CLOSED CHARACTERISTICS OF FOURTH-ORDER TWIST SYSTEMS VIA BRAIDS

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Abstract. For a large class of second order Lagrangian dynamics, one may reformulate the problem of finding periodic solutions as a problem in solving second-order recurrence relations satisfying a twist condition. We project periodic solutions of such discretized Lagrangian systems onto the space of closed braids and apply topological techniques. Under this reformulation, one obtains a gradient flow on the space of braided piecewise linear immersions of circles. We derive existence results for closed braided solutions using Morse-Conley theory on the space of singular braid diagrams.

Caractéristiques fermées pour des systèmes de Twist par les tresses

Pour une grande classe de systèmes Lagrangiens du deuxième ordre, on peut reformuler le problème de chercher solutions periodiques comme l'investigation d'une relation de récurrence qui satisfait une condition 'Twist'. On projette les solutions periodiques d'un tel système Lagrangien discretisé sur l'espace des tresses fermées. Un flot gradient et obtenu sur l'espace des tresses linéaires par morceaux. Nous dérivons des résultats d'existence pour des solutions periodiques tressées en appliquant la théorie de Morse-Conley.

Version française abrégée

Considérons un système Lagrangien du deuxième ordre (L, dt), où $L \in C^2(\mathbb{R}^3; \mathbb{R})$ est nondégénéré: $\partial_w^2 L(u, v, w) \geq \delta > 0$. Le but principal est de trouver des functions bornées $u : \mathbb{R} \to \mathbb{R}$ qui sont stationnaires pour l'intégrale d'action $J[u] = \int L(u, u', u'')dt$. Ces *caratéristiques bornées* sont contenues dans des surfaces d'énergie. Dans cette Note nous décrivons les méthodes pour étudier les orbites fermés dans le contexte présent qui est caractérisé par des surfaces d'énergie noncompactes.

Pour un niveau d'energie E qui est régulier, les extrêmes d'une caractéristique sont contenus dans les ensembles fermés $\{u \mid L(u, 0, 0) + E \geq 0\}$, dont les composantes connexes sont denotées par I_E. Pour formuler le principe variationel des caractéristiques

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en termes des extrêmes de u, l'hypothèse *Twist* a été introduite dans [6]. Cette hypothèse signifie que les pièces monotones entre les minima et les maxima sont uniques: celles-ci sont les caractéristiques (géodésiques) brisées.

Théorème 1. *On considère l'équation (1.1) pour un niveau d'energie régulier E sous l'hypothèse Twist (T). Les conditions suivantes suffisent pour l'existence d'un nombre infini de caractéristiques fermées (dans des classes de tresse distincts):*

- *Une composante connexe compacte I_E et l'existence d'une/d' d'orbite(s) fermée(s) dont la representation de tresse est nontrivialle (voir Figure 1, à gauche); ou,*
- *Une composante $I_E = \mathbb{R}$ avec du comportement asymptotique dissipatif (voir §2.2) et l'existence d'une paire d'orbites fermées dont les representations de tresse ne sont pas enlacées (non-maximalles, voir Figure 1, à droite).*

1. Fourth order twist systems

Consider a second order Lagrangian system (L, dt), where $L = L(u, u', u'')$ is the Lagrangian. We assume that $L \in C^2(\mathbb{R}^3; \mathbb{R})$ satisfies the nondegeneracy hypothesis $\partial_w^2 L(u, v, w) \geq \delta > 0$. Our aim is to find bounded functions $u : \mathbb{R} \to \mathbb{R}$, which are stationary for the action integral $J[u] = \int L(u, u', u'')dt$. Such *bounded characteristics* satisfy the energy constraint

$$\Big(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\Big)u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u'') = E = \text{ constant.} \tag{1.1}$$

By transforming to a Hamiltonian context, one finds that characteristics reside on non-compact three-dimensional energy surfaces in $\mathbb{R}^4$. In this Note we describe the tools and perspectives necessary for the study of closed characteristics (closed orbits) on non-compact energy surfaces in the present context.

For a fixed regular[1] energy value E the extrema of a characteristic are contained in the closed set $\{u \mid L(u, 0, 0) + E \geq 0\}$ whose connected components I_E are called *interval components*. In order to set up a variational principle for bounded characteristics in terms of the extrema of u the following *twist hypothesis* was introduced in [6].

(T) $\inf\{J_E[u] := \int_0^\tau \big(L(u, u', u'') + E\big)dt \mid u \in X_\tau(u_1, u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimizer $u(t; u_1, u_2)$ for all $\{(u_1, u_2) \in I_E \times I_E \mid u_1 \neq u_2\}$, and u and τ are C^1-smooth functions of (u_1, u_2).*

Here $X_\tau = X_\tau(u_1, u_2) = \{u \in C^2([0, \tau]) \mid u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = u'(\tau) = 0$ and $u'|_{(0,\tau)} > 0$ if $u_1 < u_2$, and $u'|_{(0,\tau)} < 0$ if $u_1 > u_2\}$. Hypothesis (T) assumes that the monotone laps between minima and maxima are essentially unique. Numerical evidence indicates that this is a valid hypothesis for most second order Lagrangian systems. A nondegenerate second order Lagrangian system that satisfies (T) is called a *Twist system.*[2]

[1] I.e. $\frac{\partial L}{\partial u}(u, 0, 0) \neq 0$ for all u that satisfy $L(u, 0, 0) + E = 0$.

[2] The paper [6] proves (T) for a large class of Lagrangians L, and numerics suggest that (T) is satisfied on interval components of regular energy surfaces in general.

We will restrict here to two special cases: a compact component I_E or a component $I_E = \mathbb{R}$ with *dissipative* asymptotic behavior (for a definition see §2.2).

The twist hypothesis (T) allows one to encode any characteristic by its extrema $\{u_i\}$. Assume without loss of generality that u_1 is a local minimum. We can construct a piecewise linear (PL) graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ with straight line segments. If u is a closed characteristics then its critical points are encoded in a finite sequence $\{u_i\}_{i=1}^{2p}$, where $2p$ is the [discrete] period. The PL graph is really cyclic: one restricts to $1 \leq i \leq 2p+1$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n cyclic PL graphs. We place on these diagrams a *braid structure* by assigning a crossing type (positive) to every transversal intersection of the graphs. We thus project collections of periodic sequences of extrema to closed, positive, PL braid diagrams.

Theorem 1. *Consider Equation (1.1) for a regular energy level E under the twist hypothesis (T). The following are sufficient conditions for the existence of infinitely many distinct*[3] *closed characteristics.*

- *A compact interval component I_E and the existence of any closed orbit(s) whose braid diagram is nontrivial*[4] *(e.g., Figure 1, left); or,*
- *An infinite interval component $I_E = \mathbb{R}$ with dissipative asymptotic behavior and the existence of any pair of closed orbits whose braid representations are unlinked (not maximally linked, e.g., Figure 1, right).*

2. The primary ingredients

2.1. **Discretization of the variational principle.** We recast the problem of finding smooth periodic orbits for a given energy level E into solving second-order recurrence relations. This is accomplished via a method comparable to *broken geodesics*, which in the present context are concatenations of the monotone laps given by (T) (see [6]).

A closed characteristic u at energy level E is a (C^2-smooth) function $u: [0,\tau] \to \mathbb{R}$, $0 < \tau < \infty$, which is stationary for the action $J_E[u]$ with respect to variations $\delta u \in C^2_{\rm per}([0,\tau])$, and $\delta\tau \in \mathbb{R}^+$. Using (T), a broken geodesic $u: [0,\tau] \to \mathbb{R}$ is a closed characteristic at energy level E taking values in a fixed I_E if and only if the sequence of its extrema $\mathbf{u} = (u_i)$ satisfies $\nabla W_{2p}(u_i, \ldots, u_{i+2p}) = 0$, where $W_{2p}(\mathbf{u}) = \sum_{i=1}^{2p} S(u_i, u_{i+1})$, and $S(u_i, u_{i+1})$ is the action of a lap connecting u_i and u_{i+1}. This function S is a generating function and the functional W_{2p} is a discrete action defined on the space of $2p$-periodic sequences.

Critical points of W_{2p} satisfy the recurrence relation

$$\mathcal{R}(u_{i-1}, u_i, u_{i+1}) \stackrel{\rm def}{=} \partial_2 S(u_{i-1}, u_i) + \partial_1 S(u_i, u_{i+1}) = 0, \tag{2.1}$$

[3] In particular having distinct braid types: see §3.

[4] For example, any closed orbit of minimal period larger than two, or a pair of period-2 orbits which are linked.

and[5] can be found by analyzing the gradient flow $u_i' = \mathcal{R}(u_{i-1}, u_i, u_{i+1})$ on a space of sequences, where we may assume, without loss of generality, that $(-1)^{i+1}u_i < (-1)^{i+1}u_{i+1}$, for $u_i, u_{i+1} \in I_E$. In this context, the twist hypothesis (T) translates into the *twist property* for $\mathcal{R}$:

$$\partial_1 \mathcal{R} > 0 \quad \text{and} \quad \partial_3 \mathcal{R} > 0. \tag{2.2}$$

2.2. **Boundary conditions.** In order have a smooth flow on a compact space we consider two natural boundary conditions for the generating function S, which are derived from the behavior of S near $\partial(I_E \times I_E)$. In the compact case we can find a compact interval $I \subset I_E$ such that $\partial(I \times I)$ is repelling, and in case $I_E = \mathbb{R}$ we assume (the natural condition) that there exists a compact interval $I \subset \mathbb{R}$ such that $\partial(I \times I)$ is attracting (dissipativity assumption):

(C) [compact] large amplitudes are repelling; or
(D) [dissipative] large amplitudes are attracting.

2.3. **Braids.** Via the discussion in §1 the gradient flow of (2.1) on $2p$-periodic sequences immediately translates to a flow on the space of closed positive k-strand PL-braid diagrams, denoted $\overline{\mathcal{D}}_k$ for any $k \geq 1$, completed to include certain singular braid diagrams. Property (2.2) implies that the variational flow is transversally oriented on the singular braids:

Proposition 2. *The word metric in the braid group corresponds to a (weak) Lyapunov function for the variational flow on* $\overline{\mathcal{D}}_k$.

The strategy behind Theorem 1 is to construct isolating neighborhoods for the gradient flow of (2.1) on PL-braid diagrams and compute their Conley homology [3]. Non-trivial Conley homology implies the existence of closed characteristics.[6]

3. Morse-Conley theory on the space of PL-braid diagrams

Consider the special situation of $(n+1)$-strand braid diagrams where n designated strands, the *skeleton*, corresponds to a collection of closed characteristics. This induces a flow on a $2p$-dimensional invariant subset $\overline{\mathcal{D}}_{1,n}$ of $\overline{\mathcal{D}}_{n+1}$: the *relative braid diagrams.*

We can now use the relative braid types in $\overline{\mathcal{D}}_{1,n}$ to construct various isolating neighborhoods for the induced flow. The space $\overline{\mathcal{D}}_{n+1}$ of all $(n+1)$-strand positive PL-braid diagrams is partitioned into braid classes by codimension-1 "walls" of singular braids (cf. [7]). This also induces a partitioning of $\overline{\mathcal{D}}_{1,n}$. The equivalence classes of braid types in $\mathcal{D}_{1,n}$ are candidates for isolating neighborhoods.

Under (1) either boundary condition (C) or (D), and (2) braid classes for which the $(n+1)$st strand is non-isotopic to the skeleton (i.e., none of the strands of the skeleton is contained in the boundary), Proposition 2 implies that the closure of the

[5] See also [1] where similar recurrence relations are studied for twist maps of the annulus.

[6] Compare with [2] where the Conley homology of certain knot types is computed.

braid class is a proper isolating neighborhood for the induced flow. Consequently the Conley homology is well-defined.

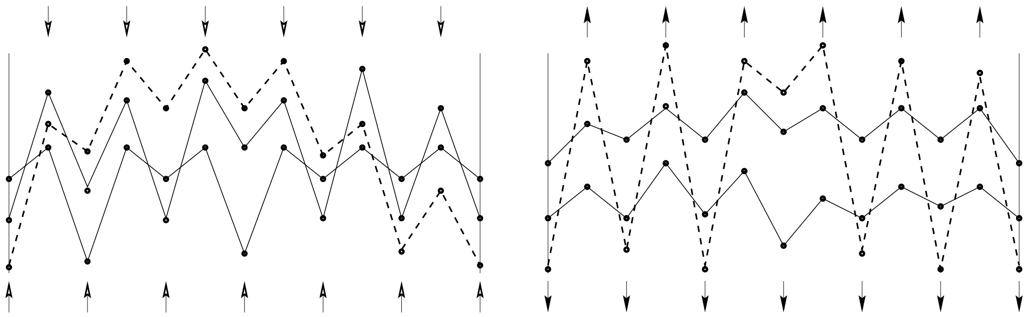

FIGURE 1. Two examples of solutions (dashed) whose Conley homology with respect to the fixed strands (solid) is nontrivial. Left: $X^r_{p,q}$, with $p = 6$, $r = 8$, and $q = 1$; Right: $Y^r_{p,q}$, with $p = 6$, $r = 0$, and $q = 5$.

In order prove the main theorem we carry out the above construction for two special braid classes: see Figure 1. In the compact case we consider a skeleton of two linked strands with period $2p \geq 4$ and crossing number $0 < r \leq 2p$. The third strand (dashed) has linking number q with the skeleton, with $0 < 2q < r \leq 2p$. We denote this braid class by $X^r_{p,q}$. In the dissipative case we consider a skeleton of two linked strands of period $2p$ with crossing number $0 \leq r < 2p$. The third strand (dashed) has linking number q with the skeleton, and satisfies $0 \leq r < 2q < 2p$. We denote this braid class by $Y^r_{p,q}$.

Proposition 3. *Consider the braid classes $X^r_{p,q}$ (with $0 < 2q < r \leq 2p$) and $Y^r_{p,q}$ (with $0 \leq r < 2q < 2p$) given in Figure 1. The Conley homology of the gradient flow of (2.1) on these braid classes is well-defined and is equal to*

$$CH_k(X^r_{p,q}) = \begin{cases} \mathbb{Z} & k = 2q-1, 2q \\ 0 & else \end{cases} \qquad CH_k(Y^r_{p,q}) = \begin{cases} \mathbb{Z} & k = 2q, 2q+1 \\ 0 & else \end{cases}$$

From this computation one easily constructs an infinite family of distinct braid types forced by the pair of not maximally linked orbits (under condition (D)) or linked orbits (condition (C)). Indeed, the above homology computation yields an infinite number of closed characteristics by taking coverings, and applying Proposition 3 iteratively, i.e. replacing p and r by $p' = kp$ and $r' = kr$, for any $k > 1$, and then projecting the orbits back to the original setting. However, many additional classes of closed characteristics can be found by considering more elaborate braid types (e.g. $q > 1$). This is the subject of ongoing research [4].

4. AN EXAMPLE: THE SWIFT-HOHENBERG MODEL

As an application of the main theorem we consider the so-called *Swift-Hohenberg model* that is used in various physical settings: phase-transitions, non-linear optics, shallow water waves, and amplitude equations. The Swift-Hohenberg model is defined via a Lagrangian of the form $L(u,u',u'') = \frac{1}{2}|u''|^2 + \frac{\alpha}{2}|u'|^2 + F(u)$, $\alpha \leq 0$. It is shown in [6] that the Swift-Hohenberg model satisfies the Twist property (T).

Suppose that F has *two* non-degenerate global minima, say at (singular) energy level E_0, and $F(u) \sim |u|^s$, with $s > 1$ as $|u| \to \infty$, for example $F(u) = \frac{1}{4}(u^2-1)^2$. Then there exists an $\epsilon > 0$ such that for each regular $E \in (E_0, E_0+\epsilon)$ there exist at least two unlinked simple closed characteristics. For all $E > E_0$, $I_E = \mathbb{R}$ and the dissipative boundary conditions are met for any sufficiently large subinterval $I \in \mathbb{R}$ (see [6]). Part 2 of Theorem 1 now yields an infinity of non-simple closed characteristics for all $E \in (E_0, E_0+\epsilon)$. These characteristics still exist in the limit $E = E_0$, $\alpha \leq 0$.

As for an example of the compact case, consider any F that has a compact interval component I_{E_0} for energy E_0, which contains an interior local minimum (non-degenerate) for F at E_0, for example $F(u) = -\frac{1}{4}(u^2-1)^2$ with $E_0 = \frac{1}{4}$. As before, there exists an $\epsilon > 0$ such that for each regular $E \in (E_0, E_0+\epsilon)$ there exists at least one non-simple closed characteristic. This follows from results obtained in [4, 5]. Part 1 of Theorem 1 now yields an infinity of non-simple closed characteristics for all $E \in (E_0, E_0+\epsilon)$, and also in the limit $E = E_0$, $\alpha \leq 0$.

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