Vanishing beyond all orders: Stokes lines in a water-wave model equation

Jan Bouwe van den Berg* and John R. King†

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Abstract

We study the existence of solutions homoclinic to a saddle centre in a family of singularly perturbed fourth order differential equations, originating from a water wave model. Due to a reversibility symmetry, the occurrence of such embedded solitons is a codimension-1 phenomenon. By varying a parameter a countable family of solitary waves is found. We examine the asymptotic frequency at which this phenomenon of persistence in the singular limit occurs, by performing a refined Stokes line analysis. In the limit where the parameter tends to infinity, each Stokes line splits into a pair, and the contributions of these two Stokes lines cancel each other for a countable set of parameter values. More generally, we derive the full leading order asymptotics for the Stokes constant, which governs the (exponentially small) amplitude of the (minimal) oscillations in the tails of nearly homoclinic solutions. True homoclinic trajectories are characterised by the Stokes constant vanishing. This formal asymptotic analysis is supplemented with numerical calculations.

1 Introduction

In many singularly perturbed problems, exponentially small terms have to be taken into account to study the persistence of solutions. Often an algebraic expansion suggests the existence of certain (types of) solutions, which eventually — beyond all algebraic orders — are destroyed by an exponentially small term. However, sometimes the coefficient of the exponentially small term vanishes, and the conclusion derived from the algebraic expansion is then valid after all. In this paper we consider an instructive model problem, illustrating how matched asymptotic expansions can be used to study how frequently the relevant exponentially small term vanishes. The method will be illustrated for a well-known model equation, introduced below, where the objects of interest are solutions homoclinic to saddle centres, also known as embedded solitons.

It is by now well established that the homoclinic solution of the second order equation

$$u'' - u + u^2 = 0 \tag{1}$$

does not persist for the (singularly perturbed) fourth order problem

$$\varepsilon^2 u'''' + u'' - u + u^2 = 0. \tag{2}$$

To be more precise, (2) has *no* solution homoclinic to the origin for small positive ε, as was observed in [1, 9] and proved in [2]. There are numerous explanations of this phenomenon using matched asymptotics in the complex plane [12, 15], optimal truncation [14, 7] and more rigorous

*Department of Mathematics, VU University Amsterdam, janbouwe@few.vu.nl

†School of Mathematical Sciences, University of Nottingham, john.king@nottingham.ac.uk

methods [13, 16], all these approaches being closely related. In particular, one may write down a formal homoclinic solution

$$u = \sum_{n=0}^{N-1} \varepsilon^{2n} u_n(x)$$

to any finite algebraic order N, but the remainder term, at any order of truncation, contains an exponentially small, rapidly oscillating term

$$2\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}\sin(x/\varepsilon), \tag{3}$$

where $\Lambda \neq 0$.[1] In §2 we recall one route to calculating this term, namely via a matched asymptotic expansion analysis of the Stokes line in the complex plane. As it turns out, Λ can be described in terms of the limit of a certain recursively defined sequence.

The nonexistence of a homoclinic solution to the origin of (2) does not come as a surprise, since the stable and unstable manifolds of the origin are both one dimensional. Taking the reversal symmetry $x \to -x$ into account, one concludes that a homoclinic connection is a codimension-1 phenomenon. In this paper, we therefore consider the two parameter problem

$$\varepsilon^2 u'''' + u'' - u + u^2 + \mu\varepsilon^2(u'^2 + 2uu'') = 0, \tag{4}$$

which appears in the study of water waves [11, 8]. We are interested in the asymptotic problem $\varepsilon \to 0$, $\mu \to +\infty$, the limits being taken in that order so that, in particular, we require $\mu \ll 1/\varepsilon^2$. The additional term does not break the reversibility, and one may thus expect homoclinic solutions for a discrete set of μ-values (for each $\varepsilon > 0$). We shall denote those values for which a homoclinic orbit exists by $\mu_m(\varepsilon)$, and we write $\mu_m = \lim_{\varepsilon\to 0}\mu_m(\varepsilon)$ for simplicity. The number of curves in the (μ, ε) parameter plane for which homoclinic orbits exist, can a priori be zero, finite or infinite. Such orbits homoclinic to a saddle centre are embedded in a family of solutions that limit to small periodic solutions in their tails, hence the name *embedded solitons*. Solutions that are homoclinic to a small periodic orbit come in one parameter families (for fixed ε and μ), and exist for any value of μ for small ε. The minimal amplitude of the periodic solutions in such a family is $\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}$, asymptotically for small ε. It is thus quite natural that (nondegenerate) zeros of $\Lambda = \Lambda(\mu)$ correspond to homoclinic solutions in the limit $\varepsilon \to 0$ [5, 13].

For the water wave equation (4), numerical calculations [6] show the existence of an *infinite* number of curves $\{\mu_n(\varepsilon)\}_{n=1}^{\infty}$ in the (μ, ε) parameter plane, for which homoclinic solutions exist and continue to exist in the limit $\varepsilon \to 0$. In terms of the asymptotics beyond all orders, this is characterised by the fact that the amplitude Λ of the exponentially small term has infinitely many zeros as a function of μ.

As was noted in [6], the first of these branches of solutions is explicit:

$$\mu_1(\varepsilon) = \frac{5}{2}\frac{5(1+4\varepsilon^2)^{1/2} - 3}{4 + 25\varepsilon^2}, \qquad \mu_1 = \frac{5}{4},$$

with corresponding solution

$$u(x) = \frac{75}{8(\mu_1 + 5)\cosh^2\left(x\left(\frac{5\mu_1}{4(\mu_1+5)}\right)^{1/2}\right)}.$$

For $m \geq 2$ the values μ_m cannot be expressed in closed form, but can be approximated numerically using numerical continuation of the homoclinic orbit [6]. This is a nontrivial task, since in the limit $\varepsilon \to 0$ one has to distinguish true homoclinic orbits from orbits that limit to exponentially small limit cycles. Alternatively, one may approximate $\Lambda(\mu)$ using a recursion formula, and calculate

[1] The reason why the factor 2π has not been absorbed into the constant Λ will become clearer later.

the zeros of the resulting polynomial numerically, see §4.1. Indeed, this is a much more efficient method for determining μ_m.

Although $\Lambda(\mu)$ can be calculated (approximately) for every value of μ (see §4.1), the dependence on μ is not captured, in general, by an explicit formula. Hence, one cannot determine explicitly all values of μ for which homoclinic orbits exist for small ε, but studying the asymptotics for large μ is a feasible proposition, as noted by Champneys in [5]. In other words, one may wonder how often homoclinic orbits occur for large μ. This question was not just posed in [5] but also studied using normal forms and Melnikov functions (see also [13]). A drawback of the method presented there is that one needs to invoke an "unjustified hypothesis" (cited from [5]). Here we present an alternative method for computing the asymptotic behaviour of the sequence μ_m. It relies on a matched asymptotic expansion method, which is fairly straightforward computationally, and has the additional benefit of being fully systematic and hence self-consistent. In particular, for the problem (4) the values μ_m for which there is persistence of the homoclinic orbit for $\varepsilon > 0$ behave asymptotically as

$$\mu_{m+1} - \mu_m \sim \frac{\pi^2}{24} m \qquad \text{as } m \to \infty. \tag{5}$$

The asymptotic result (5), obtained via matched asymptotic expansions, confirms the spacing between homoclinic orbits predicted in [5]. In fact, we derive the following more refined asymptotic expansion for the amplitude $\Lambda = \Lambda(\mu)$ appearing in (3), in terms of the parameter μ:

$$\Lambda(\mu) \sim \Omega \mu^{-19/16} \sin\Big(4\sqrt{3}\mu^{1/2} + \frac{\pi}{16}\Big) \qquad \text{as } \mu \to +\infty, \tag{6}$$

where Ω is defined as the limit of a recursively defined sequence, and its numerical value is approximately -0.152.

We derive (5) via two routes. By way of introduction, in §2 we derive the recurrence relation that leads to the definition of $\Lambda(\mu)$, and we show how this is related to the Stokes phenomenon (which turns on the fast oscillations). In §3 we derive (5) directly via matched asymptotic expansions and a detailed analysis of the asymptotic behaviour of the Stokes lines as $\mu \to +\infty$. The delicate asymptotics that lead to (6) are established in §3.2. We find that each $\varepsilon \to 0$ Stokes line splits into two in the limit $\mu \to +\infty$. The combined effect of the Stokes lines leads to the expression (6). In particular, the two terms turned on when crossing these Stokes lines cancel each other for a countable set of μ-values. The analysis of the Stokes lines for the asymptotic problem $\mu \to +\infty$ builds on the one for finite μ, but is considerably more involved, which motivates us to first go over all the necessary steps for fixed μ in §2, before moving on to the large μ asymptotics in §3.

In §4 we take the complementary approach of analysing the recurrence relation directly. Numerically obtained results for the function $\Lambda(\mu)$ and its limit behaviour as $\mu \to +\infty$ are presented in §4.1. Next, we show that the main asymptotic behaviour of the zeros of $\Lambda(\mu)$, see (5), can be derived from the recurrence relation itself. This alternative is more lengthy than the delicate analysis of the Stokes lines preformed in §3, but it leads to additional understanding about the relation between the matched asymptotic expansions near the pole in the complex plane and the asymptotics of the recurrence relation that defines Λ, and is of interest in its own right in terms of the asymptotic analysis of difference equations. Finally, in §5 we conclude with some open problems.

2 Role and definition of the amplitude Λ

2.1 Initial Stokes line analysis

We recall the Stokes lines analysis using optimal truncation from [7], which we reformulate slightly to develop a framework to which we can refer in the subsequent sections, and apply to the current example. For $\varepsilon = 0$ the equation has the homoclinic solution

$$u_0 = \frac{3}{2} \frac{1}{\cosh^2(x/2)}. \tag{7}$$

In the complex plane, this function has poles on the imaginary axis; the ones closest to the real axis are at $\pm\pi i$. We expand the sought for homoclinic solution u as an algebraic series in ε, i.e.

$$u \sim \sum_{n=0}^{\infty} \varepsilon^{2n} u_n, \tag{8}$$

where the functions u_n tend to 0 as $x \to \pm\infty$, and they have poles at the same positions as u_0. As detailed in [7] they are dominated (as $n \to \infty$, $x \in \mathbb{R}$) by

$$u_n \sim \Gamma(2n+2) \left[\chi(x - i\pi)^{-2n-2} + \overline{\chi}(x + i\pi)^{-2n-2} \right],$$

for some constant $\chi \in \mathbb{C}$. Let us truncate the formal algebraic series after N terms:

$$u = \sum_{n=0}^{N-1} \varepsilon^{2n} u_n + w_N$$

and study the remainder w_N, which asymptotically (for small ε, large N) satisfies (neglecting quadratic terms in w_N and higher order terms in ε)

$$\varepsilon^2 w_N'''' + w_N'' + (2u_0 - 1) w_N \sim -\varepsilon^{2N} u_{N-1}''''.$$

A WKB ansatz shows that the solutions of the homogeneous version have asymptotic behaviour (as $\varepsilon \to 0$)

$$w \sim e^{ix/\varepsilon}, \qquad w \sim e^{-ix/\varepsilon}, \qquad w \sim u_0', \qquad w \sim u_0' \int (u_0')^{-2}.$$

We will see that terms of the form $e^{\pm ix/\varepsilon}$ are turned on across the Stokes lines, which run *down* the imaginary axis from the pole at $x = \pi i$ and *up* the imaginary axis from the pole $x = -\pi i$ (the contributions from the other poles are negligible on the real line). As will be explained below, the analysis near the poles also reveals the amplitude of the oscillatory term that is switched on across the Stokes lines.

As it will turn out, in the strip $0 < \mathrm{Im}(x) < 2\pi$ in the complex plane a term $e^{-ix/\varepsilon}$ is the dominant exponentially small term. It should match near $x = \pi i$ with an inner scale, as detailed in §2.2. Although introducing this inner scale is not strictly necessary at this point, it separates the order of magnitude (in terms of ε) from the Stokes constant Λ, and paves the way for later generalisation (such as in §3.2).

From those local considerations in the inner scale emerges a global Stokes line across which a term of the leading order asymptotic form $A(\varepsilon)e^{-ix/\varepsilon}$ is in general switched on. The position of the Stokes line is then found by identifying where the exponent has vanishing imaginary part (and negative real part), see e.g. [14].

2.2 Inner expansion near the pole $x = \pi i$

We recall from (7) that

$$u_0 \sim -6(x \mp \pi i)^{-2} \qquad \text{as } x \to \pm\pi i.$$

Substituting the expansion $u = \sum \varepsilon^{2n} u_n$ into (4) and evaluating in a neighbourhood of the pole $x = \pm\pi i$, one sees that

$$u_n(x) \sim e_n(x \mp \pi i)^{-2n-2} \qquad \text{as } x \to \pm\pi i.$$

This inspires us to introduce the following new variables for an inner scale near the pole $x = \pm\pi i$:

$$z = \varepsilon^{-1}(x \mp \pi i) \qquad \text{and} \qquad u = \varepsilon^{-2} v. \tag{9}$$

The equation for $v(z)$ at leading order is

$$v'''' + v'' + v^2 + \mu(v'^2 + vv'') = 0, \tag{10}$$

with

$$v \sim -6z^{-2} \qquad \text{for large } z. \tag{11}$$

We remark that the scaling (9) and equation (10) can also be found by looking directly for scales on which a nontrivial balance of terms arises.

Equation (10) has a formal solution of the form

$$v(z) = \sum_{n=0}^{\infty} A_n (-1)^{n+1} (2n+1)! z^{-2n-2}, \tag{12}$$

where $A_0 = 6$ and the $A_n = A_n(\mu)$, which are related to e_n via $A_n = (-1)^{n+1} e_n$, satisfy the recurrence relation

$$\begin{aligned}
[(2n+4)(2n+5) - 2A_0]A_{n+1} = (2n+4)(2n+5)A_n + \sum_{k=0}^{n-1} \frac{(2k+3)!(2n-2k+1)!}{(2n+3)!} A_{k+1}A_{n-k} \\
- \mu \sum_{k=0}^{n} (4n-2k+8) \frac{(2k+1)!(2n-2k+2)!}{(2n+3)!} A_k A_{n-k}.
\end{aligned} \tag{13}$$

In particular, we see that $A_n = A_n(\mu)$ is a polynomial of degree n in μ. Finally, we define

$$\Lambda = \Lambda(\mu) = \lim_{n\to\infty} A_n(\mu). \tag{14}$$

We note that the factor $(-1)^{n+1}(2n+1)!$ in (12) was chosen such that the remaining coefficient A_n tends to a constant as $n \to \infty$.

Clearly, the series (12) is an asymptotic one, diverging for fixed z. Hence, we need to truncate it:

$$v = \sum_{n=0}^{N-1} A_n (-1)^{n+1} (2n+1)! z^{-2n-2} \; + \; R_N, \tag{15}$$

where the remainder satisfies to leading order (for large z, large N)

$$R_N'''' + R_N'' = 0. \tag{16}$$

As will be explained in more detail below, using the *optimal* truncation formalism from [7], we can calculate that a term of the form e^{-iz} is turned on across the Stokes line that runs down the imaginary axis from the pole at $z = 0$. Similarly a Stokes line runs up the imaginary axis from $z = 0$ across which a term of the form e^{+iz} is turned on. As it turns out, Λ determines the amplitude of the oscillatory terms that are switched on across the Stokes lines.

Before going into detailed calculations, let us remark that by matching for large z to the outer scale $x \mp \pi i = O(1)$, we see that the term $v = ce^{-iz}$, turned on across the Stokes line running down the imaginary axis from $z = \pi i$, corresponds to an exponentially small contribution $c\varepsilon^{-2}e^{-\pi/\varepsilon}e^{-ix/\varepsilon}$ to u, and similarly for the term $v = ce^{iz}$ that is turned on when crossing the Stokes line running up the imaginary axis from $-\pi i$. Hence, the scaling with ε of these oscillations has already been inferred without any involved calculations, with only the coefficient c, which depends on μ but not on ε, remaining to be determined.

2.3 Optimal truncation

While the leading order equation for the remainder is (16), we need additional terms to perform optimal truncation. Hence, to next order we have

$$R_N'''' + R_N'' - 12z^{-2}R_N \sim -v_{N-1}'''', \tag{17}$$

where v_{N-1} is the final term in the sum in (15).

We concentrate on the term e^{-iz} and obtain the result for e^{iz} by using the symmetry $z \to -z$. The following steps can also be found in [7]. First we *optimally* truncate the right-hand side of (17):

$$v_{N-1}'''' \sim \frac{(-1)^{N+1}A_{N-1}(2N+3)!}{z^{2N+4}}$$

(i.e., we choose for fixed large z the value of N for which this is smallest in magnitude). Using polar coordinates $z = re^{i\theta}$ we get

$$v_{N-1}'''' \sim (-1)^{N+1}\Lambda\sqrt{2\pi}e^{-(2N+3)}(2N+3)^{2N+7/2}r^{-(2N+4)}e^{-i\theta(2N+4)},$$

which is smallest when $N = r/2 + s$, where s is bounded as $r \to \infty$. Hence, when optimally truncated equation (17) reduces to (choosing $-1 = e^{-\pi i}$ for convenience, rather than $e^{+\pi i}$; this is not clear a priori, but the other choice is equally valid if we keep in mind that $r + 2s$ is always an even integer)

$$R'''' + R'' - 12z^{-2}R \sim \sqrt{2\pi}\Lambda r^{-1/2}e^{-i\pi(r+2s+2)/2}e^{-r}e^{-i\theta(r+2s+4)}.$$

Anticipating that a term of the form e^{-iz} is turned on, we write

$$R = F(z)e^{-iz}.$$

By using that the derivative in polar coordinates can be written as

$$\frac{d}{dz} = -\frac{ie^{-i\theta}}{r}\frac{d}{d\theta}, \tag{18}$$

to highest non-vanishing order (for large r) one obtains

$$2r^{-1}e^{-i\theta}e^{-ire^{i\theta}}\frac{dF}{d\theta} \sim \sqrt{2\pi}\Lambda r^{-1/2}e^{-i\pi(r+2s+2)/2}e^{-r}e^{-i\theta(r+2s+4)},$$

hence

$$\frac{dF}{d\theta} \sim \sqrt{\pi/2}\Lambda r^{1/2}e^{-i\pi(r+2s+2)/2}e^{-r(1-ie^{i\theta})}e^{-i\theta(r+2s+3)}.$$

The right-hand side is exponentially small, except near $\theta = -\pi/2$. Let us introduce $\theta = -\frac{\pi}{2} + \tau\tilde{\theta}$, where τ is small (to be chosen a little later). Then, expanding in τ, many terms in the exponentials cancel and we are left with

$$\frac{d\tilde{F}}{d\tilde{\theta}} \sim \sqrt{\pi/2}\Lambda\tau r^{1/2}e^{i\pi/2}e^{-r\tau^2\tilde{\theta}^2/2}.$$

Finally, choosing $\tau = r^{-1/2}$ (i.e. the scale becomes small for large r) leads to

$$\frac{d\tilde{F}}{d\tilde{\theta}} \sim \pi i 2^{-1/2}\Lambda e^{-\pi\tilde{\theta}^2/2},$$

embodying the familiar error function smoothing that occurs across a Stokes line [3]. By integrating one obtains $\tilde{F}(+\infty) - \tilde{F}(-\infty) = \int_{-\infty}^{\infty} \pi i 2^{-1/2}\Lambda e^{-\pi\tilde{\theta}^2/2}d\tilde{\theta} = i\pi\Lambda$. Hence, crossing the ray $\theta = -\pi/2$ clockwise for large r, a term $i\pi\Lambda e^{-iz}$ in v is turned on. On the real x-axis, where $z = (x - \pi i)/\varepsilon$, this corresponds to an exponentially small term in u of size

$$i\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}e^{-ix/\varepsilon}. \tag{19}$$

For the other term e^{iz} we use the symmetry $z \to -z$. Hence, crossing the ray $\theta = -\pi/2$, again in the clockwise direction, a term $i\pi\Lambda e^{iz}$ is turned on for large r. Translating this back to the original variables $u(x)$ and noticing that for increasing x the ray emanating downwards from $z = \pi i$ is crossed in the clockwise direction, whereas the ray emanating upwards from $z = -\pi i$ is crossed in the anti-clockwise direction, we see that on crossing the origin $x = 0$ in total an exponentially small term of the form

$$u \sim \pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}(ie^{-ix/\varepsilon} - ie^{ix/\varepsilon}) = 2\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}\sin(x/\varepsilon)$$

is turned on.

Let us remark that the above analysis does not imply that there exists a solution in the unstable manifold of $u = 0$ (as $x \to -\infty$) which connects for large x to an exponentially small periodic solution. In fact, the differential equation is conservative, the conserved quantity, the *energy*, being

$$E = \varepsilon^2 u'''u' - \frac{\varepsilon^2}{2}u''^2 + \frac{1}{2}u'^2 - \frac{1}{2}u^2 + \frac{1}{3}u^3 + \mu\varepsilon^2 uu'^2.$$

It is not hard to infer that solutions that tend to 0 as $x \to -\infty$, and thus have energy $E = 0$, cannot have any extremal values smaller that $\frac{3}{2}$. In particular no oscillations near 0 are possible for such solutions. This implies that for solutions that start, for $x \to -\infty$, in the unstable manifold of 0 and that (at least initially) stay close to (7), the exponentially small rapid modulations never become dominant as $x \to +\infty$. Instead an exponentially growing term (in x), with a coefficient which is exponentially small in ε but of an exponential order smaller than the one considered in the arguments above, becomes dominant before the algebraic terms in (8) have decreased sufficiently for the exponentially small modulations to become visible.

On the other hand, solutions that are asymptotic to a small periodic orbit in *both* limits $x \to \pm\infty$ do exist, and the amplitude of the limit cycles is bounded below by $\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}$, see e.g. [13, 10]. Asymptotically, those limit cycles are of the form $P_\pm \cos(x/\varepsilon) + Q_\pm \sin(x/\varepsilon)$ as $x \to \pm\infty$, with $P_+ \sim P_-$ and $Q_+ \sim Q_- + 2\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}$, as the Stokes line analysis above shows. The asymptotic energy of the limit cycles is $E_\pm = -\frac{1}{2}(P_\pm^2 + Q_\pm^2)$, with, since the energy is conserved, $E_+ = E_-$. This implies that connections are only possible when $Q_- = -\pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}$ and $Q_+ = \pi\Lambda\varepsilon^{-2}e^{-\pi/\varepsilon}$. Furthermore, for each energy $E < E_{\min} \sim -\pi^2\Lambda^2\varepsilon^{-4}e^{-2\pi/\varepsilon}/2$ there are *two* connections between periodic orbits, one with $P_\pm \sim [2(E_{\min} - E)]^{1/2}$ and one with $P_\pm \sim -[2(E_{\min} - E)]^{1/2}$.

3 Matched asymptotics for the Stokes line

3.1 Large μ asymptotics

In this section we derive the behaviour of $\Lambda(\mu)$ for large μ. Although $\Lambda(\mu)$ can be computed (e.g. numerically, see §4.1) from its definition via the recurrence relation, such recurrence relations can be difficult to work with, and it is more convenient to stick, for as long as possible, with differential equations, for which a matched asymptotic analysis can be performed more easily.

As is discussed in detail below, in the limit $\mu \to +\infty$ an additional inner scale appears near the poles $x = \pm\pi i$, and the Stokes lines split in pairs, see Figure 1.

Near the pole $x = \pi i$ of the homoclinic solution (7) of (1), let us, as in §2.2, introduce

$$z = \frac{x - \pi i}{\varepsilon} \qquad \text{and} \qquad v = \varepsilon^2 u.$$

The differential equation for $v(z)$ at leading order is (10), with asymptotic behaviour (11). To

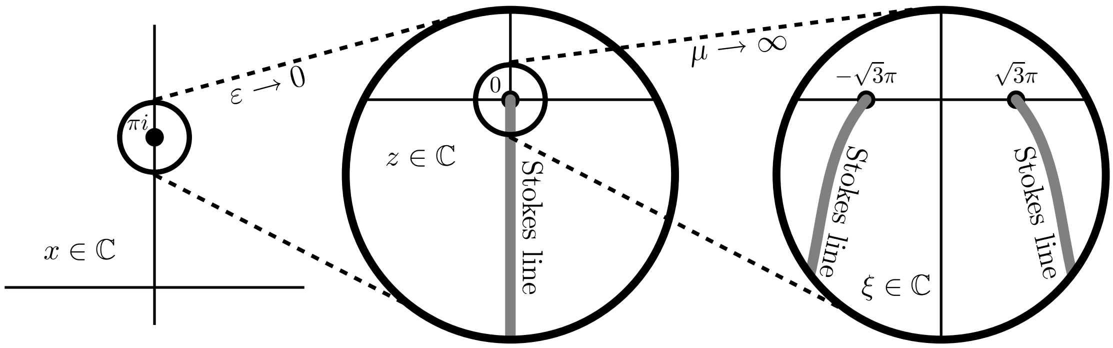

Figure 1: In the limit $\varepsilon \to 0$ a Stokes line appears, which emanates from the pole $x = \pi i$. A blowup near the pole allows an analysis of the exponentially small term (and its amplitude) switched on across the Stokes line, see §2.2 and §2.3. Moreover, in the limit $\mu \to +\infty$ another scale appears in which the Stokes line splits into a pair emanating from $\xi = \pm\sqrt{3}\pi$.

consider the asymptotic limit $\mu \to +\infty$, we use rescaled variables[2]

$$\xi = \mu^{-1/2} z \qquad \text{and} \qquad \psi = \mu v_0.$$

This transforms (10) into the (singularly perturbed) problem

$$\mu^{-1}\psi'''' + \psi'' + \psi^2 + \psi'^2 + 2\psi\psi'' = 0, \tag{20}$$

so at leading order

$$\psi_0'' + \psi_0^2 + {\psi_0'}^2 + 2\psi_0\psi_0'' = 0, \tag{21}$$

still with

$$\psi_0 \sim -6\xi^{-2} + O(\xi^{-4}) \qquad \text{as } |\xi| \to \infty. \tag{22}$$

Thanks to its Hamiltonian nature, this differential equation can be solved in two steps: a first integral gives

$$\frac{1}{2}{\psi_0'}^2 + \frac{1}{3}\psi_0^3 + \psi_0{\psi_0'}^2 = 0,$$

and hence, again using (22),

$$\left(\frac{1+2\psi_0}{2\psi_0}\right)^{1/2} - \ln\left((1+2\psi_0)^{1/2} + (2\psi_0)^{1/2}\right) = \frac{i}{2\sqrt{3}}\,\xi.$$

Here ψ_0 is real and positive for ξ purely imaginary (the left-hand side decreases from $+\infty$ to $-\infty$ as a function of $\psi_0 > 0$). In particular, we choose the branch on which $\psi_0^{1/2} \sim -i\sqrt{6}\xi^{-1}$ for large ξ. Note also that ψ_0 is real and negative for ξ real with $|\xi| > \sqrt{3}\pi$.

There are branch points when $\psi_0 = -\frac{1}{2}$, at $\xi = \pm\sqrt{3}\pi$:

$$\begin{aligned} \psi_0 &\sim -\frac{1}{2} + \left(\tfrac{3}{32}\right)^{1/3}\left(\xi - \sqrt{3}\pi\right)^{2/3} && \text{as } \xi \to \sqrt{3}\pi, \\ \psi_0 &\sim -\frac{1}{2} + \left(\tfrac{3}{32}\right)^{1/3}\left(-\sqrt{3}\pi - \xi\right)^{2/3} && \text{as } \xi \to -\sqrt{3}\pi. \end{aligned} \tag{23}$$

These are most easily derived from the identity

$$(1+2\psi_0)^{1/2}\psi_0' = -i\left(\tfrac{2}{3}\right)^{1/2}\psi_0^{3/2}, \tag{24}$$

and the fact that $\psi_0 < 0$ for real ξ with $|\xi| > \sqrt{3}\pi$. We will come back to the behaviour near $\pm\sqrt{3}\pi$ in §3.2.

To determine what gets turned on across the two Stokes lines, which emanate from the singular points $\xi = \pm\sqrt{3}\pi$, see Figure 1, we use, in the spirit of §2 and without further ado, a formal WKB

[2]Note that $\mu \ll 1/\varepsilon^2$ is crucial here since the ξ scaling needs to be much smaller than that of the outer region.

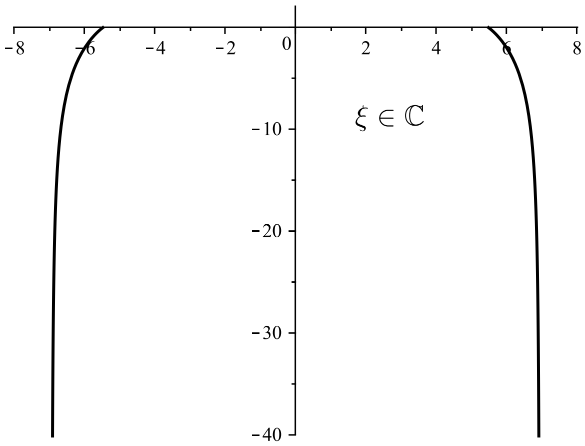

Figure 2: The two Stokes lines emanating from $\xi = \pm\sqrt{3}\pi$. They run down towards $\pm 4\sqrt{3} - i\infty$. The angle the Stokes line makes with the real axis at $\xi = \sqrt{3}\pi$ is $-3\pi/8$.

ansatz and set

$$\psi \sim \psi_0 + \mu^{-1}\psi_1 + \cdots + \mu^{-n}\psi_n + \ldots\ldots + K(\xi)\,\mu^{\zeta}\,e^{\mu^{1/2}\sigma(\xi)},$$

with the constant $\zeta \in \mathbb{R}$ to be determined later (by matching to the inner scale in §3.2). Substituting this into (20) we get

$$\sigma'^4 + \sigma'^2 + 2\psi_0\sigma'^2 = 0,$$

while the next order gives

$$2\left(2\sigma'^3 + (1+2\psi_0)\sigma'\right)K' + \left(6\sigma'^2\sigma'' + (1+2\psi_0)\sigma'' + 2\psi_0'\sigma'\right)K = 0.$$

Hence

$$\sigma' = -i(1+2\psi_0)^{1/2}, \tag{25}$$

with the minus sign required, as we shall see later, for the term to be exponentially small as $\xi \to -i\infty$ (the Stokes lines corresponding to the other sign choice run in the opposite direction and are thus not relevant to this analysis at the singularity in the upper half of the x-plane). The equation for the amplitude is also nicely integrable:

$$K = \frac{\tilde{K}_0}{\left(2\sigma'^3 + (1+2\psi_0)\sigma'\right)^{1/2}} = \frac{\tilde{K}_0}{(\sigma')^{3/2}} = \frac{K_0}{(1+2\psi_0)^{3/4}},$$

for constants $\tilde{K}_0$ and K_0.

Using (24), we can rewrite (25) as

$$\sigma' = -i(1+2\psi_0)^{1/2} = (\tfrac{3}{2})^{1/2}\psi_0'\psi_0^{-3/2}(1+2\psi_0).$$

In §3.2 we will need to be able to match the exponentially small term to an inner region (ξ near $\sqrt{3}\pi$), hence we require $\sigma = 0$ at $\xi = \sqrt{3}\pi$ (where $\psi_0^{1/2} = -(\frac{1}{2})^{1/2}i$). This implies that

$$\sigma_+ = -\sqrt{6}(\psi_0^{-1/2} - 2\psi_0^{1/2} - 2\sqrt{2}i),$$

while for the branch point at $\xi = -\sqrt{3}\pi$ (where $\psi_0^{1/2} = -(\frac{1}{2})^{1/2}i$)

$$\sigma_- = -\sqrt{6}(\psi_0^{-1/2} - 2\psi_0^{1/2} + 2\sqrt{2}i).$$

The two Stokes lines are where $\mathrm{Im}(\sigma) = 0$ and $\mathrm{Re}(\sigma) < 0$, for each of these σ's. They are depicted in Figure 2. Since $\psi_0^{1/2} \sim -i\sqrt{6}\xi^{-1} + O(\xi^{-3})$ as $|\xi| \to \infty$, the asymptotic behaviour of σ is

$$\sigma_\pm \sim -i\xi \pm 4\sqrt{3}i \qquad \text{as } |\xi| \to \infty. \tag{26}$$

For later reference we note that the exponentially small term behaves like

$$K(\xi)\mu^{\zeta}e^{\mu^{1/2}\sigma_+(\xi)} \sim \frac{K_0\,\mu^{\zeta}}{(\frac{3}{4})^{1/4}(\xi-\sqrt{3}\pi)^{1/2}}e^{-i\mu^{1/2}(\frac{3}{4})^{7/6}(\xi-\sqrt{3}\pi)^{4/3}} \qquad \text{as } \xi \to \sqrt{3}\pi. \tag{27}$$

The two terms which are turned on in passing from left of the left-hand Stokes line to right of the right-hand one, take the form

$$\frac{\mu^{\zeta}}{(1+2\psi_0)^{3/4}}e^{-\sqrt{6}(\psi_0^{-1/2}-2\psi_0^{1/2})\mu^{1/2}}(K_0e^{4\sqrt{3}i\mu^{1/2}}-\overline{K_0}e^{-4\sqrt{3}i\mu^{1/2}}),$$

where the calculation of K_0 requires the inner problem to be solved. Symmetry arguments (in particular, $\psi_0(-\overline{\xi})=\overline{\psi_0}(\xi)$: reflection in the imaginary ξ-axis) have been used in expressing the other coefficient as $-\overline{K_0}$. Taking into account the asymptotics as $\xi \to -i\infty$, and converting back to (z,v) variables, we conclude that on crossing these Stokes lines an exponential term

$$2iS\mu^{-1+\zeta}\sin(4\sqrt{3}\mu^{1/2}+\Theta)e^{-iz}, \tag{28}$$

where $K_0 = Se^{i\Theta}$, is switched on for large μ. Comparing with (19) we see that

$$\Lambda(\mu) \sim \frac{2S}{\pi\mu^{1-\zeta}}\sin(4\sqrt{3}\mu^{1/2}+\Theta) \qquad \text{as } \mu \to +\infty. \tag{29}$$

Hence the true homoclinic connections correspond to

$$4\sqrt{3}\mu^{1/2} \sim l\pi - \Theta \qquad \text{as } l \in \mathbb{N}, l \to \infty, \tag{30}$$

implying (5). The value of Θ is determined in §3.2 to be $\pi/16$.

One interpretation of these results is that for a countable set of values of μ the contribution from the left-hand Stokes line (originating from $\xi = -\sqrt{3}\pi$) is cancelled (or absorbed) by the right-hand Stokes line. It follows from (26) that in the limit $\xi \to -i\infty$ (where the real x-axis is located) the terms $\exp(\mu^{1/2}\sigma_\pm(\xi))$ represent exponentially small oscillations in the real ξ direction with frequency $\mu^{1/2}/(2\pi)$. On the other hand, the distance between the Stokes lines (in terms of ξ) is $8\sqrt{3}$ for $\xi \to -i\infty$, hence (30) implies that exactly l oscillations fit between the two Stokes lines, as $\xi \to -i\infty$ and $l \to \infty$, with the oscillatory contributions being abruptly turned off as the Stokes lines are crossed. These oscillations are not directly observed in the corresponding homoclinic solutions because of their exponentially small size, but one could argue that remnants of these oscillations become visible when the solutions are continued far away from $\varepsilon = 0$, as observed in [6].

3.2 Inner expansion near $\xi = \sqrt{3}\pi$

We concentrate on the Stokes line emanating from $\xi = \sqrt{3}\pi$ and running down the complex plane. We expand the solution in an inner region near $\xi = \sqrt{3}\pi$ and match to the exponentially small term $K(\xi)\mu^{\zeta}e^{\mu^{1/2}\sigma_+(\xi)}$ from §3.1, to obtain an expression for the (complex) pre-exponential constant K_0.

Close to $\xi = \sqrt{3}\pi$ we set, being inspired by (23),

$$\psi = -\frac{1}{2} + \mu^{-p}\Phi \qquad \text{and} \qquad \xi = \sqrt{3}\pi + \mu^{-q}Z.$$

Looking at (23) we impose $p = \frac{2}{3}q$. On the other hand, for the fourth order term in (20) to balance the nonlinear terms, we need $-1-p+4q = -2p+2q$. Hence we set

$$\psi = -\frac{1}{2} + \mu^{-1/4}\Phi \qquad \text{and} \qquad \xi = \sqrt{3}\pi + \mu^{-3/8}Z.$$

This gives for $\Phi(Z) \sim \Phi_0(Z)$

$$\Phi_0'''' + {\Phi_0'}^2 + 2\Phi_0\Phi_0'' = 0,$$

with

$$\Phi_0 \sim \left(\frac{3}{32}\right)^{1/3} Z^{2/3} \qquad \text{as } Z \to -i\infty.$$

Let us expand Φ_0 as

$$\Phi_0 \sim \sum_{n=0}^{\infty} \alpha_n Z^{-8n/3+2/3} \qquad \text{as } Z \to -i\infty.$$

The recurrence relation for α_n is, with $\alpha_0 = (\frac{3}{32})^{1/3}$,

$$(\tfrac{10}{3} - \tfrac{8n}{3})(\tfrac{7}{3} - \tfrac{8n}{3})(\tfrac{4}{3} - \tfrac{8n}{3})(\tfrac{1}{3} - \tfrac{8n}{3})\alpha_{n-1} \\ + \sum_{k=0}^{n} \left[(\tfrac{2}{3} - \tfrac{8k}{3})(\tfrac{2}{3} - \tfrac{8(n-k)}{3}) + 2(\tfrac{2}{3} - \tfrac{8k}{3})(-\tfrac{1}{3} - \tfrac{8k}{3})\right] \alpha_k \alpha_{n-k} = 0,$$

hence

$$\tfrac{16n}{3}(\tfrac{8n}{3} - 1)\alpha_n\alpha_0 = -(\tfrac{10}{3} - \tfrac{8n}{3})(\tfrac{7}{3} - \tfrac{8n}{3})(\tfrac{4}{3} - \tfrac{8n}{3})(\tfrac{1}{3} - \tfrac{8n}{3})\alpha_{n-1} - \sum_{k=1}^{n-1}(\tfrac{2}{3} - \tfrac{8k}{3})(-\tfrac{8n}{3} - \tfrac{8k}{3})\alpha_k\alpha_{n-k}.$$

From the balance

$$\begin{aligned} \alpha_n &= -\alpha_0^{-1}\alpha_{n-1}\left[\frac{(\frac{10}{3}-\frac{8n}{3})(\frac{7}{3}-\frac{8n}{3})(\frac{4}{3}-\frac{8n}{3})(\frac{1}{3}-\frac{8n}{3})}{\frac{16n}{3}(\frac{8n}{3}-1)} + O(1)\right] \\ &= -\alpha_0^{-1}\alpha_{n-1}\left[\tfrac{32}{9}n^2 - \tfrac{76}{9}n + O(1)\right] \\ &= -\tfrac{8}{9}\alpha_0^{-1}\alpha_{n-1}\left[(2n - \tfrac{7}{8} - 1)(2n - \tfrac{7}{8} - 2) + O(1))\right], \end{aligned}$$

we conclude that

$$\alpha_n \sim \omega\,\Gamma(2n - 7/8)\left(\frac{-8}{9\alpha_0}\right)^n \qquad \text{as } n \to \infty, \tag{31}$$

for some $\omega \in \mathbb{R}$, which can be determined only by using the full sequence α_n satisfying the recurrence relation; by computing, from the recurrence relation above, the first few hundred terms of the sequence $(-9\alpha_0/8)^n\Gamma(2n - 7/8)\alpha_n$ and extrapolating, one establishes that $\omega \approx -0.0608$.

As in §2.3 we need to perform optimal truncation to evaluate the amplitude of the oscillations that are turned on. We thus write $\Phi_0 = \sum_{n=0}^{N-1} \alpha_n Z^{-8n/3+2/3} + R_N$, where the remainder R_N satisfies, to leading order (large Z, large N),

$$R'''' + \alpha_0(\tfrac{4}{3}Z^{-1/3}R' + 2Z^{2/3}R'' - \tfrac{4}{9}Z^{-4/3}R) = -\alpha_{N-1}(\tfrac{8N}{3})^4 Z^{-8N/3-2/3}. \tag{32}$$

Looking at the large Z behaviour of solutions by performing a WKB analysis on the homogeneous equation, we see that what is turned on across the Stokes line is

$$R = d\,Z^{-1/2}e^{-i(9\alpha_0/8)^{1/2}Z^{4/3}}, \tag{33}$$

for some $d \in \mathbb{C}$, which nicely matches with the exponential term in (27), provided that

$$\zeta = -\frac{3}{16} \qquad \text{and} \qquad K_0 = d\Big(\frac{3}{4}\Big)^{1/4}. \tag{34}$$

While this already fixes the (algebraic) dependence on μ of the amplitude of exponential term, the remaining task is to determine the complex constant d.

For large N and large Z the right-hand side of (32) needs to be optimally truncated. To combat formula bloat, we introduce the constants

$$\beta \stackrel{\text{def}}{=} \Big(\frac{9\alpha_0}{8}\Big)^{1/2} = \Big(\frac{3}{4}\Big)^{7/6} \qquad \text{and} \qquad \omega_1 \stackrel{\text{def}}{=} \omega\sqrt{2\pi}(\tfrac{8}{3})^4 2^{-27/8}\tfrac{9\alpha_0}{8}.$$

Using (31) and Stirling's formula, and writing $Z = re^{i\theta}$, we obtain (for N and r large)

$$-\alpha_{N-1}(\tfrac{8N}{3})^4 Z^{-8N/3-2/3} \sim (-1)^N\omega_1 r^{-2/3}e^{-2i\theta/3}e^{-8iN\theta/3}N^{2N+5/8}(e^2r^{8/3}\beta^2/4)^{-N}.$$

Optimal truncation is then found to be at

$$N \sim N_{\text{opt}} = \frac{1}{2}\beta r^{4/3},$$

and the asymptotically minimal right-hand side becomes

$$\omega_1(-1)^{N_{\rm opt}} r^{-2/3} e^{-2i\theta/3} e^{-8N_{\rm opt} i\theta/3} e^{-\beta r^{4/3}} (\tfrac{\beta}{2})^{5/8} r^{5/6}.$$

Substituting

$$R = D(Z) Z^{-1/2} e^{-i\beta Z^{4/3}}$$

into (32), and using the expression (18) for the derivative in polar coordinates, the equation for $D(Z)$ becomes, to highest non-vanishing order (for large r),

$$\alpha_0^{3/2} 2^{5/2} r^{-1/2} e^{-i\theta/2} e^{-i\beta r^{4/3} e^{i4\theta/3}} \frac{dD}{d\theta} = \omega_1(\tfrac{\beta}{2})^{5/8} e^{-\pi i\beta r^{4/3}/2} r^{1/6} e^{-2i\theta/3} e^{-4i\beta r^{4/3}\theta/3} e^{-\beta r^{4/3}}.$$

This simplifies to

$$\frac{1}{r^{2/3}} \frac{dD}{d\theta} = \omega_2 e^{-i\theta/6} e^{\beta r^{4/3}(ie^{i4\theta/3} - 1 - 4i\theta/3 - \pi i/2)}, \tag{35}$$

with

$$\omega_2 \stackrel{\rm def}{=} \omega_1(\tfrac{\beta}{2})^{5/8} \alpha_0^{-3/2} 2^{-5/2} = \omega\sqrt{\pi}(\tfrac{8}{9})^{1/2} \beta^{-3/8}.$$

The right-hand side in (35) is exponentially small unless $\theta \approx -\frac{3\pi}{8}$, and the relevant inner scale can be easily read off to be $\theta = -\frac{3\pi}{8} + r^{-2/3}\hat{\theta}$. In the new variables the equation becomes

$$\frac{d\hat{D}}{d\hat{\theta}} = \omega_2 e^{i\pi/16} e^{-(8\beta/9)\hat{\theta}^2}.$$

Hence, on crossing the ray $\theta = -\frac{3\pi}{8}$ (see also Figure 2) in the clockwise direction for large r, the term (33) is turned on via the usual error-function smoothing, with

$$d = \omega_2 e^{i\pi/16} \int_{-\infty}^{\infty} e^{-(8\beta/9)\hat{\theta}^2} d\hat{\theta} = \omega_2 e^{i\pi/16} (\tfrac{9\pi}{8\beta})^{1/2} = \omega\pi\beta^{-7/8} e^{i\pi/16}$$

Combining this with (28) and (34) we finally obtain

$$S = \omega\pi\Big(\frac{3}{4}\Big)^{-37/48} \qquad \text{and} \qquad \Theta = \frac{\pi}{16},$$

so that (29) leads to

$$\Lambda(\mu) \sim \omega\, 2(\tfrac{3}{4})^{-37/48} \mu^{-19/16} \sin(4\sqrt{3}\mu^{1/2} + \pi/16) \qquad \text{as } \mu \to +\infty, \tag{36}$$

with $\omega \approx -0.0608$ (see above) determined by the recurrence relation via (31). This establishes (6).

4 The recurrence relation

Here we take an alternative approach to the splitting Stokes lines from §3.1. Namely, we investigate what information can be obtained directly from the recurrence relation (13). In particular, we present numerical results on the Stokes constant Λ, and we derive (5) directly from the recurrence relation, see §4.3. We emphasise, however, that the results of §3 are, for our purposes, complete and that what we present here is an instructive alternative approach.

4.1 Numerics for $\Lambda(\mu)$

Our aim is to study the properties of the Stokes constant $\Lambda = \Lambda(\mu)$ numerically, and compare the results to the asymptotic expression (6). We recall that $\Lambda(\mu) = \lim_{n\to\infty} A_n(\mu)$, see (14) and §2.3. Using computer algebra (MAPLE) we have computed the first few polynomials $A_n(\mu)$ from the recurrence relation (13).[3] They are depicted in Figure 3 and can be seen to converge as $n \to \infty$ for fixed μ. The zeros μ_m of the limit function $\Lambda(\mu)$ are well approximated by the zeros $\tilde{\mu}_m$ of $A_{n_0}(\mu)$

[3]We recall that $A_n(\mu)$ is an n-th order polynomial.

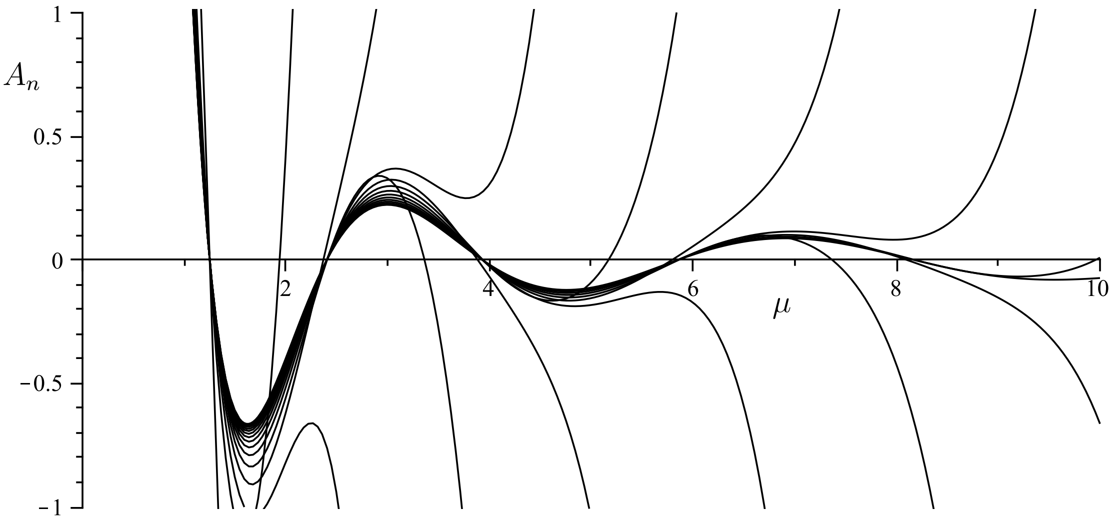

Figure 3: The polynomials $A_n(\mu)$, $n = 1 \ldots 15$, which converge to $\Lambda(\mu)$, see (14).

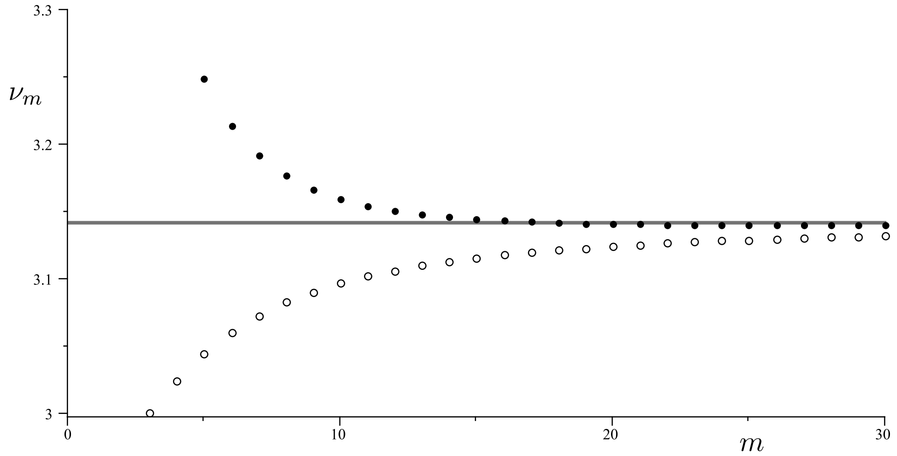

Figure 4: The normalised differences ν_m between square roots of consecutive zeros $\tilde{\mu}_m$ and $\tilde{\mu}_{m-1}$ are indicated by open circles. The extrapolated data $\nu_m^{(2)}$ are depicted by dots, while the grey line indicates the limit value π.

for large n_0 (provided $\mu \ll n^2$). For the numerical results presented here we have used $n_0 = 197$. According to (30) we expect the differences

$$\nu_m = 4\sqrt{3}\Big(\sqrt{\tilde{\mu}_m} - \sqrt{\tilde{\mu}_{m-1}}\Big)$$

to approach π as $m \to \infty$. Indeed, this is what can be observed in Figure 4 for the first 30 zeros of $A_{n_0}(\mu)$. To make the evidence for convergence even more convincing, we have also used Richardson extrapolation

$$\nu_m^{(j)} = m\nu_m^{(j-1)} - (m-1)\nu_{m-1}^{(j-1)}, \qquad j = 1, 2, \ldots \text{ and } m = j+1, j+2, \ldots \tag{37}$$

with $\nu_m^{(0)} = \nu_m$. For example, Figure 4 depicts the twice extrapolated data $\nu_m^{(2)}$ as well as ν_m, and the acceleration in the convergence to π is clearly visible.

In Figure 5 one can compare the shape of the function $\Lambda(\mu)$ for large μ, as expressed by (36), with the numerically computed polynomial A_{n_0}. One sees that the spacing between the zeros, as well as the decay for μ not too large, correspond very well. For the phase shift the agreement seems to be less obvious. To investigate numerically the phase shift Θ, we define

$$\rho_m = 4\sqrt{3}\sqrt{\tilde{\mu}_m} - (m+1)\pi,$$

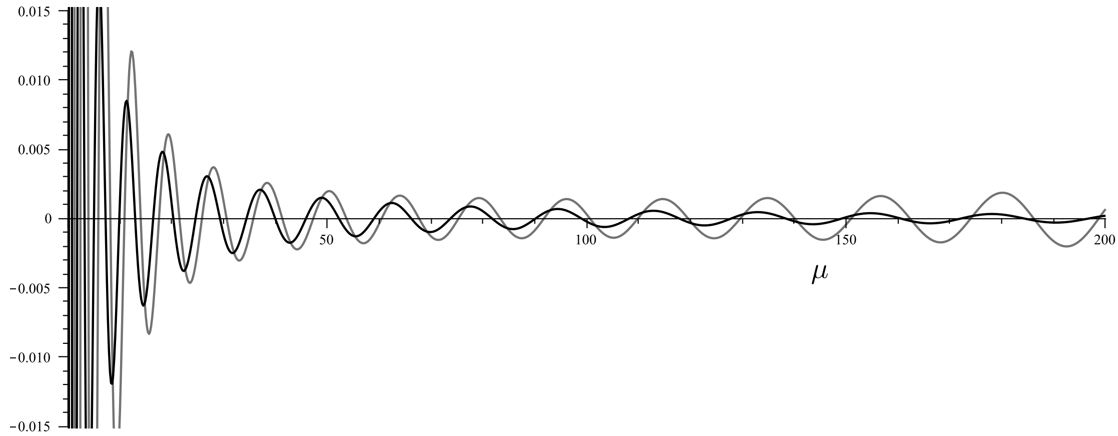

Figure 5: The polynomial $A_{197}(\mu)$ in grey, and the predicted asymptotic behaviour for large n and μ, given by (36), in black. For μ larger than 100 one can see the two starting to diverge.

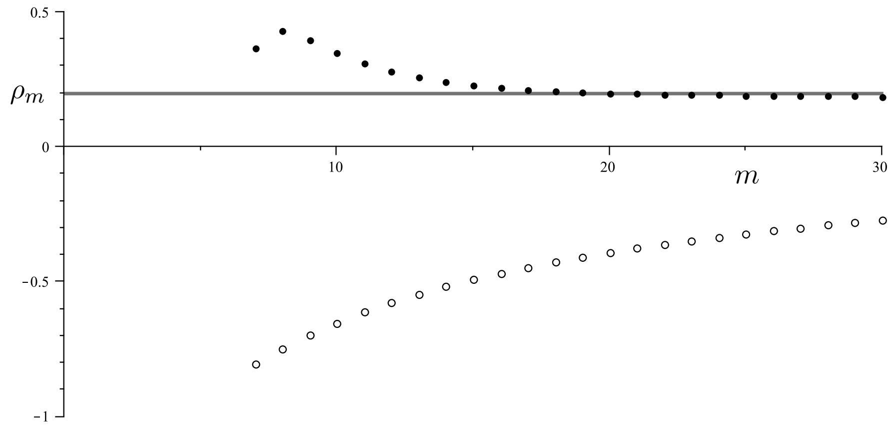

Figure 6: The phase shifts ρ_m of the zeros $\tilde{\mu}_m$ are indicated by open circles; their behaviour seems transient (not clearly converging). The fifth order extrapolated data $\rho_m^{(5)}$, depicted by dots, converge to a limit value for the phase shift. The grey line indicates $\Theta = \pi/16$.

where we have shifted the index compared to (30) in order to fit with $4\sqrt{3}\sqrt{\mu_1} = 2\sqrt{15}$ being closer to 2π than to π. We expect ρ_m to converge to $\Theta = \pi/16$ as $m \to \infty$, but this is not evident in Figure 6. However, using repeated Richardson extrapolation, see (37), the sequence $\rho_m^{(5)}$ converges quite convincingly, see Figure 6 again.

4.2 The scale $\mu \gg n^2$

The recurrence relation (13) can be slightly reshuffled to read

$$A_{n+1} - A_n = \sum_{k=0}^{n} \frac{(2k+3)!(2n-2k+1)!}{(2n+5)!} A_{k+1} A_{n-k} \\ - \mu \sum_{k=0}^{n} (4n-2k+8) \frac{(2k+1)!(2n-2k+2)!}{(2n+5)!} A_k A_{n-k}. \tag{38}$$

In order to investigate the asymptotic behaviour of $\Lambda(\mu) = \lim_{n\to\infty} A_n(\mu)$, we thus need to understand the behaviour of $A_n(\mu)$ for large μ and large n, where we first let n tend to infinity and subsequently μ. As it turns out, the critical scaling is $\mu = O(n^2)$, and we need to start with the easier scale $\mu \gg n^2$ and the go through the scales to arrive at $\mu \ll n^2$.

Introducing $A_n = \frac{\mu^n}{(2n+1)!} a_n$ leads to the following recurrence relation for a_n:

$$(2n+4)(2n+5)[a_{n+1} - \frac{(2n+2)(2n+3)}{\mu} a_n] = \sum_{k=0}^{n+1} a_k a_{n+1-k} - \sum_{k=0}^{n} (4n-2k+8)(2n-2k+2) a_k a_{n-k},$$

with $a_0 = 6$. For $\mu \gg n^2$ this simplifies somewhat to

$$(2n+4)(2n+5) b_{n+1} = \sum_{k=0}^{n+1} b_k b_{n+1-k} - \sum_{k=0}^{n} (4n-2k+8)(2n-2k+2) b_k b_{n-k}, \tag{39}$$

where $a_n \sim b_n$ for $\mu \gg n^2$. The sequence b_n behaves as $\beta_1(-\beta_2)^n n^{-\beta_3}$ when $n \to \infty$. The constants β_1 and β_2 cannot be calculated from asymptotic considerations (the full sequence b_n is needed), whereas β_3 can be determined with some effort to be $\frac{5}{3}$ (this involves going to second order in the appropriate balance and replacing all summations by corresponding integrals), but that turns out not to be relevant for the leading order asymptotics.

The recurrence relation (39) can be "solved" using the generating function

$$\Psi(\eta) = \sum_{k=0}^{\infty} \frac{b_k}{\eta^{2k+2}},$$

where the function Ψ is the solution of

$$\Psi'' - \Psi^2 + (\Psi')^2 + 2\Psi\Psi'' = 0, \tag{40}$$

with asymptotic behaviour

$$\Psi \sim \frac{6}{\eta^2} \text{ as } \eta \to \infty.$$

With these definitions we thus have

$$A_n \sim \frac{\mu^n}{(2n+1)!} b_n \qquad \text{as } \mu, n \to \infty \text{ and } \mu \gg n^2. \tag{41}$$

The differential equation (40) is (for obvious reasons) the same as (21), up to the change of variables $\eta = i\xi$, i.e. $\Psi(\eta) = \psi_0(\xi)$. We find it a bit easier to work with real variables here. The solution is given implicitly by

$$\frac{1}{2\sqrt{3}} \eta = \left(\frac{1+2\Psi(\eta)}{2\Psi(\eta)}\right)^{1/2} - \ln\left(\left(1+2\Psi(\eta)\right)^{1/2} + \left(2\Psi(\eta)\right)^{1/2}\right). \tag{42}$$

4.3 The scale $\mu = O(n^2)$

We now want to consider μ and n^2 going to infinity and being of the same order of magnitude. It is useful to introduce new variables, containing an artificial small parameter δ, namely

$$n = \frac{X}{\delta}, \qquad \mu = \frac{Y}{\delta^2} \qquad \text{and} \qquad A_n(\mu) = B(X, Y).$$

Then the recurrence relation (38) turns into

$$B(X+\delta, Y) - B(X, Y) = \sum_{k=0}^{n} \frac{(2k+3)!(2n-2k+1)!}{(2n+5)!} B((k+1)\delta, Y) B(X - k\delta, Y)$$
$$- \mu \sum_{k=0}^{n} (4n-2k+8) \frac{(2k+1)!(2n-2k+2)!}{(2n+5)!} B(k\delta, Y) B(X - k\delta, Y).$$

However, this is not the most convenient way to write it in the scale $\mu = O(n^2)$. We anticipate that the sums are endpoint dominated, hence we rewrite the sums in (38) as

$$\sum_{k=0}^{n} \frac{(2k+3)!(2n-2k+1)!}{(2n+5)!} A_{k+1} A_{n-k} \approx 2 \sum_{k=0}^{n/2} \frac{(2k+3)!(2n-2k+1)!}{(2n+5)!} A_{k+1} A_{n-k}$$

and

$$\sum_{k=0}^{n}(4n-2k+8)\frac{(2k+1)!(2n-2k+2)!}{(2n+5)!}A_kA_{n-k}\approx$$

$$\sum_{k=0}^{n/2}\frac{(2k+1)!(2n-2k+1)!}{(2n+5)!}[(4n-2k+8)(2n-2k+2)+(2n+2k+8)(2k+2)]A_kA_{n-k}.$$

For $k \ll n = O(\sqrt{\mu})$ the terms $A_{k+1}(2k+3)! \sim \mu^{k+1}b_{k+1}$ and $\mu A_k(2k+1)! \sim \mu^{k+1}b_k$ are of the same order. We may therefore neglect the first of these two sums (the second one has an additional multiplicative factor n^2). Also, Stirling's formula implies that $\frac{(2n-2k+1)!}{(2n+5)!} \sim (2n)^{-2k-4}$. We thus obtain

$$\begin{aligned}
B(X+\delta,Y)-B(X,Y) &\sim -\mu\sum_{k=0}^{n/2}A_k(2k+1)!\,(2n)^{2k-4}8n^2\,B(X-k\delta,Y)\\
&\sim -2\sum_{k=0}^{X/2\delta}b_k\mu^{k+1}(2n)^{-2k-2}\,B(X-k\delta,Y)\\
&= -2\sum_{k=0}^{X/2\delta}b_k\left(\frac{Y}{4X^2}\right)^{k+1}B(X-k\delta,Y).
\end{aligned}$$

Using a WKB ansatz

$$B(X,Y)=M(X,Y)e^{\phi(X,Y)/\delta}, \tag{43}$$

we get at leading order

$$e^{\phi_X}-1=-2\sum_{k=0}^{\infty}b_k\left(\frac{Y}{4X^2}\right)^{k+1}e^{-k\phi_X}=-2e^{\phi_X}\sum_{k=0}^{\infty}b_k\left(\frac{Y^2}{4X}e^{-\phi_X}\right)^{k+1}=-2e^{\phi_X}\Psi\left(\frac{2X}{\sqrt{Y}}e^{\phi_X/2}\right),$$

or, with Ψ defined in §4.2,

$$e^{-\phi_X}-1=2\Psi\left(\frac{2X}{\sqrt{Y}}e^{\phi_X/2}\right).$$

It inspires us to, once again, introduce new variables

$$W=\frac{X}{\sqrt{Y}}\qquad\text{and}\qquad \phi_X(X,Y)=g\left(\frac{X}{\sqrt{Y}}\right),$$

for which we then get

$$e^{-g}-1=2\Psi(2We^{g/2}).$$

Substituting this into the implicit expression (42) for Ψ, we obtain

$$W=\sqrt{3}\,e^{-g/2}\left[\left(\frac{e^{-g}}{e^{-g}-1}\right)^{1/2}-\ln\left(\left(e^{-g}\right)^{1/2}+\left(e^{-g}-1\right)^{1/2}\right)\right].$$

This defines W as a function of g, but also g as a function of W. We note that

$$g(W)\sim -2\ln W\qquad \text{as } W\to 0,$$

while

$$g(W)\sim -3W^{-2}\qquad \text{as } W\to\infty. \tag{44}$$

Our goal is to calculate the exponent

$$\phi(X,Y)=\int^{X}g\left(\frac{\hat{X}}{\sqrt{Y}}\right)d\hat{X}=\sqrt{Y}\int^{W}g(\hat{W})\,d\hat{W}$$

in the WKB term (43), but it easier to compute $\int W(g)dg$ first, namely

$$\int W(g)\,dg=\sqrt{3}\left[-4\sqrt{e^{-g}-1}+2e^{-g/2}\ln\left(e^{-g/2}+\sqrt{e^{-g}-1}\right)\right]+C.$$

It follows from the inverse function theorem that

$$\int g(W)\,dW = -\int W(g)\,dg + Wg + C.$$

Hence

$$\begin{aligned} G(W) &\stackrel{\text{def}}{=} \int^{W} g(\hat{W})\,d\hat{W} \\ &= \sqrt{3}\left[-4\sqrt{e^{-g}-1} + 2e^{-g/2}\ln\left(e^{-g/2} + \sqrt{e^{-g}-1}\right)\right] + Wg + \widehat{C} \\ &= -2W - \sqrt{3}\frac{4-2e^{-g}}{\sqrt{e^{-g}-1}} + Wg + \widehat{C}, \end{aligned}$$

where the final constant of integration is denoted by $\widehat{C}$ for definiteness and will be determined by matching to the scale $\mu \gg n^2$. We first infer that

$$G(W) \sim \pm i4\sqrt{3} + \widehat{C} - 2W\ln W \qquad \text{as } W \to 0, \tag{45}$$

since $W \sim \pm\sqrt{3}i\pi/2e^{-g/2}$ for small W. Next, we note that, see (41),

$$A_n(\mu) = \frac{b_n\mu^n}{(2n+1)!} \sim \frac{b_n}{\sqrt{2\pi}}e^{2n}(2n)^{-2n-3/2}\mu^n,$$

with $b_n \sim \beta_1(-\beta_2)^n n^{-\beta_3}$ as $n \to \infty$, so that in the new variables

$$A_n(\mu) = e^{-2n\ln n + n\ln\mu + O(n)} \sim e^{-(2\sqrt{Y}\,W\ln W)/\delta}.$$

We conclude from (45) that

$$B(x,Y) \sim Me^{\sqrt{Y}\,G(W)/\delta}$$

matches to $A_n(\mu)$ for small W ($\mu \gg n^2$) provided that

$$\widehat{C} = \mp i4\sqrt{3}.$$

Finally, taking the limit $W \to \infty$, i.e. $\mu \ll n^2$, we obtain, using (44), that

$$G(W) \to \widehat{C} = \pm i4\sqrt{3} \qquad \text{as } W \to \infty,$$

which implies that[4]

$$A_n(\mu) \sim B(X,Y) \sim e^{\pm i4\sqrt{3}\sqrt{Y}/\delta} = e^{\pm i4\sqrt{3}\sqrt{\mu}} \qquad \text{for } \mu \text{ and } n \text{ large, with } \mu \ll n^2.$$

Indeed, this is the analogue of (30) and thus leads us once again to (5).

We note that the main difficulty in the above analysis is caused by the fact that we need to pass through the entire scale $\mu = O(n^2)$ to go from the relatively easy scale $\mu \gg n^2$ to the desired scale $\mu \ll n^2$. In particular, the problem for b_n in this intermediate scale needs to be solved completely. The explicit "solution" of the recurrence relation for b_n involves generating function techniques. This constitutes a link with the direct Stokes line approach in §3.1. In particular, equation (40) is the same as (21) modulo a straightforward change of variables.

We conclude that working with the recurrence relation is much more involved than working with the differential equation directly, as in §3. We shall therefore not proceed with calculation of the pre-exponential term via this route, but conclude that we have illustrated the principle that the main asymptotic behaviour can also be derived using the recurrence relation.

[4]Here only $\sim$ means that the exponential dependence upon μ, but not the pre-exponential, has been captured.

5 Conclusion

We have sought in this paper to provide a systematic approach, the need for which was noted in [5], to constructing homoclinic connections in a class of ODEs. The key ingredient of our approach is to extend the techniques of [7] by exploiting a second small parameter (here $1/\mu$, in addition to ε) to subdivide, and make analytically tractable, the Stokes line structure. In the course of the analysis, we have also expored some novel asymptotic techniques applicable to certain classes of difference equations. The scope for applications of the approach is significantly broader than the model problem treated here, no special properties of the nonlinearities having been exploited. Indeed, we anticipate that the asymptotic structure, elucidated above, whereby the (exponentially small) oscillations are confined to a finite range of x, will be generic in many problems of the current class.

More specifically, for the (singularly perturbed) model equation (4) we have determined the asymptotic dependence of the Stokes constant on the parameter μ. In particular, a vanishing amplitude implies the occurrence of an embedded soliton, or solitary wave. The precise asymptotics of the amplitude for large μ are given by (6). The asymptotic analysis reveals that, asymptotically, an additional inner scale appears when $\mu \to +\infty$, where each Stokes line splits into a pair, and the contributions of each pair of Stokes lines cancel for a countable set of μ-values. Furthermore, the relevant Stokes constant can also be defined via a limit in a certain recurrence relation. This allows for a complementary, partly numerical, investigation, which corroborates the asymptotic analysis, as well as providing an alternative approach to finding the frequency at which embedded solutions occur for large μ, shedding additional light on asymptotic methods for recurrence relations.

These results lead to several open questions. First, can they be proved, e.g. in the spirit of [17]? Second, the leading order asymptotics (5) concur with the normal form approach in [5], but what is the precise relation between the two methods? Third, what role is played by the conservative nature of the differential equation? It is certainly used in the analysis, but is it essential to the asymptotic structure? A related question is whether it is worthwhile to analyse higher order exponentially small asymptotic terms in order to quantitatively study the phenomenon described at the end of §2.3 (this could require the treatment of higher-order Stokes lines, cf. [4]). Such analyses may also shed light on the occurrence of multi bump solitons (arising in related equations). Finally, for solitary waves, an integer number of oscillations occur in between the pair of Stokes lines in the ξ-plane ($\mu \to +\infty$), as noted in the final paragraph of §3.1. Although these are exponentially small as $\varepsilon \to 0$ and thus not visible in this limit, oscillations do appear in the homoclinic profile as ε becomes large, as observed in [6]. It would be interesting to obtain insight into the connection (if any) between these oscillations of very different size.

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