Global branches of multi bump periodic solutions of the Swift-Hohenberg equation

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1. Introduction

In this paper we present new families of global branches of single and multi bump periodic solutions of the fourth order equation

$$\frac{d^4u}{dx^4} + q\frac{d^2u}{dx^2} + u^3 - u = 0, \qquad q \in \mathbf{R}. \tag{1.1}$$

This equation arises in a variety of problems in mathematical physics and mechanics. As an important example we mention that (1.1) describes stationary solutions of the Swift- Hohenberg (SH) equation:

$$\frac{\partial U}{\partial t} = -\Big(1 + \frac{\partial^2}{\partial x^2}\Big)^2 U + \alpha U - U^3, \qquad \alpha > 0. \tag{1.2}$$

Equation (1.2) was first introduced by Swift & Hohenberg [SH] in studies of Rayleigh-Bénard convection, and was proposed by Pomeau & Manneville [PM] as a good description of cellular flows just past the onset of instability. Of particular interest in these studies was the formation of stationary periodic patterns, and the selection of their wave-lengths. For further references about the SH equation we refer to the book by Collet & Eckmann [CE] and the survey by Cross & Hohenberg [CH]. If $\alpha > 1$, then stationary solutions U of equation (1.2), when suitably scaled, are readily seen to be solutions u of equation (1.1). Specifically, U and u are related through

$$u(x) = \frac{1}{\sqrt{\alpha - 1}}U((\alpha - 1)^{-1/4}x) \qquad \text{and} \qquad q = \frac{2}{\sqrt{\alpha - 1}}. \tag{1.3}$$

It is clear from (1.3) that in this example, q only takes *positive* values. An example where it takes *negative* values of q is the Extended Fisher-Kolmogorov (EFK) equation [CER,DS],

$$\frac{\partial U}{\partial t} = -\gamma\frac{\partial^4 U}{\partial x^4} + \frac{\partial^2 U}{\partial x^2} + U - U^3, \qquad \gamma > 0, \tag{1.4}$$

which yields (1.1) if we set

$$u(x) = U(\gamma^{-1/4}x) \qquad \text{and} \qquad q = -\gamma^{-1/2}. \tag{1.5}$$

Both, the Swift-Hohenberg equation and the Extended Fisher-Kolmogorov are gradient systems involving functionals $I(u)$ of the form

$$I(u) = \int \left\{ \frac{1}{2}(u'')^2 - \frac{q}{2}(u')^2 + F(u) \right\} dx,$$

and equation (1.1) is the corresponding Euler-Lagrange equation. Such as furnctionals also arise in variational problems arising in the study of layering phenomena in second order materials [LM,CMM,MPT]. We mention in particular in this context the governing equation of a strut [HB, TG] with stiffness EI under an axial compression P and subjected to a load $Q(y)$:

$$EIy^{iv} + Py'' + Q(y) = 0.$$

Here, y denotes the deflection of the strut in a direction perpendicular to its axis.

The investigation in this paper is part of a study of complex patterns in physics and mechanics in whose description equation (1.1) plays an important role. A typical example of such a pattern is the phenomenon of *localized buckling* in mechanics. In this type of buckling, the deflections are confined to a small portion of the otherwise unperturbed material. In Figure 1.1 we give an example of such a pattern due to Ramsay [R]. It shows of the effect of compression on a layered material in which the layers have different stiffness. Because the stiffer layer (black, in the center) will not contract as easily as the more ductile material that surrounds it, the stiff layer deflects sideways and produces folds.

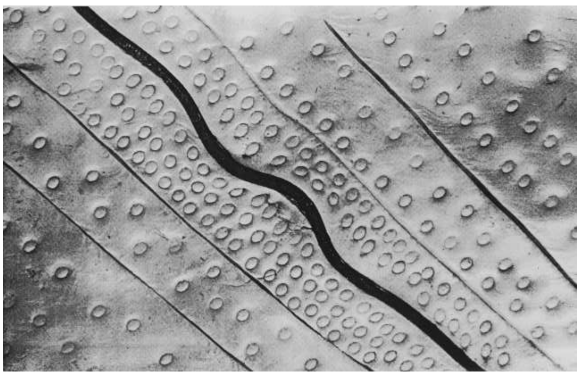
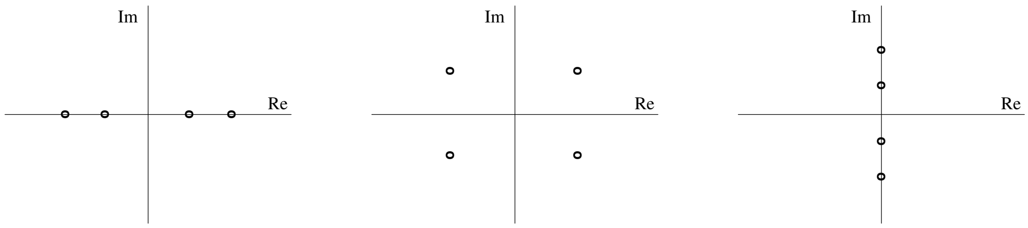


Fig. 1.1. Folding of a stiff layer in a ductile material [R]

Patterns are described by bounded solutions of equation (1.1) on the real line. Thus, mathematically this study amounts to an investigation of the different types of bounded solutions equation (1.1) possesses.

In recent years a great deal has been learnt about the structure of the set of bounded solutions of equation (1.1) on the real line. It turns out to depend very much on the value of the parameter q. In particular, one can identify two critical values of q: $+\sqrt{8}$ and $-\sqrt{8}$. At these values the linearisation around the constant solutions $u = \pm 1$, i.e. the points $P_{\pm} = (\pm 1, 0, 0, 0)$ in (u, u', u'', u''') phase space, changes character, as indicated in Figure 1.2.



(a) $q \le -\sqrt{8}$ (b) $q \in (-\sqrt{8}, \sqrt{8})$ (c) $q \ge \sqrt{8}$

Fig. 1.2. The spectrum of the linearisation around P_+ and P_-

For $q \le -\sqrt{8}$, the set of bounded solutions is very limited, and consists – modulo translations – of a one parameter family of single bump periodic solutions, which are even with respect to their extrema and odd with respect to their zeros, and two heteroclinic orbits, or kinks, connecting P_+ and P_- [Be2,3, PT1,2]: both are odd, one is strictly increasing and one is strictly decreasing. Because we shall often need to refer to it, we denote the odd increasing kink for $q = -\sqrt{8}$ by $\varphi(x)$.

As q increases beyond $-\sqrt{8}$ the set of bounded solutions becomes much richer. It has been proved that if $-\sqrt{8} < q \le 0$ it includes a great variety of multi bump periodic solutions, heteroclinic orbits and homoclinic orbits leading to P_+ and P_-, as well as chaotic solutions. For detailed results we refer to [KV,KKV,PT2-4]. For $q > 0$ the results are more tentative and incomplete: although numerical studies for equation (1.1) [Be1] and related equations [BCT,CT] suggest an abundance of bounded solutions in this parameter range as well, much of this still remains unproved.

The object of this paper is to investigate the existence and qualitative properties of *multi bump* periodic solutions of equation (1.1). By this we mean here solutions which have more than one critical point in each period. When q lies in a right neighbourhood of $-\sqrt{8}$, then all local extrema of the periodic solutions lie near the constant solutions $u = \pm 1$, and solutions have transitions between these uniform states. In this regime the term 'multi bump' corresponds to the way it is commonly used in dynamical systems theory. However, as q moves away from $-\sqrt{8}$, local extrema are no longer tied to $u = \pm 1$, and it is not easy to identify the transitions. However, we shall still describe such solutions as multi bump periodic solutions. Thus, this work extends previous results [PT2,MPT] in which the properties of *single bump* periodic solutions were studied. In particular, we will investigate the existence and global behaviour of families of *odd* and *even*, single and multi bump periodic solutions which bifurcate from the strictly increasing kink φ at $q = -\sqrt{8}$. Odd solutions may also be even with respect to some of their critical points. On the other hand, by even solutions we mean solutions, which are not odd with respect to any of their zeros. Below we indicate some of our findings:

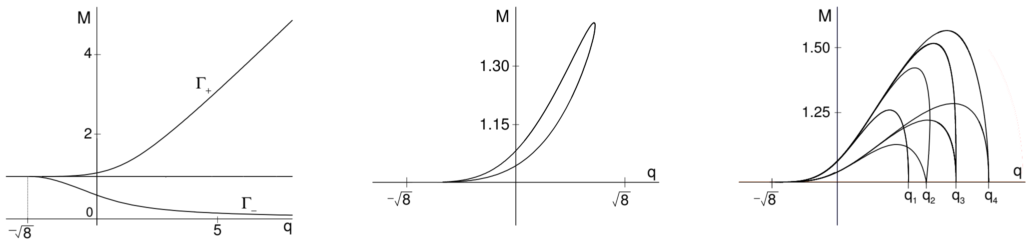
(I) We obtain a family of periodic solutions, bifurcating from the kink φ at $q = -\sqrt{8}$ and extending to infinity, i.e, these solutions exist for all $q > -\sqrt{8}$. The family consists of a countable infinity of distinct periodic solutions. The simplest examples of these are shown in Figures 1.4-6. In the bifurcation diagram we graph the supremum norm $M = ||u||_\infty$ against q.

(II) In addition, another *pair* of families, both consisting of a countable infinity of distinct periodic solutions, are proven to exist for $q \in (-\sqrt{8}, 0]$. These solutions continue to exist for some, but not all, positive values of q. Numerical evidence suggests that the solutions from both families pairwise lie on *loops* in the (q, M) plane of which the projection on the q-axis is of the form $(-\sqrt{8}, q^*]$ (See Figure 1.7), and one solution lies on the top of the loop whilst the other solution lies on the bottom. At $q^* > 0$ the two solutions coalesce.

(III) Finally, we find a third kind of periodic solutions. These again come as a family of countable many distinct periodic solutions which bifurcate from the kink φ at $q = -\sqrt{8}$. However, this family does not extend to infinity nor do they lie on loops. Instead, our numerical results indicate that these periodic solutions bifurcate from the constant solution $u = 1$ as q tends to a critical value q_n which is of the form

$$q_n = \sqrt{2}\Big(n + \frac{1}{n}\Big), \qquad n = 1, 2, \ldots. \tag{1.6}$$

Note that $q_1 = \sqrt{8}$, that $q_{n+1} > q_n$ for every $n \geq 1$, and that $q_n \to \infty$ as $n \to \infty$. For $n \geq 2$ these solutions come in pairs. Graphs of some of them are shown in Figures 1.8, 1.9 and 1.11. The critical values q_n arise when the moduli of the eigenvalues of the linearisation around $u = 1$ are a multiple of one another. Further details of the derivation of (1.6) are given in Section 5. (see also [CTh]). Samples of the bifurcation curves in the (q, M) plane of these types of solutions are shown in Figure 1.3 below.



Type (I) Type (II) Type (III)

Fig. 1.3. The three types of bifurcation graphs

Equation (1.1) admits a first integral, often referred to as the *Energy*,

$$\mathcal{E}(u) \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + F(u), \tag{1.7a}$$

where

$$F(u) = \frac{1}{4}(u^2 - 1)^2, \tag{1.7b}$$

which is constant if u is a solution. For the solutions $u(x) = \varphi(x)$ and $u(x) = 1$ it is clear that $\mathcal{E}(u) = 0$. In this paper we focus on branches of periodic solutions which bifurcate from either $u = \varphi$ or $u = 1$ or both. This motivates us to consider solutions which have zero energy, that is for which

$$\mathcal{E}(u) = 0. \tag{1.8}$$

In constructing periodic solutions, we make extensive use of symmetry properties of solutions: if $a \in \mathbf{R}$ is a point where $u' = 0$ as well as $u''' = 0$, then thanks to the reversibility of equation (1.1), it is easily verified that u is even with respect to a:

$$u(a - y) = u(a + y) \quad \text{for all} \quad y \in \mathbf{R}.$$

Also, since the function F is even, it follows that if $b \in \mathbf{R}$ is a point where $u = 0$ as well as $u'' = 0$, then u is odd with respect to b:

$$u(b - y) = -u(b + y) \quad \text{for all} \quad y \in \mathbf{R}.$$

We begin with a brief summary of previous results [PT2,MPT] in which the existence of two families of single-bump periodic solutions u_+ and u_- was proved.

Theorem A. *For every $q > -\sqrt{8}$ there exist two periodic solutions u_+ and u_- of equation (1.1) such that $\mathcal{E}(u_\pm) = 0$. Both u_+ and u_- are odd with respect to their zeros and even with respect to their critical points, and*

$$M_+ = \max\{u_+(x) : x \in \mathbf{R}\} > 1 \qquad \textit{and} \qquad M_- = \max\{u_-(x) : x \in \mathbf{R}\} < 1.$$

We denote these two families of solutions by

$$\Gamma_\pm = \{u_\pm(\cdot, q) : q > -\sqrt{8}\}.$$

Numerical computations of these branches made with AUTO97 [D] are shown in Figure 1.4. As will be done throughout this paper when depicting solution branches, we set q along the horizontal axis and the supremum norm of the solution,

$$M \stackrel{\text{def}}{=} \|u\|_\infty,$$

along the vertical axis.

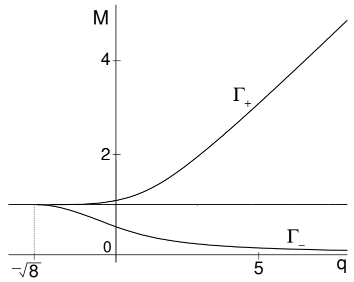


Fig. 1.4. The branches Γ_+ and Γ_-

Graphs of u_+ and u_- made with Phaseplane [E] are given in Figure 1.5.

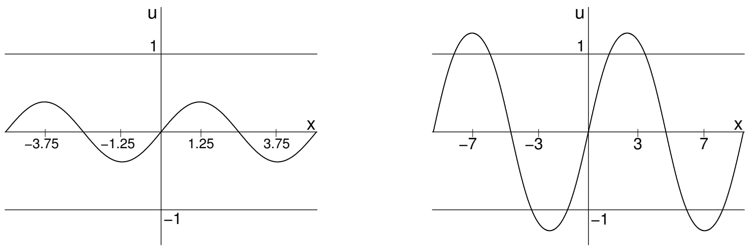


Fig. 1.5. Small and large single bump periodic solutions ($q = 1$)

The two families Γ_+ and Γ_- will be used to construct families of multi bump periodic solutions of greater complexity. We shall characterise these solutions by the critical points and the critical values of their graphs. We label the positive critical points corresponding to local maxima by $\{\xi_k\}$ and the positive critical points corresponding to local minima by $\{\eta_k\}$. For *odd* solutions with $u'(0) > 0$ these points satisfy

$$0 < \xi_1 \le \eta_1 \le \xi_2 \le \dots .$$

In fact, since $-u(x)$ is a solution whenever $u(x)$ is, we will assume throughout that $u'(0) > 0$ for odd solutions. For *even*, nonconstant solutions such that $\mathcal{E}(u) = 0$, we find that $u(0) \in \mathbf{R} \setminus \{-1, 1\}$. In this case the function u' also has infinitely many positive zeros and these satisfy

$$\begin{aligned} &0 < \xi_1 \le \eta_1 \le \xi_2 \le \dots \quad \text{if} \quad u''(0) > 0, \\ &0 < \eta_1 \le \xi_2 \le \eta_2 \le \dots \quad \text{if} \quad u''(0) < 0. \end{aligned}$$

Starting from the solutions u_+ and u_-, we use a shooting technique to obtain a countable family of odd multi bump periodic solutions which also exists on the entire q-interval $-\sqrt{8} < q < \infty$. The solutions of this family obey the rule that all their local maxima lie *above* $u = +1$ and all their local minima lie *below* $u = -1$. However, the first point of symmetry ζ is an exception: at such a point $u(\zeta)$ lies *below* $u = -1$ if it is a *maximum* and *above* $u = +1$ if it is a *minimum*:

$$u(\zeta) \begin{cases} < -1 & \text{if} \quad \zeta = \xi_k \quad \text{for some} \quad k \ge 1, \\ > +1 & \text{if} \quad \zeta = \eta_k \quad \text{for some} \quad k \ge 1. \end{cases}$$

We denote by T_N the branch of odd periodic solutions of this family, of which the N^{th} critical point ζ_N is the first point of symmetry:

$$T_N = \{u(\cdot, q) : q > -\sqrt{8}, \quad u(\cdot, q) \text{ is symmetric with respect to } \zeta_N\}.$$

In Figure 1.6 we give the numerically computed branches T_2 and T_3, as well as specific solutions which lie on T_2 and T_3 at $q = 2$.

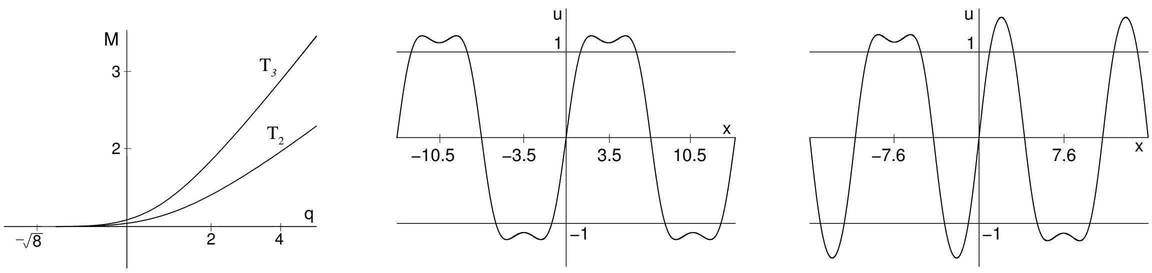


Fig. 1.6. The branches T_2 and T_3, and corresponding solutions at $q = 1.5$

In Section 3 the precise result for this family is formulated in Theorem 3.2.

In addition to these branches which exist on the entire q-interval $(-\sqrt{8}, \infty)$, we prove the existence of a second family of odd periodic solutions on $(-\sqrt{8}, 0]$. They exist in pairs, and our numerical experiments indicate that they sit on loop shaped branches, which extend well into the regime $q > 0$. An example of such a loop, together with the corresponding two solutions, is given in Figure 1.7. For this particular family, the solutions are symmetric with respect to η_1 with $-1 < u(\eta_1) < 1$. The precise description of these solutions is given in Theorems 3.3 and 3.4.

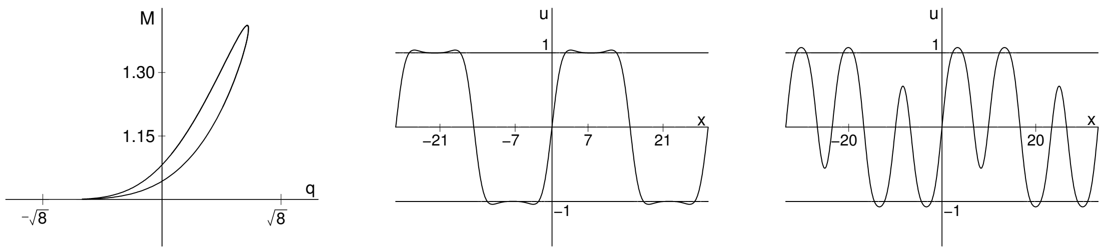


Fig. 1.7. Loop shaped branch and odd periodic solutions at $q = -\frac{1}{10}$

The techniques used to prove the existence of these solutions for $-\sqrt{8} < q \leq 0$ have been developed in [PT2,3,4]. In the present paper we show how to obtain several families of single and multi bump periodic solutions via this method. However, we do not aim at completeness, and there are many more branches of periodic solutions that can be established using this technique for $-\sqrt{8} < q \leq 0$ (see also [PT3,4]) than those presented here. We are not able to extend this existence proof to the regime $q > 0$. An essential difficulty seems to be that the solutions cease to exist at a coalescence point q^*, which is different for every branch of solutions.

A different method for proving the existence of periodic solutions with energy $\mathcal{E}(u) = 0$ in the regime $-\sqrt{8} < q < 0$ has been presented in [KKV]. There, a minimisation procedure is used to obtain periodic solutions both with and without symmetry with respect to a zero or an extremum. Since only the *minimisers* of an associated functional are considered, the variational method establishes (for example) the existence of only one of the two solutions in Figure 1.8.

In Sections 4, 5 and 6 we turn to *even* periodic solutions of equation (1.1). Here we find a third type of branching phenomenon: solutions existing on finite q-intervals of the form $(-\sqrt{8}, q_n)$, still bifurcating from the kink φ at the lower end, and according to numerical evidence, also bifurcating from the constant solution $u = 1$ at the top end. Our results here extend those obtained in our analysis [PT5] of the equation

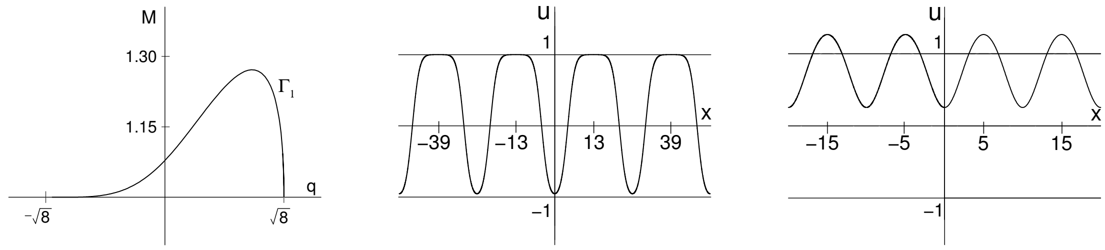
$$u^{iv} + qu'' + e^u - 1 = 0, \qquad q = c^2, \tag{1.9}$$

proposed in [LMcK] in connection with the study of travelling waves (with speed c) in suspension bridges. Concerning even single bump periodic solutions of (1.1) we prove the following existence theorem:

Theorem B. *For every $q \in (-\sqrt{8}, \sqrt{8})$ there exists a periodic solution u of equation (1.1), such that $\mathcal{E}(u) = 0$, which is even with respect to all its critical points, with the properties:*

$$-1 < \min\{u(x) : x \in \mathbf{R}\} < +1 < \max\{u(x) : x \in \mathbf{R}\}.$$

Two such solutions, at $q = -2$ and $q = +2$, are shown in Figures 1.8b and 1.8c, and the branch of solutions on which they lie is presented in Figure 1.8a.



(a) Branch (b) $q = -2$ (c) $q = +2$

Fig. 1.8. Branch of even single bump even periodic solutions

In order to formulate our results about even *multi bump* periodic solutions we need to introduce the notion of an *n-lap solution.* If u is an even periodic solution for which all the critical points are local maxima or minima, we say that u is an n-lap solution if it is symmetric with respect to its n^{th} critical point, so that its graph will have $2n$ monotone segments in one period.

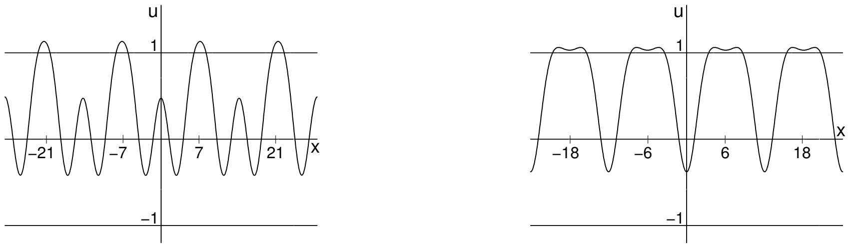
Theorem C. *For each $n \geq 2$ there exist two families of even periodic n-lap solutions when $q \in (-\sqrt{8}, q_n)$. At the points of symmetry, ζ_n, we have*

$$u(\zeta_n) > 1 \quad \text{for every} \quad n \geq 1.$$

For $n = 2$ and $n = 3$, and $q \in (-\sqrt{8}, q_n)$ it is possible to show in addition that the critical values of the solutions all lie in the interval $(-1, 1)$ with the exception of the point of symmetry, and if the solution is symmetric with respect to a minimum, its neighbouring maxima.

Note that the case $n = 1$ is discussed in Theorem B, and corresponding 1-lap solutions are shown in Figure 1.8.

In Figure 1.9 we present two 2-lap solutions and in Figure 1.10 we show the branches of the two solutions, as well as a blowup near the point $(q, M) = (q_2, 1)$.



(a) On the upper branch Γ_{2a} (b) On the lower branch Γ_{2b}

Fig. 1.9. Two even periodic 2-lap solutions at $q = 2$

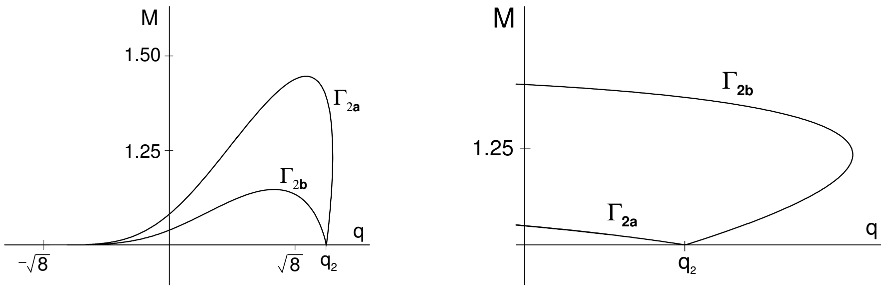
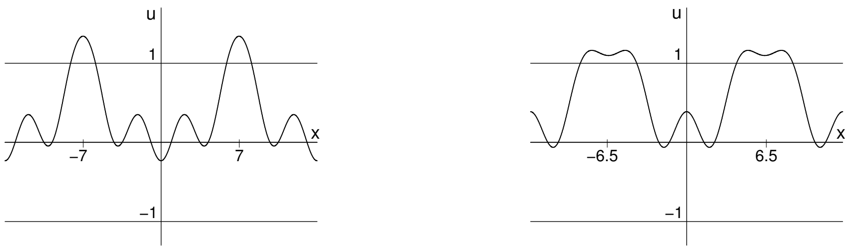


Fig. 1.10. Branches of even periodic 2-lap solutions, and a blowup at q_2

In Figure 1.10, the solutions on the upper branch Γ_{2a} satisfy $u''(0) < 0$ and are symmetric with respect to ξ_2. Along the lower branch Γ_{2b} the solutions satisfy $u''(0) > 0$ and are symmetric with respect to η_1. Thus, both branches consist of 2-lap solutions. The existence of these solutions is proved in Theorem 6.2.

Corresponding results for 3-lap solutions are presented in Figures 1.11 and 1.12.



(a) On the upper branch Γ_{3a} (b) On the lower branch Γ_{3b}

Fig. 1.11. Two even periodic 3-lap solutions at $q = 2$

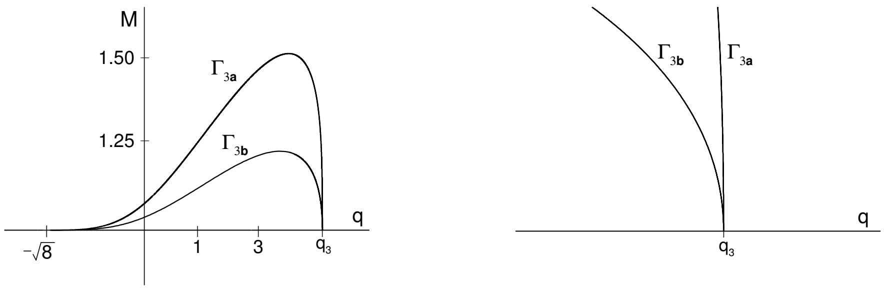


Fig. 1.12. Branches of even periodic 3-lap solutions, and a blowup at q_3

Solutions on the upper branch Γ_{3a} satisfy $u''(0) > 0$ and are symmetric with respect to ξ_2. On the lower branch Γ_{3b} the solutions satisfy $u''(0) < 0$ and are symmetric with respect to η_2. Thus both branches consist of 3-lap solutions. The existence of these solutions is proved in Theorem 6.4.

It is interesting to note the difference in the local behaviour of the solution branches near the points $(q_2, 1)$ and $(q_3, 1)$ in the (q, M)-plane (see Figures 1.10 and 1.12). Although a detailed analysis of this local behaviour is beyond the scope of this paper, we do present a local analysis of the branches Γ_{2a} and Γ_{2b} near $(q_2, 1)$. This yields the angles θ_a and θ_b between the branches Γ_{2a} and Γ_{2b} and the q axis at $(q_2, 1)$. They are given by

$$\tan\theta_a = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta_b = -\frac{\sqrt{2}}{3}.$$

In addition to the branches of n-lap solutions extending over $(-\sqrt{8}, q_n)$, which bifurcate at $q = -\sqrt{8}$ and at $q = q_n$, it is possible to construct branches of even periodic solutions which bifurcate at $q = -\sqrt{8}$ and at $q = q_{m,n}$ where

$$q_{m,n} = \sqrt{2}\Big(\frac{n}{m} + \frac{m}{n}\Big), \qquad m,\, n \geq 1. \tag{1.10}$$

In this paper we do not study these solution exhaustively, but merely prove the existence of two banches of 3-lap periodic solutions which connect $q = -\sqrt{8}$ and $q = q_{2,3}$. This is done in Theorem 6.5.

As an example of the general phenomenon we present in Figures 1.13 and 1.14 solutions which lie on the branches that bifurcate at $q = q_{m,5}$ for $m = 1, 2, 3, 4$. In each case only solutions on one of the two branches bifurcating from each bifurcation point are shown. All depicted solutions are at $q = 2$. Without going into details, we observe that n is the number of monotone laps, while m is the number of laps that cross the constant solution $u = 1$ between two points of symmetry.

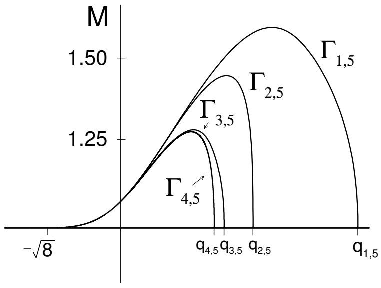
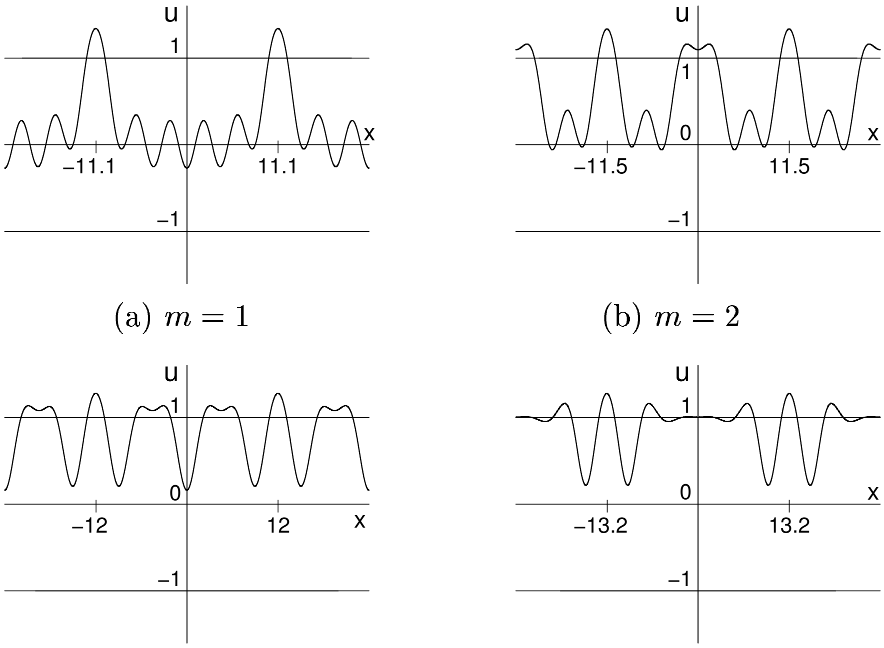


Fig. 1.13. Four branches of periodic 5-lap solutions.



(a) $m = 1$ (b) $m = 2$

(c) $m = 3$ (d) $m = 4$

Fig. 1.14. Periodic 5-lap solutions on branches that bifurcate at $q = q_{m,5}$ for $m = 1, 2, 3, 4$. All solutions are at $q = 2$.

The organisation of the paper is the following. In Section 2 we introduce some notation and recall the important properties of critical points obtained in earlier papers [PT1-5]. In Section 3 we establish the existence of several families of odd periodic solutions, some of which exist for all $q > -\sqrt{8}$ and some on finite q-intervals only. In Section 4 we begin our analysis of even periodic solutions with a discussion of the interval $-\sqrt{8} < q < \sqrt{8}$. After a brief section on the local behaviour of solutions near $u = 1$, this study is continued in Section 6 for values of q in the interval $(-\sqrt{8}, q_3)$. In this interval it is possible to obtain information about the location of the critical values of the solution graphs thanks to a Comparison Lemma which is valid for $-\sqrt{8} < q < q_3$. In Section 7, we study the local behaviour of solution branches near q_2, and in Section 8, we establish of families of periodic solutions with an arbitrary large number of laps (Theorem C). In Section 9, we conclude with the rather technical proof of the Comparison Lemma used in Section 6.

2. Critical points

To establish the existence of new families of periodic solutions we shall further develop the topological shooting method established in [PT2-5]. For this, we begin with a summary of the key properties of critical points in Lemmas 2.1, 2.2 and 2.3. Then, in Lemma 2.4, we prove a new global result. Finally, Lemmas 2.5 and 2.6 summarize two previously obtained global results. These properties will allow us to extend our shooting method and enables us to obtain further families of solutions and more detailed information about their qualitative properties.

The solutions we discuss in this paper will be either even or odd, and thus we shall study equation (1.1), which we restate here for convenience:

$$u^{iv} + qu'' + u^3 - u = 0 \tag{2.1}$$

and supply appropriate initial conditions. When looking for *odd* solutions we impose the conditions

$$u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = \beta, \tag{2.2a}$$

and when looking for *even* solutions we set

$$u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0. \tag{2.2b}$$

In both cases we shall assume that the first integral is zero, i.e.

$$\mathcal{E}(u) \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + \frac{1}{4}(u^2-1)^2 = 0. \tag{2.3}$$

This means that the constants α and β are related by

$$\beta \stackrel{\text{def}}{=} \beta(\alpha) = \begin{cases} -\dfrac{q\alpha}{2} - \dfrac{1}{4\alpha} \quad (\alpha \neq 0) & \text{for odd solutions,} \quad (2.4\text{a}) \\ \pm \dfrac{1}{\sqrt{2}}|\alpha^2 - 1| & \text{for even solutions.} \quad (2.4\text{b}) \end{cases}$$

For brevity we denote Problem (2.1), (2.2a), (2.3), (2.4a) by *Problem A* and Problem (2.1), (2.2b), (2.3), (2.4b) by *Problem B*. We observe that if u is a solution of equation (2.1), then so is $-u$. Therefore, when discussing odd solutions of Problem A, we restrict our attention to solutions with a *positive* initial slope, i.e. $\alpha > 0$.

For any given $\alpha \in \mathbf{R}^+$ there exists a unique local solution of Problem A and for any given $\alpha \in \mathbf{R}$ there exists a unique local solution of Problem B. In both cases we denote it by $u(x,\alpha)$. The critical points of the solution graphs of $u(x,\alpha)$, that is the zeros of $u'(x,\alpha)$, will play a pivotal role in the construction and classifiation of the different families of periodic solutions. Below we summarize the most important properties of these points. They were derived in [PT1-5].

We begin with a preliminary lemma which implies that all critical points are isolated.

Lemma 2.1. [PT2] *Suppose that u is a nonconstant solution of (1.1) such that $\mathcal{E}(u)=0$, and that $u'(x_0)=0$ at some $x_0 \in \mathbf{R}$.*

(a) *If* $u''(x_0)=0$ *then* $u(x_0)=\pm 1$ *and* $u'''(x_0)\neq 0$;

(b) *If* $u(x_0)=\pm 1$ *then* $u''(x_0)=0$ *and* $u'''(x_0)\neq 0$.

Lemma 2.1 implies that, unless u is a constant solution, we can number the critical points of the graph of $u(x,\alpha)$. We denote the positive local maxima by ξ_k and the minima by η_k with $k=1,2,\ldots$. At inflection points these points coincide. To start the sequences, we need to distinguish two cases:

$$(i)\;\; u'>0 \;\text{ in }\; (0,\delta) \qquad \text{and} \qquad (ii)\;\; u'<0 \;\text{ in }\; (0,\delta),$$

for some small $\delta>0$.

Case *(i)*. When $u'>0$ in a right-neighbourhood of the origin, we define

$$\xi_1=\sup\{x>0: u'>0 \text{ on } [0,x)\}. \tag{2.5a}$$

If $u''(\xi_1)<0$ we set

$$\eta_1=\sup\{x>\xi_1: u'<0 \text{ on } (\xi_1,x)\}. \tag{2.5b}$$

When $u''(\xi_1)=0$, and so $u(\xi_1)=1$ by Lemma 2.1, we set

$$\eta_1=\xi_1. \tag{2.5c}$$

Case *(ii)*. When $u'<0$ in a right-neighbourhood of the origin, we *skip* ξ_1 and define

$$\eta_1=\sup\{x>0: u'<0 \text{ on } [0,x)\}. \tag{2.5d}$$

In (2.5a-c) we have defined the first terms in the sequences $\{\xi_k\}$ and $\{\eta_k\}$ in both cases. We can now continue formally to larger values of k. For $k\geq 2$ we define

$$\xi_k=\begin{cases}\sup\{x>\eta_{k-1}: u'>0 \text{ on } (\eta_{k-1},x)\}, & \text{if } u'>0 \text{ in } (\eta_{k-1},\eta_{k-1}+\delta_1),\\ \eta_{k-1} & \text{otherwise,}\end{cases} \tag{2.6a}$$

where δ_1 is some small positive number. Similarly, we set

$$\eta_k = \begin{cases} \sup\{x > \xi_k : u' < 0 \text{ on } (\xi_k, x)\}, & \text{if } u' < 0 \text{ in } (\xi_k, \xi_k + \delta_2), \\ \xi_k & \text{otherwise}, \end{cases} \tag{2.6b}$$

in which δ_2 is some small positive number. It is readily seen that

$$\xi_k \le \eta_k \le \xi_{k+1}, \qquad k \ge 1. \tag{2.7}$$

We will make extensive use of the following observation:

Remark. If u is a nonconstant solution, then one of the inequalities in (2.7) must be strict.

To see this, suppose that

$$\xi_k = \eta_k.$$

Then, because the zeros of u' are isolated by Lemma 2.1, it follows from (2.6b) that $u' > 0$ in a right-neighbourhood of ξ_k, so that u has an inflection point at ξ_k, where $u''' > 0$. Hence, by (2.6a),

$$\eta_k < \xi_{k+1}.$$

On the other hand, if

$$\eta_k = \xi_{k+1},$$

then by Lemma 2.1 and (2.6a), $u' < 0$ in a right-neighbourhood of ξ_k and u has an inflection point at η_k, where $u''' < 0$. Therefore, by (2.6b),

$$\xi_k < \eta_k.$$

In particular, this implies that

$$\xi_k < \xi_{k+1} \quad \text{and} \quad \eta_k < \eta_{k+1} \quad \text{for every} \quad k \ge 1. \tag{2.8}$$

In the following lemma we present the important continuity properties of the critical points. In particular, we emphasize that, as α changes, critical points are preserved and cannot disappear by coalescing with one another.

Lemma 2.2. [PT2,3] *Suppose that $q > -\sqrt{8}$. For every $\alpha \in I$, where $I = \mathbf{R}^+$ in Problem A, and $I = \mathbf{R} \setminus \{-1, +1\}$ in Problem B, and for every $k \ge 1$,*

(a) *$\xi_k(\alpha) < \infty$ and $\eta_k(\alpha) < \infty$;*

(b) *$u'(\xi_k(\alpha), \alpha) = 0$ and $u'(\eta_k(\alpha), \alpha) = 0$;*

(c) *$\xi_k \in C(I)$ and $\eta_k \in C(I)$.*

Remarks. (1) In [PT2,3], Lemma 2.2 has been proved for solutions of Problem A. For solutions of Problem B the proof is similar (see also [PT5]).

(2) For odd solutions we have the following result: If $q \le -\sqrt{8}$, then there exists a unique value $\alpha_0 > 0$ for which the corresponding solution $u(x, \alpha_0)$ tends monotonically to 1 (the *Kink*), so that $\xi_1(\alpha_0) = \infty$ and the sequence $\{\xi_k\}$ is not well defined [Be2,3,PT1].

In order to proceed with the construction of new families of periodic solutions with complex structure, we need to determine the precise local behaviour of $u(\xi_k)$ and $u(\eta_k)$ when they cross the level $u = 1$ or $u = -1$ as α changes. This is the subject of the next lemma.

Lemma 2.3. [PT3,5] *Let $q \in \mathbf{R}$. Suppose that for some $k \ge 1$,*

$$u(\xi_k) = 1 \quad \text{and} \quad u''(\xi_k) = 0 \quad \text{at} \quad \alpha = \alpha^*, \tag{2.9}$$

and for some $\delta > 0$,

$$u(\xi_k(\alpha), \alpha) > 1 \quad \text{for} \quad \alpha^* < \alpha < \alpha^* + \delta. \tag{2.10}$$

(a) *If $u'''(\xi_k) > 0$ at α^*, then there exists an $\varepsilon > 0$ such that*

$$u(\xi_k(\alpha), \alpha) > u(\eta_k(\alpha), \alpha) > 1 \quad \text{for} \quad \alpha^* < \alpha < \alpha^* + \varepsilon. \tag{2.11a}$$

(b) *If $u'''(\xi_k) < 0$ at α^*, then there exists an $\varepsilon > 0$ such that*

$$u(\xi_k(\alpha), \alpha) > u(\eta_{k-1}(\alpha), \alpha) > 1 \quad \text{for} \quad \alpha^* < \alpha < \alpha^* + \varepsilon. \tag{2.11b}$$

Remarks. (1) In [PT3], Part (a) of Lemma 2.3 was first proved for $q \le 0$, and in [PT5] this restriction on q was subsequently removed. The proof of Part (b) is completely analogous to that of Part (a).

(2) A similar result applies when $u(\xi_k)$ and $u(\eta_k)$ cross the line $u = +1$ from below, or when $u(\xi_k)$ and $u(\eta_k)$ cross the line $u = -1$ from above or below.

(3) It is clear from Lemma 2.3 that critical values cross the lines $u = \pm 1$ *in pairs*. This is a property which we shall very much exploit in Section 8.

The next three lemmas give important *global* properties of solutions of equation (1.1). The first one applies to solutions which have a critical point on the line $u = 1$ or on $u = -1$. Thus, let u be a solution of equation (1.1), and let $a \in \mathbf{R}$ be a critical point where u has the following properties:

$$u(a) = 1, \quad u'(a) = 0, \quad u''(a) = 0 \quad \text{and} \quad u'''(a) > 0. \tag{2.12}$$

Then $u' > 0$ in a right-neighbourhood of a so that the point

$$b = \sup\{x > a : u' > 0 \quad \text{on} \quad (a, x)\} \tag{2.13}$$

is well defined. By Lemma 2.2, it is also finite. We now derive some properties of u and its derivatives at b.

Lemma 2.4. *Suppose that*

$$-\sqrt{8} < q < q_3 = \sqrt{2}\Big(3 + \frac{1}{3}\Big).$$

Let u be a solution of equation (1.1) which at a point $a \in \mathbf{R}$ has the properties listed in (2.12). Then

$$u(b) > 1, \quad u'(b) = 0, \quad u''(b) < 0 \quad \textit{and} \quad u'''(b) < 0. \tag{2.14}$$

The proof of Lemma 2.4 is given in Section 9. This result will play an important role in the analysis of n-lap periodic solutions given in Section 6.

The second lemma applies to solutions for which $0 \le u \le \frac{1}{\sqrt{3}}$ at a critical point, and yields properties of the subsequent maxima and minima. We emphasize that it is only valid for *nonpositive* values of q.

Lemma 2.5. [PT4] *Suppose that $-\sqrt{8} < q \le 0$. Let u be a solution of equation (1.1) on $\mathbf{R}$ such that $\mathcal{E}(u) = 0$, and that for some $a \in \mathbf{R}$*

$$0 \le u(a) \le \frac{1}{\sqrt{3}}, \quad u'(a) = 0, \quad u''(a) > 0 \quad \text{and} \quad u'''(a) \ge 0.$$

Then

$$|u| > \sqrt{2} \quad \textit{whenever} \quad u' = 0 \quad \textit{on} \quad (a, \omega).$$

Here $[a, \omega)$ is the maximal interval in $[a, \infty)$ on which u exists.

We conclude with a universal bound for bounded solutions.

Lemma 2.6. [Be2,3,PT4] *Suppose that $-\sqrt{8} < q \le 0$. Let u be a solution of equation (1.1), which is uniformly bounded on $\mathbf{R}$. Then*

$$\|u\|_\infty < \sqrt{2}.$$

3. Odd periodic solutions

In this section we investigate the existence and qualitative properties of odd periodic solutions u of equation (1.1) for which $\mathcal{E}(u) = 0$. In previous studies (cf. [Be] and [PT2]) is was shown that for $q \le -\sqrt{8}$ there are no such odd *zero energy* periodic solutions for which $\mathcal{E}(u) = 0$. However, for $q > -\sqrt{8}$ odd zero energy periodic solutions do exist. In fact, it was proved in [PT2] that as q increases from $-\sqrt{8}$, two families of odd, single bump periodic solutions emerge as the result of a bifurcation from the unique increasing kink φ which exists at $q = -\sqrt{8}$, and it was shown in [PT2] and [MPT] that they continue to exist for all

$q > -\sqrt{8}$. A precise description of these results is given below in Theorem 3.1. As we shall see, these families of single bump periodic solutions will form a basis for our topological shooting arguments, which lead to the construction of multi bump periodic solutions with a more complicated structure.

Theorem 3.1. *For every $q > -\sqrt{8}$ there exist two odd periodic solutions u_+ and u_- of equation (1.1), such that $\mathcal{E}(u_\pm) = 0$, with the following properties:*

(a)
$$\|u_+\|_\infty > 1 \quad \textit{and} \quad \|u_-\|_\infty < 1.$$

(b) *If $u_\pm(a) = 0$ for some $a \in \mathbf{R}$, then*
$$u_\pm(a - y) = -u_\pm(a + y) \quad \textit{for} \quad y \in \mathbf{R}.$$

(c) *If $u'_\pm(a) = 0$ for some $a \in \mathbf{R}$, then*
$$u_\pm(a - y) = u_\pm(a + y) \quad \textit{for} \quad y \in \mathbf{R}.$$

By way of convention we choose the origin such that $u'_\pm(0) > 0$.

It was also shown in [PT2] that, as q decreases to $-\sqrt{8}$, both families of periodic solutions tend to the unique odd increasing kink φ at $q = -\sqrt{8}$:

$$u_\pm(\cdot, q) \to \varphi \quad \text{as} \quad q \to -\sqrt{8} \tag{3.1}$$

uniformly on compact sets. On the other hand, as q tends to infinity, the small amplitude solutions u_- tend to zero uniformly on $\mathbf{R}$, whilst the amplitude of the large solutions u_+ tends uniformly to infinity. More specifically,

$$u_-(x, q) \sim \frac{1}{q\sqrt{2}} \sin(x\sqrt{q}) \quad \text{as} \quad q \to \infty, \tag{3.2a}$$

and

$$u_+(x, q) \sim q\, V(x\sqrt{q}) \quad \text{as} \quad q \to \infty, \tag{3.2b}$$

where V is an odd solution of the equation

$$v^{iv} + v'' + v^3 = 0, \tag{3.3}$$

which possesses the symmetry properties listed in Theorem 3.1, and

$$\max\{|V(t)| : t \in \mathbf{R}\} \in \big(0, \frac{1}{2\sqrt{2}}\big).$$

At present it is not known whether the solution V is unique. Thus, the convergence in (3.2b) is along sequences, and the function V may possibly depend on the choice of sequence.

A numerically obtained plot of the two branches Γ_+ and Γ_- of odd single bump periodic solutions:

$$\Gamma_\pm = \{u_\pm(\cdot, q) : q > -\sqrt{8}\},$$

is presented in Figure 3.1. Along the vertical axis we set $M = \|u\|_\infty$.

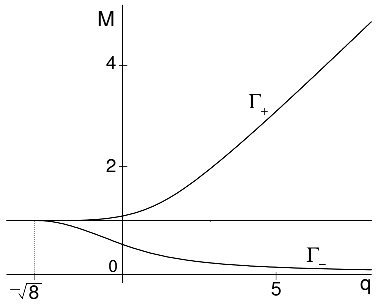


Fig. 3.1. The branches Γ_+ and Γ_- of odd single bump periodic solutions

In the figure below we give graphs of solutions on the branches Γ_+ and Γ_- at $q = 1$.

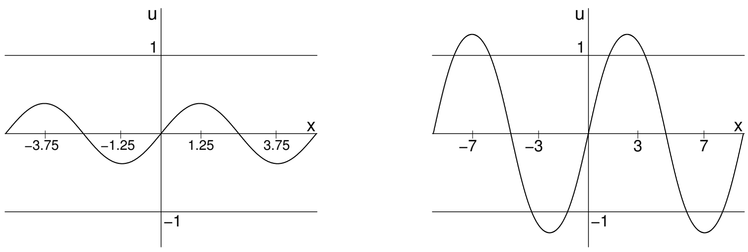


Fig. 3.2. Small and large single bump periodic solutions $(q = 1)$

As mentioned earlier, the two families of single bump periodic solutions will be used to construct further families of periodic solutions. In the following theorem we present a family which also exists on the entire interval $-\sqrt{8} < q < \infty$. These solutions look like u_+ in that their local maxima lie *above* $u = +1$ and all the local minima lie *below* $u = -1$, with the exception of the *first* point of symmetry $x = \zeta$. At that point these extrema lie on the *wrong* side of the constant solution $u = +1$, in case of a maximum, or $u = -1$ in case of a minimum:

$$u(\zeta) \begin{cases} < -1 & \text{if} \quad \zeta = \xi_k \quad \text{for some} \quad k \geq 2, \\ > +1 & \text{if} \quad \zeta = \eta_k \quad \text{for some} \quad k \geq 1. \end{cases}$$

We denote by T_N $(N \geq 2)$ the branch of odd periodic solutions of this family, of which the N^{th} critical point is the first point of symmetry:

$$T_N = \{u(\cdot, q) : q > -\sqrt{8}, \quad u'(\zeta_N, q) = 0, \quad u'''(\zeta_N, q) = 0\}.$$

The branches T_2 and T_3, as well as solutions on these branches at $q = 1.5$ are presented in Figure 1.6.

The existence of this family is the content of the next theorem.

Theorem 3.2. *Let $q > -\sqrt{8}$.*
(a) *For each $N \geq 1$ there exists an odd periodic solution u of equation (1.1) such that $\mathcal{E}(u) = 0$ and $u'(0) > 0$, which is symmetric with respect to η_N, and has the properties*

$$\begin{array}{lll} u(\xi_k) > 1 & \textit{for} \quad 1 \leq k \leq N & \\ u(\eta_k) < -1 & \textit{for} \quad 1 \leq k \leq N-1 \quad (N \geq 2) & \end{array} \quad \textit{and} \quad u(\eta_N) > 1. \tag{3.4a}$$

(b) *For each $N \geq 2$ there exists an odd periodic solution u of equation (1.1), such that $u'(0) > 0$, which is symmetric with respect to ξ_N, and has the properties*

$$\begin{array}{ll} u(\xi_k) > 1 & \textit{for} \quad 1 \leq k \leq N-1 \\ u(\eta_k) < -1 & \textit{for} \quad 1 \leq k \leq N-1 \end{array} \quad \textit{and} \quad u(\xi_N) < -1. \tag{3.4b}$$

Proof. We use an iterative type of argument, and begin by proving the existence of a periodic solution which is symmetric with respect to η_1. This is the case $N = 1$ of Part (a). Such a solution is illustrated in Figure 1.6.

We denote the initial slopes of the zero energy odd single bump periodic solutions u_+ and u_-, constructed in Theorem 3.1, by α_+ and α_- respectively, that is $\alpha_\pm = u'_\pm(0)$. From the construction in [PT2] we know that $0 < \alpha_- < \alpha_+$. Plainly (see Figure 3.2),

$$u(\xi_k) > 1 \ \text{and} \qquad u(\eta_k) < -1 \quad \text{for} \ k \geq 1 \quad \text{when} \ \alpha = \alpha_+, \tag{3.5a}$$
$$0 < u(\xi_k) < 1 \ \text{and} \quad -1 < u(\eta_k) < 0 \quad \text{for} \ k \geq 1 \quad \text{when} \ \alpha = \alpha_-, \tag{3.5b}$$

where $u(\xi_k) = u(\xi_k(\alpha), \alpha)$ and $u(\eta_k) = u(\eta_k(\alpha), \alpha)$. In view of (3.5a) we can define

$$a_1 = \inf\{\alpha > 0 : u(\xi_1) > 1 \ \text{ on } \ (\alpha, \alpha_+)\},$$

and it follows from (3.5b) that $a_1 \in (\alpha_-, \alpha_+)$. By Lemma 2.2, the location of the first critical point, $\xi_1(\alpha)$, depends continuously on α, and by standard theory, the solution $u(x, \alpha)$ of Problem (2.1), (2.2) depends continuously on α for x in compact sets. Since $\mathcal{E}(u) = 0$, it follows from Lemma 2.1 that

$$u(\xi_1) = 1, \qquad \eta_1 = \xi_1 \quad \text{and} \quad u'''(\xi_1) > 0 \qquad \text{if} \ \ \alpha = a_1. \tag{3.6}$$

By Lemma 2.3 this implies that $u(\eta_1(\alpha), \alpha) > 1$ for $\alpha \in (a_1, a_1 + \delta)$, where $\delta > 0$ is a small positive constant. Hence, we can define

$$a_1^+ = \sup\{\alpha > a_1 : u(\eta_1) > 1 \ \text{ on } \ (a_1, \alpha)\},$$

As we saw in (3.5a), $u(\eta_1) < -1$ at α_+, so that $a_1^+ \in (a_1, \alpha_+)$. Invoking the continuity of $\eta_1(\alpha)$ and $u(x, \alpha)$, we deduce that

$$u(\eta_1) = 1, \qquad \eta_1 = \xi_2 \quad \text{and} \quad u'''(\eta_1) < 0 \qquad \text{if} \ \ \alpha = a_1^+. \tag{3.7}$$

Using the continuity of $\eta_1(\alpha)$ and of u and its derivatives we see that (3.6) and (3.7) imply that there must exist a point $\alpha_1^* \in (a_1, a_1^+)$ where $u'''(\eta_1)$ vanishes, and so

$$u'(\eta_1(\alpha_1^*), \alpha_1^*) = 0 \quad \text{and} \quad u'''(\eta_1(\alpha_1^*), \alpha_1^*) = 0.$$

This means that the solution $u(x, \alpha_1^*)$ is symmetric with respect to $\eta_1(\alpha_1^*)$. Since it is also odd with respect to the origin, we conclude that $u(x, \alpha_1^*)$ is a periodic solution with period $4\eta_1$. It is readily verified that it has the desired properties.

We continue with the construction of the periodic solution which is symmetric with respect to ξ_2. This is the case $N = 2$ of Part (b), and such a solution is illustrated in Figure 1.6.

Because $u(\eta_1) < -1$ at α_+, we can define

$$b_1 = \inf\{\alpha < \alpha_+ : u(\eta_1) < -1 \quad \text{on} \quad (\alpha, \alpha_+)\},$$

and it follows from (3.6) that $b_1 \in (a_1^+, \alpha_+)$. Like at a_1^+, we once again invoke the continuity of η_1 and u and the fact that $\mathcal{E}(u) = 0$ to conclude from Lemma 2.1 that

$$u(\eta_1) = -1, \qquad \eta_1 = \xi_2 \quad \text{and} \quad u'''(\eta_1) < 0 \qquad \text{at} \quad b_1. \tag{3.8}$$

By a result similar to Lemma 2.3 we find that $u(\xi_2) < -1$ in an interval $(b_1, b_1 + \delta)$, where $\delta > 0$ is sufficiently small. Thus, we can define

$$b_1^+ = \sup\{\alpha > b_1 : u(\xi_2) < -1 \quad \text{on} \quad (b_1, \alpha)\}.$$

Remembering (3.5a), we see that $b_1^+ \in (b_1, \alpha_+)$, and using the continuity properties of ξ_2 and u, we conclude that

$$u(\xi_2) = -1, \qquad \xi_2 = \eta_2 \quad \text{and} \quad u'''(\xi_2) > 0 \qquad \text{at} \quad b_1^+. \tag{3.9}$$

Another application of the continuity of ξ_2 and u and its derivatives implies the existence of a point $b_1^* \in (b_1, b_1^+)$ such that

$$u(\xi_2) < -1, \quad u'(\xi_2) = 0 \quad \text{and} \quad u'''(\xi_2) = 0 \quad \text{at} \quad b_1^*.$$

As in the previous case, this means that $u(x, b_1^*)$ is a periodic solution with period $4\xi_2$. Recall that $b_1^* > a_1$, so that $u(\xi_1) > 1$. Thus, this solution has the desired properties.

Continuing in this manner, we successively prove the existence of all the periodic solutions listed in Theorem 3.2.

In addition to these branches of solutions, which exist for all $q > -\sqrt{8}$, there exists a multitude of odd zero energy periodic solutions for $-\sqrt{8} < q \leq 0$. In Theorems 3.3 and 3.4 we present a few of these families. They exist in pairs. Those corresponding to Theorem 3.3 are shown in Figures 3.3 and 3.4, and those obtained in Theorem 3.4 in Figure 3.6.

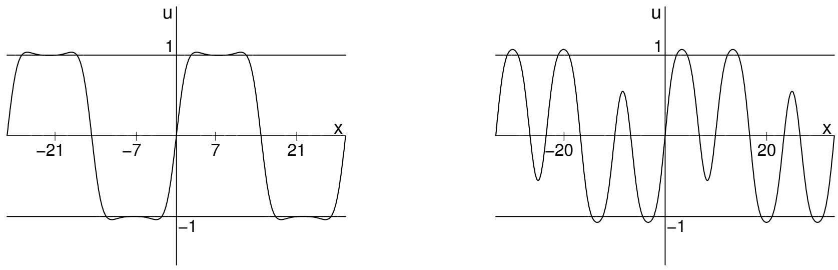


Fig. 3.3. Solutions symmetric with respect to η_1 from Theorem 3.3(a) $(N = 1,\ q = -\frac{1}{10})$

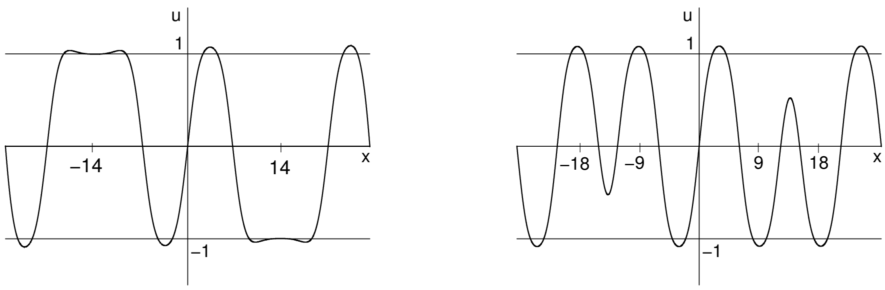


Fig. 3.4. Solutions symmetric with respect to ξ_2 from Theorem 3.3(b) $(N = 2,\ q = -\frac{1}{10})$

A numerical study shows that the solutions obtained in Theorem 3.3 lie on branches which are loop shaped. The branch for Part (a) is shown in Figure 3.5.

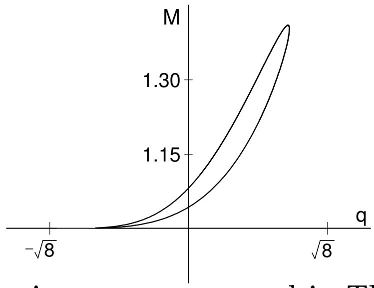


Fig. 3.5. Branch of solutions as constructed in Theorem 3.3(a) for $N = 1$

Theorem 3.3. *Let* $-\sqrt{8} < q \leq 0$.

(a) *For each* $N \geq 1$ *there exist two odd periodic solutions* u_1 *and* u_2 *of equation (1.1) such that* $\mathcal{E}(u_i) = 0$ *and* $u_i'(0) > 0$ $(i = 1, 2)$, *which are symmetric with respect to* η_N, *and have the properties*

$$
\begin{aligned}
&u_1(\xi_k) > 1, \quad u_2(\xi_k) > 1 \quad &&\textit{for} \quad 1 \leq k \leq N \\
&u_1(\eta_k) < -1, \quad u_2(\eta_k) < -1 \quad &&\textit{for} \quad 1 \leq k \leq N-1 \ \ (\textit{if } N \geq 2) \\
&-1 < u_1(\eta_N) < 0 < u_2(\eta_N) < 1. &&
\end{aligned} \tag{3.10a}
$$

(b) *For each $N \geq 2$ there exist two odd periodic solutions u_1 and u_2 of equation (1.1), such that $\mathcal{E}(u_i) = 0$ and $u_i'(0) > 0$ $(i = 1, 2)$, which are symmetric with respect to ξ_N, and have the properties*

$$\begin{aligned} &u_1(\xi_k) > 1, \quad u_2(\xi_k) > 1 \quad &&\textit{for} \quad 1 \leq k \leq N-1 \\ &u_1(\eta_k) < -1, \quad u_2(\eta_k) < -1 \quad &&\textit{for} \quad 1 \leq k \leq N-1 \\ &-1 < u_1(\xi_N) < 0 < u_2(\xi_N) < 1. \end{aligned} \tag{3.10b}$$

Proof. We pick up the line of argument in the proof of Theorem 3.2, and consider the interval $[a_1, b_1]$. We recall that

$$u(\eta_1) = 1 \quad \text{at} \quad a_1 \quad \text{and} \quad u(\eta_1) = -1 \quad \text{at} \quad b_1.$$

By continuity this implies that $u(\eta_1)$ has a zero on (a_1, b_1). Let c_1 be the smallest zero of $u(\eta_1)$ on (a_1, b_1), and let

$$c_1^- = \inf\{\alpha < c_1 : u(\eta_1) < 1 \;\text{ on }\; (\alpha, c_1)\}. \tag{3.11a}$$

Similarly, let d_1 be the largest zero of $u(\eta_1)$ on (a_1, b_1), and let

$$d_1^+ = \sup\{\alpha > d_1 : u(\eta_1) > -1 \;\text{ on }\; (d_1, \alpha)\}. \tag{3.11b}$$

Plainly, $c_1^- \in [a_1^+, c_1)$ and $d_1^+ \in (d_1, b_1]$, and

$$u(\eta_1) = 1 \;\text{ at }\; c_1^- \qquad \text{and} \qquad u(\eta_1) = -1 \;\text{ at }\; d_1^+.$$

Since $u(\xi_1) > 1$ on $(a_1, b_1]$, it follows that $u'''(\eta_1) < 0$ at c_1^- as well as at d_1^+.

We deduce from Lemma 2.5 that $u'''(\eta_1) > 0$ at c_1 and at d_1. For suppose to the contrary that $u'''(\eta_1) \leq 0$ at c_1 or at d_1. Then we deduce from Lemma 2.5 that $u > \sqrt{2}$ at every critical point on the interval $[-\eta_1, \eta_1]$. But $u(-\eta_1) = 0$ because u is odd, a contradiction. Thus, $u'''(\eta_1)$ changes sign on (c_1^-, c_1) and on (d_1, d_1^+), so that there exist a point $c_1^* \in (c_1^-, c_1)$ and a point $d_1^* \in (d_1, d_1^+)$ such that

$$u'''(\eta_1) = 0 \quad \text{at} \quad c_1^* \;\text{ and }\; d_1^*.$$

Writing $u_1(x) = u(x, d_1^*)$ and $u_2(x) = u(x, c_1^*)$, we conclude that u_1 and u_2 are odd periodic solutions which are symmetric with respect to η_1, and that they have the properties

$$u_1(\xi_1) > 1, \quad u_2(\xi_1) > 1 \quad \text{and} \quad -1 < u_1(\eta_1) < 0 < u_2(\eta_1) < 1.$$

This completes the proof of the case $N = 2$ of Part (a).

Next, we construct a pair of odd periodic solutions which are symmetric with respect to ξ_2. This corresponds to the case $N = 2$ of Part (b). To this end, we define

$$a_2 = \inf\{\alpha < \alpha_+ : u(\xi_2) > 1 \;\text{ on }\; (\alpha, \alpha_+)\}.$$

Plainly, $b_1 < a_2 < \alpha_+$. Because $u(\xi_2) = -1$ at b_1 and $u(\xi_2) = +1$ at a_2, $u(\xi_2)$ has a zero on (b_1, a_2). Let e_1 be the smallest zero of $u(\xi_2)$ on (b_1, a_2) and f_1 the largest. Then, proceeding as in the previous case we show that $u'''(\xi_2)$ has two zeros, e_1^* and f_1^*, on (b_1^+, a_2) such that $u_1(x) = u(x, e_1^*)$ and $u_2(x) = u(x, f_1^*)$ are periodic solutions with the following properties:

$$u_i(\xi_1) > 1, \qquad u_i(\eta_1) < -1 \quad (i = 1, 2),$$

and

$$-1 < u_1(\xi_2) < 0 < u_2(\xi_2) < 1.$$

This completes the proof of the case $N = 2$ of Part (b).

For the next step we define

$$b_2 = \inf\{\alpha < \alpha_+ : u(\eta_2) < -1 \ \text{ on } \ (\alpha, \alpha_+)\}.$$

Since $u(\eta_2) = 1$ at a_2, it follows that $b_2 > a_2$. By repeating the arguments applied to the interval $[a_1, b_1]$ to $[a_2, b_2]$, we prove Part (a) for $N = 2$. We can continue this process indefinitely, and so successively prove all the cases in Parts (a) and (b) of Theorem 3.3.

Remark. Using a continuity argument, we can show that the families of periodic solutions described in Theorem 3.3 continue to exist for small values of $q > 0$. This is consistent with the numerically computed bifurcation branch shown in Figure 3.5 for the case $N = 1$ of Part (a).

In the next theorem we obtain a different family of periodic solutions. To explain the difference, let u be a periodic solution which is symmetric with respect to its n^{th} critical point ζ_n. Then solutions of Theorem 3.3 have the property that

$$|u(\zeta_k)| > 1 \ \text{ for } \ 1 \le k \le n-1 \quad \text{and} \quad |u(\zeta_n)| < 1.$$

In contrast, the solutions of Theorem 3.4 have the property that

$$|u(\zeta_k)| > 1 \ \text{ for } \ 1 \le k \le n-2 \quad \text{and} \quad |u(\zeta_{n-1})| < 1, \ \ |u(\zeta_n)| < 1.$$

Thus, whereas in the first family, the point of symmetry is the only critical point at which $u \in (-1, 1)$, in the second family the value of u at the point of symmetry, *as well as at the two adjacent critical points*, lie in the interval $(-1, 1)$. A pair of such solutions is shown in Figure 3.6. Since the characteristics of these solutions are not very clear when $q \le 0$, we have set $q = 1.5$.

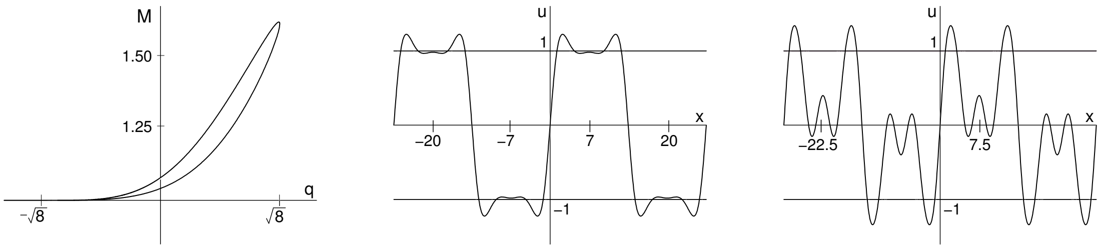


Fig. 3.6. Solutions symmetric with respect to ξ_2 from Theorem 3.4(a)
$(N = 2, q = 1.5)$

Theorem 3.4. *Let* $-\sqrt{8} < q \leq 0$.
(a) *For each* $N \geq 2$ *there exist two odd periodic solutions* u_1 *and* u_2 *of equation (1.1) such that* $\mathcal{E}(u_i) = 0$ *and* $u_i'(0) > 0$ $(i = 1, 2)$, *which are symmetric with respect to* ξ_N, *and have the following properties:*

$$\left.\begin{array}{lll} u_i(\xi_k) > 1 & \textit{for} & 1 \leq k \leq N-1 \\ u_i(\eta_k) < -1 & \textit{for} & 1 \leq k \leq N-2 \ \ (\textit{if } N \geq 3) \end{array}\right\} \quad \textit{for} \quad i = 1, 2, \tag{3.12a}$$

and

$$\begin{array}{c} 0 < u_i(\xi_N) < 1 \quad \textit{for} \quad i = 1, 2, \\ -1 < u_1(\eta_{N-1}) < 0 < u_2(\eta_{N-1}) < 1. \end{array} \tag{3.12b}$$

(b) *For each* $N \geq 2$ *there exist two odd periodic solutions* u_1 *and* u_2 *of equation (1.1) such that* $\mathcal{E}(u_i) = 0$ *and* $u_i'(0) > 0$ $(i = 1, 2)$, *which are symmetric with respect to* η_N, *and have the following properties:*

$$\left.\begin{array}{lll} u_i(\xi_k) > 1 & \textit{for} & 1 \leq k \leq N-2 \ \ (\textit{if } N \geq 3) \\ u_i(\eta_k) < -1 & \textit{for} & 1 \leq k \leq N-1 \end{array}\right\} \quad \textit{for} \quad i = 1, 2, \tag{3.12c}$$

and

$$\begin{array}{c} 0 < u_i(\eta_N) < 1 \quad \textit{for} \quad i = 1, 2, \\ -1 < u_1(\xi_N) < 0 < u_2(\xi_N) < 1. \end{array} \tag{3.12d}$$

Proof. (a) We begin with the proof for $N = 2$. To that end, we return to the interval $[a_1, b_1]$ defined in the proof of Theorem 3.3, and consider the two subintervals:

$$I_1^- = [a_1, c_1] \quad \text{and} \quad I_1^+ = [d_1, b_1],$$

where c_1 and d_1 are, respectively, the smallest and the largest zero of $u(\eta_1)$ on (a_1, b_1). and we observe that by Lemma 2.5,

$$u(\xi_k) > 1 \quad \text{and} \quad u(\eta_k) < -1 \quad \text{for} \quad k \geq 2 \quad \text{at} \ \ c_1 \text{ and } d_1. \tag{3.13}$$

We first consider the interval I_1^-. As in (3.11a) we set

$$c_1^- = \inf\{\alpha < c_1 : u(\eta_1) < 1 \ \ \text{on} \ \ (\alpha, c_1)\}.$$

Then $a_1^+ \leq c_1^- < c_1$. By Lemma 2.3, $u(\xi_2) < 1$ in a right-neighbourhood of c_1^-, so that we can define

$$c_1^+ = \sup\{\alpha > c_1^- : u(\xi_2) < 1 \ \ \text{on} \ \ (c_1^-, \alpha)\}.$$

It is clear that $c_1^+ \in (c_1^-, c_1)$ and that

$$u(\xi_2) = u(\eta_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \quad \text{at} \ c_1^+.$$

Since $u'''(\xi_2) < 0$ at c_1^-, it follows that $u'''(\xi_2)$ has a zero $c_1^* \in (c_1^-, c_1^+)$. Thus $u_2(x) = u(x, c_1^*)$ is an even periodic solution with the properties

$$u_2(\xi_1) > 1, \quad 0 < u_2(\xi_2) < 1, \quad 0 < u_2(\eta_1) < 1 \quad \text{and} \quad u_2'''(\xi_2) = 0. \tag{3.14a}$$

For the second solution we consider the interval $I_1^+ = [d_1, b_1]$. Because $u(\xi_2) = -1$ at b_1 we can define

$$b_1^- = \inf\{\alpha < b_1 : u(\xi_2) < 1 \ \text{ on } \ (\alpha, b_1)\},$$

and in view of (3.13) it follows that $b_1^- \in (d_1, b_1)$. Since $u(\eta_1) \le 0$ on I_1^+ and $u(\xi_2) = 1$ at b_1^-, we conclude that $u'''(\xi_2) > 0$ at b_1^-. Let

$$\tilde{b}_1 = \sup\{\alpha > b_1^- : u(\xi_2) > 0 \ \text{ on } \ (b_1^-, \alpha)\}.$$

Plainly, $b_1^- < \tilde{b}_1 < b_1$. As in the proof of Theorem 3.3, we deduce from Lemma 2.5 that $u'''(\xi_2) < 0$ at $\tilde{b}_1$. Therefore, $u'''(\xi_2)$ changes sign on $(b_1^-, \tilde{b}_1)$, and hence there exists a point $b_1^* \in (b_1^-, \tilde{b}_1)$ where $u'''(\xi_2)$ vanishes. This means that $u_1(x) = u(x, b_1^*)$ is a periodic solution endowed with the properties

$$u_1(\xi_1) > 1, \quad 0 < u_1(\xi_2) < 1, \quad -1 < u_1(\eta_1) < 0 \quad \text{and} \quad u_1'''(\xi_2) = 0. \tag{3.14b}$$

This completes the proof of Theorem 3.4(a) for $N = 2$.

To prove Part (a) for $N = 3$, we repeat the above arguments for the interval $[a_2, b_2]$ defined in the proof of Theorem 3.3. For $N = 4$ we consider the corresponding interval $[a_3, b_3]$ and generally, we consider the interval $[a_{N-1}, b_{N-1}]$ for arbitrary $N \ge 2$.

For the proof of Part (b), say for $N = 2$, we consider the interval $[b_1, a_2]$. Proceeding as in the proof of Part (a) ($N = 2$) we now find solutions which are symmetric with respect to η_2 for values of α on (b_1, e_1) and (f_1, a_2), where e_1 and f_1 have been defined in the proof of Theorem 3.3. The argument, and its generalisation to higher values of N is very similar to the arguments involved in the proof of Part (a), and we shall therefore omit the details.

4. Even periodic solutions: $-\sqrt{8} < q < \sqrt{8}$

In this section we establish the existence of an infinite sequence of countable families of even periodic solutions of equation (1.1) for $q \in (-\sqrt{8}, \sqrt{8})$, distinguished by the number and location of local maxima and minima of their graphs. Thus, whereas some of the results obtained for for odd solutions were only valid for $q \le 0$, the results proved in this section are also valid for *positive* values of q up to $\sqrt{8}$. In the next five sections we go even beyond this number, and show how the branches of multi bump periodic solutions obtained in this section extend to higher values of q. As we stated in Section 2, we shall use a shooting technique to establish the existence of these solutions, and hence, thanks to symmetry with respect to $x = 0$, we will study the initial value problem

$$\begin{cases} u^{iv} + qu'' + u^3 - u = 0 \qquad \text{for} \quad x > 0 & \text{(4.1a)} \\ u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0. & \text{(4.1b)} \end{cases}$$

Again, we only discuss solutions for which the first integral is zero, i.e.

$$\mathcal{E}(u) \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + F(u) = 0, \tag{4.2a}$$

where

$$F(s) = \frac{1}{4}(s^2 - 1)^2 \quad \text{and} \quad F'(s) = f(s) = s^3 - s. \tag{4.2b}$$

This means that

$$\beta \stackrel{\text{def}}{=} \beta(\alpha) = \pm\frac{1}{\sqrt{2}}|\alpha^2 - 1|. \tag{4.3}$$

The cases $u''(0) > 0$ and $u''(0) < 0$ will be dealt with in succession. We refer to them by, respectively, Case I and Case II:

$$\text{Case I}: \ u''(0) > 0 \qquad \text{and} \qquad \text{Case II}: \ u''(0) < 0.$$

In both cases we shall denote the solution of problem (4.1) by $u(x, \alpha)$.

Note that the single bump periodic solutions of Section 3 become even solutions after a shift over a quarter of a period. Thus if the period of $u_\pm$ is $4\ell_\pm$, and $M_\pm = \|u_\pm\|_\infty$, then

$$\begin{aligned} u(x, -M_\pm) &= u_\pm(x - \ell_\pm) \qquad \text{in Case I,} \\ u(x, +M_\pm) &= u_\pm(x - \ell_\pm) \qquad \text{in Case II.} \end{aligned} \tag{4.4}$$

These solutions provide the point of departure for the shooting arguments which will yield new families of even periodic solutions with more complicated structure. For convenience we provide the graphs of $u(x, -M_\pm)$ in Figure 4.1.

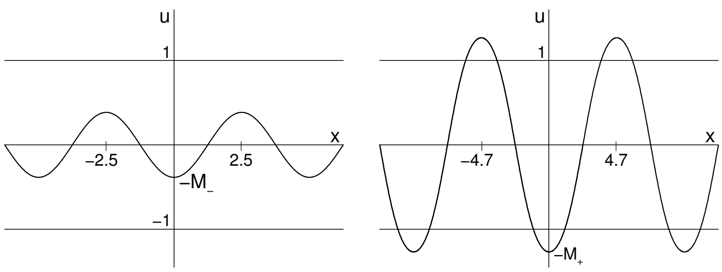
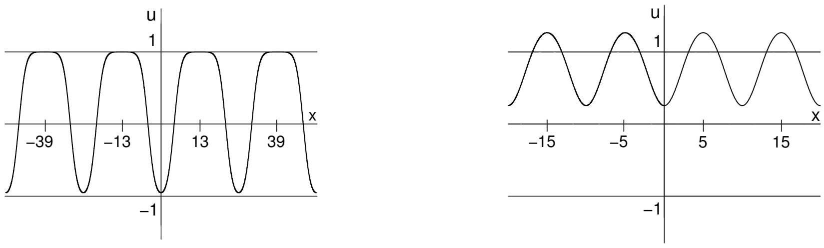


Fig. 4.1. The small and the large single bump periodic solutions for Case I

We begin by establishing the existence of a new family of even, single bump periodic solutions whose maxima lie above the line $u = +1$, and whose minima lie *between* the lines $u = -1$ and $u = +1$. In Figure 4.2 we give examples of two such solutions computed at $q = -2$ and $q = 2$.



$q = -2$ $\qquad\qquad$ $q = 2$

Fig. 4.2. Even single bump periodic solutions for $q = -2$ and $q = 2$

It is readily apparent that these solutions are qualitatively different from those shown in Figure 4.1, because $|u(x, -M_-)| < 1$ for every $x \in \mathbf{R}$ and $|u(x, -M_+)| > 1$ at every critical point. Like Γ_- and Γ_+, this new family of solutions forms a branch Γ_1 which bifurcates from the unique odd kink $\varphi(x)$ at $q = -\sqrt{8}$. However, as the bifurcation diagram in Figure 4.3 below shows, in contrast to the branches Γ_- and Γ_+, which extend all the way to $q = +\infty$ (see Fig. 1.3), our computations indicate that Γ_1 only extends over the *finite* q-interval $(-\sqrt{8}, \sqrt{8})$, and bifurcates at $q = \sqrt{8}$ from the constant solution $u = +1$. In Theorem 4.1 we prove that the new solutions indeed exist for every $q \in (-\sqrt{8}, \sqrt{8})$.

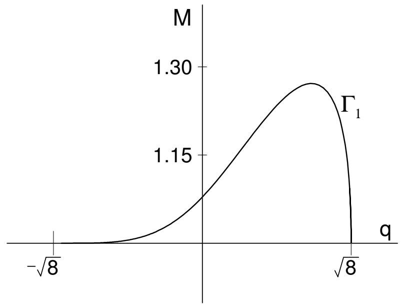


Fig. 4.3. Branch of even single bump periodic solutions

Theorem 4.1. *Let $q \in (-\sqrt{8}, \sqrt{8})$. Then there exists an even, single bump periodic solution u such that $\mathcal{E}(u) = 0$, and*

$$-1 < \min\{u(x) : x \in \mathbf{R}\} < +1 < \max\{u(x) : x \in \mathbf{R}\}.$$

Proof. The proof uses ideas developed in [PT5]. Since we seek a single bump periodic solution, it follows that $u''(0) > 0$, so that we are in Case I. We follow $u(\xi_1)$ as α varies, and seek a value of $\alpha \in (-1, 1)$ such that

$$u(\xi_1) > 1 \qquad \text{and} \qquad u'''(\xi_1) = 0. \tag{4.5}$$

Then, by symmetry, u is a periodic solution with half-period $L = \xi_1$ whose maxima and minima are located as indicated above:

$$\min\{u(x) : x \in \mathbf{R}\} = \alpha \in (-1,1) \quad \text{and} \quad \max\{u(x) : x \in \mathbf{R}\} = u(\xi_1) > 1.$$

To find such a value of α, we use the auxiliary functional

$$\mathcal{H}(u) = \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + F(u), \tag{4.6}$$

where F has been defined in (4.2b). Let $u(x)$ be a smooth function. Then we write

$$H(x) \stackrel{\text{def}}{=} \mathcal{H}(u(x)).$$

Differentiation yields

$$H' = u''u''' + qu'u'' + f(u)u', \tag{4.7}$$

and if, in addition, u is a solution of equation (1.1), then

$$H'' = (u''')^2 + qu'u''' + f'(u)(u')^2. \tag{4.8}$$

The right hand side of (4.8) is a second order polynomial in u''', with discriminant

$$D = \{q^2 - 4f'(u)\}(u')^2. \tag{4.9}$$

Thus, H'' will be nonnegative whenever

$$q^2 < 4(3u^2 - 1) \qquad \text{or} \qquad u^2 > \frac{q^2+4}{12}. \tag{4.10}$$

Define

$$\alpha_0 = \sqrt{\frac{q^2+4}{12}}. \tag{4.11}$$

Then $\alpha_0 \in (0,1)$ as long as $q^2 < 8$.

Lemma 4.2. *Let $q^2 < 8$, and let $\alpha \in [\alpha_0, 1)$. Then*

$$u(\xi_1) > 1 \qquad \textit{and} \qquad u'''(\xi_1) < 0.$$

Proof of Lemma 4.2. Let $\alpha \in [\alpha_0, 1)$. Then $u''(0) > 0$, so that $u'(x,\alpha) > 0$ and $u(x,\alpha) > \alpha_0$ for $0 < x \le \xi_1$. This implies, by (4.8) and (4.10), that $H''(x) > 0$ for $0 < x \le \xi_1$, and so

$$H'(x) > H'(0) = 0 \qquad \text{for} \quad x \in (0, \xi_1]. \tag{4.12}$$

Hence

$$H(\xi_1) > H(0). \tag{4.13}$$

According to the identity (4.2),

$$(u'')^2 = 2F(u) \quad \text{if} \quad u' = 0. \tag{4.14}$$

Therefore, at any critical point ζ of u we have

$$H(\zeta) = 2F(u(\zeta)). \tag{4.15}$$

Combining (4.13) and (4.15), we conclude that

$$F(u(\xi_1)) > F(\alpha_0).$$

Because $F' = f < 0$ on $(0,1)$ and $\alpha_0 \in (0,1)$, this implies that $u(\xi_1) > 1$. From (4.14) we deduce that $u''(\xi_1) < 0$. Because $H'(\xi_1) > 0$ by (4.12), we conclude from (4.7) that $u'''(\xi_1) < 0$, and the proof of Lemma 4.2 is complete.

We now continue with the proof of Theorem 4.1. By Lemma 4.2, $u(\xi_1) > 1$ at $\alpha = \alpha_0$. Thus, remembering that $u(\xi_1) = M_- < 1$ when $\alpha = -M_-$, we can introduce the point

$$\alpha_1 = \inf\{\alpha < \alpha_0 : u(\xi_1) > 1 \quad \text{on} \quad (\alpha, \alpha_0)\},$$

and conclude that $\alpha_1 \in (-M_-, \alpha_0)$. It follows from the continuity of $\xi_1(\alpha)$ and $u(\xi_1(\alpha), \alpha)$, established in Lemma 2.1 and Lemma 2.2, which we can apply because $\mathcal{E}(u) = 0$, that

$$u(\xi_1) = 1, \quad u'''(\xi_1) > 0 \quad \text{and} \quad \xi_1 = \eta_1 \quad \text{at} \quad \alpha_1. \tag{4.16}$$

The fact that $u'''(\xi_1)$ is positive at α_1 follows from Lemma 2.1 and the definition of ξ_1. Thus $u'''(\xi_1)$ has changed sign on (α_1, α_0). Since $u'''(\xi_1(\alpha), \alpha)$ depends continuously on α by Lemma 2.2, there must be a point $\alpha_1^* \in (\alpha_1, \alpha_0)$ where $u'''(\xi_1)$ vanishes, i.e.

$$u'''(\xi_1(\alpha_1^*), \alpha_1^*) = 0. \tag{4.17}$$

Remembering that $u'(\xi_1) = 0$ as well, it follows by symmetry that the function $u(x, \alpha_1^*)$ is an even, single bump periodic solution, which is symmetric with respect to ξ_1, such that $u(\xi_1) > 1$ and the period is $2\xi_1$. This completes the proof of Theorem 4.1.

We continue with the construction of an even, 1-bump periodic solution, which is symmetric with respect to ξ_2. We have seen in (4.16) that when $\alpha = \alpha_1$, then $u = 1$ and $u''' > 0$ at ξ_1. This means that $u(\xi_2) > 1$ at α_1. We now define the point

$$\tilde{\alpha}_1 = \sup\{\alpha > -M_- : u(\xi_1) < 1 \quad \text{on} \quad (-M_-, \alpha)\}.$$

Then $\tilde{\alpha}_1 \in (-M_-, \alpha_0)$, and (4.16) holds, but now at $\tilde{\alpha}_1$. Thus, $u(\xi_2) > 1$ at $\tilde{\alpha}_1$, and we define

$$\alpha_2 = \inf\{\alpha < \tilde{\alpha}_1 : u(\xi_2) > 1 \quad \text{on} \quad (\alpha, \tilde{\alpha}_1)\},$$

Since $u(\xi_2, -M_-) = M_- < 1$, we can conclude again that $\alpha_2 \in (-M_-, \tilde{\alpha}_1)$.

As before, $u'''(\xi_2) > 0$ at α_2. Thus, it remains to determine the sign of $u'''(\xi_2)$ when $\alpha = \tilde{\alpha}_1$. We have

$$H'(\xi_1) = 0 \quad \text{at} \quad \tilde{\alpha}_1.$$

Because $u(x) > 1$ for $x \in (\xi_1, \xi_2]$, it follows that $H'' > 0$ on (ξ_1, ξ_2), and

$$H'(\xi_2) = u''(\xi_2)\, u'''(\xi_2) > 0 \quad \text{at} \quad \tilde{\alpha}_1.$$

Plainly, $u(\xi_2) > 1$, and hence, by the first integral, $u''(\xi_2) < 0$. Thus, $u'''(\xi_2) < 0$ at $\tilde{\alpha}_1$, and $u'''(\xi_2)$ has changed sign on $(\alpha_2, \tilde{\alpha}_1)$, and therefore has a zero α_2^* in this interval:

$$u'''(\xi_1(\alpha_2^*), \alpha_2^*) = 0. \tag{4.18}$$

By symmetry this yields an even periodic solution $u(x, \alpha_2^*)$ which is symmetric with respect to ξ_2, so that $u(\xi_2) > 1$, $u(\xi_1) < 1$ and the period is $2\xi_2$ (cf. Fig 4.4(a)).

We can now construct an N-bump periodic solution for any $N \geq 2$ by continuing the above process in an iterative manner. This yields a decreasing sequence of numbers $\{\alpha_k^*\}$ such that the solutions $u_k(x) = u(x, \alpha_k^*)$ are even and periodic with period $2\xi_k$. They have the properties

$$u_k(\xi_j) < 1 \quad \text{for} \quad j = 1, \ldots, k-1 \quad \text{and} \quad u_k(\xi_k) > 1. \tag{4.19}$$

Thus we have proved:

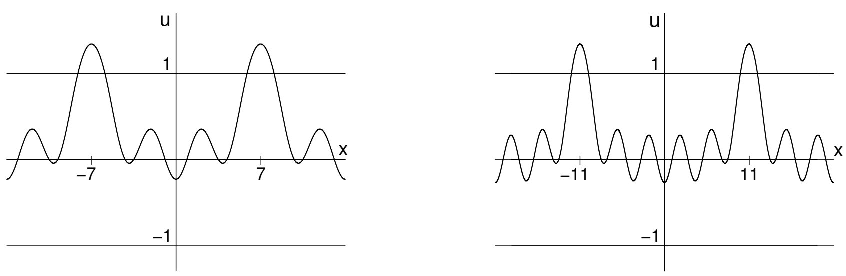
Theorem 4.3. *Let $-\sqrt{8} < q < \sqrt{8}$. Then for any $N \geq 2$ there exists an even periodic solution u of equation (1.1) such that $\mathcal{E}(u) = 0$ and $u''(0) > 0$, which is symmetric with respect to ξ_N and has the properties:*

(a) *u has $N-1$ local maxima on the interval $(0, \xi_N)$.*

(b) *All the local maxima of u, except the one at the point of symmetry, ξ_N, lie below the line $u = 1$, but $u(\xi_N) > 1$, i.e.*

$$u(\xi_k) < 1 \quad \textit{for} \quad 1 \leq k \leq N-1 \quad \textit{and} \quad u(\xi_N) > 1.$$

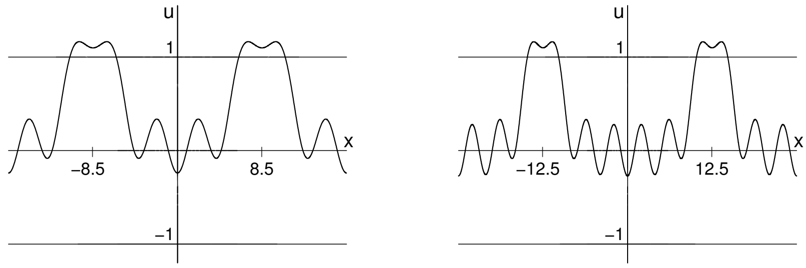
For $N = 2$ and $N = 3$ such solutions are shown below in Figure 4.4.



(a) $N = 2$ (b) $N = 3$

Fig. 4.4. Multi bump periodic solutions of Theorem 4.3; $u''(0) > 0$ and $q = 2$

In addition to the family of periodic solutions u_N described in Theorem 4.1 and Theorem 4.3, which are symmetric with respect to ξ_N for some $N \geq 1$, there exists a corresponding family of periodic solutions which are similar to u_N, but they are symmetric with respect to η_N. Such solutions are shown in Figure 4.5. The existence of this family of solutions is established in Theorem 4.4.



(a) $N = 2$ (b) $N = 3$

Fig. 4.5. Multi bump periodic solutions of Theorem 4.4; $u''(0) > 0$ and $q = 2$

Theorem 4.4. *Let $-\sqrt{8} < q < \sqrt{8}$. Then for every $N \geq 1$ there exists an even periodic solution u of equation (1.1) such that $\mathcal{E}(u) = 0$ and $u''(0) > 0$, which is symmetric with respect to η_N and has the properties:*

(a) *u has N local maxima on the interval* $(0, \eta_N)$.

$$\text{(b)} \qquad u(\xi_k) < 1 \quad \text{for} \quad 1 \leq k \leq N-1 \quad \text{if} \quad N > 2,$$

$$\text{(c)} \qquad u(\xi_N) > 1 \quad \text{and} \quad u(\eta_N) > 1.$$

Proof. We give the proof for $N = 2$. For $N = 1$ and for $N \geq 3$ it is similar. For further details we refer to [PT5]. We recall the point α_2 defined in the proof of Theorem 4.3, and in particular that

$$u(\xi_2) = 1, \qquad u'''(\xi_2) > 0 \quad \text{and} \quad \xi_2 = \eta_2 \qquad \text{at} \quad \alpha_2. \tag{4.20}$$

Therefore, by Lemma 2.3, $u(\eta_2) > 1$ for $\alpha \in (\alpha_2, a_2 + \delta)$ for some small $\delta > 0$. At the point $\alpha_2^* > \alpha_2$ – also defined in the proof of Theorem 4.3 – the solution $u(x, \alpha_2^*)$ is symmetric with respect to ξ_2, and hence

$$u(\eta_2) = u(\eta_1) < u(\xi_1) < 1 \qquad \text{at} \quad \alpha_2^*.$$

Thus

$$\overline{\alpha}_2 = \sup\{\alpha > \alpha_2 : u(\eta_2) > 1 \quad \text{on} \quad (\alpha_2, \alpha)\}$$

is well defined, and $\overline{\alpha}_2 \in (\alpha_2, \alpha_2^*)$. We have

$$u(\xi_2) > 1, \qquad u(\eta_2) = 1 \quad \text{and} \quad u'''(\eta_2) < 0 \qquad \text{at} \quad \overline{\alpha}_2.$$

Remembering from (4.20) that $u'''(\eta_2) > 0$ at α_2, we conclude that there must be a point $\alpha_2^{**} \in (\alpha_2, \overline{\alpha}_2)$ where $u'''(\eta_2)$ vanishes, so that $u(x, \alpha_2^{**})$ is a periodic solution of equation (1.1) with the properties listed in Theorem 4.4 for $N = 2$.

For any $q \in (-\sqrt{8}, \sqrt{8})$, there exists yet another family $\{u_n\}$ of even periodic solutions. They are characterised by the properties:

(P1) For every $n \geq 1$, u_n is symmetric with respect to the n^{th} critical point ζ_n;

$$\text{(P2)} \qquad \left.\begin{array}{l} u_n(\zeta_k) > 1 \text{ if } \zeta_k \text{ is a maximum} \\ u_n(\zeta_k) < 1 \text{ if } \zeta_k \text{ is a minimum} \end{array}\right\} \quad \text{if} \quad k = 1, \dots, n-1.$$

and these inequalities are *reversed* at $k = n$. This family was first investigated in some detail in [PT5], where the proof of their existence can be found. We show three solutions of this family in Figure 4.6.

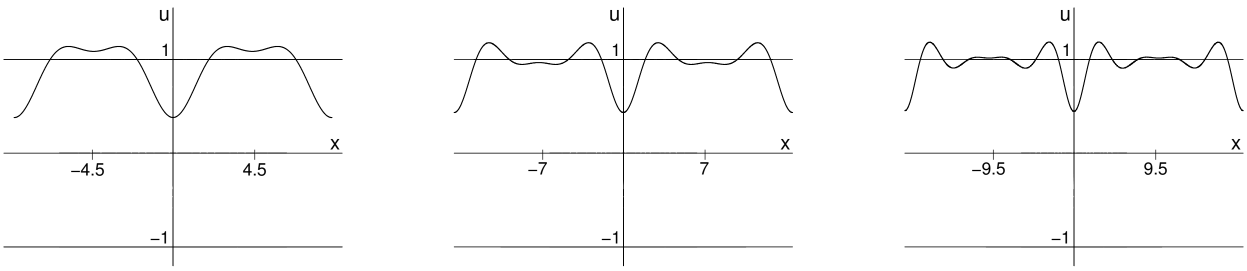


Fig. 4.6. Multi bump periodic solutions at $q = 2.5$

These solutions are interesting because of the following conjecture:

Conjecture. *Let u_n be a sequence of even periodic solutions with the properties P1 and P2, and let $u_n(0) = \alpha_n$. Then $\alpha_n \to \alpha^*$ as $n \to \infty$, and $u(x, \alpha^*)$ is an even homoclinic solution of equation (1.1) with $u(0, \alpha^*) < 1$.*

Remark. Following the construction described in [PT5], it is possible to obtain an infinite sequence of periodic solutions w_n such that $w_n(0) > 1$ and $w_n''(0) < 0$. This leads one to conjecture that there exists a homoclinic solution $w(x)$ of equation (1.1) with $w(0) > 1$.

We now investigate Case II, where we assume that $u''(0) < 0$, and establish results similar to those obtained in Theorems 4.1, 4.3 and 4.4 for Case I. We emphasize that since the first critical point is now a minimum, which is denoted η_1, we skip ξ_1 and number the critical points as follows:

$$0 < \eta_1 < \xi_2 < \eta_2 < \dots .$$

We begin our analysis of Case II by proving a result analogous to Lemma 4.2. For this recall the definition of α_0 given in (4.11), and the fact that $0 < \alpha_0 < 1$ for $-\sqrt{8} < q < \sqrt{8}$.

Lemma 4.5. *Let $q^2 < 8$, and $u''(0) < 0$. Then there exists a point $\tilde{\alpha} \in (\alpha_0, 1)$ such that if $\alpha \in [\tilde{\alpha}, 1)$, then*

$$u(\xi_2) > 1 \quad \textit{and} \quad u'''(\xi_2) < 0 \quad \textit{at} \quad \alpha.$$

Proof. From a linear analysis at $u = 1$, of which the details can be found in the Appendix, we see that

$$\xi_2(\alpha) \to \frac{3\pi}{\sqrt{q+\sqrt{8}}} \qquad \text{and} \qquad u(\eta_1(\alpha), \alpha) \sim 1 - (1-\alpha)\frac{a}{b}\sinh\Big(\frac{\pi a}{2b}\Big) \quad \text{as} \quad \alpha \to 1,$$

where a and b are positive constants which are independent of α, and given in equation (A.4) of the Appendix. Thus, there exists an $\overline{\alpha} \in (\alpha_0, 1)$ such that if $\alpha \in (\overline{\alpha}, 1)$, then

$$u(x, \alpha) > \alpha_0 \quad \text{if} \quad 0 < x \le \xi_2.$$

Hence, by (4.10),

$$H''(x) > 0 \quad \text{if} \quad 0 < x \le \xi_2.$$

Because $H'(0) = u''(0)u'''(0) = 0$, this implies that

$$H'(x) > 0 \quad \text{if} \quad 0 < x \le \xi_2. \tag{4.21}$$

Thus $H(0) < H(\eta_1) < H(\xi_2)$, and hence, by (4.15), $F(u(0)) < F(u(\eta_1)) < F(u(\xi_2))$. This means that $u(\xi_2) > 1$. We also deduce from (4.21) that $H' = u''u''' > 0$ at ξ_2. Since $u''(\xi_2) < 0$, we conclude that $u'''(\xi_2) < 0$. This completes the proof.

Thus, for $\alpha \in (0, 1)$ sufficiently close to 1, we have $u(\xi_2) > 1$ and $u'''(\xi_2) < 0$. On the other hand, when $\alpha = M_-$, we have $u(\xi_2) < 1$. Therefore, we can define the number

$$\alpha_2 = \inf\{\alpha < 1 : u(\xi_2) > 1 \quad \text{on} \quad (\alpha, 1)\}$$

and $\alpha_2 \in (M_-, \tilde{\alpha})$. Plainly, $u(\xi_2) = 1$ and $u'''(\xi_2) > 0$ at α_2, so that $u'''(\xi_2)$ must have a zero for some $\alpha_2^* \in (M_-, 1)$, which yields the first of a family of even periodic solutions with $u''(0) < 0$. As in the proof of Theorems 4.3 and 4.4, we can continue inductively and prove the existence of two families of periodic solutions:

Theorem 4.6. *Let $-\sqrt{8} < q < \sqrt{8}$. Then for any $N \ge 2$ there exists an even periodic solution u of equation (1.1) such that $\mathcal{E}(u) = 0$ and $u''(0) < 0$, which is symmetric with respect to ξ_N and has the properties:*

(a) *u has $N - 2$ local maxima on the interval* $(0, \xi_N)$.

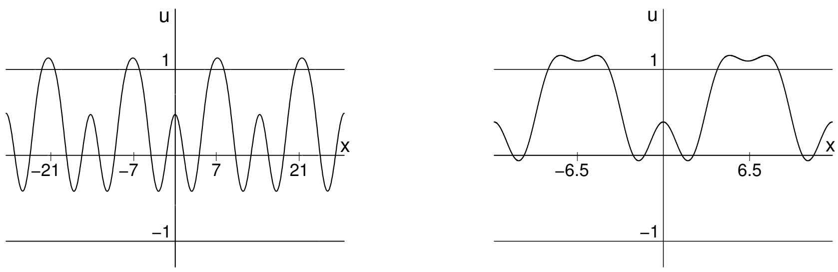
(b) $\qquad u(\xi_k) < 1 \quad \textit{for} \quad 2 \le k \le N-1 \quad \textit{if} \quad N \ge 3,$

(c) $\qquad u(\xi_N) > 1 \qquad\qquad\qquad\qquad \textit{if} \quad N \ge 2.$

Theorem 4.7. *Let $-\sqrt{8} < q < \sqrt{8}$. Then for every $N \geq 2$ there exists an even periodic solution u of equation (1.1) such that $\mathcal{E}(u) = 0$ and $u''(0) < 0$, which is symmetric with respect to η_N and has the the properties:*
(a) *u has $N-1$ local maxima on the interval $(0, \eta_N)$.*

(b) $\qquad u(\xi_k) < 1 \quad$ *for* $\quad 2 \leq k \leq N-1 \quad$ *if* $\quad N \geq 3$

(c) $\qquad u(\xi_N) > 1 \quad$ *and* $\quad u(\eta_N) > 1 \qquad$ *if* $\quad N \geq 2.$



(a) See Theorem 4.6: $N = 2$ (b) See Theorem 4.7: $N = 2$

Fig. 4.7. Multi bump periodic solutions: $u''(0) < 0$ and $q = 2$

5. Local analysis near $u = 1$

In order to extend the existence results of the previous section to the range $q \geq \sqrt{8}$, we need to develop further analytical techniques. This is because Lemma 4.2 no longer holds for $q \geq \sqrt{8}$. These techniques will rely on a detailed analysis of the local behaviour of solutions near $u = 1$. Thus, we substitute $u = 1 + \varepsilon v$ into (1.1) and require that $u(0) = 1 - \varepsilon$ After omitting the higher order terms in ε, we then obtain the linear equation

$$v^{iv} + qv'' + 2v = 0 \tag{5.1a}$$

and at the origin, the initial conditions become

$$v(0) = -1, \qquad v'(0) = 0, \qquad (v'')^2(0) = 2, \quad \text{and} \quad v'''(0) = 0. \tag{5.1b}$$

The fact that v should be even implies that v' and v''' vanish at the origin, and the assumption that the energy $\mathcal{E}$ is zero leads to the condition on $(v'')^2$. A detailed analysis of this problem is given in the Appendix of this paper and in Appendix B of [PT5]. For easy reference, we give here the main results of this analysis. As in Section 4, it is necessary to distinguish two cases:

$$\text{Case I}: \; v''(0) = \sqrt{2} \qquad \text{and} \qquad \text{Case II}: \; v''(0) = -\sqrt{2},$$

and we denote the solution of Problem (5.1) in these two cases by $v_\pm(x)$, so that $v''_\pm(0) = \pm\sqrt{2}$.

The roots $\pm\lambda$ and $\pm\mu$ of the corresponding characteristic equation are defined by

$$\lambda = ia \quad \text{and} \quad \mu = ib, \tag{5.2a}$$

in which $a > 0$ and $b > 0$ are defined by

$$a^2 = \frac{1}{2}(q + \sqrt{q^2 - 8}) \quad \text{and} \quad b^2 = \frac{1}{2}(q - \sqrt{q^2 - 8}). \tag{5.2b}$$

In what follows, the values of q at which resonance occurs (cf. [GH], p. 397), i.e.

$$\frac{a}{b} = \frac{n}{m}, \qquad m, n \in \mathbf{N} \quad (n \geq m) \tag{5.3}$$

will play a special role. These values are readily computed to be

$$q_{m,n} = \sqrt{2}\left(\frac{n}{m} + \frac{m}{n}\right). \tag{5.4a}$$

For convenience we write $q_n = q_{1,n}$, i.e.

$$q_n = \sqrt{2}\left(n + \frac{1}{n}\right). \tag{5.4b}$$

In the following two lemmas we state the main results about the solutions $v_\pm(x)$ of Problem (5.1). In the first one we present the explicit expressions of these solutions.

Lemma 5.1. *The solutions $v_\pm(x)$ of Problem (5.1) are given by*

$$v_\pm(x) = A_\pm \cos(ax) + B_\pm \cos(bx), \tag{5.5a}$$

where

$$A_\pm = \frac{b^2 \mp \sqrt{2}}{a^2 - b^2} \qquad \text{and} \qquad B_\pm = -\frac{a^2 \mp \sqrt{2}}{a^2 - b^2}. \tag{5.5b}$$

In the second lemma we focus on the critical points of the solutions $v_\pm$ of Problem (5.1). In particular, it will be important for our shooting arguments in Section 6 that we know the location of these point with respect to the $v = 0$ axis – *above* or *below* – and the sign of the third derivative $v_\pm'''$ at these points.

Lemma 5.2. *Let ζ be a critical point of the solution v of Problem (5.1), i.e. $v'(\zeta) = 0$. Then*

$$\sin(a\zeta) + \sin(b\zeta) = 0 \qquad \text{in Case I}, \tag{5.6a}$$

$$\sin(a\zeta) - \sin(b\zeta) = 0 \qquad \text{in Case II}, \tag{5.6b}$$

and

$$v_\pm(\zeta) = \frac{a^2 \mp \sqrt{2}}{a^2 - b^2}\left\{\mp\frac{b}{a}\cos(a\zeta) - \cos(b\zeta)\right\} \tag{5.6c}$$

$$v_\pm'''(\zeta) = b(a^2 \mp \sqrt{2})\sin(b\zeta). \tag{5.6d}$$

In both cases,

$$\operatorname{sgn} v(\zeta) = -\operatorname{sgn}\big(\cos(b\zeta)\big) \tag{5.6e}$$

and

$$\operatorname{sgn} v'''(\zeta) = \operatorname{sgn}\big(\sin(b\zeta)\big). \tag{5.6f}$$

The proof of Lemma 5.2 is elementary, and makes use of the observation that $A_\pm/B_\pm = \pm\, b/a$ and that $ab = \sqrt{2}$.

6. Even periodic solutions: $-\sqrt{8} \le q < q_3$

In this section, and in Section 8, we investigate the existence of even periodic solutions u for which $\mathcal{E}(u) = 0$. As was explained in the Introduction, we find it convenient to label the solutions according to the number of monotone segments, or *Laps*, that go in a half-period. Thus, the solutions of Theorem 4.1 are called 1-lap solutions. In the subsequent theorems of Section 4 we have shown that for every $q \in (-\sqrt{8}, \sqrt{8})$ there exist n-lap solutions for any $n \ge 1$.

Numerical evidence suggest that 1-lap solutions no longer exist for $q > \sqrt{8}$, but that 2-lap solutions still exist for some values of $q > \sqrt{8}$, and that n-lap solutions exist for n large enough. Specifically, we prove the following result. Let

$$q_n = \sqrt{2}\Big(n + \frac{1}{n}\Big), \qquad n \ge 1. \tag{6.1}$$

Theorem 6.1. *For each $n \ge 2$ there exist two families of even periodic n-lap solutions when $q \in (-\sqrt{8}, q_n)$. At the points of symmetry, ζ_n, we have*

$$u(\zeta_n) > 1 \quad \text{for every} \quad n \ge 1.$$

Whereas in Section 4 we could use the properties of the functional $\mathcal{H}(u)$, when $q > \sqrt{8}$ this is no longer possible. The main ingredients used in the proof will now be the linear analysis given in Section 5, a powerful Comparison Lemma (Lemma 2.4) valid for $q \in (-\sqrt{8}, q_3)$ and, in Section 8, a counting argument which is reminiscent of a topological degree argument.

The Comparison Lemma enables us to obtain information about the location of all the critical values of the solutions with respect to the line $u = 1$. In the present section we assume that $-\sqrt{8} < q < q_3$, so that the Comparison Lemma holds. In Section 8 we allow q to be arbitrary large, and develop the counting argument.

In the first result of this section we show that when $-\sqrt{8} < q < q_2$, then there exist two families of n-lap solutions with $n \ge 2$. We begin with a pair of 2-lap solutions.

Theorem 6.2. *Let $-\sqrt{8} < q < q_2$. Then there exist two even 2-lap periodic solutions u_{2a} and u_{2b} of equation (1.1), such that $\mathcal{E}(u_{2a}) = 0$ and $\mathcal{E}(u_{2b}) = 0$, with the following properties:*

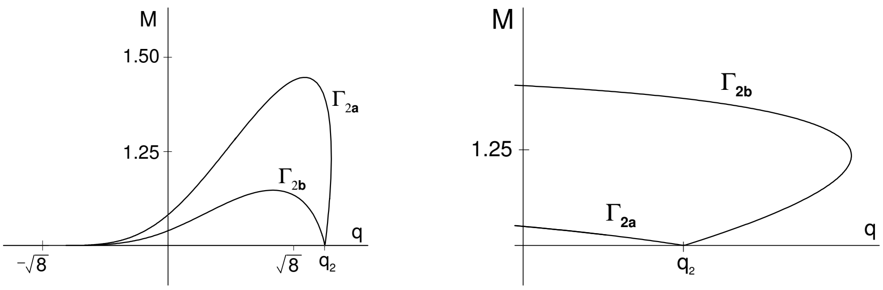
(a) *The solution* u_{2a} *is even with respect to* ξ_2,

$$u''_{2a}(0) < 0 \qquad \textit{and} \qquad u_{2a}(\xi_1) > 1.$$

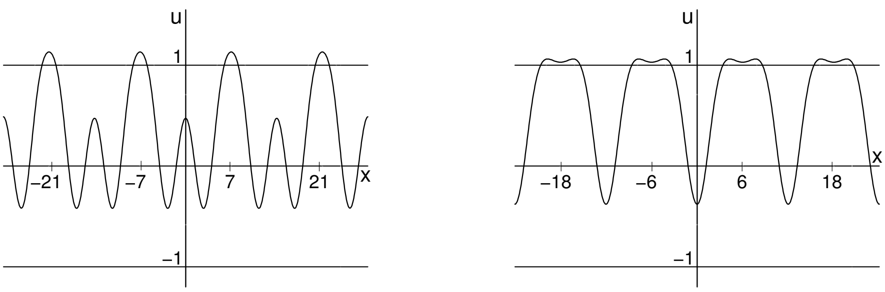
(b) *The solution* u_{2b} *is even with respect to* η_1,

$$u''_{2b}(0) > 0 \qquad \textit{and} \qquad u_{2b}(\eta_1) > 1.$$

We denote the branches of these solutions by, respectively, Γ_{2a} and Γ_{2b}. These branches, as well as graphs of two specific solutions u_{2a} and u_{2b}, are presented in Figure 6.1. Recall that $M = \|u\|_\infty$.



(a) The branches Γ_{2a} and Γ_{2b} and a blowup at q_2



(b) Graph of u_{2a} at $q = 2$ (c) Graph of u_{2b} at $q = 3$

Fig. 6.1. Solutions of Theorem 6.2

Proof. We first consider Part (b) and prove the existence of the solution u_{2b} for which $u''(0) > 0$ (Case I). For convenience we have dropped the subscript $2b$. We distinguish three cases:

$$(i) \;\; \sqrt{8} \le q < q_{3,5} \qquad (ii) \;\; q = q_{3,5}, \qquad (iii) \;\; q_{3,5} < q < q_2.$$

According to (5.3) and (5.4a), these cases correspond to

$$(i) \;\; 1 \le \frac{a}{b} < \frac{5}{3}, \qquad (ii) \;\; \frac{a}{b} = \frac{5}{3}, \qquad (iii) \;\; \frac{5}{3} < \frac{a}{b} < 2.$$

Case *(i)*: Recall from Section 5 that v denotes the solution of equation (5.1a), the linearisation of equation (1.1) around $u = 1$. It follows from Lemma 5.2 that in this case

$$v(\xi_1) > 0 \qquad \text{and} \qquad v(\eta_1) < 0.$$

Hence, it follows from continuity that there exists a $\delta > 0$ such that

$$u(\xi_1) > 1 \quad \text{and} \quad u(\eta_1) < 1 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.2}$$

Recall that $u(\xi_1) = M_- < 1$ when $\alpha = -M_-$, and define

$$a_1 = \inf\{\alpha < 1 : u(\xi_1) > 1 \;\; \text{on} \;\; (\alpha, 1)\}.$$

Then, by Lemma 2.3, $u(\eta_1) > 1$ for $a_1 < \alpha < a_1 + \nu$ for some small $\nu > 0$. However, $u(\eta_1) < 1$ for α close to 1. Therefore

$$b_1 = \sup\{\alpha > a_1 : u(\eta_1) > 1 \;\; \text{on} \;\; (a_1, \alpha)\} \in (a_1, 1).$$

Plainly, at b_1 we have $u(\eta_1) = 1$ and by Lemma 2.1, $u'''(\eta_1) < 0$. Since $u'''(\eta_1) > 0$ at a_1 and $u'''(\eta_1) < 0$ at b_1, it follows that $u'''(\eta_1)$ has a zero for some $\alpha^* \in (a_1, b_1)$, and again we use symmetry to conclude that $u(x, \alpha^*)$ is a periodic solution with the desired properties.

Case *(iii)*: From Lemma 5.2 we see that in this case

$$v(\xi_1) > v(\eta_1) > 0 \qquad \text{and} \qquad v'''(\eta_1) < 0.$$

Hence, there exists a $\delta > 0$ such that

$$u(\xi_1) > u(\eta_1) > 1 \quad \text{and} \quad u'''(\eta_1) < 0 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.3}$$

As before, we define

$$a_1 = \inf\{\alpha < 1 : u(\xi_1) > 1 \;\; \text{on} \;\; (\alpha, 1]\}.$$

and

$$b_1 = \sup\{\alpha > a_1 : u(\eta_1) > 1 \;\; \text{on} \;\; (a_1, \alpha)\} \in (a_1, 1].$$

Recall that $u'''(\eta_1) > 0$ at a_1, so that $b_1 > a_1$ by Lemma 2.3. If $b_1 = 1$, then (6.3) implies that $u'''(\eta_1) < 0$ for α near b_1. On the other hand, if $b_1 < 1$, then $u'''(\eta_1) < 0$ at b_1. Thus, in both cases $u'''(\eta_1)$ changes sign on (a_1, b_1). Once again the existence of a periodic solution of the desired type follows.

Case *(ii)*: From Lemma 5.2 we see that in this case,

$$v(\xi_1) > v(\eta_1) = 0 \qquad \text{and} \qquad v'''(\eta_1) < 0.$$

Thus, there exists a $\delta > 0$ such that

$$u(\xi_1) > 1 \quad \text{and} \quad u'''(\eta_1) < 0 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.4}$$

Fix $\alpha \in (1-\delta, 1)$. If, for this value of α, $u(\eta_1) < 1$ we can complete the proof as in Case (i), and if $u(\eta_1) \geq 1$, we can complete it as in Case (iii). This finishes the proof of Part (b).

Next, we prove Part (a). Recall that ξ_2 denotes the first positive local maximum of u since $u''(0) < 0$. From Lemma 5.2 we conclude that

$$v(\eta_1) < 0 \quad \text{and} \quad v(\xi_2) > 0, \qquad v'''(\xi_2) < 0$$

for the entire interval $q_1 \leq q < q_2$. Hence, there exists a $\delta > 0$ such that

$$u(\eta_1) < 1 \quad \text{and} \quad u(\xi_2) > 1 \qquad u'''(\xi_2) < 0 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.5}$$

Set

$$a_2 = \inf\{\alpha < 1 : u(\xi_2) > 1 \quad \text{on} \quad (\alpha, 1)\}.$$

Then $M_- < a_2 < 1$. Plainly, $u'''(\xi_2) > 0$ at a_2. Since by (6.5), $u'''(\xi_2) < 0$ for α close to 1, it follows that there exists a point $\alpha_2^* \in (a_2, 1)$ such that $u'''(\xi_2) > 0$ at α_2^*, and hence $u(x, \alpha_2^*)$ is an even periodic solution with period $2\xi_2(\alpha_2^*)$. Finally, $u(\xi_2, \alpha_2^*) > 1$ since $a_2 < \alpha_2^* < 1$. This completes the proof of Theorem 6.2.

Remark. We have now shown that the solutions u_{2a} and u_{2b} exist over the entire interval $(-\sqrt{8}, q_2)$. Our numerical investigation (Figure 6.1(a)) indicates that u_{2b} does not exist past q_2 although it appears that u_{2a} does exist on a small q-interval beyond q_2. Our experiments also indicate that the branches Γ_{2a} and Γ_{2b}, on which these solutions sit, disappear at q_2 as a result of a bifurcation from the constant solution $u = 1$. In Section 7 we shall give an explanation for the behaviour of the two branches Γ_{2a} and Γ_{2b} near the point $(q, M) = (q_2, 1)$.

Theorem 6.2, together with Lemma 2.4, enables us to establish the existence of two families of even n-lap periodic solutions, for any $n \geq 2$.

Theorem 6.3. *Let $-\sqrt{8} < q < q_2$, and let $n \geq 2$. Then there exist two families of even n-lap periodic solutions u of equation (1.1), such that $\mathcal{E}(u) = 0$, one for which $u''(0) > 0$ and one for which $u''(0) < 0$. They are symmetric with respect to the n^{th} critical point ζ_n. The critical values have the properties*

$$u(\zeta_k) < 1 \quad \text{if} \quad k \leq n-2 \ (n \geq 3) \quad \text{and} \quad u(\zeta_n) > 1,$$

and

$$u(\zeta_{n-1}) < 1 \quad \text{if} \quad \zeta_n \quad \text{is a maximum,}$$
$$u(\zeta_{n-1}) > 1 \quad \text{if} \quad \zeta_n \quad \text{is a minimum.}$$

Proof. The proof proceeds very much along the lines of the proofs of Theorems 4.3 and 4.4. To show how Lemma 2.4 is used, we give the proof for the 3-lap solution when $u''(0) > 0$. This solution is symmetric with respect to ξ_2. Otherwise, we leave the proof as an exercise.

Let

$$a_1^- = \sup\{\alpha > -M_- : u(\xi_1) < 1 \quad \text{on} \quad (-M_- \alpha)\}.$$

It follows from Theorem 6.2 that $a_1^- \in (-M_-, 1)$. In addition,

$$u(\xi_1) = 1 \quad \text{and} \quad u'''(\xi_1) > 0 \quad \text{at} \quad a_1^- .$$

Thus, in view of Lemma 2.4,

$$u(\xi_2) > 1 \quad \text{and} \quad u'''(\xi_2) < 0 \quad \text{at} \quad a_1^- .$$

Next, let

$$\tilde{\alpha}_2 = \inf\{\alpha < a_1^- : u(\xi_2) > 1 \quad \text{on} \quad (\alpha, a_1^-)\}.$$

Then $\tilde{\alpha}_2 \in (-M_-, \tilde{\alpha}_2)$, and

$$u(\xi_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \quad \text{at} \quad \tilde{\alpha}_2.$$

Therefore, $u'''(\xi_2)$ changes sign, and thus has a zero α_2^*, on $(\tilde{\alpha}_2, a_1^-)$. It is clear from the construction that the solution $u(x, \alpha_2^*)$ has the required properties.

Next, we turn our attention to the interval $(-\sqrt{8}, q_3)$, and show that n-lap solutions continue to exist up to q_3 for $n \geq 3$. We first focus on 3-lap solutions.

Theorem 6.4. *Let $-\sqrt{8} < q < q_3$. Then there exist two even 3-lap periodic solutions u_{3a} and u_{3b} of equation (1.1), such that $\mathcal{E}(u_{3a}) = 0$ and $\mathcal{E}(u_{3b}) = 0$, with the following properties:*

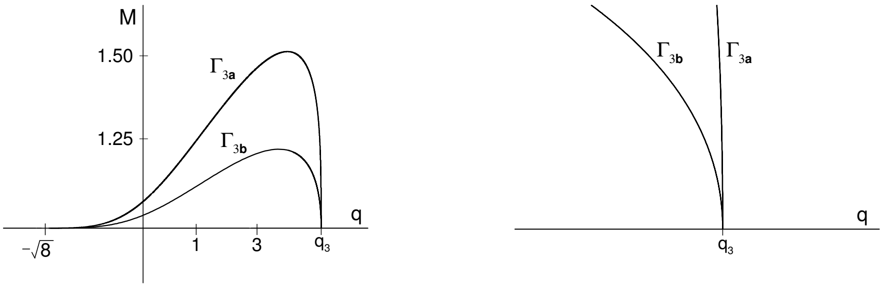
(a) *The solution u_{3a} is even with respect to ξ_2,*

$$u_{3a}''(0) > 0 \qquad \textit{and} \qquad u_{3a}(\xi_1) < 1, \qquad u_{3a}(\xi_2) > 1.$$

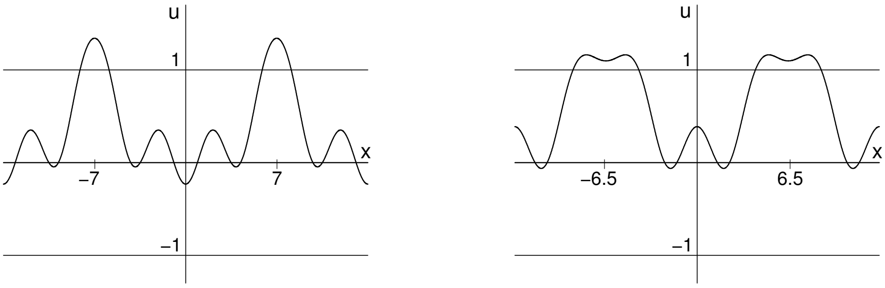
(b) *The solution u_{3b} is even with respect to η_2,*

$$u_{3b}''(0) < 0 \qquad \textit{and} \qquad u_{3b}(\eta_2) > 1.$$

The branches Γ_{3a} and Γ_{3b} of these solutions, as well as graphs of the solutions u_{3a} and u_{3b} at a specific value of q are presented in Figure 6.2.



(a) The branches Γ_{3a} and Γ_{3b} and a blowup at q_3



(b) Graph of u_{3a} (c) Graph of u_{3b}

Fig. 6.2. Solutions of Theorem 6.3 at $q = 2$

Proof. For $-\sqrt{8} < q < q_2$, the existence of solutions such as u_{3a} and u_{3b} has been established in Theorems 4.3, 4.7 and 6.3. Thus, it suffices to prove Theorem 6.4 for $q_2 \le q < q_3$.

(a) Set

$$a_1 = \sup\{\alpha > -M_- : u(\xi_1) < 1 \quad \text{on } (-M_-, \alpha)\}.$$

From Lemma 5.2 we know that when $q_2 \le q < q_3$, then

$$v(\xi_1) > 1 \quad \text{and} \quad v(\eta_2) < 1.$$

Hence, by continuity, there exists a constant $\delta > 0$ such that

$$u(\xi_1) > 1 \quad \text{and} \quad u(\eta_2) < 1 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.6}$$

Therefore $a_1 \in (-M_-, 1)$, and

$$u(\xi_1) = 1, \quad u''(\xi_1) = 0 \quad \text{and} \quad u'''(\xi_1) > 0 \quad \text{at} \quad \alpha = a_1.$$

This implies that $u(\xi_2) > 1$ and, by Lemma 2.4, that $u'''(\xi_2) < 0$. Let

$$a_2 = \inf\{\alpha < a_1 : u(\xi_2) > 1 \quad \text{on } (\alpha, a_1)\}.$$

Then

$$u(\xi_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \quad \text{at} \quad \alpha = a_2,$$

because $u(\eta_1) < u(\xi_1) < 1$. Therefore, $u'''(\xi_2)$ changes sign on (a_2, a_1), so that there exists a point $a_2^* \in (a_2, a_1)$ such that $u'''(\xi_2) = 0$ at a_2^*, and $u(x, a_2^*)$ is a periodic solution which is symmetric with respect to ξ_2. By construction $u(\xi_1, a_2^*) < 1$ and $u(\xi_2, a_2^*) > 1$, as required.

(b) We analyse the following three cases separately:

$$(i) \;\; q_2 \le q < q_{3,7} \qquad (ii) \;\; q = q_{3,7}, \qquad (iii) \;\; q_{3,7} < q < q_3.$$

Again, according to (5.3) and (5.4a), these cases correspond to

$$(i)\ \ 2 \le \frac{a}{b} < \frac{7}{3}, \qquad (ii)\ \ \frac{a}{b} = \frac{7}{3}, \qquad (iii)\ \ \frac{7}{3} < \frac{a}{b} < 3.$$

Case (i): It follows from Lemma 5.2 that in this case

$$v(\xi_2) > 0, \qquad v(\eta_2) < 0 \quad \text{and} \quad v'''(\eta_2) < 0,$$

so that for some small $\delta > 0$,

$$u(\xi_2) > 1 \quad \text{and} \quad u(\eta_2) < 1 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.7}$$

Define

$$a_1 = \inf\{\alpha < 1 : u(\xi_2) > 1 \ \text{ on } \ (\alpha, 1)\}.$$

Because $u(\xi_2) = M_- < 1$ when $\alpha = M_-$, it follows that $M_- < a_1 < 1$, and hence

$$u(\xi_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \quad \text{at} \quad a_1,$$

since $u(\eta_1) < M_- < 1$ at a_1. This means, according to Lemma 2.3, that $u(\eta_2) > 1$ for $a_1 < \alpha < a_1 + \nu$ where $\nu > 0$ is some small constant. Define

$$b_1 = \sup\{\alpha > a_1 : u(\eta_2) > 1 \ \text{ on } \ (a_1, \alpha)\}. \tag{6.8}$$

As we have seen in (6.7), $u(\eta_2) < 1$ for α close to 1. Therefore $b_1 \in (a_1, 1)$. Since $u(\xi_2) > 1$ at b_1 it follows that $u'''(\eta_2) < 0$ at b_1. Hence, in view of (6.8), $u'''(\eta_2)$ changes sign, and therefore has a zero, at a point $\alpha_1^* \in (a_1, b_1)$. Thus, $u(x, \alpha_1^*)$ is an even periodic solution which, by construction, is symmetric with respect to η_2, and $u(\eta_2, \alpha_1^*) > 1$, as required.

Case (iii): In this case

$$v(\xi_2) > 0, \qquad v(\eta_2) > 0 \quad \text{and} \quad v'''(\eta_2) < 0,$$

so that

$$u(\xi_2) > 1, \qquad u(\eta_2) > 1 \quad \text{and} \quad u'''(\eta_2) < 0 \quad \text{for} \quad (1 - \delta, 1) \tag{6.9}$$

when $\delta > 0$ is sufficiently small. As before, we define

$$a_2 = \inf\{\alpha < 1 : u(\xi_2) > 1 \ \text{ on } \ (\alpha, 1)\},$$

and since $u(\xi_2) = M_-$ when $\alpha = M_-$, it follows that $a_2 \in (M_-, 1)$. Because in this case, $u(\eta_1) < \alpha < 1$ we deduce that

$$u(\xi_2) = u(\eta_2) = 1 \quad \text{and} \quad u'''(\eta_2) > 0 \quad \text{if} \quad \alpha = a_2. \tag{6.10}$$

By Lemma 2.3, $u(\eta_2) > 1$ in a right-neighbourhood of a_2. Let

$$b_2 = \sup\{\alpha > a_2 : u(\eta_2) > 1 \quad \text{on} \quad (a_2, \alpha)\}.$$

We claim that $u'''(\eta_2)$ changes sign on (a_2, b_2).
– If $b_2 = 1$, then this assertion follows at once from (6.9), (6.10), and continuity.
– If $b_2 < 1$, then $u(\eta_2) = 1$ at b_2, and because $u(\xi_2) > 1$ at b_2, it follows that $u'''(\eta_2) < 0$, so that in view of (6.10), $u'''(\eta_2)$ also changes sign on (a_2, b_2).
Thus, there exists a point $\alpha_2^* \in (a_2, b_2)$ such that $u(x, a_2^*)$ is an even periodic solution which is symmetric with respect to η_2, such that

$$u(\eta_1) < 1, \quad u(\xi_2) > 1 \quad \text{and} \quad u(\eta_2) > 1.$$

Case *(ii)*: By Lemma 5.2,

$$v(\xi_2) > 0, \qquad v(\eta_2) = 0 \quad \text{and} \quad v'''(\eta_2) < 0,$$

so that for some small $\delta > 0$,

$$u(\xi_2) > 1 \quad \text{and} \quad u'''(\eta_2) < 0 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{6.11}$$

If $u(\eta_2) < 1$ we continue as in the proof of Case (i) and if $u(\eta_2) \geq 1$ we continue the proof as in Case (iii). This completes the proof of Theorem 6.4.

About the families of n-lap solutions we can repeat the claim of Theorem 6.3 for the larger range of q-values: $q \in (-\sqrt{8}, q_3)$. However, the minimum number of laps is now raised from 2 to 3.

Up till now we have studied branches of even multibump periodic solutions with $\mathcal{E} = 0$ on intervals of the form $(-\sqrt{8}, q_n)$ for $n \geq 1$. These branches appear to bifurcate from the points $(q_n, 1)$. In addition to these solutions there exist families of even periodic solutions which exist on intervals of the form $(-\sqrt{8}, q_{m,n})$, where $q_{m,n}$ is given in (1.10), and $n > m \geq 2$. The corresponding branches appear to bifurcate from the points $(q_{m,n}, 1)$. We shall not go into a general analysis of such solutions. Instead, we provide the details for one example, and choose $m = 2$ and $n = 3$. We denote the two branches of even periodic solutions by $\Gamma_{2,3a}$ and $\Gamma_{2,3b}$, and the solutions that sit on these branches by $u_{2,3a}$ and $u_{2,3b}$, respectively. Two such solutions are presented in Figure 6.3b.c, at $q = 2$.

We note that $q_{2,3} = \frac{13}{6}\sqrt{2} \in (q_1, q_2)$. Therefore, we may again use Lemma 2.4 to prove the existence of these two new families of solutions.

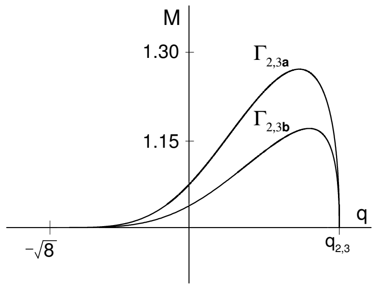
Theorem 6.5. *Let $-\sqrt{8} < q < q_{2,3}$. Then there exist two even 3-lap periodic solutions $u_{2,3a}$ and $u_{2,3b}$ of equation (1.1) such that $\mathcal{E}(u_{2,3a}) = 0$ and $\mathcal{E}(u_{2,3b}) = 0$ with the following properties:*

(a) *The solution $u_{2,3a}$ is symmetric with respect to η_2, and*

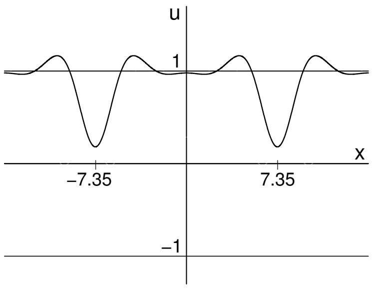
$$u_{2,3a}(0) < 1, \quad u''_{2,3a}(0) < 0, \quad u_{2,3a}(\xi_2) > 1, \quad u_{2,3a}(\eta_2) < 1.$$

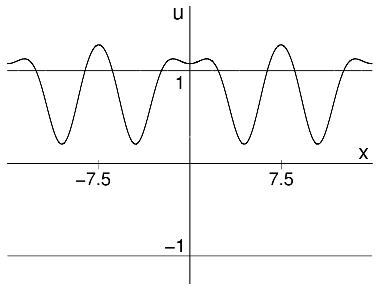
(b) *The solution $u_{2,3b}$ is symmetric with respect to ξ_2, and*

$$u_{2,3b}(0) > 1, \quad u''_{2,3b}(0) > 0, \quad u_{2,3b}(\eta_1) < 1, \quad u_{2,3a}(\xi_2) > 1.$$



(a) The branches $\Gamma_{2,3a}$ and $\Gamma_{2,3b}$ and a blowup at $q_{2,3}$





(b) Graph of $u_{2,3a}$ (c) Graph of $u_{2,3b}$

Fig. 6.3. Solutions of Theorem 6.5 at $q = 2$

Proof. (a) If $q_1 < q < q_{2,3}$, then, according to Lemma 5.2,

$$v(0) = -1, \quad v(\eta_1) < 0, \quad v(\xi_2) > 0, \quad v(\eta_2) < 0 \quad \text{and} \quad v'''(\eta_2) > 0. \tag{6.11}$$

Hence, for any $q \in (q_1, q_{2,3})$ there exists by continuity a $\delta > 0$ such that for $\alpha \in (1 - \delta, 1)$

$$u(\eta_1) < 1, \quad u(\xi_2) > 1, \quad u(\eta_2) < 1, \quad \text{and} \quad u'''(\eta_2) > 0. \tag{6.12}$$

Let

$$a_1 = \inf\{\alpha < 1 : u(\xi_2) > 1 \ \text{ on } \ (\alpha, 1)\}.$$

Because $u(\xi_2) < 1$ when $\alpha = M_-$, where M_- is the amplitude of the *small* single bump periodic solution u_- (cf Fig. 4.1), it follows that $a_1 \in (M_-, 1)$, and

$$u(\xi_2) = 1 \quad \text{and} \quad u(\eta_2) = 1 \quad \text{at} \quad a_1.$$

Next, let

$$a_2 = \inf\{\alpha < 1 : u(\eta_2) < 1 \ \text{ on } \ (\alpha, 1)\}.$$

Then $a_2 > a_1$ by Lemma 2.3 and hence

$$u(\eta_2) = 1, \quad u(\xi_2) > 1 \quad \text{and} \quad u'''(\eta_2) < 0 \quad \text{at} \quad a_2. \tag{6.13}$$

Remembering (6.12), we conclude that $u'''(\eta_2)$ changes sign on $(a_2, 1)$. Thus, by continuity there exists an $a_2^* \in (a_2, 1)$ such that $u'''(\eta_2) = 0$ at a_2^*. Therefore, $u^*(x) = u(x, a_2^*)$ is a periodic solution which is symmetric with respect to η_2 and has the properties

$$u^*(0) < 1, \quad u^*(\eta_1) < 1, \quad u^*(\xi_2) > 1, \quad u^*(\eta_2) < 1,$$

as required.

(b) If $q_1 < q < q_{2,3}$, then, according to Lemma 5.2, or (6.11) when we multiply by a factor -1 and relabel the critical points,

$$v(0) = 1, \quad v(\xi_1) > 0, \quad v(\eta_1) < 0, \quad v(\xi_2) > 0 \quad \text{and} \quad v'''(\xi_2) < 0. \tag{6.14}$$

We now proceed exactly as in Case (a), except that instead of the small single bump periodic solution u_-, we now use that *large* single bump periodic solution u_+ with amplitude M_+ (cf Fig. 4.1) as a reference. We omit the details.

Proving that the solutions $u_{2,3a}$ and $u_{2,3b}$ exist for $-\sqrt{8} < q \leq q_1$ as well is also left as an exercise.

7. Local behaviour near q_2

In Figure 1.10 (or 6.1) we saw that the numerically computed branches Γ_{2a} and Γ_{2b} of even 2-lap periodic solutions approach the bifurcation point $(q_2, 1)$ in the (q, M)-plane from different directions. In this section we present a local analysis at the point $(q_2, 1)$, and compute the angles θ_a and θ_b which these branches make with the positive q-axis at the bifurcation point. Specifically we obtain the following result.

Proposition 7.1. *Let θ_a and θ_b be the angles with which the branches Γ_{2a} and Γ_{2b} approach $(q_2, 1)$. Then*

$$\tan\theta_a = \frac{2\sqrt{2}}{3} \qquad \textit{and} \qquad \tan\theta_b = -\frac{\sqrt{2}}{3}.$$

We shall discuss the two branches in succession.

(1) The branch Γ_{2a}. The solutions that sit on Γ_{2a} have the properties

$$u''(0) < 0 \quad \text{and} \quad u'''(\xi_2) = 0,$$

and ξ_2 is the first point of symmetry. We use $\varepsilon = 1 - u(0) > 0$ as a small parameter, and we make the *Ansatz*:

$$u(\varepsilon) = 1 + \varepsilon v + \varepsilon^2 w + O(\varepsilon^3), \tag{7.1a}$$

$$q(\varepsilon) = q_0 + \varepsilon q_1 + O(\varepsilon^2), \tag{7.1b}$$

$$\xi(\varepsilon) = \xi_0 + \varepsilon \xi_1 + O(\varepsilon^2). \tag{7.1c}$$

To keep the notation simple, we have denoted the *zero*th order terms in the expansion of q and ξ_2 by, respectively, q_0 and ξ_0. Thus, in this notation, $q_0 = q_2$ and $\xi_0 = \xi_2|_{\varepsilon=0}$. We recall that

$$q_0 = \frac{5}{\sqrt{2}} \quad \text{and hence} \quad a = 2^{3/4} \quad \text{and} \quad b = 2^{-1/4} \quad (a = 2b), \tag{7.2a}$$

and we obtain from Lemma 5.1 that

$$\xi_0 = \frac{\pi}{b} = 2^{1/4}\pi. \tag{7.2b}$$

In this notation, we need to compute

$$\tan\theta_a = \frac{v(\xi_0)}{q_1}.$$

When we substitute the Ansatz (7.1) into equation (1.1), use the initial conditions

$$u(0) = 1 - \varepsilon, \quad u'(0) = 0, \quad u''(0) = -\frac{1}{\sqrt{2}}(1 - u^2), \quad u'''(0) = 0,$$

and equate terms of equal order in ε, we obtain

$$\begin{cases} v^{iv} + q_0 v'' + 2v = 0 & (7.3a) \\ v(0) = -1, \quad v'(0) = 0, \quad v''(0) = -\sqrt{2}, \quad v'''(0) = 0, & (7.3b) \end{cases}$$

and

$$\begin{cases} w^{iv} + q_0 w'' + 2w = -q_1 v''(x) - 3v^2(x) & (7.4a) \\ w(0) = 0, \quad w'(0) = 0, \quad w''(0) = +\frac{1}{\sqrt{2}}, \quad w'''(0) = 0. & (7.4b) \end{cases}$$

The problem for v has been discussed in Section 5, where we found that

$$v(x) = \cos(ax) - 2\cos(bx). \tag{7.5}$$

Note that v does not depend on the value of q_1. For later use we note that

$$v(\xi_0) = 3, \quad v'(\xi_0) = 0, \quad v''(\xi_0) = -6b^2, \quad v'''(\xi_0) = 0, \quad v^{iv}(\xi_0) = 18b^4. \tag{7.6}$$

Thus, we find that

$$\begin{aligned} u(\xi, \varepsilon, q) &= 1 + \varepsilon v(\xi_0 + \varepsilon\xi_1) + \varepsilon^2 w(\xi_0 + \varepsilon\xi_1, q_1) + \dots \\ &= 1 + \varepsilon v(\xi_0) + O(\varepsilon^2) \\ &= 1 + 3\varepsilon + O(\varepsilon^2). \end{aligned} \tag{7.7}$$

In order to compute ξ_1 we write

$$\begin{aligned} u'(\xi,\varepsilon,q) &= \varepsilon v'(\xi_0+\varepsilon\xi_1)+\varepsilon^2 w'(\xi_0+\varepsilon\xi_1,q_1) \\ &= \varepsilon v'(\xi_0)+\varepsilon^2\xi_1 v''(\xi_0)+\varepsilon^2 w'(\xi_0,q_1)=0. \end{aligned}$$

Hence

$$\xi_1 = -\frac{w'(\xi_0,q_1)}{v''(\xi_0)}. \tag{7.8}$$

To compute q_1 we observe that we can write

$$w(x,q_1) = w_0(x)+q_1 w_1(x), \tag{7.9}$$

and an easy computation shows that

$$w_0(x) = f(x) - 3g(x), \tag{7.10a}$$

where f is given by

$$f(x) = \frac{1}{\sqrt{2}(a^2-b^2)}\{-\cos(ax)+\cos(bx)\}. \tag{7.10b}$$

and g is given by

$$g(x) = \frac{1}{a^2-b^2}\int_0^x K(x-t)v^2(t)\,dt, \tag{7.10c}$$

in which

$$K(x) = \frac{\sin(bx)}{b} - \frac{\sin(ax)}{a}. \tag{7.10d}$$

For $w_1(x)$ we find

$$w_1(x) = -\frac{1}{a^2-b^2}\int_0^x K(x-t)v''(t)\,dt. \tag{7.11}$$

We wish to choose q_1 in such a way that $u'''(\xi,\varepsilon,q)=0$. Differentiating u we obtain

$$\begin{aligned} u'''(\xi,\varepsilon,q) &= \varepsilon v'''(\xi_0)+\varepsilon^2\xi_1 v^{iv}(\xi_0)+\varepsilon^2\{w_0'''(\xi_0)+q_1 w_1'''(\xi_0)\}+O(\varepsilon^3) \\ &= \varepsilon^2\big\{\xi_1 v^{iv}(\xi_0)+w_0'''(\xi_0)+q_1 w_1'''(\xi_0)\big\}+O(\varepsilon^3), \end{aligned} \tag{7.12}$$

because $v'''(\xi_0)=0$ by (7.6). Remembering from (7.8) the expression for ξ_1 we write

$$\xi_1 v^{iv}(\xi_0) = -\frac{v^{iv}(\xi_0)}{v''(\xi_0)}\{w_0'(\xi_0)+q_1 w_1'(\xi_0)\}. \tag{7.13}$$

We saw in (7.6) that $v^{iv}(\xi_0)/v''(\xi_0) = -\frac{3}{\sqrt{2}}$. When we use this in (7.13) and substitute the result into (7.12), we find that

$$u'''(\xi,\varepsilon,q) = \varepsilon^2 X(q_1)+O(\varepsilon^3),$$

where

$$X(q_1) = w_0'''(\xi_0) + \frac{3}{\sqrt{2}} w_0'(\xi_0) + q_1 \Big\{ w_1'''(\xi_0) + \frac{3}{\sqrt{2}} w_1'(\xi_0) \Big\}. \tag{7.14}$$

Thus we need to choose q_1 so that $X(q_1) = 0$. Using the expression for w_0 given in (7.10a) and for w_1 given in (7.11), we find that

$$w_0'(\xi_0) = 0 \qquad \text{and} \qquad w_0'''(\xi_0) = -\frac{3\pi}{b}$$

$$w_1'(\xi_0) = -\frac{\pi}{3b} \qquad \text{and} \qquad w_1'''(\xi_0) = \frac{7\pi}{3} b.$$

Therefore

$$X(q_1) = -\frac{3\pi}{b} + q_1 \Big(\frac{7\pi}{3} b - \frac{3}{\sqrt{2}} \frac{\pi}{3b} \Big) = -\frac{3\pi}{b} + q_1 \frac{4\pi}{3} b,$$

so that $X(q_1) = 0$ if

$$q_1 = \frac{9}{4b^2} = \frac{9}{2\sqrt{2}}.$$

Remembering (7.7), we conclude that the branch Γ_{2a} leaves the point $(q, M) = (q_2, 1)$ under an angle θ_a given by

$$\tan \theta_a = \frac{3}{q_1} = \frac{2\sqrt{2}}{3}.$$

(2) The branch Γ_{2b}. The solutions that sit on Γ_{2b} have the properties

$$u''(0) > 0 \quad \text{and} \quad u'''(\eta_1) = 0.$$

In the notation introduced in (7.1), we now need to compute

$$\tan \theta_b = \frac{v(\xi_0)}{q_1},$$

where now ξ_0 denotes the location of the first maximum of v when $q = q_0$.

For solutions on this branch, η_1 is the first point of symmetry. We expand u and q as in (7.1) drop the subscript 1 from η_1, and write

$$\eta(\varepsilon) = \eta_0 + \varepsilon \eta_1 + O(\varepsilon^2). \tag{7.15}$$

We emphasize that in the remainder of this section, η_1 will denote the first order term in the expansion for $\eta(\varepsilon)$. From Lemma 5.1 we find that

$$v(x) = -\frac{1}{3} \{ \cos(ax) + 2 \cos(bx) \},$$

and

$$\xi_0 = \frac{\pi}{3b}, \qquad \eta_0 = \frac{\pi}{b}, \qquad v(\xi_0) = \frac{1}{2}, \qquad v''(\eta_0) = \frac{\sqrt{2}}{3}, \qquad v^{iv}(\eta_0) = -\frac{7}{3},$$

so that

$$\frac{v^{iv}(\eta_0)}{v''(\eta_0)} = -\frac{7}{\sqrt{2}}.$$

We now proceed exactly as in the previous case, and we find that

$$u'''(\xi, \varepsilon, q) = \varepsilon^2 Y(q_1) + O(\varepsilon^3),$$

where

$$Y(q_1) = w_0'''(\eta_0) + \frac{7}{\sqrt{2}} w_0'(\eta_0) + q_1 \Big\{ w_1'''(\eta_0) + \frac{7}{\sqrt{2}} w_1'(\eta_0) \Big\}. \tag{7.16}$$

For w_0 and w_1 and their derivatives we obtain at η_0:

$$w_0'(\eta_0) = \frac{2\pi}{9b^3} \qquad \text{and} \qquad w_0'''(\eta_0) = -\frac{5\pi}{9b},$$
$$w_1'(\eta_0) = \frac{\pi}{3b} \qquad \text{and} \qquad w_1'''(\eta_0) = -\pi b.$$

Therefore

$$Y(q_1) = \frac{\pi}{b} + q_1 \frac{4\pi}{3} b,$$

and $Y(q_1) = 0$ if

$$q_1 = -\frac{3}{2\sqrt{2}}.$$

Since $v(\xi_0) = \frac{1}{2}$, it follows that

$$\tan\theta_b = \frac{1}{2}\frac{1}{q_1} = -\frac{\sqrt{2}}{3}.$$

Near $q = q_2$ and $\alpha = 1$ ($\varepsilon = 0$), we may use (7.1a) and the information about v and w obtained in this section to make the following observations.
(*i*) There exists a $\delta > 0$ such that in the case that $u''(0) < 0$ we have for $q = q_2$

$$u(\xi_2) > 1 \;\text{ and }\; u'''(\xi_2) < 0 \quad \text{for } 1 - \delta < \alpha < 1.$$

Therefore, the proof of the existence of u_{2a} in Theorem 6.1 also holds for $q = q_2$.
(*ii*) Again in the case that $u''(0) < 0$, for every $\delta > 0$ we have

$$u'''(\xi_2) < 0 \quad \text{for } q = q_2 + \varepsilon(q_1 + \delta)$$
$$u'''(\xi_2) > 0 \quad \text{for } q = q_2 + \varepsilon(q_1 - \delta)$$

for $\varepsilon > 0$ sufficiently small. Hence, fixing $\delta > 0$, there exists an $\varepsilon_0 > 0$ such that for $q_2 < q < q_2 + \varepsilon_0$ there are *two* periodic solutions u_{2a} and $\tilde{u}_{2a}$ which are symmetric with respect to ξ_2 and such that $u(\xi_2) > 1$. The first one has $1 - \frac{q-q_2}{q_1-\delta} < \alpha < 1 - \frac{q-q_2}{q_1+\delta}$, and it is found by varying α between these values and searching for an α such that $u'''(\xi_2) = 0$. The second one has $\alpha < 1 - \frac{q-q_2}{q_1-\delta}$ and can be found as in the proof of Theorem 6.1. A similar statement holds for the solutions of type u_{2b}, but there is only one solution of this type. Continuity of the branches near $q = q_2$ and $\alpha = 1$ can be proved using the Implicit Function Theorem, but we will not going into that here.

8. Even periodic solutions: $q \geq q_3$

In Section 4 we have exhibited the existence of an even single bump, or 1-lap periodic solution for $-\sqrt{8} < q < q_1 = \sqrt{8}$. In Section 6, we extended this result and showed that there exist *two* even 2-lap periodic solutions for $-\sqrt{8} < q < q_2$ and *two* even 3-lap periodic solutions $-\sqrt{8} < q < q_3$. One solution is *convex* at the origin ($u''(0) > 0$) and one solution is *concave* at the origin ($u''(0) < 0$). The goal of this section is to extend these results to the parameter regime $q \geq q_3$. It is important to note that the sharp qualitative results obtained in Sections 4 and 6 were proved by means of Lemma 2.4. However, this lemma no longer applies in the range $q \geq q_3$. Thus, to determine the existence and qualitative properties of periodic solutions, we shall develop further analytical techniques.

We recall that an even periodic solution u is called an n-lap periodic solution if it is even, and symmetric with respect to its n^{th} positive critical point ζ_n (and not symmetric with respect to any of the critical points in between 0 and ζ_n). In order to determine if an n-lap periodic solution u attains a relative maximum or minimum at the point of symmetry, one needs to know whether n is odd or even, as well as whether $u''(0)$ is positive or negative. For convenience we list the correspondence in the following table:

$$\left.\begin{array}{lll} \zeta_n = \xi_{(n+1)/2} & \text{when} \quad n & \text{is odd} \\ \zeta_n = \eta_{n/2} & \text{when} \quad n & \text{is even} \end{array}\right\} \qquad \text{if} \quad u''(0) > 0,$$

$$\left.\begin{array}{lll} \zeta_n = \xi_{(n+2)/2} & \text{when} \quad n & \text{is even} \\ \zeta_n = \eta_{(n+1)/2} & \text{when} \quad n & \text{is odd} \end{array}\right\} \qquad \text{if} \quad u''(0) < 0.$$

In the present section we extend the above existence theorems to n-lap periodic solutions for arbitrary $n \geq 2$.

Theorem 8.1. *Let $n \geq 2$, and let*

$$-\sqrt{8} < q < q_n.$$

Then there exist two even n-lap periodic solutions, u_a and u_b, such that

$$u_{a,b}(\zeta_n) > 1 \quad \text{and} \quad u'''_{a,b}(\zeta_n) = 0, \tag{8.1}$$

whilst $u_a''(0) > 0$ *and* $u_b''(0) < 0$.

We prove Theorem 8.1 in two steps: first, we establish the existence of n-lap solutions which satisfy (8.1) in intervals of the form $(q_{3,2n-1}, q_n)$, and then we show that these n-lap solutions exist on the whole interval $(-\sqrt{8}, q_n)$.

Lemma 8.2. *Let* $n \geq 2$. *Then for all* $q_{3,2n-1} < q < q_n$ *there exist two even periodic solution* u_a *and* u_b *which are symmetric with respect to their* n^{th} *positive critical point* ζ_n, *such that*

$$u_{a,b}(\zeta_n) > 1 \qquad \textit{and} \quad u_a''(0) > 0, \;\; u_b''(0) < 0.$$

Proof. For the cases $n = 2$ and $n = 3$ we refer to the stronger results of Section 6. Thus, in the remainder of this proof we assume that $n \geq 4$.

We only consider the case where $u''(0) > 0$. The case $u''(0) < 0$ is analogous. Let $u(x, \alpha_-)$ be a *small* amplitude, even, single bump periodic solution for which $u''(0) > 0$. Then $\alpha_- = -M_-$, where $M_- = \|u(\cdot, \alpha_-)\|_\infty < 1$. In particular, $u(\xi_k(\alpha_-), \alpha_-) < 1$ and $u(\eta_k(\alpha_-), \alpha_-) > -1$ for all $k \geq 1$ (see Figure 4.1).

Fix $n \geq 4$ and define $m = (n+1)/2$ if n is *odd*, and $m = n/2$ if n is even. We set

$$a = \sup\{\alpha > \alpha_- : u(\xi_i(\alpha), \alpha) \leq 1 \;\; \text{for} \;\; i = 1, \ldots, m\}. \tag{8.2}$$

We stress that this definition is completely different from the ones used so far. For all $\alpha > a$ at least one of the maxima ξ_i with $i = 1, 2, \ldots m$ lies in the region $\{u > 1\}$.

We assert that $a < 1$. To see this we study the behaviour of $u(x, \alpha)$ when α is close to 1. Let v be the solution of the problem obtained by linearising around $u = 1$, which was introduced in Section 5. Then, according to Lemma 5.2,

$$v(\xi_m) > 0 \quad \text{and} \quad v'''(\xi_m) < 0 \quad \text{if} \quad q \in (q_{3,2n-1}, q_n).$$

Therefore, there exists a constant $\delta > 0$ such that

$$u(\xi_m(\alpha), \alpha) > 1 \quad \text{and} \quad u'''(\xi_m(\alpha), \alpha) < 0 \quad \text{for} \quad 1 - \delta < \alpha < 1. \tag{8.3}$$

From the first inequality in (8.3) we deduce that $a < 1$, as claimed.

It follows from the definition of a and Lemma 2.1 that

$$u(\xi_k(a), a) < 1 \;\; \text{for} \;\; k = 1, \ldots, m-1 \quad \text{and} \quad u(\xi_m(a), a) = 1. \tag{8.4}$$

From (8.4) and the energy identity (1.8) it follows that $u'''(\xi_m) \neq 0$ at a. Because $u(\eta_{m-1}) < u(\xi_{m-1}) < 1$ it follows that

$$u'''(\xi_m(a), a) = u'''(\eta_m(a), a) = u'''(\zeta_n(a), a) > 0, \tag{8.5}$$

where we remark that $\zeta_n = \xi_m$ if n is odd, and $\zeta_n = \eta_m$ if n is even. We now define

$$b = \sup\{\alpha \in (a, 1) : u(\zeta_n) > 1 \;\; \text{on} \;\; (a, \alpha)\}, \tag{8.6}$$

which is well-defined because of the definition of a, Equation (8.4) and Lemma 2.3. Besides, we define (in view of (8.5))

$$c = \sup\{\alpha \in (a, 1) : u'''(\zeta_n) > 1 \quad \text{on} \quad (a, \alpha)\}.$$

We now first consider the case that $\zeta_n = \xi_m$ (i.e. n odd). If $b = 1$, then it follows from (8.3) that $c < 1$, thus $u(x, c)$ is an even n-lap periodic solution which is symmetric with respect to ξ_m and $u(\xi_m) > 1$.

If $b < 1$ we use the following result to obtain a solution.

Lemma 8.3. *Let a and b be defined as in (8.2) and (8.6). If $b < 1$, then $u'''(\zeta_n) < 0$ at b.*

We postpone the proof of Lemma 8.3 for a moment and first finish the proof of Lemma 8.2. We conclude from Lemma 8.3 that $c < b$, and as we saw before this implies that there exists an even periodic solution which is symmetric with respect to ξ_m such that $u(\xi_m) > 1$. The case that $\zeta_n = \eta_m$ (i.e. n even) is dealt with in a similar manner, but we have to distinguish three cases (the situation is similar to the proof of Part (b) of Theorem 6.2). According to Lemma 5.2 we have

$$v'''(\eta_m) < 0 \quad \text{for all } q \in (q_{3,2n-1}, q_n),$$

and

$$\begin{aligned} v(\eta_m) &> 0 \quad \text{if } q_{3,2n+1} < q < q_n, \\ v(\eta_m) &= 0 \quad \text{if } q = q_{3,2n+1}, \\ v(\eta_m) &< 0 \quad \text{if } q_{3,2n-1} < q < q_{3,2n+1}. \end{aligned}$$

If $q_{3,2n+1} < q < q_n$ then the proof is finished in the same way as above. If $q_{3,2n-1} < q < q_{3,2n+1}$ then $b < 1$ and the proof is finished with the help of Lemma 8.3. Finally, if $q = q_{3,2n+1}$ then we have, for $\delta > 0$ small enough,

$$u(\xi_m) > 1 \quad \text{and} \quad u'''(\eta_m) < 0 \quad \text{for } 1 - \delta < \alpha < 1.$$

We now choose an $\alpha \in (1 - \delta, 1)$. If $u(\eta_m) > 1$ then we finish the proof as in the case where $q_{3,2n+1} < q < q_n$, whereas if $u(\eta_m) \leq 1$ then we finish the proof as in the case where $q_{3,2n-1} < q < q_{3,2n+1}$. This completes the proof of Lemma 8.2.

Before we give the proof of Lemma 8.3 we introduce some notation. We define the sets of maxima and minima in the region $\{u > 1\}$ by

$$\begin{aligned} \mathcal{C}_+ &= \{1 \leq k \leq m - 1 \, : \, u(\xi_k) > 1\}, \\ \mathcal{C}_- &= \{1 \leq k \leq m - 1 \, : \, u(\eta_k) > 1\}. \end{aligned}$$

The following Proposition shows that for all $\alpha \in [a, b]$ we have $u(\xi_k) > 1$ if and only if $u(\eta_k) > 1$. Besides, if $u(\xi_\ell) > 1$ for some $1 \leq \ell \leq m - 1$ then $u(\zeta) > 1$ for all critical points ζ in between ξ_ℓ and ξ_m.

Proposition 8.4. *For all* $\alpha \in [a,b]$ *we have*

$$\mathcal{C}_+ = \mathcal{C}_- = \emptyset \quad \text{or} \quad \mathcal{C}_+ = \mathcal{C}_- = \{\ell, \ell+1, \dots, m-1\} \ \text{for some } \ell \in \{1, 2, \dots m-1\}. \tag{8.7}$$

Proof. We first notice that $u(0) < 1$ and $u(\xi_m) > 1$ for all $\alpha \in (a,b)$ (note that $u(\eta_m) > 1$ implies that $u(\xi_m) > 1$). For $\alpha = a$ Equation (8.7) holds since $\mathcal{C}_+ = \mathcal{C}_- = \emptyset$. It follows from the definition of a and Lemma 2.3 that $\mathcal{C}_+ = \mathcal{C}_- = \emptyset$ for $\alpha \in (a, a+\delta)$ with $\delta > 0$ sufficiently small. We now use a continuation argument in α to show that Equation (8.7) holds for all $\alpha \in [a,b)$. We define

$$\alpha_* = \sup\{\alpha \in (a,b) : \text{Equation (8.7) holds on } (a,\alpha)\},$$

and we suppose, for contradiction, that $\alpha_* < b$. Then at α_* we have $u(\xi_n) = 1$ for some $1 \le n \le m-1$. There are now two possibilities: either $u(\eta_{n-1}) = 1$ or $u(\eta_n) = 1$.

Concerning the first case, it follows from Lemma 2.1 that at $\alpha = \alpha_*$

$$u(\xi_{n-1}) > 1 \ \text{ and } \ u(\eta_n) < 1, \tag{8.8}$$

and by continuity (8.8) holds for $\alpha \in (\alpha_* - \delta, \alpha_*)$ with $\delta > 0$ sufficiently small. However, this contradicts the fact that Equation (8.7) holds for all $\alpha \in (a, \alpha_*)$. Thus at α_* there there is no $k \in \{1, 2, \dots, m-1\}$ such that $u(\xi_k) = u(\eta_{k-1}) = 1$.

Therefore, we must have $u(\xi_n) = u(\eta_n) = 1$ at α_* for some $1 \le n \le m-1$. As before, it follows that

$$u(\xi_{n+1}) > 1 \ \text{ and } \ u(\eta_{n-1}) < 1.$$

We assert that this implies that $u(\xi_{n-1}) < 1$. Namely, the possibility $u(\xi_{n-1}) > 1$ is excluded by the definition of α_* and the fact that $u(\eta_{n-1}) < 1$. Besides, $u(\xi_{n-1}) = 1$ would imply that $u(\eta_{n-2}) = 0$ which has already been excluded above. Hence $u(\xi_{n-1}) < 1$, and thus also $u(\eta_{n-2}) < 1$. A repeated argument shows that $u(\xi_k) < 1$ for all $1 \le k \le n-1$. Analogously it is proved that $u(\eta_k) > 1$ for all $n+1 \le k \le m-1$. By continuity this also holds for α close to α_*.

Finally, by the definition of α_* there must exist a sequence $\alpha_i \downarrow \alpha_*$ such that

$$u(\xi_n(\alpha_i), \alpha_i) > 1 \quad \text{and} \quad u(\eta_n(\alpha_i), \alpha_i) < 1.$$

The existence of such a sequence is excluded by the proof of Lemma 2.3, which can be found in [PT5, Lemma 2.5]. Hence, having obtained a contradiction, we have proved that Equation (8.7) holds on the entire interval $\alpha \in [a,b)$. The case $\alpha = b$ follows by continuity.

Remark. It follows from the previous Proposition that $u(\xi_k)$ and $u(\eta_k)$ can only enter and leave the region $\{u > 1\}$ together.

Proof of Lemma 8.3. To prove Lemma 8.3 we argue by contradiction. Thus, suppose that $u'''(\zeta_n) > 0$ at b. Then $\zeta_n = \xi_m = \eta_m$, and it follows that $u(\eta_{m-1}) < 1$ at $\alpha = b$. Therefore, $m - 1 \notin \mathcal{C}_-$. By Propostion 8.4, this implies that $u(\eta_k) < 1$ for all $1 \le k \le m-1$, so that $\mathcal{C}_- = \emptyset$. Besides, since $\mathcal{C}_- = \mathcal{C}_+$ by Proposition 8.4, this also implies that $\mathcal{C}_+ = \emptyset$. Thus, we conclude that $u(\xi_k) \le 1$ for all $1 \le k \le m-1$ and $u(\xi_m) = u(\eta_m) = 1$ at $\alpha = b$. Since a was defined as the largest value of α for which this situation occurs, this situation is excluded and we have reached a contradiction.

We recall the definition

$$c = \sup\{\alpha > a : u'''(\zeta_n) > 0 \ \text{ on } \ (a, \alpha)\}.$$

Clearly $c < b \le 1$ by the proof of Lemma 8.2.

Proposition 8.5. *For all $\alpha \in [a, c]$ we have that*

$$u'''(\xi_k) > 0 \quad \textit{and} \quad u'''(\eta_k) > 0 \qquad \textit{for all } k \in \mathcal{C}_+ = \mathcal{C}_-.$$

Proof. We first notice that since $u(\xi_k)$ and $u(\eta_k)$ only enter $\{u > 1\}$ together, we must have $u'''(\xi_k) > 0$ and $u'''(\eta_k) > 0$ at the point of entry. Suppose now, for contradiction, that there exists a smallest $\alpha \in [a, c]$, for which Proposition 8.5 does not hold: let

$$d = \sup\{\alpha > a : u'''(\xi_k) > 0,\, u'''(\eta_k) > 0 \ \text{ for all } \ k \in \mathcal{C}_+\},$$

and suppose that $d \le c$. Then at $\alpha = d$ there is a critical point $\zeta \in (0, \zeta_n)$ such that $u(\zeta) > 1$ and $u'''(\zeta) = 0$, i.e., ζ is a point of symmetry.

Since by Proposition 8.4 all the critical values between ζ and ζ_n lie above $u = 1$, it follows that that $u(x) > 1$ for $x \in [\zeta, \zeta_n]$. In fact, since ζ is a point of symmetry, we have that $u(x) > 1$ for all $x \in [2\zeta - \zeta_n, \zeta_n]$. Since $u(0) < 1$, this means that $2\zeta - \zeta_n > 0$. By symmetry, $2\zeta - \zeta_n$ is a critical point, and from the definition of d we see that $u'''(2\zeta - \zeta_n) \ge 0$. Therefore, again by symmetry, $u'''(\zeta_n) \le 0$, so that from definition of c it follows that $c \le d$. Since by assumption, $d \le c$, we conclude that $d = c$, and $u'''(\zeta_n) = 0$. But then u is symmetric with respect to both ζ and ζ_n. This means that $u(x) > 1$ for all $x \in \mathbf{R}$, which contradicts the fact that $u(0) < 1$.

Remark. Another way of obtaining the above contradiction is via the observation that

$$(u''' + qu')' = u(1 - u^2) < 0 \qquad \text{on } (\zeta, \zeta_n).$$

Upon integrating over (ζ, ζ_n) we obtain that $u'''(\zeta_n) < 0$ at $\alpha = d$, contradicting the assumption that $d \le c$.

Lemma 8.6. *Let $n \ge 4$. For every $q \in [q_{n-1}, q_n)$ there exist two even n-lap solutions u_a and u_b such that $u_{a,b}(\zeta_n) > 1$ and $u'''_{a,b}(\zeta_n) = 0$, whilst $u''_a(0) > 0$ and $u''_b(0) < 0$.*

Proof. Since $q_{3,2n-1} < q_{n-1}$ for $n \ge 4$, it follows immediately from Lemma 8.2 that there exist two periodic solutions which are symmetric with respect to

ζ_n, and such that $u_{a,b}(\zeta_n) > 1$. We assert that ζ_n is the *first* point of symmetry. Proposition 8.5 ensures that ζ_n is the first point of symmetry in the region $\{u > 1\}$. Besides, there is no critical point $0 < \zeta < \zeta_n$ with $u'''(\zeta) = 0$ in the region $\{u \le 1\}$, since that would imply (by Proposition 8.4 and the symmetry with respect to 0) that $u(x) \le 1$ for all $x \in \mathbf{R}$, contradicting the fact that $u(\zeta_n) > 1$.

The following Lemma is a reformulation of Theorem 8.1. It shows that the n-lap solutions obtained in Lemma 8.6 exists for all $q \in (-\sqrt{8}, q_n)$.

Lemma 8.7. *Let $n \ge 2$ and let $-\sqrt{8} \le q < q_n$. For any $N \ge n$ there exist two even N-lap solutions u_a and u_b such that $u_{a,b}(\zeta_N) > 1$ and $u'''_{a,b}(\zeta_N) = 0$, whilst $u''_a(0) > 0$ and $u''_b(0) < 0$.*

Proof. The range $q \in (-\sqrt{8}, q_3)$ was already covered in Sections 4 and 6. For $q \in [q_{n-1}, q_n)$ with $n \ge 4$ we see from Lemma 8.6 that there exists two n-lap solutions. For $N > n$ we again restrict our attention to the case $u''(0) > 0$, the other case being completely analogous. We can define a, b and c as before for both n and N. We have $a_n < c_n < b_n \le 1$ and $a_N < c_N \le b_N \le 1$. If $c_N < 1$ then clearly we have an N-lap solution. Arguing by contradiction we assume that $c_N = 1$. It follows directly from the definition of a and (8.4) that $a_N < a_n$. Proposition 8.5 shows that $u'''(\zeta_n) > 0$ if $u(\zeta_n) > 1$ for all $\alpha \in (a_N, 1]$. However, $u'''(\zeta_n) = 0$ for $\alpha = c_n > a_n > a_N$, a contradiction.

9. Proof of Lemma 2.4

In this section we prove Lemma 2.4. We recall the setting: we assume that u is a solution of equation (1.1), and that $a \in \mathbf{R}$ is a critical point where u has the following properties:

$$u(a) = 1, \quad u'(a) = 0, \quad u''(a) = 0 \quad \text{and} \quad u'''(a) > 0. \tag{2.13}$$

Then $u' > 0$ in a right-neighbourhood of a so that the point

$$b = \sup\{x > a : u' > 0 \ \text{ on } \ (0, x)\} \tag{2.14}$$

is well defined. By Lemma 2.2, it is also finite.

Lemma 2.4. *Suppose that*

$$-\sqrt{8} < q < q_3 = \sqrt{2}\Big(3 + \frac{1}{3}\Big).$$

Let u be a solution of equation (1.1) which at a point $a \in \mathbf{R}$ has the properties listed in (2.13). Then, at its next critical point b defined by (2.14), we have

$$u(b) > 1, \quad u'(b) = 0, \quad u''(b) < 0 \quad \textit{and} \quad u'''(b) < 0. \tag{2.15}$$

Proof. If $-\sqrt{8} < q \le \sqrt{8}$ we use the function H introduced in Section 4. Since $u > 1$ on (a, b), it follows that $H'' > 0$ on (a, b). At critical points of u, we have by (4.7)

$$H' = u''u'''.$$

Hence $H' = 0$ at a, and therefore $H' > 0$ at b. Since $u''(b) \leq 0$, and even $u''(b) < 0$ by the first integral, it follows that $u'''(b) < 0$, as asserted.

If $q > \sqrt{8}$ we can no longer prove that $H'' > 0$ on (a, b) and we have to proceed differently.

Without loss of generality we may assume that $a = 0$ and consider the initial value problem

$$\begin{cases} u^{iv} + qu'' + u^3 - u = 0, & (9.1a) \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0 \quad u'''(0) = \lambda, & (9.1b) \end{cases}$$

where the initial values are derived from (2.13) and λ is a positive number. We denote the solution of Problem (9.1) by $u(x, \lambda)$, and its first critical point, corresponding to b, by $\zeta(\lambda)$:

$$\zeta(\lambda) = \sup\{x > 0 : u'(\cdot, \lambda) > 0 \text{ on } (0, x)\}.$$

We need to show that

$$u'''(\zeta(\lambda), \lambda) < 0 \quad \text{for every} \quad \lambda > 0. \tag{9.2}$$

As a first step we show that (9.2) is satisfied for λ large enough. This can be proved by means of a simple scaling argument. With the variables

$$x = \lambda^{-1/5} t \quad \text{and} \quad u(x, \lambda) = \lambda^{2/5} w(t, \lambda), \qquad \lambda > 0, \tag{9.3}$$

the initial value problem (9.1) can be written as

$$\begin{cases} w^{iv} + q\lambda^{-2/5} w'' + w^3 - \lambda^{-4/5} w = 0, \\ w(0) = \lambda^{-2/5}, \quad w'(0) = 0, \quad w''(0) = 0, \quad w'''(0) = 1. \end{cases}$$

By standard ODE arguments, $w(t, \lambda) \to W(t)$ on compact intervals, as $\lambda \to \infty$, where W is the solution of the limit problem

$$\begin{cases} W^{iv} + W^3 = 0, \\ W(0) = 0, \quad W'(0) = 0, \quad W''(0) = 0, \quad W'''(0) = 1. \end{cases}$$

Plainly,

$$T = \sup\{t > 0 : W' > 0 \text{ on } (0, t)\} < \infty$$

and $W'''(T) < 0$. Hence, by continuity, for λ large enough, $w'''(\cdot, \lambda) < 0$ at the first zero of $w'(\cdot, \lambda)$. Thus, $u'''(\zeta(\lambda), \lambda) < 0$ for λ large enough.

Proceeding with the proof of Lemma 2.4, we suppose that (9.2) is false and that $u'''(\zeta(\lambda_0), \lambda_0) \geq 0$ for some $\lambda_0 > 0$. Then there exist a constant $\lambda^* \in [\lambda_0, \infty)$ such that $u'''(\zeta(\lambda^*), \lambda^*) = 0$. At $\zeta^* = \zeta(\lambda^*)$ we then have

$$u' > 0 \text{ on } (0, \zeta^*), \qquad u'(\zeta^*) = 0 \text{ and } u'''(\zeta^*) = 0. \tag{9.4a}$$

This means that $u(\cdot,\lambda^*)$ is symmetric with respect to ζ^* and that

$$u' < 0 \;\text{ on }\; (\zeta^*, 2\zeta^*), \qquad u(2\zeta^*) = 1 \;\text{ and }\; u'(2\zeta^*) = 0. \tag{9.4b}$$

For further reference, we introduce the notation

$$u(\zeta^*) = \alpha^* \qquad \text{and hence} \qquad u''(\zeta^*) = \frac{1}{\sqrt{2}}\big(1 - (\alpha^*)^2\big). \tag{9.4c}$$

In the remainder of the proof we show that if $1 < a/b < 3$, where a and b have been defined in (5.2b) (i.e. if $\sqrt{8} < q < q_3$), then a solution $u(x,\lambda^*)$ with the properties listed in (9.4) cannot exist. We shift the origin to $x = \zeta^*$ and $u = 1$ and write

$$x = \zeta^* + \tilde{x} \quad \text{and} \quad u(x) = 1 + w(\tilde{x}). \tag{9.5}$$

Then w is the solution of the problem

$$\begin{cases} w^{iv} + qw'' + 2w = -w^2(3+w), & (9.6a)\\ w(0) = \alpha^* - 1, \quad w'(0) = 0, \quad w''(0) = \dfrac{1}{\sqrt{2}}\big(1 - (\alpha^*)^2\big), \quad w'''(0) = 0, & (9.6b)\end{cases}$$

where we used (9.4) and have dropped the tilde. We shall compare the solution $w = w(x,\alpha^*)$ of Problem (9.6) with the solution v of the linear problem

$$\begin{cases} v^{iv} + qv'' + 2v = 0, & (9.7a)\\ v(0) = \alpha - 1, \quad v'(0) = 0, \quad v''(0) = \dfrac{1}{\sqrt{2}}(1 - \alpha^2), \quad v'''(0) = 0, & (9.7b)\end{cases}$$

where $\alpha > 1$ is an arbitrary number, which eventually will be chosen equal to α^*. We denote the solution of Problem (9.7) by $v(x)$ or $v(x,\alpha)$. An elementary computation shows that $v(x)$ can be written as

$$v(x) = A\cos(ax) + B\cos(bx), \tag{9.8}$$

where we recall that a and b are the positive roots of the equation

$$k^4 - qk^2 + 2 = 0,$$

given by

$$a^2 = \frac{1}{2}\big(q + \sqrt{q^2 - 8}\,\big) \quad \text{and} \quad b^2 = \frac{1}{2}\big(q - \sqrt{q^2 - 8}\,\big). \tag{9.9}$$

The coefficients A and B are given by

$$A = -\frac{b^2(\alpha - 1) + \beta}{a^2 - b^2} \quad \text{and} \quad B = \frac{a^2(\alpha - 1) + \beta}{a^2 - b^2}, \qquad \beta = -\frac{1}{\sqrt{2}}(\alpha^2 - 1). \tag{9.10}$$

Let

$$x_0(\alpha) = \sup\{x > 0 : v(\cdot, \alpha) > 0 \ \text{ on } \ [0, x)\}.$$

In order to proceed with our comparison argument we need the following result.

Lemma 9.1. *Suppose that*

$$\sqrt{8} < q < q_3 \qquad \textit{or, equivalently,} \qquad 1 < \frac{a}{b} < 3.$$

Then for any $\alpha > 1$ *we have* $x_0(\alpha) < \infty$ *and* $v'(\cdot, \alpha) < 0$ *on* $(0, x_0(\alpha)]$.

We postpone the proof of this lemma until after the proof of Lemma 2.4 has been completed.

We now continue with the proof of Lemma 2.4. Let

$$y_0(\alpha) = \sup\{x > 0 : w(\cdot, \alpha) > 0 \ \text{ on } \ [0, x)\}.$$

Our goal is to prove that $w'(y_0(\alpha), \alpha) < 0$ for any $\alpha > 0$. Translating this result back to the variables x and u, we find in particluar that $u'(2\zeta^*) < 0$, which contradicts (9.4b) and completes the proof.

By the variation of constants formula, we find that

$$w(x) = v(x) - \frac{1}{a^2 - b^2} \int_0^x \Big\{\frac{1}{b}\sin(bt) - \frac{1}{a}\sin(at)\Big\} h(x - t)\, dt, \tag{9.11}$$

where $h(s) = w^2(s)\{3 + w(s)\} \geq 0$ as long as $w \geq 0$, i.e. on $[0, y_0]$. Note that

$$w'(x) = v'(x) - \frac{1}{a^2 - b^2} \int_0^x \{\cos(bt) - \cos(at)\} h(x - t)\, dt. \tag{9.12}$$

At the first zero x_0 of v we have

$$w(x_0) = -\frac{1}{a^2 - b^2} \int_0^x \Big\{\frac{1}{b}\sin(bt) - \frac{1}{a}\sin(at)\Big\} h(x_0 - t)\, dt.$$

Hence, if

$$K(t) \stackrel{\text{def}}{=} \frac{1}{b}\sin(bt) - \frac{1}{a}\sin(at) > 0 \quad \text{for} \quad 0 < t < x_0, \tag{9.13}$$

then $w < v$ on $(0, x_0]$, and therefore $y_0 < x_0$. In addition, if

$$K'(t) = \cos(bt) - \cos(at) > 0 \quad \text{for} \quad 0 < t < x_0, \tag{9.14}$$

then $w' < v' < 0$ on $(0, y_0]$ by (9.12). In particular, $w'(y_0) < 0$, as asserted.

The conclusion of the proof consists of a detailed analysis of the function $K(t)$ to show that (9.13) and (9.14) hold on $(0, y_0]$. An elementary computation shows that

$$K(0) = 0, \quad K'(0) = 0, \quad K''(0) = 0, \quad K'''(0) = a^2 - b^2 > 0.$$

Hence $K > 0$ and $K' > 0$ in a right-neighbourhood of the origin. We set

$$t_1 = \sup\{t > 0 : K > 0 \ \text{ on } \ (0,t)\},$$

and

$$t_0 = \sup\{t > 0 : K' > 0 \ \text{ on } \ (0,t)\}.$$

Plainly $0 < t_0 < t_1$ and $K'(t_1) \le 0$. We recall the assumption that

$$1 < \frac{a}{b} < 3 \qquad \Longleftrightarrow \qquad \frac{\pi}{2a} < \frac{\pi}{2b} < \frac{3\pi}{2a} < \frac{3\pi}{2b}. \tag{9.15}$$

Because $a/b > 1$, it follows from (9.13) and (9.15) that $t_1 \in (\frac{\pi}{2b}, \frac{3\pi}{2b})$, and hence, that $t_0 < \frac{3\pi}{2b}$. On the other hand, since $\frac{\pi}{2a} < \frac{\pi}{2b}$ it is clear that $K' > 0$ on $(p, \frac{\pi}{2a}]$ so that $t_0 > \frac{\pi}{2a}$. Observe that

$$\cos(at) < 0 \quad \text{on} \quad (\frac{\pi}{2a}, \frac{3\pi}{2a}) \quad \text{and} \quad \cos(bt) > 0 \quad \text{on} \quad (0, \frac{\pi}{2b}).$$

Because $a/b < 3$ and hence $\frac{3\pi}{2a} > \frac{\pi}{2b}$, it follows that $K' > 0$ on $[\frac{\pi}{2a}, \frac{\pi}{2b}]$, and we conclude that

$$\frac{\pi}{2b} < t_0 < \frac{3\pi}{2b}. \tag{9.16}$$

The value of v at t_0 can easily be computed by using the formula (9.14) for K' in the expression (9.8) for v. We obtain

$$v(t_0) = \frac{\alpha - 1}{a^2 - b^2} \cos(bt_0),$$

which, in view of (9.16), shows that $v(t_0) < 0$. We conclude that $x_0 < t_0 < t_1$, and hence that the properties (9.13) and (9.14) of respectively K and K' are true. This completes the proof of Lemma 2.4.

Proof of Lemma 9.1. We first show that the assertion is true for

$$\alpha > q\sqrt{2} - 1. \tag{9.17}$$

Note that

$$(v''' + qv')' = -2v < 0 \quad \text{as long as} \quad v > 0, \tag{9.18}$$

and hence, still as long as $v > 0$,

$$v''(x) + qv(x) \le v''(0) + qv(0) = (1 - \alpha)\Big(\frac{1+\alpha}{\sqrt{2}} - q\Big).$$

Therefore, if (9.17) holds then

$$v'' < -qv < 0 \quad \text{as long as} \quad v > 0,$$

so that $x_0(\alpha) < \infty$ and $v'(\cdot, \alpha) < 0$ for $0 < x \leq x_0(\alpha)$.

Let

$$\alpha^* = \inf\{\hat{\alpha} > 1 : x_0(\alpha) < \infty \ \text{ and } \ v'(\cdot, \alpha) < 0 \ \text{ for } \ \hat{\alpha} < \alpha < \infty\}.$$

If $\alpha^* = 1$, then the assertion is proved. Therefore, we suppose that $\alpha^* > 1$. We distinguish two cases:

(*i*) $x_0(\alpha^*) = \infty$ and $v'(\cdot, \alpha^*) \leq 0$ for $0 \leq x < \infty$, or

(*ii*) $x_0(\alpha^*) < \infty$ and $v'(x_0, \alpha^*) = 0$, for some $x_1 \leq x_0(\alpha^*)$.

To keep the notation as simple as possible, we shall henceforth omit the asterisk when referring to α^*.

Case (*i*): Because $v' = 0$ and $v''' = 0$ at the origin, integration of (9.18) over $(0, x)$ shows that

$$v'''(x) + qv'(x) < 0 \quad \text{for} \quad x > 0.$$

Hence, integrating (9.18) again, but now over $(1, x)$ we conclude that

$$(v''(x) + qv(x))' < v'''(1) + qv(1) = -\delta \quad \text{for} \quad x > 1,$$

where δ is a postive constant. Hence

$$v''(x) + qv(x) < C - \delta(x-1) \quad \text{and therefore} \quad v''(x) < C - \delta(x-1) \quad \text{for} \quad x > 1,$$

where $C = v''(1) + qv(1)$ is a constant. Thus $v''(x) \to -\infty$ as $x \to \infty$, which implies that $x_0(\alpha) < \infty$, a contradiction.

Case (*ii*): Suppose that $x_1 < x_0(\alpha)$. Then

$$v(x_1) > 0, \quad v'(x_1) = 0 \quad \text{and} \quad v''(x_1) = 0. \tag{9.19}$$

When we multiply equation (9.7a) by v', integrate over $(0, x)$ and use the initial conditions, we obtain

$$v'v''' - \frac{1}{2}(v'')^2 + \frac{q}{2}(v')^2 + v^2 = (\alpha - 1)^2\Big\{1 - \frac{1}{4}(1+\alpha)^2\Big\} \quad \text{for} \quad x \in [0, x_0],$$

Evaluating the left hand side at x_1, using the properties of v at x_1 listed in (9.19), we obtain

$$v^2(x_1) = (\alpha - 1)^2\Big\{1 - \frac{1}{4}(1+\alpha)^2\Big\}. \tag{9.20}$$

Since $\alpha > 1$, the right hand side of (9.20) is negative, whilst the left hand side is nonnegative, a contradiction.

It remains to consider the case that $x_1 = x_0$, so that

$$v(x_0) = 0 \quad \text{and} \quad v'(x_0) = 0. \tag{9.21}$$

We begin with a preliminary result.

Lemma 9.2. *Suppose that* $1 < a/b < 3$. *Then, if the solution* v *of Problem (9.7) has the properties (9.21), the point* x_0 *must lie in the interval* $(0, \frac{\pi}{2b})$.

Proof. We recall the formula for v:

$$v(x) = A\cos(ax) + B\cos(bx), \tag{9.22}$$

where a and b are given in (9.9) and A and B are given in (9.10). We discuss in succession the three cases:

$$(i)\quad A = 0, \qquad (ii)\quad A < 0 \quad \text{and} \quad (iii)\quad A > 0.$$

Case (*i*) $A = 0$: Since $A + B = \alpha - 1$, it follows that in this case $B = \alpha - 1$ and hence

$$v(x) = (\alpha - 1)\cos(bx).$$

Thus, $x_0 = \frac{\pi}{2b}$. However, $v'(x_0) < 0$, which contradicts (9.21), so that A cannot be zero.

Case (*ii*) $A < 0$: Observe that in this case

$$B\cos(bx_0) = |A|\cos(ax_0),$$

and $B = \alpha - 1 - A = \alpha - 1 + |A| > |A|$. Remembering that $a > b$, we conclude that

$$B\cos(bx) > |A|\cos(ax) \geq 0 \quad \text{if} \quad 0 \leq x \leq \frac{\pi}{2a}.$$

By assumption

$$\frac{\pi}{2} < \frac{a}{b}\frac{\pi}{2} < \frac{3\pi}{2}. \tag{9.23}$$

and hence

$$B\cos(bx) \geq 0 > |A|\cos(ax) \quad \text{if} \quad \frac{\pi}{2a} < x \leq \frac{\pi}{2b}.$$

Therefore

$$\frac{\pi}{2b} < x_0 < \frac{3\pi}{2b}. \tag{9.24}$$

This implies that

$$v'' + a^2 v = (a^2 - b^2)\cos(bx) < 0 \quad \text{on} \quad \left(\frac{\pi}{2b}, x_0\right),$$

and hence, because $v' < 0$ on $(0, x_0)$,

$$\{(v')^2 + a^2 v^2\}' = 2(v'' + a^2 v)v' > 0 \quad \text{on} \quad \left(\frac{\pi}{2b}, x_0\right).$$

When we integrate this inequality over $(\frac{\pi}{2b}, x_0)$, we find that

$$(v')^2(x_0) > \{(v')^2 + a^2 v^2\}\Big|_{\pi/2b} > 0,$$

which means that $v'(x_0) < 0$. This contradicts (9.21), so that the case $A < 0$ cannot occur either.

Case (iii) $A > 0$: We have

$$v\left(\frac{\pi}{2b}\right) = A\cos\left(\frac{a}{b}\frac{\pi}{2}\right).$$

But, in view of (9.23), $\cos\left(\frac{a}{b}\frac{\pi}{2}\right) < 0$. This means that $v\left(\frac{\pi}{2b}\right) < 0$, so that $x_0 < \frac{\pi}{2b}$.

Summarising, we have found that the constant A in (9.22) must be positive, and hence, that $x_0 < \frac{\pi}{2b}$.

Proof of Lemma 9.1 – Conclusion. Using the explicit expression (9.22) for v, we deduce from (9.21) that the constants A and B must satisfy the equations

$$A\cos(ax_0) + B\cos(bx_0) = 0 \quad \text{and} \quad aA\sin(ax_0) + bB\sin(bx_0) = 0. \tag{9.25}$$

Since $A + B = \alpha - 1$, they cannot both be equal to zero. Therefore, the determinant of the system (9.25) of equations must be zero, and hence

$$\cos(bx_0)\sin(ax_0) = \cos(ax_0)\sin(bx_0). \tag{9.26}$$

By Lemma 9.2, $\cos(bx_0) > 0$. Hence, if $\cos(ax_0) = 0$, then $\sin(ax_0) = 0$ as well, and this is impossible. Thus, $\cos(ax_0) \neq 0$ and we may divide (9.26) by $\cos(ax_0)\cos(bx_0)$. We thus find that x_0 must be a solution of the equation

$$\tan(bx) = \frac{a}{b}\tan(ax) \qquad \text{in} \qquad \left(0, \frac{\pi}{2a}\right) \cup \left(\frac{\pi}{2a}, \frac{\pi}{2b}\right), \tag{9.27}$$

because $\frac{3\pi}{2a} > \frac{\pi}{2b}$ according to (9.23). With $\lambda = a/b$ and $bx = t$, we can write (9.27) as

$$\phi(t) \stackrel{\text{def}}{=} \lambda\tan(\lambda t) - \tan(t) = 0 \qquad \text{in} \qquad I \stackrel{\text{def}}{=} \left(0, \frac{\pi}{2\lambda}\right) \cup \left(\frac{\pi}{2\lambda}, \frac{\pi}{2}\right). \tag{9.28}$$

Thus, we seek a root $\tau = bx_0$ of equation (9.28).

Lemma 9.3. *If* $1 < \lambda < 3$, *then equation* $\phi(t) = 0$ *has no roots in the set* I.

Proof. Observe that

$$\lambda\tan(\lambda t) > \tan(t) \quad \text{if} \quad 0 < t < \frac{\pi}{2\lambda} \tag{9.29a}$$

and

$$\lambda\tan(\lambda t) < 0 < \tan(t) \quad \text{if} \quad \frac{\pi}{2\lambda} < t < \frac{\pi}{\lambda}. \tag{9.29b}$$

If $\lambda \leq 2$, then $\pi/2\lambda \geq \pi/2$, so that it is immediately clear from (9.29) that ϕ cannot have a zero in I.

If $2 < \lambda < 3$, then $\pi/\lambda < \pi/2 < 3\pi/2\lambda$ and it follows that if $\phi(t)$ has a zero in I, then it must lie in the interval $(\frac{\pi}{\lambda}, \frac{\pi}{2})$, where both terms in $\phi(t)$ are positive. Plainly,

$$\phi\left(\frac{\pi}{\lambda}\right) = -\tan\left(\frac{\pi}{\lambda}\right) < 0 \quad \text{and} \quad \phi(t) \to -\infty \quad \text{as} \quad t \to \frac{\pi}{2}^{-}. \tag{9.30}$$

Suppose that $\phi(t)$ has a zero, and that t_0 is the largest zero on $(\frac{\pi}{\lambda}, \frac{\pi}{2})$. Then (9.30) implies that $\phi'(t_0) \leq 0$. However, an easy computation shows that

$$\phi'(t) = \lambda^2 - 1 > 0 \quad \text{when} \quad \phi(t) = 0.$$

Thus, we have a contradiction, and $\phi(t)$ cannot have a zero in I. This completes the proof of Lemma 9.1.

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Appendix

We linearize around the constant solution $u = 1$ of the equation

$$u^{iv} + qu'' + u^3 - u = 0. \tag{A.1}$$

Thus, we write $u = 1 + \varepsilon v$ and substitute into the initial value problem for u. Omitting higher order terms, we obtain

$$\begin{cases} v^{iv} + qv'' + 2v = 0. & \text{(A.2a)} \\ v(0) = -1, \quad v'(0) = 0, \quad v''(0) = \pm\sqrt{2}, \quad v'''(0) = 0. & \text{(A.2b)} \end{cases}$$

We denote the corresponding solutions by $v_\pm(x)$.

Substitution of $v(x) = e^{\lambda x}$ yields the characteristic equation

$$\lambda^4 + q\lambda^2 + 2 = 0. \tag{A.3}$$

We distinguish two cases:

$$\textbf{Case A}: \ -\sqrt{8} < q < \sqrt{8} \qquad \text{and} \qquad \textbf{Case B}: \ \sqrt{8} \le q < \infty.$$

Case A: Equation (A.3) has roots

$$\lambda = \pm a \pm ib, \tag{A.4a}$$

in which $a > 0$ and $b > 0$ are given by

$$a = \frac{1}{2}\sqrt{\sqrt{8} - q} \quad \text{and} \quad b = \frac{1}{2}\sqrt{\sqrt{8} + q}. \tag{A.4b}$$

An elementary computation shows that the solutions $v_+(x)$ and $v_-(x)$ of Problem (A.2) are given by

$$v_\pm(x) = -\cosh(ax)\cos(bx) + K_\pm \sinh(ax)\sin(bx), \tag{A.5a}$$

where

$$K_+ = \frac{a}{b} = \sqrt{\frac{\sqrt{8} - q}{\sqrt{8} + q}} \qquad \text{and} \qquad K_- = -\frac{b}{a}. \tag{A.5b}$$

Thus,

$$\begin{aligned} v_+'(x) &= (K_+ a + b)\cosh(ax)\sin(bx) \\ v_-'(x) &= (K_- b - a)\sinh(ax)\cos(bx), \end{aligned}$$

and we see that for $v_-(x)$, we obtain

$$\eta_1(\alpha) \to \frac{\pi}{\sqrt{\sqrt{8} + q}} \qquad \text{and} \qquad \xi_2(\alpha) \to \frac{3\pi}{\sqrt{\sqrt{8} + q}} \qquad \text{as} \ \ \alpha \to 1. \tag{A.6a}$$

and, since $\varepsilon = 1 - \alpha$,

$$u_-(\eta_1(\alpha), \alpha) \sim 1 - (1 - \alpha)\frac{b}{a}\sinh\Big(\frac{\pi a}{2b}\Big) \qquad \text{as} \ \ \alpha \to 1. \tag{A.6b}$$

Case B: Equation (A.3) has roots

$$\lambda = \pm ia \quad \text{and} \quad \lambda = \pm ib, \tag{A.7a}$$

where

$$a^2 = \frac{1}{2}(q + \sqrt{q^2 - 8}\,) \qquad \text{and} \qquad b^2 = \frac{1}{2}(q - \sqrt{q^2 - 8}\,). \tag{A.7b}$$

For the solutions $v_\pm(x)$ of Problem (A.2) we find

$$v_\pm(x) = A_\pm \cos(ax) + B_\pm \cos(bx), \tag{A.8a}$$

in which

$$A_\pm = \frac{b^2 \mp \sqrt{2}}{a^2 - b^2} \qquad \text{and} \qquad B_\pm = -\frac{a^2 \mp \sqrt{2}}{a^2 - b^2}. \tag{A.8b}$$

Properties of the solutions $v_\pm(x)$ and their critical points and values are given in Lemma 5.2.