 1$. Since $\dim X \geq n$ there exists an essential family

$$\tau = \{(A_i, B_i) : i \leq n\}$$

in X. We claim that A_1 and B_1 are as required. Indeed, let L be a partition between A_1 and B_1. Observe that $L \neq \emptyset$ since τ is essential. We consider two cases. If $n = 1$ then since $L \neq \emptyset$, $\dim L \geq 0$ and so we are done. If $n > 1$ then by Theorem 3.1.9, L has an essential family of cardinality $n - 1$, namely the collection

$$\{(L \cap A_i, L \cap B_i) : i = 2, \ldots, n\}.$$

And so $\dim L \geq n - 1$. □

Exercises for §3.1. A space X is called *countable dimensional* if it can be written as the union of countably many zero-dimensional subspaces (cf. Corollary 3.1.7). In addition, a space X is called *strongly infinite-dimensional* if it has an infinite essential family. See e.g., §3.13 for more information on these notions.

Let X be a space with disjoint closed subsets A and B. We say that a closed subset S of X is an *irreducible partition* between A and B if S is a partition between A and B while no proper closed subset of S shares this property.

1. Prove every subfamily of an essential family is again essential.

2. Let $\tau = \{(A_i, B_i) : i \leq n\}$ be a family of pairs of disjoint closed subsets in the space X. Prove that if there exist different indices $i, j \leq n$ such that
$$(A_i, B_i) = (A_j, B_j)$$
then τ is inessential.

3. Let $\tau = \{(A_i, B_i) : i \leq n\}$ be a family of pairs of disjoint closed subsets in the space X. Prove that if τ is essential then X can be mapped continuously onto $\mathbb{I}^n$.

4. Let X be a compact space and let $\tau = \{(A_i, B_i) : i \in \Gamma\}$ be an essential family in X. Prove that Γ is countable.

5. Prove that if X is connected and contains more than one point then X is at least one-dimensional.

6. Prove that if X is strongly infinite-dimensional then X is not countable dimensional. Give an example of a strongly infinite-dimensional space. Assuming that every finite dimensional space is countable dimensional, give an example of an infinite-dimensional countable dimensional compact space.

7. Let X be a compact space, and let $\tau = \{(A_i, B_i) : i \in \Gamma\}$ be a family of pairs of disjoint closed subsets of X. Prove that τ is essential if and only if every finite subfamily is.

8. Let X be a space. Assume that $\tau = \{(A_i, B_i) : i \in \Gamma\}$ is an inessential family of pairs of disjoint closed subsets of X. Prove that for every subspace $E \subseteq X$ the family
$$\{(A_i \cap E, B_i \cap E) : i \in \Gamma\}$$
is inessential in E.

►9. Let X be a compact space and let $\tau = \{(A_i, B_i) : i \in \Gamma\}$ be an essential family of pairs of disjoint closed subsets of X. Prove that there is a component C of X such that the family
$$\tau \restriction C = \{(A_i \cap C, B_i \cap C) : i \in \Gamma\}$$
is essential in C. Conclude that every strongly infinite-dimensional compact space has a strongly infinite-dimensional component.

10. Prove that $2^{\mathbb{I}}$, the hyperspace of $\mathbb{I}$, is infinite-dimensional.

11. Let $f\colon X \to Y$ be continuous, and let $\tau = \{(A_i, B_i) : i \in I\}$ be a collection of pairs of disjoint closed sets in X. Assume that the collection
$$\hat{\tau} = \{(\overline{f[A_i]}, \overline{f[B_i]}) : i \in I\}$$
consists of pairs of disjoint sets and is inessential. Prove that τ is inessential.

12. Let $f\colon X \to Y$ be continuous, and let $\tau = \{(A_i, B_i) : i \in I\}$ be an inessential collection of pairs of disjoint closed sets in Y. Prove that the collection
$$\hat{\tau} = \{\ f^{-1}[A_i], f^{-1}[B_i]\ \ : i \in I\}$$

is inessential in X.

13. Let X be a space and let $\{(A_i, B_i) : i \in \Gamma\}$ be essential in X. Suppose that the set $Y \subseteq X$ is such that $Y \cap \bigcap_{i\in\Gamma} L_i \neq \emptyset$ for any choice of partitions L_i of A_i and B_i in X. For each $i \in \Gamma$ let U_i and V_i be disjoint closed neighborhoods of A_i and B_i, respectively. Prove that
$$\{(U_i \cap Y, V_i \cap Y) : i \in \Gamma\}$$
is essential in Y.

14. Let X be a zero-dimensional space containing the pairwise disjoint closed subsets A, B and C. Prove that C is a partition between A and B. (So in a zero-dimensional space every irreducible partition is empty.)

▶15. Give an example of a space X containing two disjoint closed subsets A and B such that no partition between A and B is irreducible.

▶16. Prove that the Brouwer Fixed-Point Theorem and the fact that there is a space X with $\dim X = \infty$ are equivalent statements, in the sense that they are easily deduced from each other. (This complements the list in Exercise 2.4.1 with an additional statement.)

3.2. Translation into open covers

We defined the covering dimension $\dim X$ in terms of properties of the family of all closed subsets of X. In this section we see that $\dim X$ is also describable in terms of properties of the family of all open covers of X. As an application of the results derived here we present a simple proof that $\mathbb{R}^n$ is n-dimensional for every n. We conclude from this that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

Let X be a space and let $\mathcal{A}$ and $\mathcal{B} = \{B(A) : A \in \mathcal{A}\}$ be families of subsets of X. We say that $\mathcal{B}$ is a *swelling* of $\mathcal{A}$ if

(1) for every $A \in \mathcal{A}$, $A \subseteq B(A)$,
(2) for every finite $\mathcal{F} \subseteq \mathcal{A}$, $\bigcap \mathcal{F} = \emptyset$ iff $\bigcap_{A\in\mathcal{F}} B(A) = \emptyset$.

Observe that if $B(A_0), B(A_1) \in \mathcal{B}$ are distinct then so are A_0 and A_1, but the converse need not hold.

Before we state the following proposition, let us recall that a locally finite collection is countable (see the remark on Page 486 following the definition of locally finite collection).

Proposition 3.2.1. *Let $\mathcal{F}$ be a locally finite collection of closed subsets of a space X. Then $\mathcal{F}$ has a swelling consisting of open subsets of X.*

Proof. We can enumerate $\mathcal{F}$ as $\{F_i : i \in \mathbb{N}\}$. Without loss of generality, assume that $F_1 = \emptyset$. By induction on $i \in \mathbb{N}$ we shall construct an open neighborhood U_i of F_i such that
$$\mathcal{F}_i = \{\overline{U}_1, \ldots, \overline{U}_i, F_{i+1}, F_{i+2}, \ldots\}$$

is a swelling of $\mathcal{F}$. Put $U_1 = \emptyset$, and assume that for some i the sets $U_1, \dots, U_i$ have been constructed. Observe that $\mathcal{F}_i$ is locally finite. Put

$$\mathcal{B} = \Big\{ \bigcap \mathcal{E} : \mathcal{E} \text{ is a finite subcollection of } \mathcal{F}_i \text{ and } \big(\bigcap \mathcal{E}\big) \cap F_{i+1} = \emptyset \Big\}.$$

Observe that $\mathcal{B}$ is locally finite as well and put $B = \bigcup \mathcal{B}$. Then B is closed by Exercise A.7.1 and clearly $B \cap F_{i+1} = \emptyset$. By Corollary A.4.3 there consequently exists an open neighborhood U_{i+1} of F_{i+1} the closure of which misses B. It is clear that the set U_{i+1} is as required.

We claim that $\{U_i : i \in \mathbb{N}\}$ is a swelling of $\mathcal{F}$. To this end, take arbitrary $i(1), \dots, i(n) \in \mathbb{N}$ and assume that $\bigcap_{j=1}^n F_{i(j)} = \emptyset$. Let

$$m = \max\{i(1), \dots, i(n)\}.$$

Then $\{U_1, \dots, U_m\}$ is a swelling of $\{F_1, \dots, F_m\}$ by our construction. We conclude that $\bigcap_{j=1}^n U_{i(j)} = \emptyset$. □

Corollary 3.2.2. *Let $\mathcal{F}$ be a locally finite family of closed subsets of a space X. Also, for every $F \in \mathcal{F}$ let $V(F)$ be a neighborhood of F. Then there exists a swelling $\{U(F) : F \in \mathcal{F}\}$ of $\mathcal{F}$ consisting of open subsets of X such that for every $F \in \mathcal{F}$ we have $\overline{U(F)} \subseteq V(F)$.*

Proof. By Proposition 3.2.1 there exists an 'open' swelling $\{W(F) : F \in \mathcal{F}\}$ of $\mathcal{F}$. By Corollary A.4.3 there exists for every $F \in \mathcal{F}$ an open neighborhood $U(F)$ such that $\overline{U(F)} \subseteq V(F) \cap W(F)$. We claim that $\{U(F) : F \in \mathcal{F}\}$ is as required. To this end, let $\mathcal{G} \subseteq \mathcal{F}$ be finite such that $\bigcap \mathcal{G} = \emptyset$. Then clearly $\bigcap_{F \in \mathcal{G}} W(F) = \emptyset$ from which it follows that $\bigcap_{F \in \mathcal{G}} U(F) = \emptyset$. □

We now show how finite closed collections can be modified.

Lemma 3.2.3. *Let $A, A_1, \dots, A_n$ be closed subsets of a space X. Moreover, let $V_1, \dots, V_n$ be open subsets of X with $A \cap A_i \subseteq V_i$ for every $i \leq n$. If*

$$\hat{A} = A \setminus \bigcup_{i=1}^n V_i, \quad \hat{A}_i = A_i \cup \overline{V_i}$$

for $i \leq n$ then

(1) $A \cup \bigcup_{i=1}^n A_i \subseteq \hat{A} \cup \bigcup_{i=1}^n \hat{A}_i$,
(2) $\hat{A} \cap \bigcap_{i=1}^n \hat{A}_i \subseteq \bigcap_{i=1}^n \operatorname{Fr} V_i$.

Proof. Observe that (1) is trivial. For (2), simply observe that for $i \leq n$,

$$\begin{aligned} \hat{A} \cap \hat{A}_i &\subseteq (A \setminus V_i) \cap (A_i \cup \overline{V_i}) \\ &= [(A \cap A_i) \setminus V_i] \cup \big[A \cap (\overline{V_i} \setminus V_i)\big] \\ &\subseteq \operatorname{Fr} V_i. \end{aligned}$$

So we are done. □

One should think of the previous triviality in the following way. The closed sets A are changed into the closed sets $\hat{A}$. But this is not done in an arbitrary way because (1) tells us that what is covered by the A's is also covered by the $\hat{A}$'s, and (2) tells us where the intersection of the $\hat{A}$'s can be found.

The results obtained so far have nice applications in dimension theory.

Corollary 3.2.4. *Let X be a space such that $\dim X \leq n < \infty$. Then for every countable open cover $\mathcal{U}$ of X and for every subcollection $\mathcal{F} \subseteq \mathcal{U}$ of cardinality $n+2$ there exists an open shrinking $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ of $\mathcal{U}$ having the following properties:*

(1) *for every $U \in \mathcal{U}$, $V(U) \subseteq \overline{V(U)} \subseteq U$,*
(2) $\bigcap_{U \in \mathcal{F}} V(U) = \emptyset$.

Proof. Enumerate $\mathcal{U}$ as $\{U_i : i \in \mathbb{N}\}$. Without loss of generality,

$$\mathcal{F} = \{U_1, \ldots, U_{n+2}\}.$$

By Proposition A.7.1 there exists a closed shrinking $\{B_i : i \in \mathbb{N}\}$ of $\mathcal{U}$. For every $i \leq n+1$ define $A_i = X \setminus U_i$. Then $\{(B_i, A_i) : i \leq n+1\}$ is a family of $n+1$ pairs of disjoint closed subsets of X. Since $\dim X \leq n$, there exist open sets $V_i \subseteq X$ for $i \leq n+1$ such that

(3) $B_i \subseteq V_i \subseteq \overline{V_i} \subseteq X \setminus A_i = U_i$,
(4) $\bigcap_{i=1}^{n+1} \operatorname{Fr} V_i = \emptyset$.

Now consider the closed sets $B_1, \ldots, B_{n+2}$. For $i \leq n+1$ put $\hat{B}_i = \overline{V_i}$ and let $\hat{B}_{n+2} = B_{n+2} \setminus \bigcup_{i=1}^{n+1} V_i$. Observe that the $\hat{B}$'s are the 'improved' sets we get from Lemma 3.2.3 by considering the collections of closed sets

$$B_{n+2}, B_1, \ldots, B_{n+1}$$

and open sets

$$V_1, \ldots, V_{n+1}.$$

So we get by (4) that

$$\bigcap_{i=1}^{n+2} \hat{B}_i \subseteq \bigcap_{i=1}^{n+1} \operatorname{Fr} V_i = \emptyset. \tag{5}$$

Observe that

$$\hat{\mathcal{B}} = \{\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_{n+2}, B_{n+3}, B_{n+4}, \ldots\}$$

covers X since $\bigcup_{i=1}^{n+2} B_i \subseteq \bigcup_{i=1}^{n+2} \hat{B}_i$. In addition, the first $n+2$ members of $\hat{\mathcal{B}}$ have empty intersection by (5). Since $\hat{B}_i \subseteq U_i$ for every $i \leq n+2$, the desired result now follows from Corollary 3.2.2. ☐

Let $\mathcal{U}$ be a cover of a space X and let $n \geq 0$ (we do not assume $\mathcal{U}$ to be open). We say that the *order* of $\mathcal{U}$ is at most n, $\operatorname{ord}(\mathcal{U}) \leq n$, if for every $x \in X$,

$$|\{U \in \mathcal{U} : x \in U\}| \leq n + 1.$$

We now come to the following central result.

Theorem 3.2.5. *Let X be a nonempty space and let $n \geq 0$. The following statements are equivalent:*

(1) $\dim X \leq n$,
(2) *for every open cover $\mathcal{U}$ of X there exists a locally finite closed refinement $\mathcal{V}$ of $\mathcal{U}$ such that* $\operatorname{ord}(\mathcal{V}) \leq n$,
(3) *for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ with* $\operatorname{ord}(\mathcal{V}) \leq n$,
(4) *for every open cover $\mathcal{U}$ of X there exists a closed shrinking $\mathcal{V}$ of $\mathcal{U}$ with* $\operatorname{ord}(\mathcal{V}) \leq n$,
(5) *for every open cover $\mathcal{U}$ of X there exists an open shrinking $\mathcal{V}$ of $\mathcal{U}$ with* $\operatorname{ord}(\mathcal{V}) \leq n$,
(6) *for every finite open cover $\mathcal{U}$ of X there exists a closed shrinking $\mathcal{V}$ of $\mathcal{U}$ such that* $\operatorname{ord}(\mathcal{V}) \leq n$,
(7) *for every finite open cover $\mathcal{U}$ of X there exists an open shrinking $\mathcal{V}$ of $\mathcal{U}$ such that* $\operatorname{ord}(\mathcal{V}) \leq n$.

Remark 3.2.6. This theorem explains the earlier terminology *covering dimension*. The vague idea behind dimension theory is that one wishes to cover a given space with finitely many 'small' open sets in such a way that as few of the open sets as possible have points in common. If $\dim X \leq n$ then it is possible to do that in such a way that at most $n + 1$ elements intersect. If $\dim X \geq n$ then no matter how the 'small' cover is chosen, there are always at least $n + 1$ elements having a point in common. It is interesting and intriguing that these ideas can be used to distinguish between various spaces topologically.

Remark 3.2.7. Missing in the list of statements in Theorem 3.2.5 is:

(8) *for every open cover $\mathcal{U}$ of X there exists a closed refinement $\mathcal{V}$ of $\mathcal{U}$ with* $\operatorname{ord}(\mathcal{V}) \leq n$.

This is not a very interesting statement, since every open cover $\mathcal{U}$ of a space X is refined by the closed cover $\big\{\{x\} : x \in X\big\}$ of X which has order 0. The following statement seems to be more interesting:

(8′) *for every open cover $\mathcal{U}$ of X there exists a countable closed refinement $\mathcal{V}$ of $\mathcal{U}$ with* $\operatorname{ord}(\mathcal{V}) \leq n$.

It is not true that (8′) characterizes all at most n-dimensional spaces. We will show in Exercises 3.2.7 and 3.2.8 that Erdős' space described in Example 1.5.18 has the property that every open cover $\mathcal{U}$ of it has a countable *disjoint* closed refinement, yet E is one-dimensional (as will be shown in Example 3.4.13). We do not know whether spaces that satisfy condition ($8'$) are at most $(n+1)$-dimensional.

Proof. *We prove* (1) ⇒ (2).

Since a refinement of a refinement is a refinement, by Corollary A.7.3 we may assume that $\mathcal{U}$ is locally finite. Enumerate $\mathcal{U}$ as $\{U_{0,i} : i \in \mathbb{N}\}$. In addition, let $\{F(j) : j \in \mathbb{N}\}$ enumerate the collection of all subsets of $\mathbb{N}$ of cardinality precisely $n+2$. By Corollary 3.2.4, for $j \in \mathbb{N}$ there exists an open cover $\mathcal{V}_j = \{U_{j,i} : i \in \mathbb{N}\}$ of X having the following properties:

(8) for each i, $U_{j,i} \subseteq \overline{U}_{j,i} \subseteq U_{j-1,i}$,
(9) $\bigcap_{i \in F(j)} U_{j,i} = \emptyset$.

Now for each $i \in \mathbb{N}$ define

$$S_i = \bigcap_{j=1}^{\infty} \overline{U}_{j,i}.$$

Claim 1. The collection $\mathcal{V} = \{S_i : i \in \mathbb{N}\}$ covers X.

Proof. Take an arbitrary $x \in X$. Since $\mathcal{U}$ is locally finite, x is contained in finitely many elements of $\mathcal{U}$ only. Pick $m \in \mathbb{N}$ such that $x \notin \bigcup_{i>m} U_{0,i}$. We conclude that for every $j \in \mathbb{N}$ there is an index $k(j) \leq m$ such that $x \in U_{j,k(j)}$. So there exists $k \leq m$ such that x belongs to infinitely many of the $U_{j,k}$. By (8) this implies that $x \in S_k$, which is as required. ♢

We conclude that $\mathcal{V}$ is a closed shrinking of $\mathcal{U}$ and hence is locally finite. Moreover, $\operatorname{ord}(\mathcal{V}) \leq n$ by (8) and (9).

We prove (2) ⇒ (3).

By (2) there exists a locally finite closed refinement $\mathcal{S}$ of $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{S}) \leq n$. For each $S \in \mathcal{S}$ pick $U(S) \in \mathcal{U}$ containing S. By Corollary 3.2.2 there exists an open swelling $\mathcal{V} = \{V(S) : S \in \mathcal{S}\}$ of $\mathcal{S}$ such that for every element $S \in \mathcal{S}$ we have $V(S) \subseteq U(S)$. Since $\operatorname{ord}(\mathcal{S}) \leq n$ we get $\operatorname{ord}(\mathcal{V}) \leq n$, and so $\mathcal{V}$ is as required.

We prove (3) ⇒ (5).

By (3) there exists an open refinement $\mathcal{W}$ of $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{W}) \leq n$. For each $W \in \mathcal{W}$ pick $U(W) \in \mathcal{U}$ be such that $W \subseteq U(W)$. Now for each $U \in \mathcal{U}$ define

$$V(U) = \bigcup \{W \in \mathcal{W} : U(W) = U\}.$$

Clearly, $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ is an open shrinking of $\mathcal{U}$ and it therefore suffices to prove that $\operatorname{ord}(\mathcal{V}) \leq n$. Take pairwise distinct

$$V(U_1), \ldots, V(U_{n+2}) \in \mathcal{V}$$

and assume that $x \in \bigcap_{i=1}^{n+2} V(U_i)$. For each $i \le n+2$ there exists $W_i \in \mathcal{W}$ such that $U(W_i) = U_i$ and $x \in W_i$. Since the U_i are pairwise distinct, this implies that the W_i are pairwise distinct. But $\mathrm{ord}(\mathcal{W}) \le n$, and so $x \in \bigcap_{i=1}^{n+2} W_i = \emptyset$. This is a contradiction.

We prove (5) $\Rightarrow$ (4).

Observe that if $\mathcal{A}$ is a cover of X such that $\mathrm{ord}(\mathcal{A}) \le n$ and $\mathcal{B}$ is a shrinking of $\mathcal{A}$ then $\mathrm{ord}(\mathcal{B}) \le n$. So (4) follows directly from (5) and Corollary 3.2.4.

We prove (4) $\Rightarrow$ (6).

This is a triviality.

We prove (6) $\Rightarrow$ (7).

This follows immediately from Corollary 3.2.2.

We prove (7) $\Rightarrow$ (1).

Let $\{(A_i, B_i) : i \le n+1\}$ be a family of $n+1$ pairs of disjoint closed subsets of X. Observe that

$$\bigcap_{i=1}^{n+1} A_i \cap \bigcup_{i=1}^{n+1} B_i = \emptyset$$

so that the collection

$$\Big\{X \setminus A_1, X \setminus A_2, \ldots, X \setminus A_{n+1}, X \setminus \bigcup_{i=1}^{n+1} B_i\Big\}$$

covers X. By (7) there consequently exists an open cover $\mathcal{V} = \{V_i : i \le n+2\}$ of X such that

(10) $V_i \subseteq X \setminus A_i \qquad (i \le n+1)$,
(11) $V_{n+2} \subseteq X \setminus \bigcup_{i=1}^{n+1} B_i$,
(12) $\mathrm{ord}(\mathcal{V}) \le n$.

Let $\mathcal{F} = \{F_i : i \le n+2\}$ be a closed shrinking of $\mathcal{V} = \{V_i : i \le n+2\}$ (Proposition A.7.1). Now for $i \le n+1$, define $\hat{A}_i$ and $\hat{B}_i$ by

$$\hat{A}_i = A_i \cup (F_{n+2} \setminus V_i),$$
$$\hat{B}_i = B_i \cup F_i.$$

Claim 2. For every $i \le n+1$, $\hat{A}_i \cap \hat{B}_i = \emptyset$.

Proof. Simply observe that $A_i \cap B_i = \emptyset$, that $F_i \subseteq V_i$ from which it follows that $A_i \cap F_i = \emptyset$ and $(F_{n+2} \setminus V_i) \cap F_i = \emptyset$ and that $F_{n+2} \cap B_i = \emptyset$. $\diamondsuit$

Claim 3. $\bigcup_{i=1}^{n+1} (\hat{A}_i \cup \hat{B}_i) = X$.

Proof. Since $\mathcal{F}$ covers X, we are done if we show that $\bigcup_{i=1}^{n+1} F_i \subseteq \bigcup_{i=1}^{n+1} \hat{B}_i$ and $F_{n+2} \subseteq \bigcup_{i=1}^{n+1} \hat{A}_i$. The first inclusion is a triviality. For the second one observe that $F_{n+2} \subseteq V_{n+2}$ and that by (12), $V_{n+2} \cap \bigcap_{i=1}^{n+1} V_i = \emptyset$. We conclude that

$$F_{n+2} = F_{n+2} \setminus \bigcap_{i=1}^{n+1} V_i \subseteq \bigcup_{i=1}^{n+1} \hat{A}_i,$$

which proves the claim. ♢

By Corollary A.4.3 there exists for every $i \le n+1$ a partition L_i between $\hat{A}_i$ and $\hat{B}_i$. Since $A_i \subseteq \hat{A}_i$ and $B_i \subseteq \hat{B}_i$ for every i, L_i is also a partition between the sets A_i and B_i. By Claim 3 we obtain

$$\bigcap_{i=1}^{n+1} L_i \subseteq X \setminus \bigcup_{i=1}^{n+1} \big(\hat{A}_i \cup \hat{B}_i\big) = X \setminus X = \emptyset,$$

as required. □

Naturally, the reader wonders about the relation between the dimension of a space X and the various dimensions of its subspaces. We finish this section by deriving two results about that relation.

The Countable Closed Sum Theorem 3.2.8. *If X can be covered by countably many closed and at most n-dimensional sets then X is at most n-dimensional.*

Proof. It is clear that without loss of generality we may assume that $n < \infty$. Enumerate the closed cover $\mathcal{F}$ as $\{F_i : i \in \mathbb{N}\}$, and put $F_0 = \emptyset$. Let $\mathcal{U}$ be a finite open cover of X. By induction on $i \ge 0$ we shall construct an open cover $\mathcal{U}(i)$ of X such that the following conditions are satisfied:

(1) $\mathcal{U}(0) = \mathcal{U}$,
(2) if $0 \le j < i$ then $\overline{\mathcal{U}(i)}$ is a shrinking of $\mathcal{U}(j)$,
(3) $\operatorname{ord}(\overline{\mathcal{U}(i)} \restriction F_i) \le n$.

Observe that conditions (2) and (3) are satisfied for $i = 0$. Now assume that we completed the construction for all j with $0 \le j < i$. Since $\dim F_i \le n$, by assumption, the open cover $\mathcal{U}(i-1) \restriction F_i$ of F_i has an open shrinking

$$\mathcal{V} = \{V(U) : U \in \mathcal{U}(i-1)\}$$

of order at most n (Theorem 3.2.5). For each $U \in \mathcal{U}(i-1)$ put

$$W(U) = (U \setminus F_i) \cup V(U).$$

It is clear that

$$\mathcal{W} = \{W(U) : U \in \mathcal{U}(i-1)\}$$

is an open shrinking of $\mathcal{U}(i-1)$ such that $\operatorname{ord}(\mathcal{W} \restriction F_i) \le n$. An easy application of Proposition A.7.1 and Corollary 3.2.2 now shows that there exists an open cover $\mathcal{U}(i)$ of X satisfying (2) and (3). This completes the inductive construction.

Let the cardinality of $\mathcal{U}$ be $k < \infty$. It is clear that for $i \in \mathbb{N}$ we can enumerate $\mathcal{U}(i)$ as $\{U_{m,i} : m \leq k\}$ such that for $0 \leq i < j$ and $m \leq k$ we have $\overline{U}_{m,j} \subseteq U_{m,i}$. Now for each $x \in X$ there exists $m(x) \leq k$ such that x belongs to infinitely many of the $U_{m(x),i}$. So by construction we have

$$x \in \bigcap_{i=1}^{\infty} U_{m(x),i}.$$

By (2) and (3) this implies that

$$\Big\{\bigcap_{i=1}^{\infty} U_{m,i} : m \leq k\Big\} = \Big\{\bigcap_{i=1}^{\infty} \overline{U}_{m,i} : m \leq k\Big\}$$

is a closed shrinking of $\mathcal{U}$ and has order at most n. Consequently, $\dim X \leq n$ by Theorem 3.2.5. □

Since $\mathbb{R}^n$ is a countable union of homeomorphs of $\mathbb{I}^n$, Theorems 3.1.1 and 3.2.8 imply that $\dim \mathbb{R}^n \leq n$. In the remaining part of this section we shall present, among other things, two additional proofs of this inequality which are interesting in their own rights.

The Subspace Theorem 3.2.9. *Let A be a subspace of a space X. Then $\dim A \leq \dim X$.*

Proof. If $\dim X = \infty$ then there is nothing to prove. So without loss of generality assume that $n = \dim X < \infty$.

First assume that A is closed. Observe that if τ is an essential family in A then it is also an essential family in X. From this it follows immediately that $\dim A \leq n$.

Next, assume that A is open. Observe that A is an F_σ-subset of X by Exercise A.2.3. That $\dim A \leq n$ therefore follows from the above and the Countable Closed Sum Theorem 3.2.8.

Finally, assume that A is an arbitrary subspace of X. In addition, let $\mathcal{U}$ be a cover of A by sets that are open in A. For each $U \in \mathcal{U}$ pick an open subset $V(U)$ of X with $V(U) \cap A = U$. Put $V = \bigcup_{U \in \mathcal{U}} V(U)$. Then V is an open subset of X and hence, by the above, $\dim V \leq n$. Since

$$\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$$

covers V, Theorem 3.2.5 yields the existence of an open refinement $\mathcal{W}$ of $\mathcal{V}$ with $\operatorname{ord}(\mathcal{W}) \leq n$. Then $\mathcal{X} = \mathcal{W} \restriction A$ is an open refinement of $\mathcal{U}$ with order at most n. Another application of Theorem 3.2.5 now gives us that $\dim A \leq n$, as required. □

A different proof of Theorem 3.2.9 shall be given in Corollary 3.4.14(1)2.

Observe that since $\mathbb{R}^n$ is homeomorphic to $(0,1)^n$ we have by Theorems 3.1.1 and 3.2.9 that $\dim \mathbb{R}^n \leq n$.

The results in this section can also be used to prove that various spaces are zero-dimensional. We shall demonstrate this by the following example. For all $n \in \mathbb{N}$ and $0 \leq m \leq n$ let

$$\mathfrak{R}_{n,m} = \{x \in \mathbb{R}^n : \text{exactly } m \text{ coordinates of } x \text{ are rational}\}.$$

Proposition 3.2.10. *Let $n \in \mathbb{N}$. Then $\mathbb{R}^n = \bigcup_{m=0}^n \mathfrak{R}_{n,m}$ and $\mathfrak{R}_{n,m}$ is zero-dimensional for all $0 \leq m \leq n$.*

Proof. The first part of this proposition is trivial. For the second part, observe that if $m = 0$ then there is nothing to prove since $\mathfrak{R}_{n,0}$ is the product of n copies of $\mathbb{P}$. So let $m \geq 1$ and let $A = \{i(1), \dots, i(m)\}$ be a set of m indices in $\{1, 2, \dots, n\}$ and let $q_1, \dots, q_m \in \mathbb{Q}$. Put

$$X = \{x \in \mathbb{R}^n : x_{i(j)} = q_j \text{ for every } j \leq m\}.$$

Observe that X is a closed subspace of $\mathbb{R}^n$ and that

$$X \cap \mathfrak{R}_{n,m} = \{x \in \mathbb{R}^n : x_{i(j)} = q_j \text{ for every } j \leq m \text{ and } x_i \in \mathbb{P} \text{ for } i \notin A\}.$$

Consequently, $X \cap \mathfrak{R}_{n,m}$ is a closed subspace of $\mathfrak{R}_{n,m}$ which is homeomorphic to the product of $n - m$ copies of $\mathbb{P}$. We conclude that $X \cap \mathfrak{R}_{n,m}$ is zero-dimensional. Since $\mathfrak{R}_{n,m}$ is the union of countably many sets of the above form $X \cap \mathfrak{R}_{n,m}$, we conclude that $\dim \mathfrak{R}_{n,m} = 0$ by Theorem 3.2.8. □

Corollary 3.2.11. *For all $n \in \mathbb{N}$ and $0 \leq m \leq n$,*

$$\dim\{x \in \mathbb{R}^n : \textit{at most } m \textit{ coordinates of } x \textit{ are rational}\} \leq m.$$

Proof. Observe that the set of all $x \in \mathbb{R}^n$ for which at most m coordinates are rational coincides with

$$\mathfrak{R}_{n,0} \cup \mathfrak{R}_{n,1} \cup \cdots \cup \mathfrak{R}_{n,m}.$$

So we are done by Proposition 3.2.10 and Corollary 3.1.7. □

It therefore again follows that $\dim \mathbb{R}^n \leq n$.

We now come to what is sometimes called the 'Fundamental Theorem of Dimension Theory'.

Theorem 3.2.12. *For all $n \in \mathbb{N}$,*

$$\dim \mathbb{R}^n = \dim \mathbb{I}^n = \dim \mathbb{S}^n = n.$$

Proof. By Theorem 3.1.1, $\dim \mathbb{I}^n = n$. By Theorem 3.2.9 this implies that $\dim \mathbb{R}^n \geq n$. Since $\dim \mathbb{R}^n \leq n$, we obtain $\dim \mathbb{R}^n = n$. Also, $\mathbb{S}^n$ is the union of two homeomorphs of $\mathbb{I}^n$, so an appeal to Theorem 3.2.8 gives us that $\dim \mathbb{S}^n \leq n$. It now follows easily that $\dim \mathbb{S}^n = n$ as well. □

Theorem 3.2.12 is of fundamental importance since it confirms our geometric intuition that $\mathbb{R}^n$, $\mathbb{I}^n$ and $\mathbb{S}^n$ are n-dimensional. Since the covering dimension is a topological notion, we also obtain the following

Corollary 3.2.13. *If $n \neq m$ then $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.*

We finish this section with three characterizations of dimension, two of which are in the spirit of Theorem 3.4.4 and one of which is 'combinatorial'.

Let τ be an n-dimensional simplex in a linear space L. Exercise 1.1.24 gives us that τ is homeomorphic to $\mathbb{I}^n$, hence $\dim \tau = n$. Now let $X = |\mathcal{T}|$ be a polytope. Since $\mathcal{T}$ is countable, by the Countable Closed Sum Theorem 3.2.8 and the above remark we get

$$\begin{aligned}\dim X &= \sup\{\dim \tau : \tau \in \mathcal{T}\}\\ &= \sup\{k : \tau \text{ is a } k\text{-simplex}, \tau \in \mathcal{T}\}.\end{aligned}$$

We conclude that the topological dimension of X equals its 'combinatorial' dimension.

Theorem 3.2.14. *Let X be a nonempty space and let $n \geq 0$. The following statements are equivalent:*

(1) $\dim X \leq n$,
(2) *for every open cover $\mathcal{U}$ of X there exists a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that such that* $\operatorname{ord}(\mathcal{V}) \leq n$,
(3) *for every open cover $\mathcal{U}$ of X there exists a star-finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that such that* $\operatorname{ord}(\mathcal{V}) \leq n$,
(4) *for every open cover $\mathcal{U}$ of X there exist a polytope P such that $\dim P \leq n$ and a $\mathcal{U}$-mapping $f \colon X \to P$.*

Proof. We prove (1) $\Rightarrow$ (3).

Without loss of generality, $\mathcal{U}$ is star-finite (Theorem 2.3.5(1)). Now by Theorem 3.2.5 there exists an open shrinking $\mathcal{V}$ of $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{V}) \leq n$. Since each shrinking of a star-finite cover is clearly star-finite, we are done.

We prove (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

That (3) $\Rightarrow$ (2) is a triviality and that (2) $\Rightarrow$ (1) follows from Theorem 3.2.5.

We prove (3) $\Rightarrow$ (4).

Let $\mathcal{U}$ be an open cover of X. By (3) there exists a (countable) star-finite open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{V}) \leq n$. Let $P = |N(\mathcal{V})|$. Then

$$\kappa \colon X \to P$$

is a $\mathcal{V}$-mapping by Corollary 2.3.4, and hence a $\mathcal{U}$-mapping. Also, P is a polytope by Theorem 2.3.5. Finally, since $\operatorname{ord}(\mathcal{V}) \leq n$, $N(\mathcal{V})$ has no simplices of dimension greater than n. By the Countable Closed Sum Theorem 3.2.8 it follows that $\dim P \leq n$.

We prove (4) $\Rightarrow$ (1).

Let $\mathcal{U}$ be an open cover of X. By (4) there exists a polytope P such that $\dim P \leq n$ and a $\mathcal{U}$-mapping $f\colon X \to P$. For each $U \in \mathcal{U}$ let $E(U)$ be the set of all $y \in P$ with an open neighborhood V_y such that $f^{-1}[V_y] \subseteq U$. Then the collection $\mathcal{E} = \{E(U) : U \in \mathcal{U}\}$ is an open cover of P while moreover

$$f^{-1}[E(U)] \subseteq U$$

for every $U \in \mathcal{U}$. Since $\dim P \leq n$, Theorem 3.2.5 implies the existence of an open shrinking $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ of $\mathcal{E} = \{E(U) : U \in \mathcal{U}\}$ such that $\operatorname{ord}(\mathcal{V}) \leq n$. It is clear that the collection $\mathcal{W} = \{f^{-1}[V(U)] : U \in \mathcal{U}\}$ is an open shrinking of $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{W}) \leq n$. By Theorem 3.2.5 we therefore conclude that $\dim X \leq n$. □

Exercises for §3.2. A space X is called *almost zero-dimensional* if it has an open base $\mathcal{B}$ such that every $B \in \mathcal{B}$ has the property that $X \setminus \overline{B}$ is the union of clopen subsets of X.

1. Let X be a space containing more than one point and let $p \in X$. Prove that $\dim(X \setminus \{p\}) = \dim X$. (So the dimension cannot be raised by the adjunction of a single point.)
2. Let $(X_n, f_n)_n$ be an inverse sequence of compact spaces such that
$$\dim X_n \leq m < \infty$$
for all n. Prove that the inverse limit of the sequence is at most m-dimensional. (The compactness condition is superfluous in this result. But we are not in the position yet to prove this. See Exercise 3.5.2 for more information.)

▶3. Let X be a dense-in-itself space such that $\dim X \leq n < \infty$. Prove that for every cover $\{U_1, \ldots, U_m\}$ consisting of nonempty open sets there is a collection $\{F_1, \ldots, F_m\}$ consisting of closed sets having the following properties:
   (1) $\emptyset \neq F_i \subseteq U_i$ for every $i \leq m$,
   (2) $\bigcup_{i=1}^{m} F_i = X$,
   (3) if $i_1 < i_2 < \cdots < i_{n+2} \leq m$ are arbitrary then $\bigcap_{j=1}^{n+2} F_{i_j} = \emptyset$.

▶4. Prove that every n-dimensional compact space has a component which is also n-dimensional.

5. Let X be a space. Prove that X is almost zero-dimensional if and only if X has a base $\mathcal{B}$ such that for each pair G, H of elements of $\mathcal{B}$ with disjoint closures there is a clopen set W in X with $G \subseteq W \subseteq X \setminus H$.
6. Prove that every almost zero-dimensional space is totally disconnected.

▶7. Prove that Erdős' space E is almost zero-dimensional.

8. Prove that if X is almost zero-dimensional then every open cover of X has a countable refinement by pairwise disjoint closed sets.
9. Prove that $\mathcal{C}(\mathbb{I}^2)$ is infinite-dimensional.
10. Let X be a homogeneous space. Prove that every nonempty open subspace of X has the same dimension as X.

3.3. The imbedding theorem

The aim of this section is to prove that every space X with $\dim X \leq n$ can be imbedded in $\mathbb{R}^{2n+1}$. This is a result of fundamental importance.

For $n \geq 0$, define

$$\mathfrak{N}_n = \{x \in \mathbb{R}^{2n+1} : \text{ at most } n \text{ coordinates of } x \text{ are rational}\}.$$

So

$$\mathfrak{N}_n = \bigcup_{k=0}^{n} \mathfrak{R}_{2n+1,k},$$

where the sets $\mathfrak{R}_{2n+1,k} \subseteq \mathbb{R}^{2n+1}$ are as in the previous section.

Observe that $\mathfrak{N}_0$ is the space of irrational numbers $\mathbb{P}$. The space $\mathfrak{N}_n$ is called *Nöbeling's universal n-dimensional space*. For later use, we do not aim at imbeddings in $\mathbb{R}^{2n+1}$ but at imbeddings in $\mathfrak{N}_n$. This does not require extra work and explains our terminology.

Lemma 3.3.1. *For $n \geq 0$, $\dim \mathfrak{N}_n = n$ and $\mathfrak{N}_n$ is the union of $n+1$ zero-dimensional subspaces.*

Proof. By Corollary 3.2.11 we get $\dim \mathfrak{N}_n \leq n$. To see that $\dim \mathfrak{N}_n \geq n$, we claim that $\mathfrak{N}_n$ contains a homeomorph of $\mathbb{I}^n$. This is easy. Define an imbedding $i\colon \mathbb{I}^n \to \mathfrak{N}_n$ by

$$i(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sqrt{2}, \sqrt{2}, \ldots, \sqrt{2}).$$

By Theorems 3.2.9 and 3.2.12 we now obtain $\dim \mathfrak{N}_n \geq n$.

Since, as observed above, $\mathfrak{N}_n = \mathfrak{R}_{2n+1,0} \cup \mathfrak{R}_{2n+1,1} \cup \cdots \cup \mathfrak{R}_{2n+1,n}$, an application of Proposition 3.2.10 gives us that $\mathfrak{N}_n$ is the union of $n+1$ zero-dimensional subspaces. □

A finite subset $F = \{x_1, \ldots, x_n\}$ of $\mathbb{R}^m$ is said to be in *general position* if for every collection

$$i(0) < i(1) < \cdots < i(k)$$

of at most $m+1$ indices in $\{1, \ldots, n\}$, the set

$$\{x_{i(0)}, \ldots, x_{i(k)}\}$$

is geometrically independent, cf. §2.1.

Lemma 3.3.2. *Let $G = \{x_1, \dots, x_n\}$ and $F = \{y_1, \dots, y_k\}$ be subsets of $\mathbb{R}^m$ such that G is in general position, and let $\varepsilon > 0$. Then for each $i \le k$ there exists a point $x_{n+i} \in \mathbb{R}^m$ such that*

(1) $\|y_i - x_{n+i}\| < \varepsilon$,
(2) $\{x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}\}$ *is in general position.*

Proof. It is clear that the lemma follows by induction once we have established it for $k = 1$. So we assume without loss of generality that $k = 1$.

For every nonempty subset $F \subseteq \{1, \dots, n\}$ of cardinality at most m, let

$$v(F) = \operatorname{aff}(F),$$

the affine hull of F. Each $v(F)$ is a nowhere dense closed subset of $\mathbb{R}^m$ since $|F| \le m$. Since $G = \{x_1, \dots, x_n\}$ is finite, there consequently exists a point

$$x_{n+1} \in \mathbb{R}^m \setminus \bigcup\{v(F) : F \subseteq \{1, \dots, n\}, |F| \le m\}$$

such that $\varrho(y_1, x_{n+1}) < \varepsilon$. We claim that $\{x_1, \dots, x_{n+1}\}$ is in general position. Since G is, we need only prove that if $F \subseteq \{1, \dots, n\}$ has cardinality at most m then $\{x_i : i \in F\} \cup \{x_{n+1}\}$ is geometrically independent. This is clear however since by construction, $x_{n+1} \notin v(F)$ (Theorem 2.1.6). □

Now for the remaining part of this section, let X be a fixed space such that $0 \le \dim X \le n$.

Proposition 3.3.3. *Let $f \in C_\varrho(X, \mathbb{R}^{2n+1})$, let $\varepsilon > 0$, and let H be an at most n-dimensional affine subspace of $\mathbb{R}^{2n+1}$. For every finite open cover $\mathcal{U}$ of X there exists $g \in C_\varrho(X, \mathbb{R}^{2n+1})$ such that*

(1) $\hat{\varrho}(f, g) < \varepsilon$,
(2) $\overline{g[X]} \cap H = \emptyset$,
(3) g *is a $\mathcal{U}$-mapping.*

Proof. Since f is bounded, $\overline{f[X]}$ is compact. There consequently exists a finite open cover $\mathcal{V}$ of it such that $\operatorname{mesh}(\mathcal{V}) < \varepsilon/4$. Let $\mathcal{W}$ be the common refinement of $\mathcal{U}$ and $f^{-1}[\mathcal{V}]$. Observe that $\mathcal{W}$ is finite. Since $\dim X \le n$, there exists by Theorem 3.2.5 a finite open shrinking $\mathcal{E}$ of $\mathcal{W}$ of order at most n. The essential properties of the cover $\mathcal{E}$ are the following ones:

(4) $\mathcal{E}$ has order at most n,
(5) $\mathcal{E} < \mathcal{U}$,
(6) for each $E \in \mathcal{E}$, $\operatorname{diam}(f[E]) < \varepsilon/4$.

We assume without loss of generality that every member from $\mathcal{E}$ is nonempty.

Let $\mathcal{E} = \{E_1, \dots, E_m\}$. Now for each $i \le m$, pick an arbitrary point

$$z_i \in f[E_i].$$

There is a geometrically independent set $S = \{h_1, \dots, h_k\} \subseteq \mathbb{R}^{2n+1}$ of cardinality at most $n+1$ such that S spans H. By Lemma 3.3.2, for each $i \le m$ there exists $y_i \in \mathbb{R}^{2n+1}$ such that

(7) $\varrho(y_i, z_i) < \varepsilon/4$,
(8) $\{h_1, \dots, h_k, y_1, \dots, y_m\}$ is in general position.

Let $\kappa_1, \dots, \kappa_m \colon X \to \mathbb{I}$ denote the κ-functions with respect to the cover $\mathcal{E}$. Define $g\colon X \to \mathbb{R}^{2n+1}$ by

$$g(x) = \sum_{i=1}^{m} \kappa_i(x) \cdot y_i.$$

Then g is clearly continuous and we claim that it is as required. First observe that g is bounded because its range is contained in the simplex spanned by the vectors $\{y_1, \dots, y_m\}$ which is compact by Lemma 1.1.1(2).

For $F \subseteq \{1, 2, \dots, m\}$ we let $\vec{F}$ denote $\{y_i : i \in F\}$.

For every $x \in X$ let $F_x = \{i \le m : x \in E_i\}$. Observe that $\kappa_i(x) > 0$ if and only if $i \in F_x$ so that $\sum_{i \in F_x} \kappa_i(x) = 1$.

It follows by (4) and (8) that $\vec{F_x}$ is geometrically independent. Consequently, the $\kappa_i(x)$, $i \in F_x$, are the affine coordinates of $g(x)$ with respect to $\vec{F_x}$. In particular, $g(x) \in \operatorname{aff}(\vec{F_x})$.

Claim 1. $\hat{\varrho}(f, g) < \varepsilon$.

Proof. Fix an arbitrary $x \in X$. It follows by (6) and (7) that $\|f(x) - y_i\| < \varepsilon/2$ for every $i \in F_x$. Consequently,

$$\begin{aligned} \|f(x) - g(x)\| &= \Big\| \sum_{i \in F_x} \kappa_i(x) \cdot f(x) - \sum_{i \in F_x} \kappa_i(x) \cdot y_i \Big\| \\ &\le \sum_{i \in F_x} \kappa_i(x) \|f(x) - y_i\| \\ &< \sum_{i \in F_x} \kappa_i(x) \cdot \varepsilon/2 \\ &= \varepsilon/2. \end{aligned}$$

We conclude that $\hat{\varrho}(f, g) \le \varepsilon/2 < \varepsilon$, which is as required. $\diamondsuit$

Claim 2. $g[X] \subseteq \overline{g[X]} \subseteq \bigcup_{x \in X} \operatorname{aff}(\vec{F_x}) \subseteq \mathbb{R}^{2n+1} \setminus H$.

Proof. Let $x \in X$. By (4), $|F_x| \le n+1$ and so if we put $E = \vec{F_x} \cup S$ then

$$|E| \le (n+1) + (n+1) = 2n+2.$$

By (8) this implies that E is geometrically independent. Now observe that by Proposition 2.1.5 we have

$$\operatorname{aff}(\vec{F_x}) \cap H = \operatorname{aff}(\vec{F_x}) \cap \operatorname{aff}(S) = \operatorname{aff}(\vec{F_x} \cap S) = \operatorname{aff}(\emptyset) = \emptyset.$$

Since $g(x) \in \operatorname{aff}(\vec{F}_x)$ and x is arbitrary it therefore follows that

$$g[X] \subseteq \bigcup_{x \in X} \operatorname{aff}(\vec{F}_x) \subseteq \mathbb{R}^{2n+1} \setminus H.$$

But since $\operatorname{aff}(\vec{F}_x)$ is closed by Exercise 2.1.8 and there are only finitely many F_x's, we even have that

$$g[X] \subseteq \overline{g[X]} \subseteq \bigcup_{x \in X} \operatorname{aff}(\vec{F}_x) \subseteq \mathbb{R}^{2n+1} \setminus H,$$

as desired. ◊

It remains to prove that g is a $\mathcal{U}$-mapping.

Let $p \in \overline{g[X]}$. By Claim 2 there is a subset $G_p \subseteq \{1, 2, \ldots, m\}$ of cardinality at most $n+1$ such that $p \in \operatorname{aff}(\vec{G}_p)$. We may assume that G_p is minimal with respect to this property. Since $\vec{G}_p$ is geometrically independent, the affine coordinates of p with respect to $\vec{G}_p$ are all non-zero.

In this way we associated to every element $p \in \overline{g[X]}$ a subset of indices $G_p \subseteq \{1, 2, \ldots, m\}$.

Claim 3. Let $p \in \overline{g[X]}$ and $i_0 \in G_p$. If $A \subseteq \{y_1, \ldots, y_m\} \setminus \{y_{i_0}\}$ has size at most $n+1$ then $p \notin \operatorname{aff}(A)$.

Proof. Our reasoning is similar to the one in the previous claim. Striving for a contradiction, assume that $p \in \operatorname{aff}(A)$. Observe that

$$|\vec{G}_p \cup A| \leq (n+1) + (n+1) = 2n+2.$$

By (8) this implies that $\vec{G}_p \cup A$ is geometrically independent. So by Proposition 2.1.5 we get $p \in \operatorname{aff}(A) \cap \operatorname{aff}(\vec{G}_p) = \operatorname{aff}(A \cap \vec{G}_p)$. Since $i_0 \notin A$ and the affine coordinate of p corresponding to y_{i_0} is non-zero, by unicity of affine coordinates (Theorem 2.1.8), this is a contradiction. ◊

Now let B be the complement of

$$\bigcup \{\operatorname{aff}(A) : A \subseteq \{y_1, \ldots, y_m\}, |A| \leq n+1 \text{ and for some } i \in G_p, y_i \notin A\}.$$

By Claim 3, B is an open neighborhood of p in $\mathbb{R}^{2n+1}$.

Claim 4. $g^{-1}[B] \subseteq \bigcap_{i \in G_p} E_i$.

Proof. Pick an arbitrary $z \in g^{-1}[B]$ and let $i_0 \in G_p$. By the definition of g we have $g(z) \in \operatorname{aff}(\vec{F}_z)$. Since $|F_z| \leq n+1$ by (4), if $i_0 \notin F_z$ it would follow that

$$g(z) \in \operatorname{aff}(\vec{F}_z) \subseteq \mathbb{R}^n \setminus B,$$

which is a contradiction. So $i_0 \in F_z$ or, equivalently, $z \in E_{i_0}$. ◊

If $p \in U = \mathbb{R}^n \setminus \overline{g[X]}$ then U is a neighborhood of p and $g^{-1}[U] = \emptyset$.

Since by (5) we have $\mathcal{E} < \mathcal{U}$, this completes the proof. □

Proposition 3.3.4. *Let X be a space $0 \le \dim X \le n$. In addition, let*

$$\{\mathcal{U}_i : i \in \mathbb{N}\}$$

be a sequence of finite open covers of X such that

$$\operatorname{mesh}(\mathcal{U}_i) < {}^1\!/\!_i$$

for every i. Then the function space $C_\varrho(X, \mathbb{R}^{2n+1})$ contains a dense G_δ-subset $\mathcal{C}$ having the following properties:

(1) *Every $f \in \mathcal{C}$ is an imbedding.*
(2) *Every $f \in \mathcal{C}$ is a $\mathcal{U}_i$-map for every i.*
(3) *The range of every $f \in \mathcal{C}$ is contained in $\mathfrak{N}_n$.*
(4) *The range of every $f \in \mathcal{C}$ has compact closure in $\mathfrak{N}_n$.*

Proof. Let $A = \{i(1), \dots, i(n+1)\} \subseteq \{1, 2, \dots, 2n+1\}$ and $q_1, \dots, q_{n+1} \in \mathbb{Q}$ be arbitrary. Put

$$H = \{x \in \mathbb{R}^{2n+1} : x_{i(j)} = q_j \text{ for every } j \le n+1\}.$$

Observe that H is an n-dimensional affine subspace of $\mathbb{R}^{2n+1}$ and that

$$H \cap \mathfrak{N}_n = \emptyset.$$

In addition, each point of $\mathbb{R}^{2n+1} \setminus \mathfrak{N}_n$ has at least $n+1$ rational coordinates and is therefore contained in an affine subspace of the form H. We conclude that the complement of $\mathfrak{N}_n$ is the union of countably many n-dimensional affine subspaces, say $\{L_i : i \in \mathbb{N}\}$.

Now consider the function space $C_\varrho(X, \mathbb{R}^{2n+1})$. For every $i \in \mathbb{N}$, put

$$\mathcal{C}_i = \{f \in C_\varrho(X, \mathbb{R}^{2n+1}) : f \text{ is a } \mathcal{U}_i\text{-map and } \overline{f[X]} \cap L_i = \emptyset\}.$$

It follows from Proposition 3.3.3 that $\mathcal{C}_i$ is dense in $C_\varrho(X, \mathbb{R}^{2n+1})$ for every i.

Claim 1. $\mathcal{C}_i$ is open in $C_\varrho(X, \mathbb{R}^{2n+1})$ for every i.

Proof. Take an arbitrary $f \in \mathcal{C}_i$. Since f is bounded, $\overline{f[X]}$ is compact. Therefore, since $\overline{f[X]} \cap L_i = \emptyset$, there exists $\varepsilon > 0$ such that

$$\{y \in \mathbb{R}^{2n+1} : \varrho(y, \overline{f[X]}) \le \varepsilon\} \cap L_i = \emptyset \tag{5}$$

(Corollary A.5.4). Now since f is a $\mathcal{U}_i$-map, every y in the compact set $\overline{f[X]}$ has a neighborhood V_y in $\mathbb{R}^{2n+1}$ such that $f^{-1}[V_y]$ is contained in an element of $\mathcal{U}_i$. Put $\mathcal{V} = \{V_y : y \in \overline{f[X]}\}$. By Lemma A.5.3 there exists $\delta > 0$ such that every $A \subseteq \mathbb{R}^{2n+1}$ with $\operatorname{diam}(A) < 2\delta$ and which moreover intersects $\overline{f[X]}$ is contained in an element of $\mathcal{V}$. Let $\gamma = \min\{{}^\delta\!/\!_3, \varepsilon\}$. We claim that the open ball about f with radius γ is contained in $\mathcal{C}_i$. To this end, take an arbitrary function $g \in C_\varrho(X, \mathbb{R}^{2n+1})$ such that $\hat{\varrho}(f,g) < \gamma$. We shall first prove that

$$\overline{g[X]} \cap L_i = \emptyset.$$

This is easy. Take an arbitrary $x \in X$. Then $\varrho\big(f(x), g(x)\big) < \varepsilon$, which implies that $g(x)$ is contained in the compact set

$$\{y \in \mathbb{R}^{2n+1} : \varrho(y, \overline{f[X]}) \leq \varepsilon\}.$$

So by (5), $\overline{g[X]} \cap L_i = \emptyset$. We next prove that g is a $\mathcal{U}_i$-map. Take an arbitrary element $p \in \overline{g[X]}$. There exists $x \in X$ such that $g(x) \in B(p, \delta/6)$. Consider the open neighborhood

$$B = g^{-1}[B(g(x), \delta/3)]$$

of x. We claim that B is contained in an element of $\mathcal{U}_i$. If $z \in B$ then clearly

$$\varrho\big(g(x), g(z)\big) < \delta/3.$$

Since $\hat{\varrho}(f,g) < \delta/3$, we also get

$$\varrho\big(g(x), f(x)\big) < \delta/3, \quad \varrho\big(g(z), f(z)\big) < \delta/3.$$

From these inequalities it follows that

$$\begin{aligned}\varrho\big(f(z), f(x)\big) &\leq \varrho\big(f(z), g(z)\big) + \varrho\big(g(z), g(x)\big) + \varrho\big(g(x), f(x)\big) \\ &< \delta/3 + \delta/3 + \delta/3 \\ &= \delta\end{aligned}$$

and so

$$(6) \qquad g^{-1}[B(g(x), \delta/3)] \subseteq f^{-1}[B\big(f(x), \delta\big)].$$

Since by the special choice of δ the set $B(f(x), \delta)$ is contained in an element of $\mathcal{V}$, $f^{-1}[B(f(x), \delta)]$ is contained in an element of $\mathcal{U}_i$. By (6) it therefore follows that $g^{-1}[B(g(x), \delta/3)]$ is contained in an element of $\mathcal{U}_i$. But this is as desired since $B(p, \delta/6) \subseteq B(g(x), \delta/3)$. ♢

By Corollary 1.3.6, $C_\varrho(X, \mathbb{R}^{2n+1})$ is a topologically complete space, and is therefore a Baire space (Theorem A.6.6). Consequently, Claim 1 implies that

$$\mathcal{C} = \bigcap_{i=1}^{\infty} \mathcal{C}_i$$

is a dense G_δ-subset of $C_\varrho(X, \mathbb{R}^{2n+1})$. We shall prove that $\mathcal{C}$ is as desired. Observe that (2) follows by construction.

Take any function $f \in \mathcal{C}$. Then $\overline{f[X]} \cap \bigcup_{i=1}^{\infty} L_i = \emptyset$, i.e., $\overline{f[X]} \subseteq \mathfrak{N}_n$. This proves (3) and (4) for f. So it suffices to prove that f is an imbedding.

We shall prove that for every $x \in X$ and every neighborhood W of x there exists a neighborhood V of $f(x)$ such that $f^{-1}[V] \subseteq W$. From this it follows that f is one-to-one and that $f\colon X \to f[X]$ is open, i.e., f is an imbedding. Take an arbitrary $x \in X$ and a neighborhood W of x. Let $\varepsilon > 0$ be such that $B(x,\varepsilon) \subseteq W$. There exists $i \in \mathbb{N}$ with $1/i < \varepsilon$. Since f is a $\mathcal{U}_i$-map, there exists a neighborhood V of $f(x)$ such that $f^{-1}[V]$ is contained in an element U of $\mathcal{U}_i$. So $x \in U$ and since $\operatorname{diam}(U) < 1/i < \varepsilon$, we conclude that $f^{-1}[V] \subseteq U \subseteq B(x,\varepsilon) \subseteq W$, as required. □

We now come to the main result in this section.

The Imbedding Theorem 3.3.5. *Let X be a space with $0 \le \dim X \le n$. Then there exists an imbedding $i\colon X \to \mathfrak{N}_n$ such that $i[X]$ has compact closure in $\mathfrak{N}_n$.*

Proof. By Exercise A.6.5, X has an admissible metric ϱ for which there exists a sequence of finite open covers $\{\mathcal{U}_i : i \in \mathbb{N}\}$ such that for every i, $\operatorname{mesh}(\mathcal{U}_i) < 1/i$ $(i \in \mathbb{N})$. So we are in a position to apply Proposition 3.3.4 to get what we want. □

Remark 3.3.6. Notice that we proved in fact the stronger statement that for a space X with $0 \le \dim X \le n$, any bounded map $f\colon X \to \mathbb{R}^{2n+1}$ can be approximated arbitrarily closely by an imbedding in $\mathfrak{N}_n$ the range of which has compact closure in $\mathfrak{N}_n$.

Remark 3.3.7. Theorem 3.3.5 is 'best possible' for there exist n-dimensional spaces that cannot be imbedded in $\mathbb{R}^{2n}$. An example of such a space is the union X of all n-faces of a $(2n+2)$-dimensional simplex. This X is obviously n-dimensional and that it cannot be imbedded in $\mathbb{R}^{2n}$ is due to Flores [163]. The proof of this fact for $n = 1$ is simple and is based on the Jordan Curve Theorem for $\mathbb{R}^2$. For $n > 1$, the proof is more complicated. Its main ingredient is the Borsuk-Ulam Antipodal Theorem in Exercise 2.5.2(4), which is equivalent to the Lusternik-Schnirelman-Borsuk Theorem 2.5.1. See Engelking [154, p. 106] for details and additional information.

Some applications. We now present some applications of the results in this section. The proofs are based on Theorem 3.3.5 which explains our interest in imbeddings in $\mathfrak{N}_n$ instead of imbeddings in $\mathbb{R}^{2n+1}$.

Corollary 3.3.8. *If $\dim X \le n$, $n \ge 0$, then X has a compactification γX such that $\dim \gamma X \le n$.*

Proof. Let $i\colon X \to \mathfrak{N}_n$ be an imbedding such that $\gamma X = \overline{i[X]}$ is a compact subset of $\mathfrak{N}_n$ (Theorem 3.3.5). Then $\dim \gamma X \le n$ by Lemma 3.3.1 and the Subspace Theorem 3.2.9. □

Corollary 3.3.9. *Let X be a nonempty space and let $n \ge 0$. The following statements are equivalent:*

(1) $\dim X \le n$,
(2) *X is the union of at most $n+1$ zero-dimensional subspaces.*

Proof. Suppose that $\dim X \le n$. By Theorem 3.3.5 we may assume that X is contained in $\mathfrak{N}_n$. By Lemma 3.3.1, $\mathfrak{N}_n$ is the union of $n+1$ zero-dimensional subspaces. Now apply the Subspace Theorem 3.2.9.

Conversely, assume that X is the union of at most $n+1$ zero-dimensional subspaces. Then apply Corollary 3.1.7 to conclude that $\dim X \le n$. □

We now prove a generalization of Corollary 3.1.6 to n-dimensional spaces.

Corollary 3.3.10. *Let $0 \le n < \infty$ and let X be an n-dimensional subspace of the space Y. Then for all disjoint closed subsets A and B of Y there exists a partition S between A and B in Y such that* $\dim(S \cap X) \le n-1$.

Proof. Write $X = \bigcup_{i=1}^{n+1} X_i$, where $\dim X_i \le 0$ for every i (Corollary 3.3.9). Let A and B be disjoint closed subsets of X. By Corollary 3.1.6 there is a partition S between A and B such that $S \cap X_1 = \emptyset$. But then $S \cap X$ is contained in $\bigcup_{i=2}^{n+1} X_i$ and hence is at most $(n-1)$-dimensional by Corollary 3.3.9. □

By a repeated application of the previous corollary, we get the following result.

Corollary 3.3.11. *Let X be a space with subspace Y such that* $\dim Y \le n$, *where $0 \le n < \infty$. For each collection $\tau = \{(F_i, G_i) : i \le n+1\}$ of pairs of disjoint closed subsets of X there exists for every $i \le n+1$ a partition S_i between F_i and G_i such that*

$$Y \cap \bigcap_{i=1}^{n+1} S_i = \emptyset.$$

The following result is known as the *G_δ-Enlargement Theorem.*

Corollary 3.3.12. *Let M be a subspace of a space X. If* $\dim M \le n$ *then there exists a G_δ-subset $S \subseteq X$ with $M \subseteq S$ and* $\dim S \le n$.

Proof. We first prove the corollary for the special case $\dim M = 0$. Let $\mathcal{E}$ be a countable open base for $\overline{M}$. For every pair (E_0, E_1) of elements of $\mathcal{E}$ such that $\overline{E}_0 \subseteq E_1$ there exists by Corollary 3.1.6 an open set $B \subseteq \overline{M}$ such that

$$\overline{E}_0 \subseteq B \subseteq \overline{B} \subseteq E_1$$

while moreover $M \cap \operatorname{Fr} B = \emptyset$. The collection $\mathcal{B}$ of B's obtained in this way is clearly a countable base for $\overline{M}$. In addition,

$$F = \bigcup_{B \in \mathcal{B}} \operatorname{Fr} B$$

is an F_σ-subset of $\overline{M}$ which misses M. Let $S = \overline{M} \setminus F$. Then S is a G_δ-subset of the G_δ-subset $\overline{M}$ and hence is a G_δ-subset of X. It is clear that $\mathcal{B} \restriction S$ consists of (relative) clopen sets, and so S is zero-dimensional.

We will now use the just proved special case to prove the general case. To this end, write

$$M = M_0 \cup \cdots \cup M_n,$$

where M_i is zero-dimensional for every $0 \leq i \leq n$ (Corollary 3.3.9). By the above there exists for every $0 \leq i \leq n$ a zero-dimensional G_δ-subset M_i^* of X with $M_i \subseteq M_i^*$. The union of the M_i^*'s is clearly a G_δ-subset of X and is at most n-dimensional by Corollary 3.3.9. □

Exercises for §3.3.

1. Give an example of a space X, an open cover $\mathcal{U}$ of X, and an imbedding $f\colon X \to X$ such that f is not a $\mathcal{U}$-mapping.

2. Let X be a space. Prove that every continuous function $f\colon X \to Q$ can be approximated arbitrarily closely by an imbedding in Q.

3. Prove that for every $n \geq 0$ there exists a compact space X_n such that
   (1) $\dim X_n = n$,
   (2) every space Y with $\dim Y \leq n$ can be imbedded in X_n.

►4. Let X be a compact space without isolated points such that

$$0 \leq \dim X \leq n < \infty.$$

Prove that there exists a continuous surjection $f\colon \mathsf{C} \to X$ such that each fiber of f has cardinality at most $n+1$

5. Let X be a compact space and let $f\colon \mathsf{C} \to X$ be a continuous surjection such that each fiber of f has cardinality at most $n+1$. Prove that X is at most n-dimensional.

6. Let X be a compact space with $0 \leq n = \dim X < \infty$. Prove that every open cover $\mathcal{U}$ of X has a closed refinement $\mathcal{F}$ such that for every $F \in \mathcal{F}$ the collection

$$\{F' \in \mathcal{F} : F' \cap F \neq \emptyset\}$$

has cardinality at most 3^{2n+1}.

►7. Prove that every $(n+1)$-dimensional space contains an n-dimensional subspace (hence an $(n+1)$-dimensional compactum contains an n-dimensional continuum by Exercise 3.2.4).

8. Prove that A and B are subspaces of a space X then

$$\dim(A \cup B) \leq \dim A + \dim B + 1.$$

3.4. The inductive dimension functions ind and Ind

There are two additional dimension functions that are important in dimension theory, namely, the small and the large inductive dimension function, abbreviated ind and Ind, respectively. It turns out that for a given space these functions and the dimension function dim take the same value. The functions ind and Ind are important because in certain situations it is easier to deal with them than with dim. For example, it is a triviality to verify that for all X and Y,

$$\operatorname{ind}(X \times Y) \leq \operatorname{ind} X + \operatorname{ind} Y;$$

by equality of ind and dim it therefore follows that

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

However, to verify this straight from the definition of dim is unpleasant. The aim of this section is to study basic properties of the dimension functions ind and Ind and to prove equality of all three dimension functions.

We shall now give the definition of ind, which differs from the definition of dim in the sense that it is an inductive definition. Indeed, for a space X define its *small inductive dimension* $\operatorname{ind} X \in \{-1, 0, 1, \dots\} \cup \{\infty\}$ as follows:

$\operatorname{ind} X = -1$	$\Leftrightarrow$	$X = \emptyset$,
$\operatorname{ind} X \leq n$	$\Leftrightarrow$	for every $x \in X$ and every closed subset A of X not containing x there exists a partition L between $\{x\}$ and A such that $\operatorname{ind} L \leq n-1$,
$\operatorname{ind} X = n$	$\Leftrightarrow$	$\operatorname{ind} X \leq n$ and $\operatorname{ind} X \not\leq n-1$,
$\operatorname{ind} X = \infty$	$\Leftrightarrow$	$\operatorname{ind} X \neq n$ for every $n \geq -1$.

It is clear that if X and Y are homeomorphic spaces then $\operatorname{ind} X = \operatorname{ind} Y$.

The property of having small inductive dimension at most n can be expressed in terms of a special countable base.

Lemma 3.4.1. *A space X satisfies $0 \leq \operatorname{ind} X \leq n$ if and only if X has a countable base $\mathcal{B}$ such that $\operatorname{ind} \operatorname{Fr} B \leq n-1$ for all $B \in \mathcal{B}$.*

Proof. If $\operatorname{ind} X \leq n$ then X has clearly a base $\mathcal{U}$ such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{U}$. This base has a countable subcollection $\mathcal{B}$ which is still a base (Exercise A.2.12). So $\mathcal{B}$ is as required.

The converse of the lemma is trivial. □

A consequence of this lemma is that for a space X we have $\operatorname{ind} X = 0$ if and only if X is nonempty and has a base consisting of clopen sets. This means that X is zero-dimensional if and only if $\operatorname{ind} X = 0$.

We shall now derive a few additional properties of the dimension function ind. The following triviality shall be used several times in the forthcoming:

if X is a space, $Y \subseteq X$, $y \in Y$ and A is a (relatively) closed subset of Y not containing y, then y does not belong to the closure of A in X.

Proposition 3.4.2. *Let X be a space. Then* $\dim X = 0$ *iff* $\operatorname{ind} X = 0$.

Proof. This is a triviality since, as observed above, $\operatorname{ind} X = 0$ iff the clopen subsets of X form a base iff $\dim X = 0$ (see Page 41). $\square$

Lemma 3.4.3. *Let X be a space, and let A be a subspace of X. Then* $\operatorname{ind} A \leq \operatorname{ind} X$.

Proof. There is nothing to prove if $\operatorname{ind} X = \infty$, so we assume that $\operatorname{ind} X$ is finite. Again, there is nothing to prove if $\operatorname{ind} X = -1$. So assume the lemma to be true for all spaces Y with $0 \leq \operatorname{ind} Y \leq n-1$, and assume that $\operatorname{ind} X \leq n$. Take $x \in A$ and let C be a closed subset of A not containing x. Since $x \notin \overline{C}$ there is a partition L in X between x and $\overline{C}$ such that $\operatorname{ind} L \leq n-1$. Then $L \cap A$ is a partition between x and C in A so that, by our inductive assumption, $\operatorname{ind}(L \cap A) \leq n-1$. We conclude that $\operatorname{ind} A \leq n$. $\square$

This result enables us to prove the following:

The Addition Theorem 3.4.4. *If A and B are subspaces of a space X then*

$$\operatorname{ind}(A \cup B) \leq \operatorname{ind} A + \operatorname{ind} B + 1.$$

Proof. If $\operatorname{ind} A = \infty$ or $\operatorname{ind} B = \infty$ then there is nothing to prove. So we assume that $\operatorname{ind} A$ and $\operatorname{ind} B$ are both finite. We induct on $\operatorname{ind} A + \operatorname{ind} B$. If A and B are both empty, i.e., if $\operatorname{ind} A = \operatorname{ind} B = -1$, then there is again nothing to prove. So assume that the theorem is true for all subspaces E and F of X with

$$\operatorname{ind} E + \operatorname{ind} F \leq n-1,$$

where $n \geq -1$. Let $\alpha = \operatorname{ind} A$ and $\beta = \operatorname{ind} B$ and assume that $\alpha + \beta = n$. We shall prove that $\operatorname{ind}(A \cup B) \leq n+1$. Take an arbitrary $x \in A \cup B$, say $x \in A$, and let C be a closed subset of $A \cup B$ not containing x. Since $x \notin \overline{C}$, there is a closed neighborhood Z of $\overline{C}$ such that $x \notin Z$. Let L be a partition in A between $\{x\}$ and $Z \cap A$ such that $\operatorname{ind} L \leq \alpha - 1$. Write $A \setminus L$ as the union of two relatively open disjoint sets A_0 and A_1 such that $x \in A_0$ and $Z \cap A \subseteq A_1$. There exists an open neighborhood W of x in X such that $\overline{W} \cap A \subseteq A_0$ and $\overline{W} \cap \overline{Z} = \emptyset$. Now observe that L is a partition between $\overline{W} \cap A$ and $Z \cap A$. By Lemma 3.1.4, there exists a partition S in X between $\{x\}$ and $\overline{C}$ such that $S \cap A \subseteq L$. Now observe that by Lemma 3.4.3, $\operatorname{ind}(S \cap A) \leq \alpha - 1$. Similarly, $\operatorname{ind}(S \cap B) \leq \beta$ since $\operatorname{ind} B = \beta$. Consequently, we obtain by our inductive assumption,

$$\operatorname{ind}\big(S \cap (A \cup B)\big) \leq \alpha - 1 + \beta + 1 = \alpha + \beta = n.$$

Since $S \cap (A \cup B)$ is a partition in $A \cup B$ between $\{x\}$ and C, this shows that indeed $\operatorname{ind}(A \cup B) \leq n+1$. □

Remark 3.4.5. If $X = \mathbb{R}$, $A = \mathbb{Q}$ and $Y = \mathbb{P}$ then $\operatorname{ind} X = 1$ and

$$\operatorname{ind} A = 0 = \operatorname{ind} B.$$

This shows that Theorem 3.4.4 is 'best possible'.

Corollary 3.4.6. *If X can be written as the union of $n+1$ zero-dimensional subspaces then $\operatorname{ind} X \leq n$.*

Proof. Since for every space Y, $\dim Y = 0$ iff $\operatorname{ind} Y = 0$ (Proposition 3.4.2) the result follows immediately from Theorem 3.4.4. □

We want to prove the equality of dim and ind. Before being able to do that, we need to derive a preliminary lemma.

Lemma 3.4.7. *Let X be space and let $n \geq 0$. The following two statements are equivalent:*

(1) *For every $x \in X$ and for every open neighborhood U of x there exists a partition L between $\{x\}$ and $X \setminus U$ such that* $\dim L \leq n-1$.
(2) *For every pair A and B of disjoint closed subsets of X there exists a partition L between A and B such that* $\dim L \leq n-1$.

Proof. First observe that (2) ⇒ (1) is a triviality.

For (1) ⇒ (2), let A and B be disjoint closed subsets of X. By assumption, for each $x \in X$ there exist open sets $U(x)$ and $V(x)$ such that

(3) $x \in U(x)$, $U(x) \cap V(x) = \emptyset$,
(4) if $L(x) = X \setminus \big(\overline{U(x) \cup V(x)}\big)$ then $\dim L(x) \leq n-1$,
(5) $\overline{U(x)} \cap A = \emptyset$ or $\overline{U(x)} \cap B = \emptyset$.

The cover $\{U(x) : x \in X\}$ has a countable subcover, say $\mathcal{U}$ (Corollary A.2.3). Put $L = \bigcup\{L(x) : U(x) \in \mathcal{U}\}$. Observe that by (4) and the Countable Closed Sum Theorem 3.2.8 we have $\dim L \leq n-1$. Let $\mathcal{U}_0 = \{U \in \mathcal{U} : \overline{U} \cap A \neq \emptyset\}$ and $\mathcal{U}_1 = \mathcal{U} \setminus \mathcal{U}_0$. Enumerate $\mathcal{U}_0$ as $\{U_{0,i} : i \in \mathbb{N}\}$ and $\mathcal{U}_1$ as $\{U_{1,i} : i \in \mathbb{N}\}$, respectively. For each $i \in \mathbb{N}$ put

(6) $E_i = U_{0,i} \setminus \bigcup_{j<i} \overline{U}_{1,j}$ and $F_i = U_{1,i} \setminus \bigcup_{j \leq i} \overline{U}_{0,j}$.

By (5) and (6) it follows easily that $E = \bigcup_{i=1}^{\infty} E_i$ and $F = \bigcup_{i=1}^{\infty} F_i$ are disjoint open sets with $A \subseteq E$ and $B \subseteq F$. Consequently, $G = X \setminus (E \cup F)$ is a partition between A and B. We will prove that $\dim G \leq n-1$ by proving that $G \subseteq L$ and applying the Subspace Theorem 3.2.9. Therefore, G will turn out to be the required partition between A and B.

Take an arbitrary point $y \in G$. Let i be the first natural number with the property that $y \in \overline{U}_{0,i} \cup \overline{U}_{1,i}$. Observe that such an i exists. Suppose first that $y \in \overline{U}_{0,i}$. If $y \in U_{0,i}$ then $y \in E_i$, which is not the case. So $y \notin U_{0,i}$, hence $y \in \overline{U}_{0,i} \setminus U_{0,i} \subseteq L$, and we are done. Suppose therefore that $y \notin \overline{U}_{0,i}$. If $y \in U_{1,i}$ then $y \in F_i$, which is also not the case. Consequently, $y \notin U_{1,i}$, which implies that in this case $y \in L$ as well. $\square$

Theorem 3.4.8. *For every space X, $\dim X = \operatorname{ind} X$.*

Proof. We shall first prove that $\dim X \leq \operatorname{ind} X$. If $\operatorname{ind} X = \infty$ then there is nothing to prove. So assume that $\operatorname{ind} X < \infty$. We induct on $\operatorname{ind} X$. If $\operatorname{ind} X = 0$ then apply Proposition 3.4.2. Now assume that the inequality holds for all spaces Y with $\operatorname{ind} Y \leq n-1$, $n \geq 1$, and let $\operatorname{ind} X = n$. Then for each point $x \in X$ and for every open neighborhood U of x there exists a partition L between $\{x\}$ and $X \setminus U$ such that $\operatorname{ind} L \leq n-1$. By our inductive hypothesis, all these partitions have covering dimension at most $n-1$. We conclude that $\dim X \leq n$ by Lemma 3.4.7 and Corollary 3.1.10.

We shall now prove that $\operatorname{ind} X \leq \dim X$. If $\dim X = -1$ or $\dim X = \infty$ then this is a triviality. So assume that $0 \leq \dim X = n$. By Corollary 3.3.9, X is the union of at most $n+1$ zero-dimensional subspaces. So by Corollary 3.4.6 we get $\operatorname{ind} X \leq n = \dim X$, which is what we want. $\square$

Motivated by Lemma 3.4.7, we shall now present the definition of the *large inductive dimension* Ind. Indeed, for a space X put:

$\operatorname{Ind} X = -1$	$\Leftrightarrow$	$X = \emptyset$,
$0 \leq \operatorname{Ind} X \leq n$	$\Leftrightarrow$	for every pair of disjoint closed subsets A and B of X there exists a partition L between A and B such that $\operatorname{Ind} L \leq n-1$,
$\operatorname{Ind} X = n$	$\Leftrightarrow$	$\operatorname{Ind} X \leq n$ and $\operatorname{Ind} X \not\leq n-1$,
$\operatorname{Ind} X = \infty$	$\Leftrightarrow$	$\operatorname{Ind} X \neq n$ for every $n \geq -1$.

As in the case of ind, it is easy to see that if X and Y are homeomorphic spaces then $\operatorname{Ind} X = \operatorname{Ind} Y$. We shall prove that Ind = dim, thereby establishing the announced equality.

Lemma 3.4.9. *Let X be a space. Then $\operatorname{ind} X \leq \operatorname{Ind} X$.*

Proof. If $\operatorname{Ind} X = \infty$ or $\operatorname{Ind} X = -1$ then there is nothing to prove, so assume that the lemma is true for all spaces Y with $-1 \leq \operatorname{Ind} Y \leq n-1$ and $0 \leq n < \infty$, and assume that $\operatorname{Ind} X = n$. Let $x \in X$ and let U be an open neighborhood of x. Since $\{x\}$ and $X \setminus U$ are disjoint closed sets in X, there is a partition L between them such that $\operatorname{Ind} L \leq n-1$. By our inductive assumption we have $\operatorname{ind} L \leq \operatorname{Ind} L \leq n-1$. From this we consequently conclude that $\operatorname{ind} X \leq n = \operatorname{Ind} X$. $\square$

We now come to the announced:

The Coincidence Theorem 3.4.10. *For every space X we have*

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X.$$

Proof. By Theorem 3.4.8 and Lemma 3.4.9 we need only prove that

$$\operatorname{Ind} X \leq \dim X.$$

If $\dim X = \infty$ or $\dim X = -1$ then there is nothing to prove. Assume therefore that the theorem is true for all spaces Y with $\dim Y \leq n-1$, where $0 \leq n < \infty$. Assume that $\dim X = n$. Since $\operatorname{ind} X = \dim X$ (Theorem 3.4.8), for every $x \in X$ and every closed set A in X with $x \notin A$, there exists a partition L between $\{x\}$ and A such that $\operatorname{ind} L \leq n-1$. By an application of Theorem 3.4.8 it follows that all these partitions have covering dimension at most $n-1$. Consequently, by Lemma 3.4.7, for every pair of disjoint closed subsets A and B of X there exists a partition L between A and B such that $\dim L \leq n-1$. By our inductive hypothesis it follows that these partitions all have large inductive dimension at most $n-1$. From this we conclude that indeed $\operatorname{Ind} X \leq n = \dim X$. □

Remark 3.4.11. From now on we will no longer formally distinguish between dim, ind and Ind. Theorems proved for dim will be used for ind and vice versa, etc.

Let X and Y be spaces. It is natural to wonder about the relation between $\dim X$, $\dim Y$ and $\dim(X \times Y)$. Since $\dim \mathbb{R}^n = n$ for every n, one would expect the relation $\dim(X \times Y) = \dim X + \dim Y$ to hold for all nonempty X and Y. Unfortunately, this is not true, see Example 3.4.13. We shall now prove 'half' of the expected equality.

Theorem 3.4.12. *Let X and Y be nonempty spaces. Then*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

Proof. By the Coincidence Theorem 3.4.10, it suffices to prove that

$$\operatorname{ind}(X \times Y) \leq \operatorname{ind} X + \operatorname{ind} Y.$$

First observe that without loss of generality, $0 \leq \operatorname{ind} X, \operatorname{ind} Y < \infty$. We shall prove the theorem by induction on $\operatorname{ind} X + \operatorname{ind} Y$. If $\operatorname{ind} X + \operatorname{ind} Y = 0$ then both X and Y are zero-dimensional, and so is $X \times Y$. Assume that the theorem is true for all spaces X and Y with

$$\operatorname{ind} X + \operatorname{ind} Y \leq n-1,$$

where $n \geq 1$, and let X and Y be spaces such that $\operatorname{ind} X = \alpha$, $\operatorname{ind} Y = \beta$ and $\alpha + \beta = n$. By Lemma 3.4.1 there is a countable base $\mathcal{B}$ for X such that $\dim \operatorname{Fr} B \leq \alpha - 1$ for every $B \in \mathcal{B}$. Put $B' = \bigcup_{B \in \mathcal{B}} \operatorname{Fr} B$. Then B' is an F_σ-subset of X and hence by the Countable Closed Sum Theorem 3.2.8 it follows that $\dim B' \leq \alpha - 1$. Since $\operatorname{Fr} B \cap (X \setminus B') = \emptyset$ for every $B \in \mathcal{B}$ it also follows that $\mathcal{B}\restriction(X\setminus B')$ consists of relatively clopen sets, i.e., $\dim(X\setminus B')\le 0$. Similarly, construct an F_σ-subset $E\subseteq Y$ such that $\dim E\le\beta-1$ and $\dim(Y\setminus E)\le 0$. Now observe that by our inductive hypothesis and the Countable Closed Sum Theorem 3.2.8, $B'\times Y$ is an at most $(\alpha-1)+\beta$-dimensional F_σ-subset of $X\times Y$. It follows similarly that $X\times E$ is an at most $\alpha+(\beta-1)$-dimensional F_σ-subset of $X\times Y$. Again by the Countable Closed Sum Theorem 3.2.8 we get

$$\dim\big((B'\times Y)\cup\big(X\times E\big)\big)\le\alpha+\beta-1.$$

Since the complement of $(B'\times Y)\cup(X\times E)$ is zero-dimensional, being the product $(X\setminus B')\times(Y\setminus E)$, we get by Theorem 3.4.4 that

$$\dim(X\times Y)\le(\alpha+\beta-1)+0+1=\alpha+\beta,$$

as required. □

The following example shows that the other 'half' of the expected equality $\dim(X\times Y)=\dim X+\dim Y$ need not hold. See Theorem 3.9.5 for a (strong) generalization of this result to all dimensions.

Example 3.4.13. There exists a one-dimensional space E such that $E\times E$ is one-dimensional as well.

We will show that the space E in Example 1.5.18, i.e., Erdős' space, is the required example. From Exercise 1.5.14 we know that $E\approx E\times E$. So it suffices to prove that $\dim E=1$. In Example 1.5.18 we showed that $\dim E\ge 1$. We shall prove here that $\operatorname{ind}E\le 1$.

First observe that E is a subgroup of ℓ^2, and hence is a *topological group*. By homogeneity, it therefore suffices to prove that $\operatorname{ind}E\le 1$ at the zero-element of E. We claim that for every $\varepsilon>0$, the sphere S_ε consisting of all points in E with norm ε, is zero-dimensional. Indeed, since all elements of S_ε have the same norm, Lemma 1.1.12 shows that S_ε is homeomorphic to a subspace of the countably infinite product of rational numbers, which is zero-dimensional by Proposition 1.5.3 and the trivial observation that products of zero-dimensional spaces are again zero-dimensional. So S_ε is indeed zero-dimensional, being a subspace of a zero-dimensional space.

We now summarize the results obtained so far in the following

Corollary 3.4.14.

(1) *Let X be a nonempty space and let $n\ge 0$. The following statements are equivalent:*
   1. $\dim X\le n$,
   2. *for every subspace A of X, $\dim A\le n$,*
   3. *X has a compactification γX such that $\dim\gamma X\le n$,*
   4. *for every pair of disjoint closed subsets A and B of X there exists a partition L between A and B such that $\dim L\le n-1$,*

5. *X is the union of at most $n+1$ zero-dimensional subspaces.*

(2) *If X is a space and if A and B are subspaces of X then*

$$\dim(A \cup B) \leq \dim A + \dim B + 1,$$
$$\dim(A \times B) \leq \dim A + \dim B.$$

Exercises for §3.4. Let X be a space. Define the *compactness degree* $\operatorname{cmp} X$ of X, as follows:

$\operatorname{cmp} X = -1$	$\Leftrightarrow$	X is compact,
$\operatorname{cmp} X \leq n$	$\Leftrightarrow$	for every $x \in X$ and every closed subset A of X not containing x there exists a partition L between $\{x\}$ and A such that $\operatorname{cmp} L \leq n-1$,
$\operatorname{cmp} X = n$	$\Leftrightarrow$	$\operatorname{cmp} X \leq n$ and $\operatorname{cmp} X \not\leq n-1$,
$\operatorname{cmp} X = \infty$	$\Leftrightarrow$	$\operatorname{cmp} X \neq n$ for every $n \geq -1$.

So the compactness degree of a space is intuitively its small inductive dimension modulo the class of all compact spaces. It is clear that if X and Y are homeomorphic spaces then they have the same compactness degree.

Let γX be a compactification of X. We call the space $\gamma X \setminus X$ the *remainder* of the compactification γX of X.

Let the *compactness deficiency* be the least integer $n \in [-1, \infty]$ such that X has a compactification with remainder of dimension n. Observe that homeomorphic spaces have the same compactness deficiency.

Given a space X and $n \in \mathbb{N}$, we shall denote by $X_{(n)}$ the set of all points in X that have arbitrarily small neighborhoods with at most $(n-1)$-dimensional boundaries.

1. Prove that Erdős' space E is not topologically complete. Give an example of a one-dimensional totally disconnected topologically complete space F such that F and $F \times F$ are one-dimensional
2. Let X be a space. Prove that $\operatorname{cmp} X \leq \operatorname{def} X \leq \dim X$.
3. Let X be a totally disconnected space which is not compact and let γX be a compactification of X. Prove that $\dim \gamma X \setminus X \geq \dim X$, i.e., X cannot be compactified by adding a set of smaller dimension than the dimension of X.

►4. Let X be a compact space such that $0 \leq \dim X = n < \infty$, and let Y be the product $\mathbb{Q} \times X$.
   (1) Prove that if S is a topologically complete space containing Y then $S \setminus Y$ contains a homeomorph of X. As a consequence, the dimension of $S \setminus Y$ is at least n.
   (2) Prove that $\operatorname{def} Y = n$.

5. Let X be a space and $n \in \mathbb{N}$. Prove that there is a countable family open sets $\mathcal{U}$ such that
   (1) $\mathcal{U}$ is a local base at every point of $X_{(n)}$,

(2) $\dim(\overline{U} \setminus U) \leq n - 1$ for every $U \in \mathcal{U}$.
Prove that $\dim X_{(n)} \leq n$.

3.5. Dimensional properties of compactifications

In this section we collect some results on compactifications preserving certain dimensional properties.

We saw in Corollary 3.3.8 that every m-dimensional space X has an m-dimensional compactification. Such a compactification is called *dimension preserving*. The following result generalizes this and is central in this section.

Theorem 3.5.1. *Let X be a space, and let $\{(F_n, G_n) : n \in \mathbb{N}\}$ be a countable collection of pairs of disjoint closed subsets of X. Then there is a dimension preserving compactification aX of X such that for every n the sets F_n and G_n have disjoint closures in aX.*

Proof. We will prove this in the case of finite dimensional X only, the infinite-dimensional case being simpler.

Let ϱ be an admissible metric on X for which there exists a sequence

$$\{\mathcal{U}_n : n \in \mathbb{N}\}$$

of finite open covers of X such that $\operatorname{mesh}(\mathcal{U}_n) < 1/n$ for every n (Exercise A.6.5). Since $\mathcal{A}_n = \{X \setminus F_n, X \setminus G_n\}$ is an open cover of X, we can replace $\mathcal{U}_n$ by the common refinement of $\mathcal{U}_n$ and $\mathcal{A}_n$. So we may assume without loss of generality that for every n and $U \in \mathcal{U}_n$ we have that $U \cap F_n = \emptyset$ or $U \cap G_n = \emptyset$.

Let $m = \dim X$. By Proposition 3.3.4 there is an imbedding $i\colon X \to \mathfrak{N}_m$ such that

(1) $i[X]$ has compact closure in $\mathfrak{N}_m$.
(2) i is a $\mathcal{U}_n$-map for every n.

Put $aX = \overline{i[X]}$. Then aX is a dimension preserving compactification of X (cf. Corollary 3.3.8).

Fix $n \in \mathbb{N}$. We claim that $i[F_n]$ and $i[G_n]$ have disjoint closures in aX. Since i is a $\mathcal{U}_n$-map, there is an open cover $\mathcal{G}$ of $\mathbb{R}^{2m+1}$ such that

$$\{i^{-1}[G] : G \in \mathcal{G}\}$$

refines $\mathcal{U}_n$. Striving for a contradiction, assume that for certain n and p,

$$p \in \overline{i[F_n]} \cap \overline{i[G_n]}.$$

There is an element $G \in \mathcal{G}$ such that $p \in G$. Pick an element $U \in \mathcal{U}_n$ such that $i^{-1}[G] \subseteq U$. We may assume without loss of generality that $U \cap F_n = \emptyset$. But then $i^{-1}[G] \cap F_n = \emptyset$ and so $G \cap i[F_n] = \emptyset$. Since $\mathbb{R}^{2m+1} \setminus G$ is closed, this implies that $p \notin \overline{i[F_n]}$, which is a contradiction. □

▶4. Give an example of a compact space X containing two disjoint closed subsets A and B such that no partition between A and B is irreducible.

5. Let X be a subspace of the compact space Y. Prove that the following statements are equivalent:
   (1) X is L-imbedded in Y.
   (2) For every $\varepsilon > 0$ there is a neighborhood U of X in Y such that every continuum in U has diameter less than ε.

▶6. Let A and B be L-imbedded subspaces of the compacta X and Y, respectively. Prove that $A \times B$ is L-imbedded in $X \times Y$. Conclude that $A \times B$ is at most one-dimensional.

3.6. Mappings into spheres

In the previous sections we presented several characterizations of dimension, some ‘internal’ e.g., Theorem 3.2.14(2) and (3) and some ‘external’ e.g., Theorem 3.2.14(4) . In this section we shall prove that a space X is at most n-dimensional if and only if every continuous function $f\colon A \to \mathbb{S}^n$, where $A \subseteq X$ is closed, can be extended over X, thereby deducing another important ‘external’ characterization of dimension. As an application of this result we shall present a proof of the Brouwer Invariance of Domain Theorem.

We first formulate and prove two lemmas that are needed in the proof of the main result in this section.

Lemma 3.6.1. *Let X be a space and let A_1 and A_2 be disjoint closed subsets of X such that $0 \le \dim\big(X \setminus (A_1 \cup A_2)\big) \le n$. Then there exists a partition L between A_1 and A_2 in X such that $\dim L \le n-1$.*

Proof. Put $A = A_1 \cup A_2$. By Corollary A.4.3 there exist open subsets U and V of X such that $A_1 \subseteq U$, $A_2 \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$. Since $\dim(X \setminus A) \le n$ and $\overline{U} \cap (X \setminus A)$ and $\overline{V} \cap (X \setminus A)$ are disjoint closed subsets of $X \setminus A$, by Corollary 3.4.14 there exist disjoint open sets E and F in $X \setminus A$ such that

(1) $\overline{U} \cap (X \setminus A) \subseteq E$,
(2) $\overline{V} \cap (X \setminus A) \subseteq F$,
(3) if $L = (X \setminus A) \setminus (E \cup F)$ then $\dim L \le n-1$.

Since $A_1 \cup E = U \cup E$, and E is open in X being open in the subspace $X \setminus A$, we conclude that $A_1 \cup E$ is open in X. Similarly, $A_2 \cup F$ is open in X. Consequently, L is closed in X and is a partition between A_1 and A_2 in X with $\dim L \le n-1$ by (3). So L is as required. □

Corollary 3.6.2. *Let X be a space and let A_1 and A_2 be closed subspaces of X such that $0 \le \dim\big(X \setminus (A_1 \cup A_2)\big) \le n$. Then there exist closed subspaces X_1 and X_2 of X such that*

(1) $A_1 = X_1 \cap (A_1 \cup A_2)$,

(2) $A_2 = X_2 \cap (A_1 \cup A_2)$,
(3) $\dim\big((X_1 \cap X_2) \setminus (A_1 \cup A_2)\big) \le n - 1$,
(4) $X_1 \cup X_2 = X$.

Proof. Put $Y = X \setminus (A_1 \cap A_2)$. By Lemma 3.6.1 there exists a closed set L in Y with $\dim L \le n-1$, such that $Y \setminus L$ can be written as the disjoint union of two open sets E and F such that $A_1 \cap Y \subseteq E$ and $A_2 \cap Y \subseteq F$. It is easy to see that $X_1 = A_1 \cup E \cup L$ and $X_2 = A_2 \cup F \cup L$ are as required. □

These simple results enable us to derive the following:

Theorem 3.6.3. *Let X be a space and let A be a closed subspace of X such that $0 \le \dim(X \setminus A) \le n$. Then every continuous function $g\colon A \to \mathbb{S}^n$ can be extended to a continuous function $\bar{g}\colon X \to \mathbb{S}^n$.*

Proof. We shall prove the theorem by induction on $n \ge 0$. Suppose first that $n = 0$ and recall that $\mathbb{S}^0 = \{-1, 1\}$. By Corollary 3.6.2 there exists a clopen set $C \subseteq X$ such that $g^{-1}(-1) \subseteq C$ and $g^{-1}(1) \subseteq X \setminus C$. Define the function $\bar{g}\colon X \to \mathbb{S}^0$ by

$$\bar{g}(x) = \begin{cases} -1 & (x \in C), \\ 1 & (x \notin C). \end{cases}$$

An easy check shows that $\bar{g}$ is as required.

Now assume that the theorem is true for $n-1$, $n \ge 1$, and let $g\colon A \to \mathbb{S}^n$. Define $\mathbb{S}^n_1 = \{x \in \mathbb{S}^n : x_1 \ge 0\}$ and $\mathbb{S}^n_2 = \{x \in \mathbb{S}^n : x_1 \le 0\}$, respectively. Observe that $\mathbb{S}^n_1$ and $\mathbb{S}^n_2$ are both homeomorphic to $\mathbb{I}^n$ and that $\mathbb{S}^n_1 \cap \mathbb{S}^n_2$ is homeomorphic to $\mathbb{S}^{n-1}$. Now put $A_1 = g^{-1}[\mathbb{S}^n_1]$ and $A_2 = g^{-1}[\mathbb{S}^n_2]$, respectively. Let X_1 and X_2 be such as in Corollary 3.6.2 for A_1 and A_2. By our inductive assumption, we can extend $g \restriction A_1 \cap A_2$ to a continuous function

$$h\colon X_1 \cap X_2 \to \mathbb{S}^n_1 \cap \mathbb{S}^n_2.$$

Now for $i = 1, 2$ define $g_i\colon A_i \cup (X_1 \cap X_2) \to \mathbb{S}^n_i$ by $g_i = (g \restriction A_i) \cup h$. Since $\mathbb{S}^n_i$ is homeomorphic to $\mathbb{I}^n$ it is an AR (Corollary 1.2.12) and so we can extend g_i for $i = 1, 2$ to a continuous function $\bar{g}_i\colon X_i \to \mathbb{S}^n_i$. Clearly,

$$\bar{g} = \bar{g}_1 \cup \bar{g}_2\colon X \to \mathbb{S}^n$$

is a continuous extension of g. □

Corollary 3.6.4. *Let X be a space and let $f, g\colon X \to \mathbb{S}^n$ be continuous. If*

$$D(f, g) = \{x \in X : f(x) \ne g(x)\}$$

is at most $(n-1)$-dimensional then f and g are homotopic.

Proof. Consider the product $X \times \mathbb{I}$ and put

$$A = \big(X \times \{0, 1\}\big) \cup \big(\big(X \setminus D(f, g)\big) \times \mathbb{I}\big).$$

Then A is closed in $X \times \mathbb{I}$ (Exercise A.1.9) and the function $\xi\colon A \to \mathbb{S}^n$ defined by

$$\xi(x,t) = \begin{cases} f(x) & (t=0), \\ f(x) = g(x) & \big(x \in X \setminus D(f,g)\big), \\ g(x) & (t=1), \end{cases}$$

is clearly continuous. In addition, $(X \times \mathbb{I}) \setminus A$ is contained in

$$D(f,g) \times \mathbb{I}$$

and hence is at most $\big((n-1)+1\big)$-dimensional by Theorem 3.4.12. So Theorem 3.6.3 implies that ξ can be extended to a continuous function

$$\bar{\xi}\colon X \times \mathbb{I} \to \mathbb{S}^n,$$

which means that f and g are homotopic. □

Theorem 3.6.5. *Let X be a space and let $n \geq 0$. The following statements are equivalent:*

(1) $\dim X \leq n$,
(2) *for every closed subset A of X, every continuous function $f\colon A \to \mathbb{S}^n$ can be continuously extended over X.*

Proof. For (1) ⇒ (2) observe that $\dim A \leq n$ and apply Theorem 3.6.3.

For (2) ⇒ (1), let $\tau = \{(A_i, B_i) : i \leq n+1\}$ be a family of pairs of disjoint closed subsets of X. We shall prove that τ is inessential. To this end, for every $i \leq n+1$ let $\alpha_i\colon X \to \mathbb{I}$ be a Urysohn function such that $\alpha_i \restriction A_i \equiv 0$ and $\alpha_i \restriction B_i \equiv 1$ (Corollary A.4.1). Define $\alpha\colon X \to \mathbb{I}^{n+1}$ by

$$\alpha(x) = \big(\alpha_1(x), \ldots, \alpha_{n+1}(x)\big).$$

Put $A = \bigcup_{i=1}^{n+1}(A_i \cup B_i)$ and $\beta = \alpha \restriction A$. Then $\beta[A]$ is contained in the boundary B of $\mathbb{I}^{n+1}$ which is homeomorphic to $\mathbb{S}^n$ by Exercise 1.1.24. By assumption, there exists a continuous extension $\gamma\colon X \to B$ of β. Now for every $i \leq n+1$ put $E_i = (\pi_i \circ \gamma)^{-1}(1/2)$. Then clearly E_i is a partition between A_i and B_i for every i such that $\bigcap_{i=1}^{n+1} E_i = \emptyset$. Consequently, τ is inessential. □

Application 1: The Brouwer Invariance of Domain Theorem. To begin with, we shall first present an 'internal' characterization of the boundary points of an arbitrary closed subset of a fixed $\mathbb{R}^n$.

Proposition 3.6.6. *Let $n \in \mathbb{N}$ and let X be a closed subspace of $\mathbb{R}^n$. Then a point x in X belongs to the boundary $\mathrm{Fr}(X)$ of X in $\mathbb{R}^n$ if and only if x has arbitrarily small neighborhoods U in X such that every continuous function $g\colon X \setminus U \to \mathbb{S}^{n-1}$ can be extended continuously over X.*

Proof. First assume that $x \in \mathrm{Fr}(X)$. Without loss of generality, $x = \underline{0}$. If V is a neighborhood of $\underline{0}$ in X then there is $\varepsilon > 0$ such that

$$B(x, \varepsilon) \cap X \subseteq V.$$

Consequently, it suffices to consider for some $\varepsilon > 0$ a 'spherical' neighborhood $B(\underline{0}, \varepsilon)$ of $\underline{0}$ and a continuous function $g\colon X \setminus B(\underline{0}, \varepsilon) \to \mathbb{S}^{n-1}$. Without loss of generality assume that $\varepsilon = 1$ which means that the boundary of $B(\underline{0}, \varepsilon)$ is $\mathbb{S}^{n-1}$.

Since $\underline{0}$ belongs to the boundary of X, there is a point $q \in B(\underline{0}, 1) \setminus X$. For each $p \in B^n \setminus \{q\}$ let $\tau(p)$ denote the 'projection' of p on $\mathbb{S}^{n-1}$ from q.

Put $X_0 = X \cap B^n$ and $X_1 = X \setminus B(\underline{0}, 1)$, respectively. Then X_0 and X_1 are closed in X, cover X, and have the following additional property:

$$X_0 \cap \mathbb{S}^{n-1} = X_1 \cap \mathbb{S}^{n-1}.$$

Put $Y = X_0 \cap X_1$.

Since $\dim \mathbb{S}^{n-1} = n - 1$ (Theorem 3.2.12), the function $g \restriction Y$ can be extended to a continuous function $\overline{g \restriction Y}\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ by Theorem 3.6.5. In addition, define $h\colon X_0 \to \mathbb{S}^{n-1}$ by $h = \overline{g \restriction Y} \circ \tau$. Observe that $\tau(y) = y$ for $y \in Y$ so that

$$h(y) = \overline{g \restriction Y}(y) = g(y).$$

This means that the functions g and h agree on Y from which it follows that the function

$$\bar{g} = h \cup g$$

is the desired continuous extension of g.

Conversely, let $x \in X$ and assume that x is an interior point of X. There is $\varepsilon > 0$ such that $B = D(x, \varepsilon) \subseteq X$. Let U be an open neighborhood of x in X such that $U \subseteq B(x, \varepsilon)$. We identify $\mathbb{S}^{n-1}$ and the boundary of B. For each $p \in X \setminus U$ let $\tau(p)$ denote the 'projection' of p on $\mathbb{S}^{n-1}$ from x. Then τ cannot be extended over X since this would yield a retraction from B onto its boundary, which is impossible by Theorem 2.4.10. $\square$

Corollary 3.6.7. *Let $n \in \mathbb{N}$ and let X and Y be closed subspaces of $\mathbb{R}^n$. If $f\colon X \to Y$ is a homeomorphism then $f[\mathrm{Fr}(X)] = \mathrm{Fr}(Y)$.*

We are now in a position to present a proof of the following interesting:

Brouwer Invariance of Domain Theorem 3.6.8. *Let $n \in \mathbb{N}$ and let U be an open subset of $\mathbb{R}^n$. If $f\colon U \to \mathbb{R}^n$ is injective and continuous then the following statements hold:*

(1) *$f[U]$ is open in $\mathbb{R}^n$,*
(2) *$f\colon U \to f[U]$ is a homeomorphism.*

Proof. We shall first prove (1). Take an arbitrary $x \in U$. We shall prove that $f(x)$ belongs to the interior of $f[U]$. There exists $\varepsilon > 0$ such that

$$D(x,\varepsilon) = \{y \in \mathbb{R}^n : \|x - y\| \leq \varepsilon\} \subseteq U.$$

Since B is compact, by Exercise A.5.9, $f \restriction B$ is a homeomorphism. By Corollary 3.6.7 we conclude that $f(x)$ belongs to the interior of $f[B]$ and hence to the interior of $f[U]$.

We shall now prove (2). This is easy. If V is an open subset of U then (1) applied to $f \restriction V$ shows that $f[V]$ is open in $\mathbb{R}^n$ and hence in $f[U]$. We conclude that $f\colon U \to f[U]$ is an open mapping and therefore by injectivity is a homeomorphism. □

The question naturally arises whether something like Theorem 3.6.8 can also be derived for the class consisting of all *infinite-dimensional* linear spaces. As to be expected, this is not possible. For the first part of the theorem, this can be demonstrated quite easily by considering Hilbert space ℓ^2. The function $f\colon \ell^2 \to \ell^2$ defined by

$$f(x_1, x_2, \dots) = (0, x_1, x_2, \dots),$$

is an imbedding with nowhere dense range.

We now turn to the second part of Theorem 3.6.8.

Theorem 3.6.9. *For an infinite-dimensional normed linear space L there exists a bijective continuous function $f\colon L \to L$ such that f is not a homeomorphism.*

Proof. Let S be the unit sphere in L. Since L is infinite-dimensional, S is not compact by Exercise 1.1.22. Consequently, by Exercise A.5.13 there exists a continuous function $\lambda\colon S \to (0,1]$ such that $\inf \lambda(S) = 0$. Now define $f\colon L \to L$ by the formula

$$\begin{cases} f(y) = \lambda\big(\frac{y}{\|y\|}\big) \cdot y & (y \neq \underline{0}), \\ f(\underline{0}) = \underline{0}. \end{cases}$$

Claim 1. If $x \in L$ and $\alpha \in [0,\infty)$ then $f(\alpha x) = \alpha f(x)$.

Proof. This is a triviality. If $x = \underline{0}$ or $\alpha = 0$ then there is nothing to prove. In addition, if $x \neq 0$ and $\alpha \neq 0$ then

$$f(\alpha x) = \lambda\Big(\frac{\alpha x}{\|\alpha x\|}\Big) \cdot \alpha x = \alpha f(x),$$

as required. ◇

We shall now prove that f is continuous. It is clear that this need only be checked at the origin. To this end, let $(x_n)_n$ be a sequence in $L \setminus \{\underline{0}\}$ such

►1. Let $n \geq 0$ and let X be a compact space which can be written as the union of closed sets F_i, $i \in \mathbb{N}$, such that for all distinct $i, j \in \mathbb{N}$,

$$\dim(F_i \cap F_j) < n.$$

Prove that every continuous function from F_1 into $\mathbb{S}^n$ is continuously extendable over X.

2. Observe that the previous exercise is a generalization of Theorem A.10.6.

3. Let X be a compact space and let H be the union of all components of X of dimension at most n. Prove that $\dim H \leq n$.

►4. Let A be a closed subspace of a space X such that $\dim(X \setminus A) \leq n$, let $0 \leq k \leq n$, and let $f\colon A \to \mathbb{S}^k$ be continuous. Prove that there exists a closed subspace B of X such that $A \cap B = \emptyset$ and $\dim B \leq n - k - 1$, while moreover the function f can be extended over $X \setminus B$.

5. Let $n \geq 2$ and consider the n-dimensional cube $\mathbb{I}^n$. Define

$$E^{n-1} = \{(x_1, \ldots, x_{n-1}, 1) : 0 < x_i < 1, i = 1, \ldots, n-1\}.$$

Prove that if $X = \mathbb{I}^n \setminus E^{n-1}$ then $\operatorname{def} X = n - 1$.

6. Let X be an at most $(n-1)$-dimensional space. Prove that every continuous function $f\colon X \to \mathbb{S}^n$ is nullhomotopic.

7. Prove that $\mathbb{S}^n$ is unicoherent for every $n \geq 2$.

3.7. Dimension of subsets of $\mathbb{R}^n$ and certain generalizations

In this section we prove some interesting dimensional properties of subsets of the Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$.

Theorem 3.7.1. *Let $X \subseteq \mathbb{R}^n$. Then*

$$\dim X = n \iff \operatorname{Int} X \neq \emptyset.$$

Proof. First, if $\operatorname{Int} X \neq \emptyset$ then X contains a homeomorph of $\mathbb{I}^n$ and is therefore n-dimensional by Theorems 3.2.12 and Corollary 3.4.14.

Next, assume that $\operatorname{Int} X = \emptyset$. There consequently exists a countable set $D \subseteq \mathbb{R}^n \setminus X$ which is dense in $\mathbb{R}^n$.

By Proposition 3.2.10, $\mathbb{R}^n = \mathfrak{R}_{n,0} \cup \mathfrak{R}_{n,1} \cup \cdots \cup \mathfrak{R}_{n,n}$. Put

$$Y = \mathbb{R}^n \setminus \mathfrak{R}_{n,n} = \bigcup_{i=0}^{n-1} \mathfrak{R}_{n,i}.$$

Notice that $\mathfrak{R}_{n,n}$ consists of all points of $\mathbb{R}^n$ having rational coordinates only, and hence is a countable dense set. By Proposition 3.2.10 and Corollary 3.1.7 we get that $\dim Y \leq n - 1$.

By the countable dense homogeneity of $\mathbb{R}^n$ (Corollary 1.6.10), there is an element $h \in \mathcal{H}(\mathbb{R}^n)$ such that $h[D] = \mathfrak{R}_{n,n}$. This implies that $h[X] \subseteq Y$, and hence by Theorem 3.2.9 that $\dim X = \dim h[X] \leq n - 1$, as required. □

Corollary 3.7.2. *Let $U \subseteq \mathbb{R}^n$ be nonempty and open. If U is not dense then* $\dim \operatorname{Fr} U = n-1$.

Proof. Put $F = \operatorname{Fr} U$. Then by Theorem 3.7.1 it follows that $\dim F \le n-1$. Pick an arbitrary point $m \in \mathbb{R}^n \setminus \overline{U}$.

Striving for a contradiction, assume that $\dim F \le n-2$.

We will first prove that we may assume without loss of generality that $\overline{U}$ is compact. For this we will use a trick on 'exchanging' points at infinity. The one-point compactification $X = \mathbb{R}^n \cup \{\infty\}$ is homeomorphic to $\mathbb{S}^n$ (Exercise A.4.8). The boundary of U in X is F if it is compact and $F \cup \{\infty\}$ otherwise. Since $\dim F = \dim(F \cup \{\infty\})$ (Exercise 3.2.1) we arrive at the conclusion that the boundary of U in X is at most $(n-2)$-dimensional. Now observe that $Y = X \setminus \{m\}$ is homeomorphic to $\mathbb{R}^n$, that the closure of U in Y is compact, and that its boundary is at most $(n-2)$-dimensional.

So from now on assume that $\overline{U}$ is compact in $\mathbb{R}^n$ and that $\operatorname{Fr} U$ is at most $(n-2)$-dimensional. We may assume without loss of generality that the origin $\underline{0}$ of $\mathbb{R}^n$ belongs to U. By compactness of $\overline{U}$, the collection

$$\{r \cdot \overline{U} : r \in (0,1]\}$$

is a local base at $\underline{0}$ consisting of compact neighborhoods all of whose boundaries are at most $(n-2)$-dimensional. But since $\mathbb{R}^n$ is homogeneous, the same is true for all points of $\mathbb{R}^n$. By Theorem 3.4.10 (applied twice) this implies that $\dim \mathbb{R}^n \le n-1$, which contradicts Theorem 3.2.12. □

Corollary 3.7.3. *Let L be a closed subset of $\mathbb{R}^n$ with $\dim L \le n-2$. Then L does not separate $\mathbb{R}^n$.*

Proof. Suppose that $\mathbb{R}^n \setminus L = U \cup V$, where U and V are disjoint nonempty open sets. Then $\operatorname{Fr} U \subseteq L$ and so $\dim \operatorname{Fr} L \le n-2$ which contradicts Corollary 3.7.2. □

We now aim at proving in Theorem 3.7.6 a much stronger version of this corollary. First we need to do derive some elementary results about essential families and continua.

Proposition 3.7.4. *Let X be a compact space, let $\{(A_i, B_i) : i \in \Gamma\}$ be an essential family of pairs of disjoint closed subsets of X and let $\Gamma(0) \subseteq \Gamma$. If for each $i \in \Gamma(0)$, L_i is a partition between A_i and B_i in X and $n \in \Gamma \setminus \Gamma(0)$ then $L = \bigcap_{i \in \Gamma(0)} L_i$ contains a continuum from A_n to B_n.*

Proof. Suppose that L does not contain any continuum from A_n to B_n. Let

$$H = A_n \cap L, \quad G = B_n \cap L,$$

and put

$$\mathcal{E} = \{C : C \text{ is a component of } L \text{ and } C \cap H \neq \emptyset\}, \quad E = \bigcup \mathcal{E},$$

and

$$\mathcal{F} = \{C : C \text{ is a component of } L \text{ and } C \cap G \neq \emptyset\}, \quad F = \bigcup \mathcal{F},$$

respectively. From Proposition A.10.3 it follows that both E and F are closed. Also, by assumption, $E \cap F = \emptyset$. Take an arbitrary $C \in \mathcal{E}$. Since $L \setminus F$ is an open neighborhood of C in L, by Lemma A.10.1 there exists a clopen (in L) set U_C which contains C but misses F. By compactness, finitely many U_C's cover E. We conclude that $\emptyset$ is a partition in L between H and G. By Corollary 3.1.5 there is a partition T in X between A_n and B_n such that $T \cap L = \emptyset$. But this contradicts the fact that

$$\{(A_i, B_i) : i \in \Gamma(0)\} \cup \{(A_n, B_n)\}$$

is essential (Exercise 3.1.1). □

Corollary 3.7.5. *Let X be a compact space, let*

$$\{(A_i, B_i) : i \in \Gamma\}$$

be an essential family of pairs of disjoint closed subsets of X and let $n \in \Gamma$. Suppose that $Y \subseteq X$ is such that Y meets every continuum from A_n to B_n. For each $i \in \Gamma \setminus \{n\}$ let U_i and V_i be disjoint closed neighborhoods of A_i and B_i, respectively. Then

$$\big\{(U_i \cap Y, V_i \cap Y) : i \in \Gamma \setminus \{n\}\big\}$$

is essential in Y.

Proof. By Proposition 3.7.4, if for every $i \in \Gamma \setminus \{n\}$ the set L_i is a partition between A_i and B_i then $Y \cap \bigcap_{i \in \Gamma \setminus \{n\}} L_i \neq \emptyset$. Now apply Exercise 3.1.13. □

A space X is called *continuum-connected* provided that for any pair of points $x, y \in X$ there is a subcontinuum C of X containing both x and y. A continuum-connected space is obviously connected, but the converse need not be true (Exercise 3.7.1).

Theorem 3.7.6. *Let $M \subseteq G$, where $G \subseteq \mathbb{R}^n$ is open and connected. If M is at most $(n-2)$-dimensional then $G \setminus M$ is continuum-connected.*

Proof. We first prove the special case that $G = \mathbb{R}^n$. Take two arbitrary distinct points $x, y \in \mathbb{R}^n \setminus M$ and let K denote the closed ball in $\mathbb{R}^n$ with center $1/2(x+y)$ and radius $1/2\|x-y\|$. Let $f\colon \mathbb{I}^n \to K$ be a continuous surjection which maps a pair of opposite faces A and B of $\mathbb{I}^n$ to x and y, respectively, and has the property that the restriction $g = f \restriction \mathbb{I}^n \setminus (A \cup B)$ is a homeomorphism from $\mathbb{I}^n \setminus (A \cup B)$ onto $K \setminus \{x, y\}$. Such a function can easily be constructed with the technique used in the solution of Exercise 1.1.24.

The set $f^{-1}[M \cap K]$ is at most $(n-2)$-dimensional. Hence by Corollary 3.7.5 there is a continuum C from A to B which misses $f^{-1}[M \cap K]$. But then $f[C]$ is a continuum containing x and y but missing M.

Now consider an arbitrary open and connected set $G \subseteq \mathbb{R}^n$ and arbitrary points $x, y \in G \setminus M$. By Lemma 1.5.21 there exist open balls $B_1, B_2, \ldots, B_k$ in G such that $x \in B_1$, $y \in B_k$ and $B_i \cap B_{i+1} \neq \emptyset$ for all $i \leq k-1$ (here we use that G is open and connected). Since M has empty interior by Theorem 3.7.1 there exists for $i \leq k-1$ a point $z_i \in (B_i \cap B_{i+1}) \setminus M$. Put $z_0 = x$ and $z_k = y$. By the above for $i \leq k$ there is a continuum $C_i \subseteq B_i \setminus M$ containing both z_{i-1} and z_i. Then $C = \bigcup_{i=1}^{k} C_i$ is the desired continuum in $G \setminus M$ connecting x and y. □

It is easy to see that Corollary 3.7.3 follows from Theorem 3.7.6.

Corollary 3.7.3 leads to the definition of a so-called Cantor-manifold. A compact space X with $\dim X = n \geq 1$ is called an *n-dimensional Cantor-manifold* if for each closed subset $L \subseteq X$ with

$$\dim L \leq n - 2$$

the complement $X \setminus L$ is connected. The examples of n-dimensional totally disconnected spaces for all n that will be constructed in §3.9 show that a generalization of this concept to noncompact spaces makes no sense; simply observe that the singletons are the only connected subsets of a totally disconnected space. Observe that each Cantor-manifold is a continuum and that the one-dimensional Cantor-manifolds are precisely the one-dimensional continua.

Proposition 3.7.7. *Every n-dimensional compactification of $\mathbb{R}^n$ is an n-dimensional Cantor-manifold. In particular, $\mathbb{S}^n$ and $\mathbb{I}^n$ are n-dimensional Cantor-manifolds.*

Proof. Let $\gamma\mathbb{R}^n$ be an n-dimensional compactification of $\mathbb{R}^n$, and let A in γX be at most $(n-2)$-dimensional. Then $B = \mathbb{R}^n \setminus A$ is connected by Corollary 3.7.3. In addition, $A \cap \mathbb{R}^n$ has empty interior in $\mathbb{R}^n$ by Theorem 3.7.1 which implies that B is dense. We conclude that $\gamma\mathbb{R}^n \setminus A$ contains the dense connected set B and is therefore connected itself. □

We will now show that there are 'many' n-dimensional Cantor-manifolds.

Theorem 3.7.8. *Let $n \geq 1$. Every compact n-dimensional space X contains an n-dimensional Cantor-manifold.*

Proof. By Theorem 3.6.5 there exists a closed subset $A \subseteq X$ and a continuous function $f\colon A \to \mathbb{S}^{n-1}$ such that f admits no continuous extension over X. Let $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ be a countable open base for X.

We define a decreasing sequence $(X_i)_{i\geq 0}$ of closed subsets of X, as follows. First, let $X_0 = X$. If X_i has been defined then we consider two cases. If f does not admit a continuous extension over $A \cup (X_i \setminus B_{i+1})$ then put

$$X_{i+1} = X_i \setminus B_{i+1},$$

otherwise put

$$X_{i+1} = X_i.$$

We claim that $M = \bigcap_{i=0}^{\infty} X_i$ is an n-dimensional Cantor-manifold.

Observe that by construction, for no i the function f admits a continuous extension over $A \cup X_i$.

Claim 1. f does not admits a continuous extension over $A \cup M$.

Proof. Striving for a contradiction, assume that $\bar{f}\colon A \cup M \to \mathbb{S}^{n-1}$ is a continuous extension of f. By Corollary 1.2.13 it follows that there is an open neighborhood U of M such that $\bar{f}$ can be extended to a continuous function $\tilde{f}\colon A \cup U \to \mathbb{S}^{n-1}$. Pick N so large that $M \subseteq X_N \subseteq U$ (here we use that X is compact, see Exercise A.5.17). Then $\tilde{f} \restriction (A \cup X_N)$ extends f which is impossible by construction. $\diamondsuit$

So by Theorem 3.6.3 it follows that $\dim M \geq n$. Since $\dim M \leq n$ being a subspace of X we conclude that $\dim M = n$.

To complete the proof it clearly suffices to prove that if M_1 and M_2 are proper closed subsets of M with $M = M_1 \cup M_2$ then $\dim(M_1 \cap M_2) \geq n - 1$. Striving for a contradiction, assume that there exist proper closed subsets M_1 and M_2 of M with

$$M = M_1 \cup M_2, \quad \dim(M_1 \cap M_2) \leq n - 2.$$

Put $A_1 = A \cup M_1$ and $A_2 = A \cup M_2$, respectively.

Claim 2. f admits continuous extensions over A_1 and A_2, say to f_1 and f_2, respectively.

Proof. It suffices to prove this for A_1. To this end, pick $x \in M_2 \setminus M_1$ (here we use the fact that M_1 and M_2 are proper subsets of M). Pick $i \in \mathbb{N}$ so large that $x \in B_{i+1} \subseteq X \setminus M_1$. Consider the set X_{i+1}. It contains x and hence it intersects B_{i+1}. By construction this can only happen when f can be extended continuously over $A \cup (X_i \setminus B_{i+1})$. Since $A_1 \subseteq A \cup (X_i \setminus B_{i+1})$, we are done. $\diamondsuit$

Put $B = A \cup (M_1 \cap M_2)$. The functions $f_1 \restriction B$ and $f_2 \restriction B$ only differ in a subset of $M_1 \cap M_2$, and hence on a set of dimension $n - 2$ at most. We conclude that $f_1 \restriction B$ and $f_2 \restriction B$ are homotopic by Corollary 3.6.4. Since $f_2 \restriction B$ admits a continuous extension over $A \cup M_2$, the Borsuk Homotopy Extension

Theorem 1.4.2 implies that the same is true for $f_1 \restriction B$, say to the function $f_1' \colon A \cup M_2 \to \mathbb{S}^{n-1}$. Since

$$B = (A \cup M_1) \cap (A \cup M_2),$$

it follows that the formula

$$\xi(x) = \begin{cases} f_1(x) & (x \in A \cup M_1), \\ f_1'(x) & (x \in A \cup M_2), \end{cases}$$

defines a continuous extension of f over $A \cup M$. This is a contradiction. □

This yields another solution to Exercise 3.2.4.

Corollary 3.7.9. *If X is compact and* $\dim X = n$ *then one of its components is n-dimensional.*

Exercises for §3.7.

1. Give an example of a connected space which is not continuum-connected.
2. Give an example of a space X having the property that for every component $C \subseteq X$ we have $\dim C < \dim X$.
3. Let X be compact and let A and B be disjoint closed subsets of X. In addition, let L be a closed subset of X such that no continuum in L intersects both A and B. Prove that there is an open neighborhood U of L such that no continuum in $\overline{U}$ intersects both A and B.

►4. Let X be a compact space, let $\{(A_i, B_i) : i \in \Gamma\}$ be an essential family of pairs of disjoint closed subsets of X and let $\Gamma(0) \subseteq \Gamma$. Suppose that for each $i \in \Gamma(0)$, L_i is a partition between A_i and B_i in X and let

$$L = \bigcap_{i \in \Gamma(0)} L_i.$$

Finally, let $n_1, n_2 \in \Gamma \setminus \Gamma(0)$ be distinct.

Call a subset $T \subseteq X$ *small* if no continuum in T meets both A_{n_1} and B_{n_1}.

Prove that if $\mathcal{N}$ is a finite pairwise disjoint collection of small closed subsets of X then there is a continuum in

$$L \setminus \bigcup \mathcal{N}$$

from A_{n_2} to B_{n_2}.

5. Let X be a compact space such that $\dim X \geq 2$. Prove that there are real numbers $\delta > 0$ and $\varepsilon > 0$ such that if $\mathcal{N}$ is any finite collection of pairwise disjoint closed sets with $\mathrm{mesh}(\mathcal{N}) < \delta$ then there is a continuum $C \subseteq X \setminus \bigcup \mathcal{N}$ with diameter at least ε.

►6. Let X be a compact space with $\dim X \leq 1$. Prove that for every real number $\varepsilon > 0$ there is a family $\mathcal{N}$ of pairwise disjoint closed subsets of X with $\mathrm{mesh}(\mathcal{N}) < \varepsilon$ such that each continuum $C \subseteq X \setminus \bigcup \mathcal{N}$ has diameter less than ε.

15. Let X be a space and let $E \subseteq X$. For every n let $\mathcal{U}_n$ be a collection of open subsets of X such that $E \subseteq \bigcup \mathcal{U}_n$ and $\operatorname{mesh}(\mathcal{U}_n) < 1/n$. Prove that the collection $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a local base at every point of E.
16. Let X be a space. Prove that X has at most $\mathfrak{c}$ closed subsets. (Hence X has cardinality at most $\mathfrak{c}$ and X has at most $\mathfrak{c}$ open subsets.)
17. Let X and Y be spaces. Prove that the collection of all continuous functions $X \to Y$ has cardinality at most $\mathfrak{c}$.
18. Let X be a space with closed subspace K. Let $\varepsilon \colon X \setminus K \to (0, \infty)$ be continuous. Prove that there is a continuous function $\delta \colon X \to [0, \infty)$ such that
    (1) $\delta \restriction K \equiv 0$,
    (2) if $x \in X \setminus K$ then $0 < \delta(x) \leq \varepsilon(x)$.

A.3. Limits of continuous functions

Let X and (Y, ϱ) be spaces. For all $f, g \in C(X, Y)$ put

$$\hat{\varrho}(f, g) = \sup\{\varrho\big(f(x), g(x)\big) : x \in X\}.$$

Observe that $\hat{\varrho}(f, g) \in [0, \infty]$.

If $X = Y = \mathbb{N}$ with the Euclidean metric ϱ, f is the identity and g is the function $n \mapsto 2n$, then clearly $\hat{\varrho}(f, g) = \infty$.

As a consequence, $\hat{\varrho}$ need not be a metric on $C(X, Y)$, it is an 'extended' metric. Yet it is convenient to adopt some of the terminology of metrics and to treat $\hat{\varrho}$ as some sort of generalized metric. For example, we call a sequence $(f_n)_n$ in $C(X, Y)$ *Cauchy* if for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\hat{\varrho}(f_n, f_m) < \varepsilon$ for all $n, m \geq N$. This is of course nothing but the ordinary definition of a Cauchy sequence in a metric space.

Lemma A.3.1. *Let X and (Y, ϱ) be spaces. Let $(f_n)_n$ be a $\hat{\varrho}$-Cauchy sequence in $C(X, Y)$ such that for every $x \in X$, $\lim_{n\to\infty} f_n(x)$ exists. Then the function $f \colon X \to Y$ defined by $f(x) = \lim_{n\to\infty} f_n(x)$ is continuous.*

Proof. Let $x \in X$. We shall prove that f is continuous at x. To this end, let $\varepsilon > 0$ be arbitrary. There exists by assumption an $N \in \mathbb{N}$ such that $\hat{\varrho}(f_n, f_m) < \varepsilon/3$ for all $n, m \geq N$.

Claim 1. For every $y \in X$ and $m \geq N$, $\varrho\big(f(y), f_m(y)\big) \leq \varepsilon/3$.

Proof. Let $y \in X$. Since $\varrho\big(f_n(y), f_m(y)\big) < \varepsilon/3$ for all $n, m \geq N$ and since $f(y)$ is equal to $\lim_{n\to\infty} f_n(y)$, we obtain that indeed $\varrho\big(f(y), f_m(y)\big) \leq \varepsilon/3$ for every $m \geq N$. ♢

So the sequence $(f_n)_n$ converges uniformly to f on X. Fix an admissible metric d on X. Since f_N is continuous at x, there exists $\delta > 0$ such that if $d(x,z) < \delta$ then $\varrho\big(f_N(x), f_N(z)\big) < \varepsilon/3$. Now for an arbitrary $z \in X$ with $d(x,z) < \delta$ we have

$$\begin{aligned}\varrho\big(f(x), f(z)\big) &\le \varrho\big(f(x), f_N(x)\big) + \varrho\big(f_N(x), f_N(z)\big) + \varrho\big(f_N(z), f(z)\big)\\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3\\ &= \varepsilon.\end{aligned}$$

Since ε was arbitrary, we conclude that f is continuous at x. □

Exercises for §A.3.

1. Let $f_n\colon X \to \mathbb{R}$ be continuous for every n. Let $(M_n)_n$ be a sequence of real numbers such that for every $x \in X$ and $n \in \mathbb{N}$, $|f_n(x)| < M_n$. Prove that if $\sum_{n=1}^{\infty} M_n$ is convergent then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent (and hence continuous by Lemma A.3.1).
2. Give an example of a sequence $(f_n)_n \in C(\mathbb{I},\mathbb{I})$ such that for every $x \in \mathbb{I}$ the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists while the function f is not continuous. (This shows that the condition that the sequence $(f_n)_n$ in Lemma A.3.1 is Cauchy cannot be omitted.)

A.4. Normality type properties

Real-valued continuous functions are very important in topology. The following lemma shows that disjoint closed subsets can always be separated by such a function.

Lemma A.4.1. *Let X be a space. If A and B are pairwise disjoint nonempty closed subsets of X, then there exists a continuous function $u\colon X \to \mathbb{I}$ such that $u^{-1}(0) = A$ and $u^{-1}(1) = B$.*

Proof. Let ϱ be an admissible metric for X. It is easily seen that the function $\alpha\colon X \to \mathbb{I}$ defined by

$$u(x) = \frac{\varrho(x,A)}{\varrho(x,A) + \varrho(x,B)}$$

is as required. □

Let X be a topological space, and let A and B be disjoint closed subsets of X. A continuous function $f\colon X \to \mathbb{R}$ for which there are distinct elements $a, b \in \mathbb{R}$ such that $f[A] \subseteq \{a\}$ and $f[B] \subseteq \{b\}$, is called a *Urysohn function*. So the function α in Lemma A.4.1 is a 'special' Urysohn function since it has the extra property that it 'separates' A from B in a precise way.

We will need a second type of Urysohn functions in the following situation later. Suppose that A and B are closed subsets of a space X. In addition, let $A' \subseteq A \setminus B$ and $B' \subseteq B \setminus A$ be closed sets which are nonempty. Let ϱ be an admissible metric for X. Define the sets A'' and B'' in X by

$$A'' = \{x \in X : \varrho(x, A) \le \varrho(x, B)\},$$
$$B'' = \{x \in X : \varrho(x, A) \ge \varrho(x, B)\}.$$

Observe that these sets are closed in X, and that

$$A'' \cap (A \cup B) = A, \quad B'' \cap (A \cup B) = B, \quad A'' \cup B'' = X.$$

By the above there are Urysohn functions $\alpha, \beta \colon X \to \mathbb{I}$ such that

$$\alpha^{-1}(0) = A', \quad \alpha^{-1}(1) = B'', \quad \beta^{-1}(0) = B', \quad \beta^{-1}(1) = A''.$$

Define $u \colon X \to \mathbb{I}$ by the following formula:

$$u(x) = \begin{cases} 1/2\,\alpha(x) & (x \in A''), \\ 1 - 1/2\,\beta(x) & (x \in B''). \end{cases}$$

If $x \in A'' \cap B''$ then $1/2\,\alpha(x) = 1/2$ and $1 - 1/2\,\beta(x) = 1/2$. Since A'' and B'' are closed in X, this easily implies that u is well defined and continuous. So it is a Urysohn function. It is special since

$$u^{-1}(0) = A', \quad u^{-1}(1) = B',$$

and

$$u[A \setminus B] \subseteq [0, 1/2), \quad u[B \setminus A] \subseteq (1/2, 1].$$

This can be verified quite easily. So we completed the proof of the following result:

Lemma A.4.2. *Let X be a space containing the closed sets A and B. If*

$$A' \subseteq A \setminus B, \quad B' \subseteq B \setminus A$$

are nonempty closed sets then there is a Urysohn function $u \colon X \to \mathbb{I}$ such that

(1) $u^{-1}(0) = A'$ *and* $u^{-1}(1) = B'$,
(2) $u[A \setminus B] \subseteq [0, 1/2)$ *and* $u[B \setminus A] \subseteq (1/2, 1]$.

We will now present some simple consequences.

Corollary A.4.3. *Let X be a space and let A and B be disjoint closed subsets of X. Then there are disjoint open subsets U and V in X such that*

$$A \subseteq U, \quad B \subseteq V.$$

Proof. By Lemma A.4.1 there exists a continuous function $u \colon X \to \mathbb{I}$ such that $u \restriction A \equiv 0$, and $u \restriction B \equiv 1$. Put $U = u^{-1}\big[[0, 1/2)\big]$ and $V = u^{-1}\big[(1/2, 1]\big]$, respectively. Then U and V are clearly as required. □

Corollary A.4.4. *Let X be a space. Then X can be imbedded in Q and in $\mathbb{R}^\infty$.*

Proof. Let $\mathcal{B}$ be a countable open base for X. For every pair (E, F) of elements of $\mathcal{B}$ with $\overline{E} \subseteq F$ pick a Urysohn function $u\colon X \to \mathbb{I}$ such that

$$u[\overline{E}] \subseteq \{0\}, \quad u[X \setminus F] \subseteq \{1\}$$

(Lemma A.4.1). Let $\mathcal{U}$ denote the countable collection of functions obtained in this way. The function $e\colon X \to \mathbb{I}^{\mathcal{U}}$ defined by

$$e(x) = (u(x))_{u \in \mathcal{U}}$$

is easily seen to be an imbedding (Exercise A.4.1).

Since Q clearly imbeds in $\mathbb{R}^\infty$, we are done. □

We will now generalize Lemma A.4.1 in an interesting way.

Lemma A.4.5. *Let X be a space with closed subspace A, and let $g\colon A \to \mathbb{R}$ be continuous such that $|g(a)| \le K < \infty$ for every $a \in A$. Then there is a continuous function $h\colon X \to \mathbb{R}$ such that*

$$|h(x)| \le \tfrac{1}{3}K \qquad \text{for all } x \in X,$$
$$|g(a) - h(a)| \le \tfrac{2}{3}K \qquad \text{for all } a \in A.$$

Proof. Let $B_1 = g^{-1}[-K, -\tfrac{1}{3}K], B_2 = g^{-1}[\tfrac{1}{3}K, K]$. Then B_1 and B_2 are closed and disjoint in A and so in X. By Lemma A.4.1 there exists a continuous function $\bar{h}\colon X \to \mathbb{I}$ such that $\bar{h}[B_1] \subseteq \{0\}$ and $\bar{h}[B_2] \subseteq \{1\}$. Now let $h = \tfrac{2}{3}K(\bar{h} - \tfrac{1}{2})$. We claim that h is as required. Indeed, if $x \in X$ is arbitrary then

$$|h(x)| = \tfrac{2}{3}K\,\left|\bar{h}(x) - \tfrac{1}{2}\right| \le \tfrac{2}{3}K\,\tfrac{1}{2} = \tfrac{1}{3}K.$$

Moreover, if $a \in A$ then there are three cases to consider. First assume that $a \in B_1$. Then $\bar{h}(a) = 0$ from which it follows that $h(a) = -\tfrac{1}{3}K$. Since $g(a) \in [-K, -\tfrac{1}{3}K]$ this shows that $|g(a) - h(a)| \le \tfrac{2}{3}K$. A similar calculation can be made if $a \in B_2$. Finally, if $a \notin B_1 \cup B_2$ then $|g(a)| \le \tfrac{1}{3}K$, and since by the above $|h(a)| \le \tfrac{1}{3}K$, we get $|g(a) - h(a)| \le \tfrac{2}{3}K$ as well. □

This result enables us to prove that certain continuous functions can be extended.

Theorem A.4.6. *Let X be a space with closed subspace A. If $f\colon A \to \mathbb{J}$ is continuous then there is a continuous function $F\colon X \to \mathbb{J}$ such that $F \restriction A = f$.*

Remark A.4.7. For nontrivial generalizations of this result, see §1.2.

Proof. By induction on $n \ge 0$, we define functions $h_n\colon X \to \mathbb{R}$ such that

$$|h_n(x)| \le \tfrac{1}{3} \cdot (\tfrac{2}{3})^n \qquad \text{for all } x \in X,$$

$$\left|f(a) - \sum_{i=0}^{n} h_i(a)\right| \le (\tfrac{2}{3})^{n+1} \qquad \text{for all } a \in A$$

8. Prove that the admissible metric ϱ defined on $\prod_{n=1}^{\infty} X_n$ in the proof of Lemma A.6.2 is complete.

►9. Let (X, ϱ) and Y be spaces with ϱ complete and let $f\colon X \to Y$ be continuous. Prove that the following statements are equivalent:
   (1) f is not both one-to-one and closed.
   (2) For some $\varepsilon > 0$ there exist sequences $(x_k)_k$ and $(y_k)_k$ in X such that $\varrho(x_k, y_k) > \varepsilon$ for each k and $\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f(y_k)$.

10. Let X be a Baire space. Prove that if $\mathcal{S}$ is a countable family of dense G_δ-subsets of X then $\bigcap \mathcal{S}$ is dense in X. (So in a Baire space any two dense G_δ-subsets intersect.)

11. Let X_n be a Baire space for every n. Prove that $\prod_{n=1}^{\infty} X_n$ is a Baire space.

►12. Give an example of a compact space X and an open continuous surjection $f\colon X \to Y$ and a dense G_δ-subset $S \subseteq X$ such that $f[S]$ is not a G_δ-subset of Y.

A.7. A covering type property

Let X be a space and let $\mathcal{A}$ and $\mathcal{B} = \{B(A) : A \in \mathcal{A}\}$ be covers of X (not necessarily by open or closed sets). We say that $\mathcal{B}$ is a *shrinking* of $\mathcal{A}$ if for every $A \in \mathcal{A}$, $B(A) \subseteq A$. Observe that if $B(A_0), B(A_1) \in \mathcal{B}$ are distinct then so are A_0 and A_1 and that the converse need not hold. We call $\mathcal{B}$ an *open shrinking* if $\mathcal{B}$ consists of open subsets of X. A *closed shrinking* is a shrinking consisting of closed sets.

Proposition A.7.1. *Let X be a space and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ admits a closed shrinking.*

Proof. First assume that $\mathcal{U}$ is countable. Enumerate it as $\{U_n : n \in \mathbb{N}\}$. By Lemma A.4.1, for each n there exists a continuous function $f_n\colon X \to \mathbb{I}$ such that $f_n^{-1}\big[(0,1]\big] = U_n$. Define $f\colon X \to \mathbb{I}$ by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

Since

$$\sum_{n=1}^{\infty} 2^{-n} = 1,$$

we get by Exercise A.3.1 that f is continuous. As $\mathcal{U}$ covers X, $f(x) > 0$ for every $x \in X$. Put

$$A_n = \left\{x \in X : f_n(x) \geq \frac{f(x)}{2}\right\} \qquad (n \in \mathbb{N}).$$

By continuity of the functions f_n and f, it follows easily that each A_n is closed. Also, since $f(x) > 0$ for all x, $A_n \subseteq U_n$.

We claim that the collection $\{A_n : n \in \mathbb{N}\}$ covers X. To this end, assume that there exists $x \in X$ such that for every n, $x \notin A_n$. Then

$$f_n(x) < \frac{f(x)}{2},$$

for every n, so that

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x) \leq 1/2 \cdot \sum_{n=1}^{\infty} 2^{-n} f(x) = \frac{f(x)}{2},$$

which is a contradiction since $f(x) > 0$.

Now let $\mathcal{U}$ be an arbitrary open cover of X. Let $\mathcal{V}$ be a countable subcover of $\mathcal{U}$ (Corollary A.2.3). By the above, there exists a closed shrinking

$$\mathcal{W} = \{W(V) : V \in \mathcal{V}\}$$

of $\mathcal{V}$. For each $U \in \mathcal{U}$ define the subset $E(U) \subseteq X$ by

$$E(U) = \begin{cases} W(U) & (U \in \mathcal{V}), \\ \emptyset & (U \notin \mathcal{V}). \end{cases}$$

Since $\mathcal{W}$ covers X, the collection $\mathcal{E} = \{E(U) : U \in \mathcal{U}\}$ is clearly a closed shrinking of $\mathcal{U}$. □

Let $\mathcal{A}$ be a collection of subsets of a space X. We say that $\mathcal{A}$ is *locally finite* provided that for every $x \in X$ there is a neighborhood U_x of x such that the collection

$$\{A \in \mathcal{A} : A \cap U_x \neq \emptyset\}$$

is finite. Observe that $\mathcal{A}$ is countable. This can be seen as follows: every point in X has a neighborhood meeting only finitely many elements of $\mathcal{A}$ and countably many of these neighborhoods cover X by Corollary A.2.3.

Proposition A.7.2. *Let X be a space. Then for every open cover $\mathcal{U}$ of X there is an open shrinking $\mathcal{V} = \{V(U) : U \in \mathcal{U}\}$ of $\mathcal{U}$ which is locally finite.*

Proof. Let X be a space and let $\mathcal{U}$ be an open cover of X. If $\mathcal{U}$ has a finite subcover, say $\mathcal{U}'$ then there is nothing to prove. Simply put $V(U) = U$ if $U \in \mathcal{U}'$ and $V(U) = \emptyset$ if $U \notin \mathcal{U}'$. So assume without loss of generality that $\mathcal{U}$ does not have a finite subcover.

By Proposition A.7.1 there is a closed shrinking $\{W(U) : U \in \mathcal{U}\}$ of $\mathcal{U}$. For each $U \in \mathcal{U}$ let $E(U)$ be an open subset of X such that

$$W(U) \subseteq E(U) \subseteq \overline{E(U)} \subseteq U$$

(Corollary A.4.3). The cover $\{E(U) : U \in \mathcal{U}\}$ of X has a countable subcover by Corollary A.2.3, say $\mathcal{E} = \{E(U) : U \in \mathcal{U}'\}$, where $\mathcal{U}' \subseteq \mathcal{U}$ is countable.

Observe that $\mathcal{U}'$ covers X as well. Let $\{U_n : n \in \mathbb{N}\}$ be a faithful indexing of $\mathcal{U}'$ (observe that $\mathcal{U}'$ is infinite). For every n put

$$V_n = U_n \setminus \bigcup_{m<n} \overline{E(U_m)}.$$

We claim that

$$\mathcal{V} = \{V_n : n \in \mathbb{N}\}$$

is a locally finite open cover of X. Clearly $\mathcal{V}$ consists of open sets. For each $x \in X$ let $n(x)$ be the smallest integer with $x \in U_{n(x)}$. For this number $n(x)$ we clearly have $x \in V_{n(x)}$. Consequently, $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is a cover of X. We shall prove that $\mathcal{V}$ is locally finite as well. Take an arbitrary $x \in X$. Since $\mathcal{E}$ covers X, there is an n with $x \in E(U_n)$. Clearly $E(U_n) \cap V_m = \emptyset$ for all $m > n$. Consequently, $E(U_n)$ is a neighborhood of x which intersects at most n members from $\mathcal{V}$.

Now for each $U \in \mathcal{U}$ define $V(U) \subseteq X$ by

$$V(U) = \begin{cases} V_n & (U = U_n), \\ \emptyset & (U \notin \mathcal{U}'). \end{cases}$$

Since $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ is a shrinking of $\mathcal{U}' = \{U_n : n \in \mathbb{N}\}$, this assignment is clearly as required. □

A space X is called *paracompact* if for every open cover $\mathcal{U}$ of X there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is locally finite.

Corollary A.7.3. *Every space is paracompact.*

Let X be a space. A family $\mathcal{F}$ of continuous functions from X to $\mathbb{I}$ is called a *partition of unity on* X if for each $x \in X$ there exist a neighborhood U_x of x and a *finite* subset $\mathcal{F}(x)$ of $\mathcal{F}$ such that

(1) $\sum_{f \in \mathcal{F}(x)} f(y) = 1$ for each $y \in U_x$,
(2) if $f \in \mathcal{F} \setminus \mathcal{F}(x)$ and $y \in U_x$ then $f(y) = 0$.

Each partition of unity on X is countable. This is so because countably many U_x's cover X (Corollary A.2.3) and the corresponding $\mathcal{F}(x)$'s cover $\mathcal{F}$ with the possible exception of the constant function with value 0.

If $\mathcal{F}$ is a partition of unity on X then we define

$$\mathcal{U}(\mathcal{F}) = \{f^{-1}\big[(0,1]\big] : f \in \mathcal{F}\}.$$

Lemma A.7.4. *Let X be a space and let $\mathcal{F}$ be a partition of unity on X. Then $\mathcal{U}(\mathcal{F})$ is a locally finite open cover of X.*

Proof. That $\mathcal{U}(\mathcal{F})$ consists of open sets is clear. Take an arbitrary $x \in X$ and let U_x and $\mathcal{F}(x)$ be as in the above definition. By (1) there exists an $f \in \mathcal{F}(x)$ such that $f(x) \neq 0$. We conclude that $\mathcal{U}(\mathcal{F})$ covers X. That $\mathcal{U}(\mathcal{F})$ is locally finite follows immediately from (2). □

Let $\mathcal{U}$ be a locally finite open cover of a space (X, ϱ). We shall associate with $\mathcal{U}$ a certain family of continuous functions which will be useful here as well as in Chapter 3, as follows. For every $U \in \mathcal{U}$ define $\kappa_U : X \to \mathbb{R}$ by

$$(*) \qquad \kappa_U(x) = \frac{\varrho(x, X \setminus U)}{\sum_{V \in \mathcal{U}} \varrho(x, X \setminus V)}.$$

These functions are called the *κ-functions with respect to the cover* $\mathcal{U}$. Observe that the sum in the denominator of $(*)$ contains at least one but at most finitely many non-zero terms, so that κ_U is well-defined. Also observe that $\kappa_U(x) \geq 0$.

We next claim that each κ_U is continuous. This is easy. Take an arbitrary $x \in X$. There is an open neighborhood W of x such that the set

$$\mathcal{F} = \{U \in \mathcal{U} : U \cap W \neq \emptyset\}$$

is finite. Let ρ_U denote the restriction $\kappa_U \restriction W$. Then for every $y \in W$ we have

$$\rho_U(y) = \frac{\varrho(y, X \setminus U)}{\sum_{V \in \mathcal{F}} \varrho(y, X \setminus V)}.$$

Since $\mathcal{F}$ is finite, ρ_U is a continuous function on W. Since W is a neighborhood of x this implies that κ_U is continuous at x.

Finally, observe that $\sum_{U \in \mathcal{U}} \kappa_U(x) = 1$ for every $x \in X$ and that for every $U \in \mathcal{U}$,

$$\kappa_U^{-1}\big[(0,1]\big] \subseteq U.$$

We say that a partition of unity $\mathcal{F}$ on a space X is *subordinated to a cover* $\mathcal{V}$ of X if the cover $\mathcal{U}(\mathcal{F})$ is a shrinking of $\mathcal{V}$, in the following sense. We demand that $\mathcal{F}$ can be listed as $\{f_V : V \in \mathcal{V}\}$ such that

$$f_V^{-1}\big[(0,1]\big] \subseteq V$$

for every $V \in \mathcal{V}$.

Theorem A.7.5. *Let X be a space and let $\mathcal{U}$ be an open cover of X. Then there exists a partition of unity on X which is subordinated to $\mathcal{U}$.*

Proof. By Proposition A.7.2, we may assume without loss of generality that $\mathcal{U}$ is locally finite. Let $\mathcal{K} = \{\kappa_U : U \in \mathcal{U}\}$ be the set of κ-functions with respect to $\mathcal{U}$. Since $\mathcal{U}$ is locally finite, it is clear that $\mathcal{K}$ is a partition of unity on X which is subordinated to $\mathcal{U}$. ☐

Partitions of unity are very useful. We will demonstrate this in §1.1 by using them for obtaining a classical result in selection theory. Here we will present another useful application.

Let X be a space. A function (not necessarily continuous) $f : X \to \mathbb{R}$ is called *lower semi-continuous* (abbreviated **lsc**) if $f^{-1}\big[(t, \infty)\big]$ is open in X

element $x \notin U$. It suffices to prove that $h(x_n) \to h(x)$. There are two cases to consider. Suppose first that for infinitely many n we have $h(x_n) \le h(x)$. Then $g(x_n) \le g(x)$ for those n. Since g is **usc**, $g(x_n) \to g(x)$ and so

$$h(x_n) \to g(x) = h(x).$$

The case that for infinitely many n we have $h(x) \le h(x_n)$ can be completed similarly by using the fact that f is **lsc**.

So h has all desired properties. □

Exercises for §A.7. Let (X,ϱ) and (Y,d) be metric spaces. Then $f\colon X \to Y$ is called *Lipschitz* provided that

$$d\big(f(x), f(y)\big) \le \varrho(x,y)$$

for all $x, y \in X$. Observe that a Lipschitz function is continuous.

1. Let $\mathcal{F}$ be a locally finite collection of closed subsets of a space X. Prove that $\bigcup\mathcal{F}$ is closed in X.
2. Let X be a space. Prove that the following statements are equivalent:
   (1) X is compact.
   (2) Every open cover of X has a locally finite subcover.
3. Let $\mathcal{F}$ be a discrete collection of closed subsets of a space X. Prove that every $F \in \mathcal{F}$ has an open neighborhood $U(F)$ such that the collection
   $$\{U(F) : F \in \mathcal{F}\}$$
   is discrete as well.
4. Let X be a space. Prove that the following statements are equivalent:
   (1) X is compact.
   (2) Every locally finite open cover of X is finite.

►5. Let (X,ϱ) be a space and $\mathcal{U}$ a collection of open subsets of X. Prove that there is a Lipschitz function $\varepsilon\colon X \to \mathbb{I}$ such that $\varepsilon^{-1}\big[(0,1]\big] = \bigcup\mathcal{U}$ while moreover for every $x \in X$ the open ball about x with radius $\varepsilon(x)$ is contained in an element of $\mathcal{U}$.

6. Let X be a space and let $f, g\colon X \to \mathbb{R}$ be **lsc**. For every $x \in X$ put
   $$h(x) = \min\{f(x), g(x)\}.$$
   Prove that $h\colon X \to \mathbb{R}$ is **lsc**.

A.8. Extension type properties

Let X be a space. It will be convenient to let ρX denote the family of all closed subsets of X. In addition, for every $x \in X$, we put $\varrho(x, \emptyset) = \infty$.

Lemma A.8.1. *Let X be a space with subspace Y. The function*

$$\kappa\colon \rho Y \to \rho X$$

defined by

$$\kappa(A) = \{x \in X : \varrho(x, A) \leq \varrho(x, Y \setminus A)\}$$

has the following properties:

(1) $\kappa(\emptyset) = \emptyset$, $\kappa(Y) = X$,
(2) $\kappa(A) \cap Y = A$ *for every* $A \in \rho Y$,
(3) *if* $A, B \in \rho Y$ *and* $A \subseteq B$ *then* $\kappa(A) \subseteq \kappa(B)$,
(4) *if* $A, B \in \rho Y$ *then* $\kappa(A \cup B) = \kappa(A) \cup \kappa(B)$.

Proof. The straightforward verification that κ is well-defined is left to the reader. Clearly, (1), (2), and (3) hold. For (4), take $A, B \in \rho Y$. Observe that by (3), $\kappa(A) \cup \kappa(B) \subseteq \kappa(A \cup B)$. Let $x \in \kappa(A \cup B)$. Since

$$\varrho(x, A \cup B) \in \{\varrho(x, A), \varrho(x, B)\}$$

(Exercise A.1.7), without loss of generality $\varrho(x, A \cup B) = \varrho(x, A)$. We shall prove that $x \in \kappa(A)$. Since $x \in \kappa(A \cup B)$,

$$\varrho(x, A) = \varrho(x, A \cup B) \leq \varrho\big(x, Y \setminus (A \cup B)\big), \tag{5}$$

and trivially,

$$\varrho(x, A) = \varrho(x, A \cup B) \leq \varrho(x, B). \tag{6}$$

Since $Y \setminus A \subseteq \big(Y \setminus (A \cup B)\big) \cup B$, (5) and (6) and Exercise A.1.7 imply that

$$\varrho(x, A) \leq \varrho\big(x, \big(Y \setminus (A \cup B)\big) \cup B\big) \leq \varrho(x, Y \setminus A),$$

so $x \in \kappa(A)$. □

Let X be a space. Subsets A and B of X are called *separated* if

$$\overline{A} \cap B = \emptyset = A \cap \overline{B}.$$

It is clear that if A and B are disjoint and are both closed, or both open, then A and B are separated. More interesting examples of separated sets are obtained in the following way. Let Y be a subspace of X and let U and V be disjoint subsets of Y that are open *in* Y. Then U and V are separated *in* X. For let U' be an open subset of X such that $U' \cap Y = U$. Then $U' \cap V = \emptyset$, i.e., $\overline{V} \cap U = \emptyset$. It follows similarly that $\overline{U} \cap V = \emptyset$.

Corollary A.8.2. *Let A and B be separated subsets of a space X. Then A and B can be separated by disjoint open subsets of X.*

Proof. It is clear that A and B are closed in their union $A \cup B$. By Lemma A.8.1, there exist closed subsets A' and B' of X such that

$$A \subseteq A', \quad B \subseteq B', \quad B \cap A' = \emptyset, \quad B' \cap A = \emptyset, \quad A' \cup B' = X.$$

So $U = X \setminus B'$ and $V = X \setminus A'$ are disjoint open neighborhoods of A and B, respectively. □

We now turn to extendable continuous functions.

Lemma A.8.3. *Let Y be a dense subspace of X, let Z be compact, and let $f\colon Y \to Z$ be continuous. The following statements are equivalent:*

(1) *f can be extended to a continuous function $\bar{f}\colon X \to Z$.*
(2) *For all closed sets $A, B \subseteq Z$ we have*

$$A \cap B = \emptyset \quad \Longrightarrow \quad \overline{f^{-1}[A]} \cap \overline{f^{-1}[B]} = \emptyset.$$

(Here closure means closure in X.)

Proof. For (1) ⇒ (2), pick disjoint closed subsets $A, B \subseteq Z$. Since $\bar{f}$ extends f we have

$$(*) \qquad f^{-1}[A] \subseteq \bar{f}^{-1}[A] \text{ and } f^{-1}[B] \subseteq \bar{f}^{-1}[B].$$

Observe that by continuity of $\bar{f}$, $\bar{f}^{-1}[A]$ and $\bar{f}^{-1}[B]$ are disjoint closed sets of X. So we get what we want from $(*)$.

The proof of (2) ⇒ (1) is more complicated. Pick an arbitrary $x \in X$ and let $\mathcal{B}(x)$ be the collection of all its open neighborhoods in X. Put

$$\mathcal{F}(x) = \{\overline{f[U \cap Y]} : U \in \mathcal{B}(x)\}.$$

(Here closure means closure in Z.) Since for a finite subcollection $\mathcal{U} \subseteq \mathcal{B}(x)$ we have $\bigcap \mathcal{U} \in \mathcal{B}(x)$ and

$$\overline{f\big[\bigcap \mathcal{U} \cap Y\big]} \subseteq \bigcap_{U \in \mathcal{U}} \overline{f[U \cap Y]},$$

the family $\mathcal{F}(x)$ has the finite intersection property (here we use that Y is dense in X). By compactness of Z, $\mathcal{F}(x)$ consequently has non-empty intersection.

We shall prove that $\bigcap \mathcal{F}(x)$ consists of a single point. Let y_1 and y_2 be distinct elements of Z. There exist closed neighborhoods V_1 and V_2 of y_1 and y_2 respectively, such that $V_1 \cap V_2 = \emptyset$. From (2) it follows that

$$\overline{f^{-1}[V_1]} \cap \overline{f^{-1}[V_2]} = \emptyset.$$

We may assume without loss of generality that $x \notin \overline{f^{-1}[V_1]}$. Thus

$$X \setminus \overline{f^{-1}[V_1]} \in \mathcal{B}(x).$$

Since $f\big[Y \setminus f^{-1}[V_1]\big]$ misses V_1, its closure misses the interior of V_1. This means that $y_1 \notin \bigcap \mathcal{F}(x)$.

Observe that if $x \in Y$ then $f(x) \in \bigcap \mathcal{F}(x)$ so that, by what we just proved, the unique point in $\bigcap \mathcal{F}(x)$ is $f(x)$.

Exercises for §A.9.

1. Let X be a space with Wallman base $\mathcal{F}$. Prove that for every $F \in \mathcal{F}$ the set F^* coincides with the closure of F in $\omega(X, \mathcal{F})$.
2. Let X be a space with Wallman base $\mathcal{F}$. Prove that for all disjoint nonempty closed sets A and B in $\omega(X, \mathcal{F})$ there are disjoint $F_0, F_1 \in \mathcal{F}$ such that $A \subseteq F_0^*$ and $B \subseteq F_1^*$.
3. Let X be a space and let $\mathcal{S}$ be a countable collection of closed subsets of X. Prove that X has a compactification γX such that if $S_0, S_1 \in \mathcal{S}$ are disjoint then their closures in γX are disjoint as well.
4. Let $\mathcal{F}$ and $\mathcal{G}$ be Wallman bases of X. Prove that the following statements are equivalent:
   (1) $\omega(X, \mathcal{F}) \leq \omega(X, \mathcal{G})$,
   (2) for all disjoint $F_0, F_1 \in \mathcal{F}$ there exist disjoint $G_0, G_1 \in \mathcal{G}$ such that $F_0 \subseteq G_0$ and $F_1 \subseteq G_1$.

A.10. Connectivity

If $x \in X$ then it is easy to see that the *component* of x is equal to

$$\bigcup\{C \subseteq X : C \text{ connected and } x \in C\}.$$

Since if $A \subseteq X$ is connected then so is $\overline{A}$, it follows that the component of x is a closed subset of X.

Lemma A.10.1. *Let C be a component of the compact space X. Then for every neighborhood U of C there exists a clopen subset E of X such that $C \subseteq E \subseteq U$.*

Proof. Let $C(x)$ denote the component of the point $x \in X$. In addition, let $\mathcal{G}$ be the family of all clopen neighborhoods of x, and put

$$Q(x) = \bigcap \mathcal{G}.$$

Claim 1. For every neighborhood U of $Q(x)$ there exists a clopen subset E of X such that $Q(x) \subseteq E \subseteq U$.

Proof. Observe that the intersection of finitely many elements of $\mathcal{G}$ again belongs to $\mathcal{G}$. Without loss of generality we assume that U is open. Since

$$(X \setminus U) \cap \bigcap \mathcal{G} = \emptyset$$

it follows by compactness that U contains an element of $\mathcal{G}$. $\diamondsuit$

It is clear that $C(x)$ is a subset of $Q(x)$, since if A is any clopen subset of X then either $C(x) \cap A = \emptyset$ or $C(x) \subseteq A$. We will prove that $Q(x)$ is connected. If this is true then $Q(x) \subseteq C(x)$, i.e., $C(x) = Q(x)$. So then we get what we want by Claim 1.

Striving for a contradiction, assume that $Q(x)$ is not connected. We shall derive a contradiction. We can write $Q(x)$ as $E \cup F$, where both E and F are nonempty closed subsets of X and $E \cap F = \emptyset$. Without loss of generality, we may assume that $x \in E$. There exist disjoint open neighborhoods U and V of E and F, respectively (Corollary A.4.3). Since $U \cup V$ is open and contains $Q(x)$, by Claim 1 there exists a clopen subset G of X such that $Q(x) \subseteq G \subseteq U \cup V$. Observe that

$$G \cap U = G \setminus V$$

is both open and closed. But this implies that $Q(x) \subseteq G \cap U$ and so $F = \emptyset$, which is a contradiction. □

Corollary A.10.2. *Let X be a compact space every component of which has diameter less than ε. Then X can be split into finitely many clopen sets of diameter less than ε.*

Proof. Let C be an arbitrary component of X. By Lemma A.10.1 there is a clopen neighborhood U_C of C with diameter less than ε. Finitely many U_C's cover X by compactness of X. So we are done by Exercise A.1.3. □

Let $\mathcal{P}$ be a decomposition of X into pairwise disjoint nonempty closed subsets. We say that $\mathcal{P}$ is *upper semi-continuous* provided that for every closed subset A of X, $\bigcup\{P \in \mathcal{P} : P \cap A \neq \emptyset\}$ is closed in X.

Proposition A.10.3. *Let X be compact. Then the collection of all components of X is an upper semi-continuous decomposition of X.*

Remark A.10.4. The compactness of X is an essential in this result. See Exercises A.10.9 and A.10.11.

Proof. Let $\mathcal{C}$ denote the collection of all components of X and let A be closed. We shall prove that if $\mathcal{P} = \{C \in \mathcal{C} : C \cap A \neq \emptyset\}$ then $\bigcup \mathcal{P}$ is closed in X. Take $x \notin \bigcup \mathcal{P}$. Then the component $C(x)$ of X which contains x misses A, hence by Lemma A.10.1, for some clopen E in X,

$$C(x) \subseteq E \subseteq X \setminus A.$$

But then $E \cap \bigcup \mathcal{P} = \emptyset$ and so $x \notin \overline{\bigcup \mathcal{P}}$. □

A *continuum* is a space which is both compact and connected.

Corollary A.10.5. *Let A be a closed subspace of a continuum X such that $\emptyset \neq A \neq X$. Then every component C of A meets* $\operatorname{Fr} A$.

Proof. First observe that if $\operatorname{Fr} A = \emptyset$ then $A = X$ by connectivity of X. So $\operatorname{Fr} A \neq \emptyset$. Striving for a contradiction, suppose that C is a component of A such that $C \cap \operatorname{Fr} A = \emptyset$. Then $X \setminus \operatorname{Fr} A$ is a neighborhood of C. By compactness of A there exists by Proposition A.10.3 and Lemma A.10.1 a