More on the parameterization method for center manifolds

Jan Bouwe van den Berg*, Wouter Hetebrij† and Bob Rink‡

March 3, 2020

Abstract

In a previous paper, see [14], we generalized the parameterization method of Cabré, Fontich and De la Llave to center manifolds of discrete dynamical systems. In this paper, we extend this result to several different settings. The natural setting in which center manifolds occur is at bifurcations in dynamical systems with parameters. Our first results will show that we can find parameter-dependent center manifolds near bifurcation points. Furthermore, we will generalize the parameterization method to center manifolds of fixed points of ODEs. Finally, we will apply our method to a reaction diffusion equation. In our application, we will show that the freedom to obtain the conjugate dynamics in normal form makes it possible to obtain detailed qualitative information about the center dynamics.

1 Introduction

The parameterization method introduced by Cabré, Fontich and De la Llave in [4–6] is used to find (un)stable manifolds associated to hyperbolic equilibria in dynamical systems. In a previous paper, [14], we gave a generalization of the parameterization method which can be used to find center manifolds at fixed points of discrete dynamical systems.

The original method for (un)stable manifolds has been applied to delay differential equations, see [9], and partial differential equations, see [13]. Furthermore, the method is useful for computational existence proofs of for example homoclinic and heteroclinic orbits, see [3, 12, 16]. The method has also been used for constructing (un)stable manifolds of periodic orbits, see [7]. Finally, the method was generalized in [1, 2] to find invariant manifolds for parabolic fixed points.

One of the goals of this paper is to give a generalization of the parameterization method for center manifolds in systems with parameters. For a discrete dynamical system, $F : X \to X$ on a Banach space X, the parameterization

*Department of Mathematics, Vrije Universiteit Amsterdam, The Netherlands, janbouwe@few.vu.nl; partially supported by NWO-VICI grant 639033109

†Department of Mathematics, Vrije Universiteit Amsterdam, The Netherlands, w.a.hetebrij@vu.nl.

‡Department of Mathematics, Vrije Universiteit Amsterdam, The Netherlands, b.w.rink@vu.nl.

method constructs a conjugacy K, the parameterization of the center manifold, between the center subspace X_c and the the center manifold, as well as a conjugate dynamical system $R : X_c \to X_c$ on the center subspace such that $K(X_c)$ lies tangent to the center subspace and the conjugacy equation

$$F \circ K = K \circ R \tag{1.0.1}$$

is satisfied. In other words, orbits of R on the center subspace are mapped by K to orbits of F on the center manifold.

To obtain K and R such that (1.0.1) holds, in [14] we first rewrite (1.0.1) as a fixed point problem of the form $(K, R) = \Theta(K, R)$. We prove that Θ is a contraction on a suitable function space to prove the existence of both K and R. We refer the reader to [14] for the details of this proof.

The first generalization we present here is finding a smooth parameter-dependent conjugacy for a dynamical system with parameters. Center manifolds naturally occur at bifurcations, and the change in dynamical behaviour before and after the bifurcation takes place on the center manifold. Thus finding a parameter-dependent center manifold at the bifurcation point allows us to quantitatively describe changes in dynamical behaviour. Using normal form theory, see for instance [8], one can also obtain qualitative information about the dynamical behaviour near the bifurcation point. The main advantage of our method is that we have the freedom to obtain the Taylor approximation of the conjugate dynamical system in normal form and at the same time obtain explicit bounds on the difference between the conjugate dynamical system and the normal form. Those explicit bounds allow us to obtain quantitative information about the dynamical behaviour near the bifurcation point. In the proof, we will extend the original dynamical system by adding the parameters as new variables. The main challenge in the proof of this generalization consists of choosing the right norm on the extended Banach space $\Lambda \times X$, where Λ is our parameter space.

We also want to generalize our method to center manifold in ODEs, i.e continuous time dynamical systems. For continuous time dynamical systems given by an ODE $\dot{x} = f(x)$, we want to construct a conjugacy K between the center subspace and the center manifold as well as a vector field $\dot{x} = R(x)$ on the center subspace such that orbits of R are mapped by K to orbits of f, analogous to what we did for discrete dynamical systems. This means that if we take a solution $y(t)$ of the ODE $\dot{x} = R(x)$ on the center subspace, $K(y(t))$ should be a solution of the ODE $\dot{x} = f(x)$ on the center manifold, i.e.

$$f(K(y(t)) = DK(y(t)) \cdot R(y(t)). \tag{1.0.2}$$

The main difference with the conjugacy equation (1.0.1) of the discrete case, is the spatial derivative of K at the right hand side of (1.0.2). In the discrete case we rewrote (1.0.1) as a fixed point problem, $(K, R) = \Theta(K, R)$, and showed that Θ is a contraction. If we would use the same strategy in the continuous case, we would try to rewrite (1.0.2) as a fixed point problem $(K, R) = \Theta(K, R)$. However, we now encounter two problems we have to overcome. The center manifold, and therefore K, that we obtain will only be C^n, whereas Θ is a differential operator. Thus we must find a suitable function space A on which Θ is well-defined, as well as define a norm on A and find a set $B \subset A$ such that

Θ is a contraction on B. To circumvent those two problems, we will solve an equivalent problem to obtain K and R.

If K and R solve (1.0.2), then K is also a conjugacy between the time t-map on the center subspace and the time t-map on the center manifold, for any $t \geq 0$, where we will denote the latter time t-map by φ_t. Conversely, if we find a single conjugacy K and a semigroup of discrete dynamical system $\{\psi_t\}_{t\geq 0}$ on the center subspace such that

$$K \circ \psi_t = \varphi_t \circ K \qquad \text{for all } t \geq 0,$$

then we choose R to be the infinitesimal generator of $\{\psi_t\}_{t\geq 0}$ and K maps orbits on the center subspace to orbits on the center manifold., i.e. K and R solve (1.0.2) The main advantage of this equivalence is that the time t-map of f is a discrete dynamical system, for which we can use our parameterization method from [14]. On the other hand, instead of finding a conjugacy and a conjugate dynamical system for one dynamical system, we need to find a collection of conjugate dynamical systems and a single conjugacy (independent of t) for a collection of dynamical systems. Hence we will prove that the conjugacy we obtain for a single time t-map of f will be a conjugacy for *all* $t > 0$ for a judicious choice of ψ_t.

We will demonstrate (the strength of) our method with an application which is simple yet illustrates the essential steps involved. For our application, we consider the spatial dynamics of a reaction diffusion equation on a 2D grid. By varying a parameter in the system, a period-doubling bifurcation occurs. Using the freedom to obtain the conjugate dynamics in normal form, and by computing explicit bounds on the difference between the normal form and the conjugate dynamics, we obtain explicit regions in the phase space in which a period-2 orbit must lie after the bifurcation. To obtain the explicit error bounds and subsequently the explicit regions of validity, we use the Mathematica Notebook available in [15]. Furthermore, using the explicit error bounds, we also prove that a heteroclinic orbit emerges between the period-2 orbit and the stationary point.

Outline of the paper

The paper consists of three parts. We first introduce notation and restate our main theorem from [14]. In Section 2 we will give the generalization of the parameterization method to dynamical systems with parameters. In addition to this first generalization, we will also show how one can compute explicit error bounds for the Taylor approximations of K and R. We then continue in Section 3 with the generalization to center manifolds in continuous time dynamical systems. Our proof will use the equivalent problem of finding a conjugate dynamical systems for the time t-maps and a single conjugacy, which is done using multiple intermediate steps. Finally, we conclude the paper in Section 4 with the application of the parameterization method to obtain a period doubling-bifurcation in a reaction diffusion equation.

1.1 Notation and conventions

We use the following notation and conventions in this paper.

– For functions $f : X \to Y$ between Banach spaces, we denote with

$$\|f\|_n := \max_{0\leq m\leq n} \sup_{x\in X} \|D^m f(x)\|$$

the C^n norm of f for $n \geq 0$.

For X and Y Banach spaces, we denote with

$$C_b^n(X,Y) := \{f : X \to Y \mid f \text{ is } C^n \text{ and } \|f\|_n < \infty\}.$$

the Banach space of all C^n bounded functions between X and Y.

– For a bounded linear operator $A : X \to Y$ between Banach spaces, we denote with

$$\|A\|_{\text{op}} := \sup_{\|x\|=1} \|Ax\|$$

the operator norm of A.

For X and Y Banach spaces, we denote with

$$\mathcal{L}(X,Y) := \{A : X \to Y \mid A \text{ is a linear operator and } \|A\|_{\text{op}} < \infty\}$$

the Banach space of all bounded linear operators between X and Y.

– For an unbounded linear operator $A : \mathcal{D}(A) \subset X \to Y$ between Banach spaces, we denote with $\mathcal{D}(A)$ its domain, i.e. $x \in \mathcal{D}(A)$ if and only if Ax exists and lies in Y.

Furthermore, we call an unbounded linear operator A sectorial if it satisfies the following three properties.

  – The operator A is closed, i.e. the graph of A is closed in $X \times Y$, and densely defined, i.e. $\mathcal{D}(A)$ is dense in X.
  – There exist constants $w \in \mathbb{R}$ and $\lambda > 0$ such that the spectrum $\sigma(A)$ is contained in the sector $\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < \lambda \operatorname{Re}(w - z)\}$.
  – There exists a constant $C > 0$ such that for all $z \in \mathbb{C}$ outside the sector $\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < \lambda \operatorname{Re}(w - z)\}$ the linear operator $(A - z\operatorname{Id})^{-1}$ is bounded by $C/|z - w|$.

– Let $\varepsilon > 0$ and $U \subset \mathbb{R}^m$. We denote with $U^\varepsilon := \{x \in \mathbb{R}^m \mid \operatorname{dist}(x,U) < \varepsilon\}$ the ε-neighborhood of U.

1.2 Parameterization theorem for center manifolds

For the sake of completeness, we will repeat the statement of the paramaterization method for center manifolds for discrete systems in [14].

Theorem 1.1 (Parameterization of the center manifold)**.** *Let X be a Banach space and $F : X \to X$ a C^n, $n \geq 2$, discrete dynamical system on X such that* 0 *is a fixed point of F. Denote $F = A + g$ with $A := Df(0)$ and let $k_c : X_c \to X_c$ be chosen. Assume that*

1. *There exist closed A-invariant subspaces X_c, X_u and X_s such that $X = X_c \oplus X_u \oplus X_s$. We write $A = \begin{pmatrix} A_c & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_s \end{pmatrix}$ where we define $A_c := A\big|_{X_c}$, and similarly define A_u and A_s.*

1a. *The norm on $X = X_c \oplus X_u \oplus X_h$ satisfies*

$$\|x\| = \max\{\|x_c\|_c, \|x_u\|_u, \|x_s\|_s\}, \qquad \text{for } x = (x_c, x_u, x_s) \tag{1.2.1}$$

where $x_i \in X_i$ and $\|\cdot\|_i$ is the norm on X_i for $i = c, u, s$.

2. *The linear operators A_c and A_u are invertible.*

3. *The norm on X is such that*

$$\|A_c^{-1}\|_{\mathrm{op}}^{\tilde{n}} \|A_s\|_{\mathrm{op}} < 1 \quad \text{and} \quad \|A_u^{-1}\|_{\mathrm{op}} \|A_c\|_{\mathrm{op}}^{\tilde{n}} < 1 \quad \text{for all } 1 \leq \tilde{n} \leq n.$$

4. *The non-linearities g and k_c satisfy*

$$g \in \{h \in C_b^n(X, X) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 < L_g\},$$
$$k_c \in \{h \in C_b^n(X_c, X_c) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 < L_c\},$$

for L_g and L_c small enough, as defined in Remark 2.4 of [14].

Then there exist a C^n conjugacy $K : X_c \to X$ and C^n discrete dynamical system $R = A_c + r : X_c \to X_c$ such that

$$(A + g) \circ K = K \circ (A_c + r). \tag{1.2.2}$$

Furthermore, $A_c + r$ is globally invertible and $K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$ with $\iota : X_c \to X$ the inclusion map.

Remark 1.2. If F is C^n with respect to an arbitrary norm on X and condition 1 is satisfied, we can define an equivalent norm on X which satisfies condition 1a and leaves the norm unchanged on X_c, X_u and X_s. The reason for asking condition 1a is that it allows us to make our estimates explicit.

Remark 1.3. The original version of Theorem 1.1 states condition 4 as:

4. The non-linearities g and k_c satisfy

$$g \in \{h \in C_b^n(X, X) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \leq L_g\},$$
$$k_c \in \{h \in C_b^n(X_c, X_c) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \leq L_c\},$$

for L_g and L_c small enough.

Hence it seems like our statement of the theorem is slightly weaker than in [14]. However L_g and L_c are defined in terms of strict inequalities. Thus we can replace the less than or equal to signs in condition 4 with strict inequalities, if we also replace the strict inequalities in the definition of L_g and L_c with less than or equal to signs.

Remark 1.4. For a general dynamical system, the non-linearity g will be unbounded in C^n. To satisfy the fourth assumption, the usual approach in $\mathbb{R}^n$ is to multiply g with a smooth cut-off function ξ, such that $g\xi$ is bounded in C^n. By shrinking the support of the cut-off function, the norm of the derivative of $g\xi$ can then be made as small as desired. If we apply Theorem 1.1 to the map $x \mapsto Ax + g(x)\xi(x)$, we find a local center manifold for our original system, which is valid on the region where $\xi \equiv 1$. In Section 4 we will see a different trick to bound the derivative of g.

2 Parameter-dependent systems

Center manifolds naturally occur at bifurcations, which can happen in dynamical systems with parameters. Hence a natural setting to which we want to generalize Theorem 1.1, are discrete dynamical systems with parameters. For simplicity, we will assume without loss of generality that for $\lambda = 0$ there exists a center manifold.

Theorem 2.1. *Let X be a Banach space, $W \subset \mathbb{R}^m$ open and $F : W \times X \to X$ a jointly C^n, $n \geq 2$, discrete dynamical system on X with parameter space $0 \in W$ such that 0 is a fixed point of $F_0(\cdot) := F(0, \cdot)$. Denote $F(\lambda, x) = Ax + C\lambda + g_\lambda$ with $A := D_xF(0, 0)$ and $C := D_\lambda F(0, 0)$. Let $k_c : W \times X_c \to X_c$ be chosen such that $k_c(0, 0) = 0$ and $D_\lambda k_c(0, 0) = 0$. Assume that F_0 satisfies the conditions of Theorem 1.1 where we choose $k_{c,0}(\cdot) := k_c(0, \cdot) : X_c \to X_c$, with condition 3 replaced by:*

3a. The norm on X is such that

$$\max\{1, \|A_c^{-1}\|_{\mathrm{op}}\}^n \|A_s\|_{\mathrm{op}} < 1 \quad \text{and} \quad \|A_u^{-1}\|_{\mathrm{op}} \max\{1, \|A_c\|_{\mathrm{op}}\}^n < 1.$$

Furthermore, we assume that

5. Let $\tilde{W} \subset W$ and $\varepsilon > 0$ be such that $\tilde{W}^\varepsilon \subset W$ and

$$g \in \left\{ h \in C_b^n(\tilde{W}^\varepsilon \times X, X) \;\middle|\; \sup_{\lambda \in \tilde{W}^\varepsilon} \|D_x h(\lambda, \cdot)\|_0 < L_g \right\},$$

$$k_c \in \left\{ h \in C_b^n(\tilde{W}^\varepsilon \times X, X) \;\middle|\; \sup_{\lambda \in \tilde{W}^\varepsilon} \|D_x h(\lambda, \cdot)\|_0 < L_c \right\},$$

for the L_g and L_c from Theorem 1.1 for F_0.

Then there exist a jointly C^n conjugacy $K : \tilde{W} \times X_c \to X$ and jointly C^n discrete dynamical system $R_\lambda = A_c + r_\lambda : \tilde{W} \times X_c \to X_c$ such that

$$(A + g_\lambda) \circ K_\lambda = K_\lambda \circ (A_c + r_\lambda). \tag{2.0.1}$$

for all $\lambda \in \tilde{W}$. Furthermore, $A_c + r_\lambda$ is globally invertible and $K_\lambda = \iota + \begin{pmatrix} k_{c,\lambda} \\ k_{u,\lambda} \\ k_{s,\lambda} \end{pmatrix}$ with $\iota : X_c \to X$ the inclusion map for all $\lambda \in \tilde{W}$.

Remark 2.2. All norms on $\mathbb{R}^m$ are equivalent, and thus if $F : W \times X \to X$ is smooth with respect to any norm on $\mathbb{R}^m$, it is smooth with respect to all norms on $\mathbb{R}^m$. Similarly, if $F : W \times X \to X$ is bounded in C^n with respect to any norm on $\mathbb{R}^m$, it is bounded with respect to all norms. However, the C^n norm of F does depend on the norm of $\mathbb{R}^m$.

Remark 2.3. From condition 3a it follows that neither A_u nor A_s contains 1 in their spectrum. That means that $\mathrm{Id} - A_u$ and $\mathrm{Id} - A_s$ are both invertible. In particular, we have for $(x_c, x_u, x_s) \in X$

$$\begin{pmatrix} x_c \\ x_u \\ x_s \end{pmatrix} + A \begin{pmatrix} 0 \\ (\mathrm{Id} - A_u)^{-1} x_u \\ (\mathrm{Id} - A_s)^{-1} x_s \end{pmatrix} = \begin{pmatrix} 0 \\ (\mathrm{Id} - A_u)^{-1} x_u \\ (\mathrm{Id} - A_s)^{-1} x_s \end{pmatrix} + \begin{pmatrix} x_c \\ 0 \\ 0 \end{pmatrix}.$$

That is, for every $z \in X$, we can find $y \in X_c$ and $x \in X_u \oplus X_s$ such that $z + Ax = x + y$.

Proof. The idea behind the proof is to create a dynamical system $\tilde{F} : W \times X \to W \times X$ by considering the parameters as extra variables, and apply Theorem 1.1 to this new dynamical system. However, if $W \subsetneq \mathbb{R}^m$, Theorem 1.1 cannot be applied as $W \times X$ is not a Banach space. So we first have to extend $\tilde{F}$ to a dynamical system on $\mathbb{R}^m \times X$.

Let $\varepsilon > 0$ and $\tilde{W}$ be such that condition 5 is satisfied. We consider a smooth cut-off function $\xi : \mathbb{R}^m \to [0, 1]$ such that $\xi \equiv 1$ on $\tilde{W}$, and its support lies in $\tilde{W}^\varepsilon$. Consider the dynamical system

$$\tilde{F} : Y := \mathbb{R}^m \times X \to Y, (\lambda, x) \mapsto \begin{pmatrix} \mathrm{Id} & 0 \\ C & A \end{pmatrix} \begin{pmatrix} \lambda \\ x \end{pmatrix} + \begin{pmatrix} 0 \\ g_\lambda(x)\xi(\lambda) \end{pmatrix}, \tag{2.0.2}$$

with the convention that $g_\lambda(x)\xi(\lambda) = 0$ outside W. In particular we have that $\tilde{F} \equiv (\mathrm{Id}, F_\lambda)$ on $\tilde{W} \times X$.

To make Y a Banach space, we have to define a norm on Y. Denote the norm on $\mathbb{R}^m$ with $\|\cdot\|_{\mathbb{R}^m}$, which we can choose freely, and the norm on X with $\|\cdot\|_X$. We equip Y with the supremum norm $\|(\lambda, x)\|_{\tilde{Y}} := \max\{\|\lambda\|_{\mathbb{R}^m}, \|x\|_X\}$. We now want to apply Theorem 1.1 to $\tilde{F}$. Hence we want to use the norm $\|\cdot\|_{\tilde{Y}}$ to check the smoothness of $\tilde{F}$ and find the invariant subspaces Y_c, Y_u and Y_s as in assumption 1 of Theorem 1.1. We then define an equivalent norm $\|\cdot\|_Y$ on Y such that condition 1a of Theorem 1.1 is satisfied, see (2.0.7).

<u>Smoothness and linearization:</u> We first have to show that $\tilde{F}$ is C^n. The linear part of $\tilde{F}$ is smooth, and the non-linear part is C^n, since $g_\lambda : W \times X \to X$ is jointly C^n by assumption and ξ is smooth by construction. Thus their product is also C^n. Furthermore, $(0, 0)$ is a fixed point of $\tilde{F}$ and the derivative of $\tilde{F}$ at $(0, 0)$ is given by

$$\tilde{A} := D\tilde{F}(0, 0) = \begin{pmatrix} \mathrm{Id} & 0 \\ C & A \end{pmatrix}, \tag{2.0.3}$$

since $g(0, 0) = 0$, $D_\lambda g(0, 0) = 0$ and $D_x g(0, 0) = 0$. In particular we see that our non-linearity is indeed $\tilde{g}(\lambda, x) := (0, g_\lambda(x)\xi(\lambda))$.

Invariant subspaces: For the first condition of Theorem 1.1 we need to define closed $\tilde{A}$-invariant subspaces Y_c, Y_u and Y_s. Let $(e_i)_{i=1}^m \subset \mathbb{R}^m$ be a basis of $\mathbb{R}^m$, and denote $C_i = Ce_i$. By Remark 2.3, we can find elements $(x_i)_{i=1}^m \subset X_u \oplus X_s$ and $(y_i)_{i=1}^m \subset X_c$ such that $C_i + Ax_i = x_i + y_i$. We define the closed subspaces

$$Y_c := \operatorname{span}\left(\{(0,x) \in Y \mid x \in X_c\}, \{(e_i, x_i) \mid 1 \leq i \leq m\}\right), \tag{2.0.4}$$

$$Y_u := \{(0,x) \in Y \mid x \in X_u\}, \tag{2.0.5}$$

$$Y_s := \{(0,x) \in Y \mid x \in X_s\}. \tag{2.0.6}$$

Since $X_c \oplus X_u \oplus X_s = X$, we have $Y_c \oplus Y_u \oplus Y_s = \mathbb{R}^m \times X = Y$. Furthermore, by construction we have that Y_u and Y_s are invariant under $\tilde{A}$, and for an element $y = (0,x) + \sum_{i=1}^m \mu_i(e_i, x_i) \in Y_c$, with $x \in X_c$, we have

$$\tilde{A}y = \begin{pmatrix} \mathrm{Id} & 0 \\ C & A \end{pmatrix}\begin{pmatrix} 0 \\ x \end{pmatrix} + \sum_{i=1}^m \mu_i \begin{pmatrix} \mathrm{Id} & 0 \\ C & A \end{pmatrix}\begin{pmatrix} e_i \\ x_i \end{pmatrix} = \begin{pmatrix} 0 \\ Ax \end{pmatrix} + \sum_{i=1}^m \mu_i \begin{pmatrix} e_i \\ C_i + Ax_i \end{pmatrix}.$$

By definition, X_c is invariant under A, thus $(0, Ax) \in Y_c$. By construction, we have $(e_i, C_i + Ax_i) = (e_i, x_i + y_i) = (e_i, x_i) + (0, y_i)$. As $y_i \in X_c$, we have $(e_i, x_i) + (0, y_i) \in Y_c$. Now that we have found our invariant subspaces, we can define a norm $\|\cdot\|_Y$ on Y which is equivalent with $\|\cdot\|_{\tilde{Y}}$ and satisfies condition 1a of Theorem 1.1:

$$\|(\lambda, x + \tau(\lambda))\|_Y := \max\{\|\lambda\|_{\mathbb{R}^m}, \|x\|_X\}, \tag{2.0.7}$$

where $\tau(\lambda) = \sum_{i=1}^m \lambda_i x_i$ when $\lambda = \sum_{i=1}^m \lambda_i e_i$. In particular, we have that $\tilde{A}_u = A_u$ and $\tilde{A}_s = A_s$, and their operator norms are equal given the norm $\|\cdot\|_Y$ on Y.

Invertibility: For the second condition of Theorem 1.1, we have to check that $\tilde{A}$ restricted to Y_c or Y_u is invertible. As we already noted, $\tilde{A}_u = A_u$, which is invertible. One can check that the inverse of $\tilde{A}_c$ is defined by

$$\begin{cases} \tilde{A}_c^{-1}\begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ A_c^{-1}x \end{pmatrix} & \text{for } x \in X_c, \\ \tilde{A}_c^{-1}\begin{pmatrix} e_i \\ x_i \end{pmatrix} = \begin{pmatrix} e_i \\ x_i \end{pmatrix} - \begin{pmatrix} 0 \\ A_c^{-1}y_i \end{pmatrix} & 1 \leq i \leq m. \end{cases}$$

Hence we have to show that $\tilde{A}_c^{-1}$ defines a bounded linear map. We want to reiterate that we can choose the norm on $\mathbb{R}^m$ since all norms on $\mathbb{R}^m$ are equivalent. Let $M > 0$, and define

$$\left\| \sum_{i=1}^m \lambda_i e_i \right\|_{\mathbb{R}^m} := M \sum_{i=1}^m |\lambda_i|. \tag{2.0.8}$$

We will not give an explicit value for M, but we will argue during the proof that the conditions of Theorem 1.1 are satisfied for M large enough. The norm of $y = (0,x) + \sum_{i=1}^m \lambda_i(e_i, x_i)$ is given by

$$\|y\|_Y = \max\left\{ M \sum_{i=1}^m |\lambda_i|, \|x\|_X \right\}.$$

Let $y \in Y_c$ be inside the unit ball, i.e. $M\sum_{i=1}^m |\lambda_i| \leq 1$ and $\|x\|_X \leq 1$. Then we estimate

$$
\begin{aligned}
\|\tilde{A}_c^{-1}y\|_Y &= \left\| \begin{pmatrix} \sum_{i=1}^m \lambda_i e_i \\ A_c^{-1}x + \sum_{i=1}^m \lambda_i x_i - \sum_{i=1}^m \lambda_i A_c^{-1} y_i \end{pmatrix} \right\|_Y \\
&= \max\left\{ M\sum_{i=1}^m |\lambda_i|, \left\| A_c^{-1}x - \sum_{i=1}^m \lambda_i A_c^{-1} y_i \right\|_X \right\} \\
&\leq \max\left\{ 1, \|A_c^{-1}\|_{\mathrm{op}} \|x\|_X + \sum_{i\in I} |\lambda_i| \|A_c^{-1}\|_{\mathrm{op}} \|y_i\|_X \right\} \\
&\leq \max\left\{1, \|A_c^{-1}\|_{\mathrm{op}} + M^{-1}\|A_c^{-1}\|_{\mathrm{op}} \mathcal{C}_y \right\}.
\end{aligned} \tag{2.0.9}
$$

Here we define $\mathcal{C}_y := \max_{1\leq i\leq m}\{\|y_i\|_X\}$ in the last inequality, and we use that $\sum_{i=1}^m |\lambda_i| \leq 1/M$. Thus $\tilde{A}_c^{-1}$ is a bounded linear operator, and hence $\tilde{A}_c$ is invertible, which shows that the second condition of Theorem 1.1 is satisfied.

<u>*Bounds on the linearization:*</u> For the third condition of Theorem 1.1, we have to estimate the operator norm of $\tilde{A}_c$, as we have already computed a bound for the operator norm of $\tilde{A}_c^{-1}$. For an element $y \in Y_c$ in the unit ball, we have

$$
\begin{aligned}
\|\tilde{A}_c y\|_Y &= \left\| \begin{pmatrix} \sum_{i=1}^m \lambda_i e_i \\ A_c x + \sum_{i=1}^m \lambda_i x_i + \sum_{i=1}^m \lambda_i y_i \end{pmatrix} \right\|_Y \\
&= \max\left\{ M\sum_{i=1}^m |\lambda_i|, \left\| A_c x + \sum_{i=1}^m \lambda_i y_i \right\|_X \right\} \\
&\leq \max\left\{ 1, \|A_c\|_{\mathrm{op}} \|x\|_X + \sum_{i=1}^m |\lambda_i| \|y_i\|_X \right\} \\
&\leq \max\left\{1, \|A_c\|_{\mathrm{op}} + M^{-1}\mathcal{C}_y \right\}.
\end{aligned} \tag{2.0.10}
$$

Since we estimate both $\|\tilde{A}_c\|_{\mathrm{op}}$ and $\|\tilde{A}_c^{-1}\|_{\mathrm{op}}$ by at least 1, it is enough to check the third assumption for $\tilde{n} = n$. From assumption 3a we have

$$
\max\{1, \|A_c^{-1}\|_{\mathrm{op}}\}^n \|A_s\|_{\mathrm{op}} < 1 \qquad \text{and} \qquad \|A_u^{-1}\|_{\mathrm{op}} \max\{1, \|A_c\|_{\mathrm{op}}\}^n < 1.
$$

Therefore, there exists an $\varepsilon > 0$ such that also

$$
\max\{1, \|A_c^{-1}\|_{\mathrm{op}} + \varepsilon\}^n \|A_s\|_{\mathrm{op}} < 1 \qquad \text{and} \qquad \|A_u^{-1}\|_{\mathrm{op}} \max\{1, \|A_c\|_{\mathrm{op}} + \varepsilon\}^n < 1. \tag{2.0.11}
$$

In particular, we take M big enough, such that from (2.0.9) and (2.0.10) we obtain

$$
\|\tilde{A}_c^{-1}\|_{\mathrm{op}} \leq \max\{1, \|A_c\|_{\mathrm{op}} + \varepsilon\} \qquad \text{and} \qquad \|\tilde{A}_c\|_{\mathrm{op}} \leq \max\{1, \|A_c\|_{\mathrm{op}} + \varepsilon\}. \tag{2.0.12}
$$

Together with the fact that $\|\tilde{A}_u^{-1}\|_{\mathrm{op}} = \|A_u^{-1}\|_{\mathrm{op}}$ and $\|\tilde{A}_s\|_{\mathrm{op}} = \|A_s\|_{\mathrm{op}}$, we obtain from (2.0.11) and (2.0.12)

$$
\|\tilde{A}_c^{-1}\|_{\mathrm{op}}^{\tilde{n}} \|\tilde{A}_s\|_{\mathrm{op}} < 1 \qquad \text{and} \qquad \|\tilde{A}_u^{-1}\|_{\mathrm{op}} \|\tilde{A}_c\|_{\mathrm{op}}^{\tilde{n}} < 1 \qquad \text{for all } 1 \leq \tilde{n} \leq n,
$$

where we used that $\|\tilde{A}_c^{-1}\|_{\mathrm{op}}^{\tilde{n}} \leq \|\tilde{A}_c^{-1}\|_{\mathrm{op}}^{n}$ and $\|\tilde{A}_c\|_{\mathrm{op}}^{\tilde{n}} \leq \|\tilde{A}_c\|_{\mathrm{op}}^{n}$ for all $1 \leq \tilde{n} \leq n$. Thus also the third assumption of Theorem 1.1 is satisfied.

Bounds on the non-linearities: Finally, we have to check that fourth condition of Theorem 1.1 is satisfied for $\tilde{F}$. Recall that the non-linearity of $\tilde{F}$ is given by

$$\tilde{g}(\lambda, x) = (0, g_\lambda(x)\xi(\lambda)),$$

where $\xi : \mathbb{R}^m \to [0,1]$ is a cut-off function with its support on $\tilde{W}^\varepsilon$. Furthermore, we choose

$$\tilde{k}_c : Y_c \to Y_c, (0, x) + \sum_{i=1}^{m} \lambda_i(e_i, x_i) \mapsto (0, k_{c,\lambda}(x)\xi(\lambda)),$$

with ξ the same cut-off function and $\lambda = \sum_{i=1}^m \lambda_i e_i$. We will only check that $\|D\tilde{g}\|_0 < L_g$, as checking $\|D\tilde{k}_c\|_0 < L_c$ will go analogously. We have

$$\|D\tilde{g}\|_0 = \sup_{\substack{\lambda\in\mathbb{R}^m\\ x\in X}} \|D\tilde{g}(\lambda, x)\|_{\mathrm{op}} = \sup_{\substack{\lambda\in\mathbb{R}^m\\ x\in X}} \left\| \begin{pmatrix} 0 & 0 \\ B(\lambda, x) & D_x g_\lambda(x)\xi(\lambda) \end{pmatrix} \right\|_{\mathrm{op}}, \quad (2.0.13)$$

where we introduce

$$B(\lambda, x) := D_\lambda \tilde{g}(\lambda, x) = \xi(\lambda) D_\lambda g(\lambda, x) + g_\lambda(x) \frac{\partial \xi}{\partial \lambda}(\lambda). \quad (2.0.14)$$

Recall that $\xi \in [0,1]$ has support on W^ε and we defined $\tau(\mu) = \sum_{i=1}^m \mu_i x_i$ for $\mu = \sum_{i=1}^m \mu_i e_i$. We estimate (2.0.13) by

$$\begin{aligned}
\|D\tilde{g}\|_0 &= \sup_{\substack{\lambda\in\mathbb{R}^m\\ x\in X}} \sup_{\substack{\|\mu\|_{\mathbb{R}^m}\leq 1\\ \|y-\tau(\mu)\|_X\leq 1}} \left\| \begin{pmatrix} 0 & 0 \\ B(\lambda, x) & D_x g_\lambda(x)\xi(\lambda) \end{pmatrix} \begin{pmatrix} \mu \\ y \end{pmatrix} \right\|_Y \\
&\leq \sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}} \sup_{\|\mu\|_{\mathbb{R}^m}\leq 1} \|B(\lambda, x)\mu\|_X + \sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}} \sup_{\substack{\|\mu\|_{\mathbb{R}^m}\leq 1\\ \|y-\tau(\mu)\|_X\leq 1}} \|D_x g_\lambda(x)\xi(\lambda) y\|_X \\
&\leq \sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}} \sup_{\|\mu\|_{\mathbb{R}^m}\leq 1} \|B(\lambda, x)\mu\|_X + \sup_{\lambda\in\tilde{W}^\varepsilon} \|D_x g_\lambda\|_0 \left(1 + M^{-1}\mathcal{C}_x\right), \quad (2.0.15)
\end{aligned}$$

where we used the triangle inequality in the last line to estimate

$$\|y\|_X \leq \|y - y(\mu)\|_X + \|y(\mu)\|_X \leq 1 + \sum_{i=1}^m |\mu_i| \|x_i\|_X \leq 1 + M^{-1}\mathcal{C}_x.$$

for all $\mu \in \mathbb{R}^m$ and $y \in X$ such that $\|y - \tau(\mu)\|_X \leq 1$ and $\|\mu\|_{\mathbb{R}^m} \leq 1$. Here we introduce the constant $\mathcal{C}_x := \max_{1\leq i\leq m}\{\|x_i\|_X\}$. From assumption 5 we have that $\sup_{\lambda\in\tilde{W}^\varepsilon} \|D_x g_\lambda\|_0 < L_g$. Thus we must show that we can make $\|B(\lambda, x)\|_{\mathrm{op}}$ arbitrary small uniformly on $\tilde{W}^\varepsilon \times X$. From the definition of B in (2.0.14) we obtain

$$\begin{aligned}
&\sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}}\ \sup_{\|\mu\|_{\mathbb{R}^m}\le 1}\|B(\lambda,x)\mu\|_X\\
&\qquad\le \sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}}\ \sup_{\|\mu\|_{\mathbb{R}^m}\le 1}\|\xi(\lambda)D_\lambda g(\lambda,x)\mu\|_X+\left\|g_\lambda(x)\frac{\partial\xi}{\partial\lambda}(\lambda)\mu\right\|_X\\
&\qquad\le \sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}}\|D_\lambda g(\lambda,x)\|_{\text{op}}+\sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}}\|g(\lambda,x)\|\left\|\frac{\partial\xi}{\partial\lambda}\right\|_{\text{op}}. \qquad (2.0.16)
\end{aligned}$$

Here we used that $\xi(\lambda)\in[0,1]$ for $\lambda\in\tilde{W}^\varepsilon$. From condition 5 it follows that both $D_\lambda g$ and g are uniformly bounded on $\tilde{W}^\varepsilon\times X$. We need to show that we can take the norm $\|\cdot\|_{\mathbb{R}^m}$ such that both terms of (2.0.16) are sufficiently small. As we mentioned in Remark 2.2, the operator norm of $D_\lambda g$ depends on the norm on $\mathbb{R}^m$. Recall that our norm is defined as $\|\sum_{i=1}\mu_i e_i\|_{\mathbb{R}^m}=M\sum_{i=1}^m|\mu_i|$, then there exists a constant $\mathcal{C}$ independent of M such that

$$\|D_\lambda g(\lambda,x)\|_{\text{op}}=\frac{\mathcal{C}}{M}.$$

Likewise, we have that the operator norm of $\frac{\partial\xi}{\partial\lambda}$ scales as $1/M$. Furthermore, we remark that the supremum norm of g does not depend on the norm on $\mathbb{R}^m$. Hence we can make (2.0.16) as small as desired if we take M sufficiently large. Therefore, we have a constant $\delta(M)\downarrow 0$ as $M\to\infty$ such that

$$\sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}}\ \sup_{\|\mu\|_{\mathbb{R}^m}\le 1}\|B(\lambda,x)\mu\|_X\le\delta(M).$$

Simultaneously we have that $1+M^{-1}\mathcal{C}_x\downarrow 1$ as $M\to\infty$. Together with the strict upper bound $\sup_{\lambda\in\tilde{W}^\varepsilon}\|D_x g_\lambda\|_{\text{op}}<L_g$, we see that we can choose M large enough such that (2.0.15) becomes

$$\|D\tilde{g}\|_0\le\sup_{\substack{\lambda\in\tilde{W}^\varepsilon\\ x\in X}}\ \sup_{\|\mu\|_{\mathbb{R}^m}\le 1}\|B(\lambda,x)\mu\|_X+\sup_{\lambda\in\tilde{W}^\varepsilon}\|D_x g_\lambda\|_{\text{op}}(1+M^{-1}\mathcal{C}_x)<L_g.$$

In a similar fashion, we can choose M large enough such that $\|D\tilde{k}_c\|_0<L_c$. In particular, we see that assumption 4 of Theorem 1.1 is satisfied for $\tilde{F}$.
Verifying the conjugacy equation: We can apply Theorem 1.1 to the dynamical system $\tilde{F}$. Thus we find $\tilde{K}:Y_c\to Y$ and $\tilde{R}:Y_c\to Y_c$ such that

$$\tilde{F}\circ\tilde{K}=\tilde{K}\circ\tilde{R}, \qquad (2.0.17)$$

and $\tilde{K}=\iota+\begin{pmatrix}\tilde{k}_c\\ \tilde{k}_u\\ \tilde{k}_s\end{pmatrix}$ and $\tilde{R}=\tilde{A}_c+\tilde{r}$. However, we claimed that for $\lambda\in\tilde{W}$ we have

$$(A+g_\lambda)\circ K_\lambda=K_\lambda\circ(A_c+r_\lambda)$$

which we still have to prove. So let $\lambda=\sum_{i=1}^m\lambda_i e_i\in\tilde{W}$ and $x\in X_c$. Recall that $\tau(\lambda)=\sum_{i=1}^m\lambda_i x_i$ lies in $X_u\oplus X_s$, thus we have $\tau_u(\lambda)\in X_u$ and $\tau_s(\lambda)\in X_s$

such that $\tau(\lambda) = (\tau_u(\lambda), \tau_s(\lambda))$. Then we have

$$\tilde{F}(\tilde{K}(\lambda, x + \tau(\lambda))) = \tilde{F}\begin{pmatrix} \lambda \\ x + k_{c,\lambda}(x) \\ \tau_u(\lambda) + \tilde{k}_u(\lambda, x + \tau(\lambda)) \\ \tau_s(\lambda) + \tilde{k}_s(\lambda, x + \tau(\lambda)) \end{pmatrix}.$$

Here we used that $\tilde{k}_c(\lambda, x + \tau(\lambda)) = (0, k_{c,\lambda}(x))$. We recall the definition of $\tilde{F}(\lambda, x) = (\lambda, F_\lambda(x))$, and we define $k_{u,\lambda}(x) := \tau_u(\lambda) + \tilde{k}_u(\lambda, x + \tau(\lambda))$ and $k_{s,\lambda}(x) := \tau_s(\lambda) + \tilde{k}_s(\lambda, x + \tau(\lambda))$. Then we have

$$\tilde{F}(\tilde{K}(\lambda, x + \tau(\lambda))) = \begin{pmatrix} \lambda \\ F_\lambda \begin{pmatrix} x + k_{c,\lambda}(x) \\ k_{u,\lambda}(x) \\ k_{s,\lambda}(x) \end{pmatrix} \end{pmatrix}. \tag{2.0.18}$$

On the other hand, we write $y(\lambda) = \sum_{i=1}^m \lambda_i y_i \in X_c$ and obtain

$$\tilde{R}(\lambda, x + x(\lambda)) = \begin{pmatrix} \lambda + \tilde{r}_{\mathbb{R}^m}(\lambda, x + \tau(\lambda)) \\ A_c x + \tau(\lambda) + y(\lambda) + \tilde{r}_{X_c}(\lambda, x + \tau(\lambda)) \end{pmatrix} \in Y_c. \tag{2.0.19}$$

From the $\mathbb{R}^m$ component of (2.0.17) to (2.0.19) we obtain

$$\lambda = \lambda + \tilde{r}_{\mathbb{R}^m}(\lambda, x + \tau(\lambda)),$$

thus $\tilde{r}_{\mathbb{R}^m}(\lambda, x + \tau(\lambda)) = 0$ and $R_\lambda(x) := A_c x + y(\lambda) + \tilde{r}_{X_c}(\lambda, x + \tau(\lambda)) \in X_c$. Then (2.0.17) becomes

$$\begin{aligned} \begin{pmatrix} \lambda \\ F_\lambda \begin{pmatrix} x + k_{c,\lambda}(x) \\ k_{u,\lambda}(x) \\ k_{s,\lambda}(x) \end{pmatrix} \end{pmatrix} &= \tilde{F}(\tilde{K}(\lambda, x + \tau(\lambda))) \\ &= \tilde{K}(\tilde{R}(\lambda, x + \tau(\lambda))) \\ &= \tilde{K}\begin{pmatrix} \lambda \\ R_\lambda(x) + \tau(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} \lambda \\ R_\lambda(x) + k_{c,\lambda}(R_\lambda(x)) \\ \tau_u(\lambda) + \tilde{k}_u(\lambda, R_\lambda(x) + \tau(\lambda)) \\ \tau_s(\lambda) + \tilde{k}_s(\lambda, R_\lambda(x) + \tau(\lambda)) \end{pmatrix}. \end{aligned} \tag{2.0.20}$$

With the definition of $k_{u,\lambda}$ and $k_{s,\lambda}$, (2.0.20) restricted to X becomes

$$F_\lambda \begin{pmatrix} x + k_{c,\lambda}(x) \\ k_{u,\lambda}(x) \\ k_{s,\lambda}(x) \end{pmatrix} = \begin{pmatrix} R_\lambda(x) + k_{c,\lambda}(R_\lambda(x)) \\ k_{u,\lambda}(R_\lambda(x)) \\ k_{s,\lambda}(R_\lambda(x)) \end{pmatrix} = \left(\iota + \begin{pmatrix} k_{c,\lambda} \\ k_{u,\lambda} \\ k_{s,\lambda} \end{pmatrix}\right) \circ R_\lambda(x).$$

Thus (2.0.1) holds for all $\lambda \in \tilde{W}$. Furthermore, the smoothness of K_λ and R_λ follows from the smoothness of $\tilde{K}$ and $\tilde{R}$. Finally, R_λ is invertible since $\tilde{R}$ is invertible. □

As a consequence of Theorem 2.1, we can prove the existence of bifurcations in dynamical systems with parameters. Since the solution of (2.0.1) is unique, the Taylor expansion of $R(\lambda, x)$ is also unique and determined by (2.0.1) and our choice of k_c. Thus if we choose k_c such that the Taylor approximation of R is the normal form of a bifurcation, we know qualitatively how the dynamical behaviour of F changes on the center manifold as the parameter changes. To obtain quantitative information of the dynamical behaviour on the center manifold, we want local bounds on the Taylor approximation of R. That is, given the conjugate dynamical system R and its Taylor expansion P_R, we are interested in local bounds on $R - P_R$.

Proposition 2.4. *Let $X = \mathbb{R}^m$ and $F : X \to X$ a C^n, $n \geq 2$, discrete dynamical system on X such that 0 is a fixed point of F. Let $k_c : X_c \to X_c$ be C^n and such that Theorem 1.1 holds. Let P_R and P_K be the Taylor expansion of R and K of order $n-1$. Then there exist a neighborhood $\mathcal{U}$ of 0 such that for every neighborhood $0 \in U \subset \mathcal{U}$ there exists constants C_R and C_K such that*

$$\|R(x) - P_R(x)\| \leq C_R \|x\|^n \qquad \text{and} \qquad \|K(x) - P_K(x)\| \leq C_K \|x\|^n$$

for all $x \in U$.

Remark 2.5. We first remark that our constants C_R and C_K only depend on the neighborhood U, the dynamical system F and our choice of k_c. Furthermore, the constants C_R and C_K can explicitly be computed, see Remark 2.7.

Remark 2.6. For a dynamical system $F : W \times \mathbb{R}^m \to \mathbb{R}^m$ with parameter space $W \subset \mathbb{R}^k$, we can apply Proposition 2.4 to the extended dynamical system $\tilde{F} : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$, where $\tilde{F}$ is defined as in the proof of Theorem 2.1. Alternatively, for fixed $\lambda \in \tilde{W}$, we can apply Proposition 2.4 to (2.0.1).

Proof. Before we start the proof, let us recall what it means that k_c is chosen such that Theorem 1.1 holds, which can be found in Remark 2.4 from [14]. First, we have bounds L_c, L_u and L_s for the derivatives of k_c, k_u and k_s respectively. We note that $L_u, L_s \leq 1 + L_c$. Furthermore, we have that $R = A_c + r$ is invertible with inverse $T = A_c^{-1} + t$, and the derivatives of R and T are bounded by $\|A_c\|_{\text{op}} + L_r$ and L_{-1} respectively. Finally, we have the inequalities

$$\begin{aligned} \|A_u^{-1}\|_{\text{op}} \left((\|A_c\|_{\text{op}} + L_r)^n + L_g + L_u \right) &< 1, \\ L_{-1}^n \left(\|A_s\|_{\text{op}} (1 + L_{-1} L_s) + L_g \left(1 + L_{-1} (1 + L_c)\right) \right) &< 1, \end{aligned}$$

where L_g is an upper bound for the derivative of g. Hence in particular we have the weaker bounds $\|A_u^{-1}\|_{\text{op}} (\|A_c\|_{\text{op}} + L_r)^n < 1$ and $L_{-1}^n \|A_s\|_{\text{op}} < 1$.

As F is C^n, we write $F = A + P_F + h_F$, where $A + P_F$ is the Taylor expansion up to order $n-1$ of F, and h_F is of order n around 0. Likewise, we write $K = \iota + P_K + h_K$ and $R = A_c + P_R + h_R$. We expand the conjugacy equation using the Taylor expansions to obtain

$$A h_K(x) - \iota h_R(x) + Q(x, h_K(x), h_R(x)) + h_F(K(x)) - h_K(R(x)) = 0.$$

Here we define $Q := A P_K + P_F(\iota + P_K + h_K) - \iota P_R - P_K(A_c + P_R + h_R)$. We write the equation component-wise, i.e. using the splitting $X = X_c \oplus X_u \oplus X_s$

we obtain the three equations

$$\begin{cases} h_R = A_c h_{K,c} + Q_{1,c} + Q_{2,c} h_K + Q_{3,c} h_R + h_{F,c}(K) - h_{K,c}(R), \\ A_u h_{K,u} = Q_{1,u} + Q_{2,u} h_K + Q_{3,u} h_R + h_{F,u}(K) - h_{K,u}(R), \\ A_s h_{K,s} = Q_{1,s} + Q_{2,s} h_K + Q_{3,s} h_R + h_{F,s}(K) - h_{K,s}(R), \end{cases} \tag{2.0.21}$$

where we write Q as a polynomial in x, h_K and h_R

$$Q(x, h_K(x), h_R(x)) = Q_1(x) + Q_2(x, h_K(x))h_K(x) + Q_3(x, h_R(x))h_R(x).$$

We note that Q_1 is of order $\|x\|^n$ at the origin. Furthermore, we have that A_u and R are both invertible, and recall that we defined $T = R^{-1}$. Hence we rewrite the second and third equation of (2.0.21) as

$$\begin{cases} h_R = A_c h_{K,c} + Q_{1,c} + Q_{2,c} h_K + Q_{3,c} h_R + h_{F,c}(K) - h_{K,c}(R), \\ h_{K,u} = A_u^{-1} \left(Q_{1,u} + Q_{2,u} h_K + Q_{3,u} h_R + h_{F,u}(K) - h_{K,u}(R) \right), \\ h_{K,s} = \left(Q_{1,s} + Q_{2,s} h_K + Q_{3,s} h_R + h_{F,s}(K) - A_s h_{K,s} \right) \circ T. \end{cases} \tag{2.0.22}$$

To obtain local bounds on h_R and h_K, we consider a bounded neighborhood $0 \in U$. Then we can find intervals $[a_i, b_i]$ such that $U \subset \bigtimes_{i=1}^m [a_i, b_i] =: \mathcal{I}_U$. We consider cut-off functions of the form

$$\varphi_a^b(x) = \begin{cases} a & x \leq a \\ x & a \leq x \leq b \\ b & b \leq x \end{cases}$$

which we will generalize to a C^n cut-off function in Section 4. We then consider the cut-off function on $\mathbb{R}^m$ defined by

$$\varphi_U(x)_i := \varphi_{a_i}^{b_i}(x_i). \tag{2.0.23}$$

Thus in particular it follows from (2.0.22) that, where we will write φ instead of φ_U,

$$\begin{cases} h_R \circ \varphi = \left(A_c h_{K,c} + Q_{1,c} + Q_{2,c} h_K + Q_{3,c} h_R + h_{F,c}(K) - h_{K,c}(R) \right) \circ \varphi, \\ h_{K,u} \circ \varphi = A_u^{-1} \left(Q_{1,u} + Q_{2,u} h_K + Q_{3,u} h_R + h_{F,u}(K) - h_{K,u}(R) \right) \circ \varphi, \\ h_{K,s} \circ \varphi = \left(Q_{1,s} + Q_{2,s} h_K + Q_{3,s} h_R + h_{F,s}(K) - A_s h_{K,s} \right) \circ T \circ \varphi. \end{cases} \tag{2.0.24}$$

Here we define $Q_2 \circ \varphi(x) := Q_2(\varphi(x), h_K(\varphi(x)))h_K(\varphi(x))$, and likewise define $Q_3 \circ \varphi$. We make the following observations:

- Any bound we find on $h_R \circ \varphi$ and $h_K \circ \varphi$ will be a local bound for h_R and h_K on U.
- The functions $h_{K,c/s/u} \circ \varphi$, $h_R \circ \varphi$ and $h_F \circ \varphi$ are bounded and of order $\|x\|^n$ at the origin. Therefore, there exists constants $C_{K,c/s/u}$, C_R and C_F depending on U such that $\|h_{K,c/s/u}(\varphi(x))\| \leq C_{K,c/s/u}\|x\|^n$, $\|h_R(\varphi(x))\| \leq C_R\|X\|^n$ and $\|h_F(\varphi(x))\| \leq C_F\|X\|^n$.

– The derivatives of K, R and T are bounded by $1 + L_c$, $\|A_c\|_{\mathrm{op}} + L_r$ and L_{-1} respectively. Hence we have the bounds $\|K(x)\| \leq (1 + L_c)\|x\|$, $\|R(x)\| \leq (\|A_c\|_{\mathrm{op}} + L_r)\|x\|$ and $\|T(x)\| \leq L_{-1}\|x\|$.

– There exists a constant C_Q depending on U such that $Q_2 \circ \varphi$ and $Q_3 \circ \varphi$ are bounded by C_Q. In particular, the constant C_Q goes to zero when the diameter of the neighborhood U goes to zero.

– The polynomial $Q_1 \circ \varphi$ is bounded and is of order $\|x\|^n$ at the origin. Therefore, there exists a constant C_P depending on U such that $\|Q_1(\varphi(x))\| \leq C_P\|x\|^n$.

We can estimate $h_R \circ \varphi$, $h_{K,u} \circ \varphi$ and $h_{K,s} \circ \varphi$ using those observations and (2.0.24). We apply the triangle inequality and substitute the estimates from our observations in (2.0.24) to obtain

$$\begin{cases} \|h_R(\varphi(x))\| \leq (\|A_c\|_{\mathrm{op}} C_{K,c} + C_P + C_Q\,(C_{K,c} + C_{K,s} + C_{K,u} + C_R) \\ \qquad\qquad + C_F(1 + L_c)^n + C_{K,c}(\|A_c\|_{\mathrm{op}} + L_r)^n)\|x\|^n, \\ \|h_{K,u}(\varphi(x))\| \leq \|A_u^{-1}\|_{\mathrm{op}}(C_P + C_Q\,(C_{K,c} + C_{K,s} + C_{K,u} + C_R) \\ \qquad\qquad + C_F(1 + L_c)^n + C_{K,u}(\|A_c\|_{\mathrm{op}} + L_r)^n)\|x\|^n, \\ \|h_{K,s}(\varphi(x))\| \leq L_{-1}^n(C_P + C_Q(C_{K,c} + C_{K,s} + C_{K,u} + C_R) \\ \qquad\qquad + C_F(1 + L_c)^n + \|A_s\|_{\mathrm{op}} C_{K,s})\|x\|^n. \end{cases}$$

Here we used the crude estimate $\|h_K(\varphi(x))\| \leq C_{K,c}\|x\|^n + C_{K,s}\|x\|^n + C_{K,u}\|x\|^n$.

To obtain the constants C_R, $C_{K,u}$ and $C_{K,s}$ we will show that the map

$$\begin{pmatrix} C_R \\ C_{K,u} \\ C_{K,s} \end{pmatrix} \mapsto \begin{pmatrix} \|A_c\|_{\mathrm{op}} C_{K,c} + C_P + C_Q\,(C_{K,c} + C_{K,s} + C_{K,u} + C_R) \\ \qquad + C_F(1 + L_c)^n + C_{K,c}(\|A_c\|_{\mathrm{op}} + L_r)^n \\ \|A_u^{-1}\|_{\mathrm{op}}(C_P + C_Q\,(C_{K,c} + C_{K,s} + C_{K,u} + C_R) \\ \qquad + C_F(1 + L_c)^n + C_{K,u}(\|A_c\|_{\mathrm{op}} + L_r)^n) \\ L_{-1}^n(C_P + C_Q(C_{K,c} + C_{K,s} + C_{K,u} + C_R) \\ \qquad + C_F(1 + L_c)^n + \|A_s\|_{\mathrm{op}} C_{K,s}) \end{pmatrix} \tag{2.0.25}$$

has a unique component-wise positive fixed point. We first note that (2.0.25) is of the form $C \mapsto \mathcal{A}C + b$. We find

$$\|\mathcal{A}\|_\infty \leq \max\left\{\|A_u^{-1}\|_{\mathrm{op}}(\|A_c\|_{\mathrm{op}} + L_r)^n, L_{-1}^n\|A_s\|_{\mathrm{op}}\right\} + 3\max\{1, \|A_u^{-1}\|_{\mathrm{op}}, L_{-1}^n\}C_Q. \tag{2.0.26}$$

As we mentioned at the beginning of the proof, we have $\|A_u^{-1}\|_{\mathrm{op}}(\|A_c\|_{\mathrm{op}} + L_r)^n < 1$ and $L_{-1}^n\|A_s\|_{\mathrm{op}} < 1$. Furthermore, C_Q can be made as small as desired by decreasing the diameter of U. Hence there exists a neighborhood $\mathcal{U}$ of 0 such that $C \mapsto \mathcal{A}C + b$ is a contraction for every neighborhood $U \subset \mathcal{U}$. We know from the smoothness of R and K that there exist large, non-explicit bounds C_R, $C_{K,u}$ and $C_{K,s}$. From this starting point, by iterating the inequalities above we conclude that the unique fixed point of (2.0.25) provides an explicit bound. □

Remark 2.7. We again write $C \mapsto \mathcal{A}C + b$ for (2.0.25), and note that $\mathcal{A}$ and b are both component-wise positive. In practice, we do not compute the right hand side of (2.0.26) and show that it is strictly less than 1. Instead, we find

$\tilde{C}$ component-wise positive such that $\mathcal{A}\tilde{C} + b < \tilde{C}$ component-wise. From the Min-max CollatzWielandt formula it follows that (2.0.25) is a contraction, so it has a unique fixed point, explicitly given by $C := (\mathcal{A} - \mathrm{Id})^{-1}b$. We can then either use C or our Ansatz $\tilde{C}$, which is component-wise larger than C, to obtain the constants C_R, $C_{K,u}$ and $C_{K,s}$.

Remark 2.8. In the proof Proposition 2.4 we find constants C_R and C_K such that $\|h_R(x)\| \leq C_R\|x\|^n$ and $\|h_K(x)\| \leq C_K\|x\|^n$ for $h_R := R - A_c - P_R$ and $h_K := K - \iota - P_K$. We note that under the assumptions of Proposition 2.4 we can also find constants $C_{R,m}$ and $C_{K,m}$ such that $\|D^m h_R(x)\| \leq C_{R,m}\|x\|^{n-m}$ and $\|D^m h_K(x)\| \leq C_{K,m}\|x\|^{n-m}$. To obtain those bounds, we have to modify the proof slightly. Before we introduce the cut-off function φ_U in (2.0.23), we take the m^{th} derivative on both sides of (2.0.22). We then obtain an expression similar to (2.0.24) for the composition of the m^{th} derivative of h and φ_U, except the right hand side of the equation will be more complicated due to taking derivatives of compositions and products. We can still use the triangle inequality to obtain a system like the system in (2.0.25) we have to solve to obtain $C_{R,m}$ and $C_{K,m}$. That is, the right hand side of the system will be of the form

$$\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \|A_u^{-1}\|_{\mathrm{op}}(\|A_c\|_{\mathrm{op}} + L_r)^{n-m} & 0 \\ 0 & 0 & \|A_s\|_{\mathrm{op}} L_{-1}^{n-m} \end{pmatrix} + \mathcal{E}_U\right) \begin{pmatrix} C_{R,m} \\ C_{K,m,u} \\ C_{K,m,s} \end{pmatrix} + \mathcal{D}_U.$$

Here the norm of the matrix $\mathcal{E}_U$ goes to 0 as the diameter of U goes to zero, and the vector $\mathcal{D}_U$ is continuous as function of the diameter of U. Since both $\|A_u^{-1}\|_{\mathrm{op}}(\|A_c\|_{\mathrm{op}} + L_r)^{n-m} < 1$ and $\|A_s\|_{\mathrm{op}} L_{-1}^{n-m} < 1$, there exists a neighborhood $\mathcal{U}$ of 0 such that (2.0.25) has a positive solution for $C_{K,m,u}$, $C_{K,m,s}$ and $C_{R,m}$ for every neighborhood $U \subset \mathcal{U}$.

Finally, if X is higher dimensional, we do not have to assume that our Taylor expansions are of the same order in every variable. For example, we may also obtain a Taylor expansion in two variables x and y of the form $c_{0,0} + c_{1,0}x + c_{2,0}x^2 + c_{3,0}x^3 + c_{0,1}y + c_{1,1}xy$. In that case our bound on $R - P_R$ will reflect the different orders of the Taylor expansion, i.e. in our example the bound will be of the form $c_{4,0}x^4 + c_{0,2}y^2 + c_{2,1}x^2y + c_{1,2}xy^2$.

Remark 2.9. In the proof of Proposition 2.4 we used the crude estimate $\|R(x)\| \leq (\|A_c\|_{\mathrm{op}} + L_r)\|x\|$ in the observations after (2.0.24). The bound we find for C_R and C_K by solving (2.0.25) depends on this estimate. That is, if we have a better estimate for $R(x)$, the bound we find for C_R and C_K improves. Since we know the Taylor expansion of R, and we know that the constant C_R exists on U as R is C^n, we can improve the bound on $R(x)$ using

$$\|R(\varphi(x))\| = \|A_c\varphi(x) + P_R(\varphi(x)) + h_R(\varphi(x))\| \leq \|A_c\varphi(x) + P_R(\varphi(x))\| + C_R\|x\|^n.$$

We can similarly improve the bounds for $\|K(x)\|$ and $\|T(x)\|$ on U in the same observation.

3 Continuous time dynamical systems

Besides center manifolds for discrete time systems, we are also interested in center manifolds for ODEs. Hence our second generalization of Theorem 1.1

will be a parameterization theorem for ODEs. To prove the existence of center manifolds, we will use the time t-maps of the ODE. That is, we will show that there exists a time-independent conjugacy between the time t-maps of the ODE and time t-maps of a conjugate vector field on the center subspace. To do so, the time t-maps of our ODE have to exist and have to be sufficiently smooth. We first state an existence result. We recall that we defined sectorial operators in Section 1.1 such that $\exp(At)$ exists for all $t \geq 0$ if A is sectorial.

Proposition 3.1. *Let $A : X \to X$ be a sectorial operator and $g : X \to X$ a uniformly bounded C^n function. Then there exists a jointly C^n flow map $\varphi : (0, \infty) \times X \to X$ such that*

$$\varphi(t, x) = \exp(At)x + \int_0^t \exp(A(t-s))g(\varphi(s, x))ds.$$

In particular, $\varphi(t, x)$ is a solution to the ODE

$$\dot{x}(t) = Ax(t) + g(x(t)) \qquad \text{for } t > 0.$$

Proof. The existence of the flow map for all $t > 0$ is Theorem 3.3.4 of [10] and the smoothness of the flow map is Corollary 3.4.6 of [10]. ☐

Remark 3.2. The flow φ exists for all $t \geq 0$, but if A is not bounded then φ is not necessarily differentiable at $t = 0$.

Beside the existence result of the time t-map, we also need that the non-linearity G_t of the time t-map is uniformly bounded. In fact, we have an explicit bound on the derivative of G_t in terms of A, t and $\|Dg\|_0$.

Proposition 3.3. *Let $A : X \to X$ be a sectorial operator and $g : X \to X$ a uniformly bounded C^n function. We can write the flow map φ_τ as*

$$\varphi_\tau(x) = \exp(A\tau)x + G_\tau(x).$$

Furthermore, if $g(0) = 0$ and $Dg(0) = 0$, then

$$G_\tau \in \{h \in C_b^n(X) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \leq L_G(\|Dg\|_0, \tau)\},$$

where we define

$$L_G(\|Dg\|_0, \tau) := \tau\|Dg\|_0 \sup_{s\in[0,\tau]} \|\exp(As)\|_{\text{op}}^2 \exp\left(\|Dg\|_0 \int_0^\tau \|\exp(A(\tau-s))\|_{\text{op}} ds\right). \tag{3.0.1}$$

Proof. From Proposition 3.1, we know that the time τ-map satisfies

$$\varphi_\tau(x) = \exp(A\tau)x + \int_0^\tau \exp(A(\tau - s))g(\varphi(s, x))\mathrm{d}s.$$

Thus G_τ is defined as

$$G_\tau(x) := \int_0^\tau \exp(A(\tau - s))g(\exp(As)x + G_s(x))\mathrm{d}s, \tag{3.0.2}$$

which is jointly C^n on $(0, \infty)$. We will use Grönwall's inequality to show that $G_\tau(0) = 0$. Using the smoothness of G_τ, we can show that also $DG_\tau(0) = 0$. Finally, we will only prove that $\|DG_\tau\|_0 \leq L_G(\|Dg\|_0, \tau)$, as the proof that G_τ is uniformly bounded in the C^n norm is similar.

By taking the norm on both sides of (3.0.2) for $x = 0$, we obtain

$$\|G_\tau(0)\| \leq \int_0^\tau \|\exp(A(\tau - s))\|_{\mathrm{op}} \|g(G_s(0))\| \mathrm{d}s.$$

Both $s \mapsto \|g(G_s(0))\|$ and $s \mapsto \|\exp(A(\tau - s))\|_{\mathrm{op}}$ are continuous, thus with Grönwall's inequality we obtain $\|G_\tau(0)\| \leq 0$, and therefore $G_\tau(0) = 0$.

To show that $DG_\tau(0) = 0$, we take the derivative of (3.0.2) and obtain

$$\begin{aligned} DG_\tau(x) = &\int_0^\tau \exp(A(\tau - s)) Dg(\exp(As)x + G_s(x)) \exp(As) \mathrm{d}s \\ &+ \int_0^\tau \exp(A(\tau - s)) Dg(\exp(As)x + G_s(x)) DG_s(x) \mathrm{d}s. \end{aligned} \tag{3.0.3}$$

For $x = 0$ we obtain

$$DG_\tau(0) = \int_0^\tau \exp(A(\tau - s)) Dg(0) \left(\exp(As) + DG_s(0)\right) \mathrm{d}s = 0,$$

since $Dg(0) = 0$.

For the upper bound on DG_τ, we take the norm on both sides of (3.0.3) and obtain

$$\begin{aligned} \|DG_\tau(x)\| \leq &\int_0^\tau \|\exp(A(\tau - s))\|_{\mathrm{op}} \|Dg\|_0 \|\exp(As)\|_{\mathrm{op}} \mathrm{d}s \\ &+ \int_0^\tau \|\exp(A(\tau - s))\|_{\mathrm{op}} \|Dg\|_0 \|DG_s(x)\| \mathrm{d}s \\ \leq &\ \tau \|Dg\|_0 \sup_{s \in [0,\tau]} \|\exp(As)\|_{\mathrm{op}}^2 + \int_0^\tau \|\exp(A(\tau - s))\|_{\mathrm{op}} \|Dg\|_0 \|DG_s(x)\| \mathrm{d}s. \end{aligned}$$

Grönwall's inequality then gives the upper bound

$$\|DG_\tau(x)\| \leq \tau \|Dg\|_0 \sup_{s \in [0,\tau]} \|\exp(As)\|_{\mathrm{op}}^2 \exp\left(\|Dg\|_0 \int_0^\tau \|\exp(A(\tau - s))\|_{\mathrm{op}} \mathrm{d}s\right).$$

Since the right hand side does not depend on x, we find

$$\begin{aligned} \|DG_\tau\|_0 &\leq \tau \|Dg\|_0 \sup_{s \in [0,\tau]} \|\exp(As)\|_{\mathrm{op}}^2 \exp\left(\|Dg\|_0 \int_0^\tau \|\exp(A(\tau - s))\|_{\mathrm{op}} \mathrm{d}s\right) \\ &=: L_G(\|Dg\|_0, \tau). \end{aligned}$$

We can use analogous estimates to recursively show that G_τ is uniformly bounded in C^n norm. That is, we take the m-th derivative of (3.0.2) for $m \leq n$, take the norm on both sides of this new equation and apply Grönwall's inequality. $\square$

3.1 Parameterization theorem for ODEs

The previous two propositions allow us to use Theorem 1.1 on a single time t-map, provided the non-linearity g is small enough. We would like to apply Theorem 2.1 to a collection of time t-maps. If we consider the dynamical system $(t,x) \mapsto \varphi_{t+\tau}(x)$ on $\mathbb{R} \times X$ as we would do in Theorem 2.1, then the non-linearity of our dynamical system is given by $(t,x) \mapsto (\exp(At) - \exp(A\tau))\, x + G_{\tau+t}(x)$. In particular, we see that the time derivative of the non-linearity will be unbounded, hence we can not apply Theorem 2.1 to a collection of time t-maps. However, we can construct a collection of discrete dynamical systems on the center subspace, and prove that these discrete dynamical systems are precisely the time t-maps of a conjugate vector field on the center subspace.

Theorem 3.4 (Center manifolds for flows)**.** *Let X be a Banach space and $f = A + g$ a vector field on X. Here $A : \mathcal{D}(A) \subset X \to X$ is a sectorial operator and $g : X \to X$ is a C^n, $n \geq 2$, vector field such that $g(0) = 0$ and $Dg(0) = 0$. Let $k_c : X_c \to X_c$ be chosen. Assume that there exists a $\tau > 0$ such that*

1. *There exists closed subspace X_c, X_u and X_s such that $X = X_c \oplus X_u \oplus X_s$ for which we have $A : \mathcal{D}(A) \cap X_c \to X_c$, $A : \mathcal{D}(A) \cap X_u \to X_u$ and $A : \mathcal{D}(A) \cap X_s \to X_s$. We write $A = \begin{pmatrix} A_c & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_s \end{pmatrix}$ where we define $A_c := A\big|_{\mathcal{D}(A)\cap X_c}$, and similarly define A_u and A_s. In particular, we then have*
$$\exp(A\tau) = \begin{pmatrix} \exp(A_c\tau) & 0 & 0 \\ 0 & \exp(A_u\tau) & 0 \\ 0 & 0 & \exp(A_s\tau) \end{pmatrix}.$$
2. *The linear operators $\exp(A_c\tau)$ and $\exp(A_u\tau)$ are invertible.*
3. *The norm on X is such that*
$$\|\exp(A_c\tau)^{-1}\|_{\mathrm{op}}^{\tilde{n}} \|\exp(A_s\tau)\|_{\mathrm{op}} < 1 \qquad \text{and} \qquad \|\exp(A_u\tau)^{-1}\|_{\mathrm{op}} \|\exp(A_c\tau)\|_{\mathrm{op}}^{\tilde{n}} < 1$$
for all $1 \leq \tilde{n} \leq n$.
4. *The non-linearities g and k_c satisfy*
$$\begin{aligned} g &\in \{h \in C_b^n(X,X) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 < L_g\}, \\ k_c &\in \left\{h \in C_b^{n+1}(X_c,X_c) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 < L_c\right\}, \end{aligned}$$
such that $L_G(L_g, \tau)$, defined in (3.0.1), and L_c are small enough in the sense of Theorem 1.1.

Then there exist a C^n conjugacy $K : X_c \to X$ and C^n vector field $R = A_c + r : X_c \to X_c$ such that the flow $\psi : [0,\infty) \times X_c \to X_c$ of R is mapped to the flow of f under K:
$$\varphi_t \circ K = K \circ \psi_t \qquad \text{for all } t \geq 0.$$

Furthermore, K and R have the following properties:

A) The vector field $R = A_c + r$ has forward and backward flow which satisfies the ODE $\dot{y}(t) = R(y(t))$ for all initial conditions $y(0) = y_0 \in X_c$. Furthermore, we have

$$r \in \{h \in C_b^n(X_c, X_c) \mid h(0) = 0 \text{ and } Dh(0) = 0\}.$$

B) The conjugacy K is given by

$$K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$$

with $\iota : X_c \to X$ the inclusion map and

$$\begin{aligned} k_u &\in \{h \in C_b^n(X_c, X_u) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \le L_u\}, \\ k_s &\in \{h \in C_b^n(X_c, X_s) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \le L_s\}. \end{aligned}$$

The constants L_u and L_s depend on $L_G(L_g, \tau)$ and L_c.

Remark 3.5. If we assume that k_c is C^n instead of C^{n+1}, then the vector field R on X_c would only be C^{n-1}.

Remark 3.6. We note that we again assume that the non-linearity g is bounded in C^n. We want to remark that this is a bound on the non-linearity of the ODE $\dot{x} = Ax + g(x)$, and not on the non-linearity of the time τ-map.

Remark 3.7. Similar to how we generalized Theorem 1.1 to dynamical systems with parameters in Theorem 2.1, we can generalize Theorem 3.4 to ODEs with parameters. That is, we expand our ODE to include the parameters as variables, i.e., we consider $(\dot{\lambda}, \dot{x}) = (0, Ax + C\lambda + g(\lambda, x))$. Then the proof would be a combination of the proofs of Theorems 2.1 and 3.4.

3.2 Obtaining the conjugacy

As we mentioned before, we want to construct the time t-maps of a conjugate vector field on the center subspace. Instead of giving a single long proof of Theorem 3.4, we will split it into several lemmas from which Theorem 3.4 will follow. As a starting point, we assume in Theorem 3.4 that for a specific time $\tau > 0$ the time τ-map satisfies the conditions of Theorem 1.1. From now on, $\tau > 0$ will always refer to the specific time from Theorem 3.4.

Lemma 3.8. *Under the conditions of Theorem 3.4, there exists a C^n conjugacy $K : X_c \to X$ and C^n discrete dynamical system $\Psi_\tau = \exp(A_c\tau) + \psi_\tau : X_c \to X_c$ such that*

$$\varphi_\tau \circ K = K \circ \Psi_\tau.$$

Furthermore, K and Ψ_τ have the following properties:

A) The dynamical system $\Psi_\tau = \exp(A_c\tau) + \psi_\tau$ is globally invertible and

$$\psi_\tau \in \{h \in C_b^n(X_c, X_c) \mid h(0) = 0,\ Dh(0) = 0, \text{ and } \|Dh\|_0 \le L_r\}.$$

B) The conjugacy K is given by

$$K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$$

with $\iota : X_c \to X$ the inclusion map and

$$k_u \in \{h \in C_b^n(X_c, X_u) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 < L_u\},$$
$$k_s \in \{h \in C_b^n(X_c, X_s) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 < L_s\}.$$

Proof. From Propositions 3.1 and 3.3 and the assumptions of Theorem 3.4 it follows that φ_τ together with k_c satisfy the assumptions of Theorem 1.1, from which the lemma follows. $\square$

As mentioned already, we want to construct the time t-maps of a conjugate vector field on the center subspace. For the time t-map of the vector field f on X, we will prove the existence of a conjugate discrete dynamical system Ψ_t on the center subspace. We will then show that the dynamical systems Ψ_t are in fact the time t-maps of a conjugate vector field on the center subspace.

Lemma 3.9. *Under the conditions of Theorem 3.4, there exists a $T > 0$ such that for all $t \in [0, T)$ there exists a conjugate dynamical system $\Psi_t : X_c \to X_c$ such that $\varphi_t \circ K = K \circ \Psi_t$.*

Before we prove the lemma, we will give the idea behind the construction of the conjugate dynamical systems Ψ_t. We want to solve the conjugacy equation $\varphi_t \circ K = K \circ \Psi_t$ where the conjugacy K is given by the equation $\varphi_\tau \circ K = K \circ \Psi_\tau$. With the flow property of φ_t, we have

$$\varphi_\tau \circ (\varphi_t \circ K) = \varphi_t \circ (\varphi_\tau \circ K) = (\varphi_t \circ K) \circ \Psi_\tau.$$

Hence we see that $\varphi_t \circ K$ is also a conjugacy between φ_τ and Ψ_τ. In [14] we proved that the center manifold is unique under some mild conditions. In particular, we have that for any two conjugacies K and $\tilde{K}$ there exists a diffeomorphism $\zeta : X_c \to X_c$ such that $K \circ \zeta = \tilde{K}$. In our case, this means that we obtain diffeomorphisms ζ_t such that $K \circ \zeta_t = \varphi_t \circ K$. Thus we have to prove that the conditions which imply a unique center manifold are satisfied. For the sake of completeness, we will repeat the uniqueness results from [14].

Lemma 3.10 (Lemma 6.2 in [14])**.** *Let $F : X \to X$ and $k_c : X_c \to X_c$ satisfy the conditions of Theorem 1.1. Then the conjugacy equation*

$$F \circ \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} = \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} \circ (A_c + r)$$

has a unique solution for $k_u \in C_b^0(X_c, X_u)$, $k_s \in C_b^0(X_c, X_s)$ and $r \in C_b^0(X_c, X_c)$ with the property that $A_c + r$ is a homeomorphism.

Lemma 3.11 (Proposition 6.3 in [14])**.** *Let $F : X \to X$ and $k_c, \tilde{k}_c : X_c \to X_c$ satisfy the conditions of Theorem 1.1. Then the image of $K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$ and*

$\tilde{K} = \iota + \begin{pmatrix} \tilde{k}_c \\ \tilde{k}_u \\ \tilde{k}_s \end{pmatrix}$ *are the same, for* $K, \tilde{K}$ *the (unique) conjugacy obtained from Theorem 1.1. In particular, there exists a diffeomorphism* $\zeta : X_c \to X_c$ *such that* $K \circ \zeta = \tilde{K}$.

We are now ready to prove Lemma 3.9.

Proof of Lemma 3.9. We first note that from the invertiblity of $\exp(A_c\tau)$ it follows that A_c is bounded. We have

$$A_c = (A_c \exp(A_c\tau)) (\exp(A_c\tau))^{-1}.$$

By definition, $(\exp(A_c\tau))^{-1}$ is bounded and $A_c \exp(A_c\tau)$ is the time derivative of $t \mapsto \exp(A_c t)$ at $\tau > 0$, hence bounded as A_c is a sectioral operator. So the right hand side is a bounded operator, thus A_c is also a bounded operator. In particular, this means that $\exp(A_c t)$ exists for all $t \in \mathbb{R}$.

For $t \geq 0$ we define the conjugacy $K_t := \varphi_t \circ K \circ \exp(-A_c t) : X_c \to X$ and dynamical system $R_t := \exp(A_c t)\Psi_\tau \circ \exp(-A_c t) : X_c \to X_c$. From the flow property, $\varphi_{t+s} = \varphi_t \circ \varphi_s$ it follows that

$$\begin{aligned}
\varphi_\tau \circ K_t &= \varphi_\tau \circ (\varphi_t \circ K \circ \exp(-A_c t)) \\
&= \varphi_t \circ \varphi_\tau \circ K \circ \exp(-A_c t) \\
&= \varphi_t \circ K \circ \Psi_\tau \circ \exp(-A_c t) \\
&= (\varphi_t \circ K \circ \exp(-A_c t)) \circ (\exp(A_c t)\Psi_\tau \circ \exp(-A_c t)) \\
&= K_t \circ R_t.
\end{aligned}$$

As we already mentioned, we want to show that there exists a diffeomorphism $\zeta_t : X_c \to X_c$ such that $\varphi_t \circ K \circ \exp(-A_c t) = K \circ \zeta_t$, or equivalent

$$\varphi_t \circ K = K \circ \zeta_t \circ \exp(A_c t).$$

This would follow directly from Lemma 3.11 if its condition is satisfied. That is, K_t should be the conjugacy obtained from Theorem 1.1 when we choose

$$\tilde{k}_c := K_{t,c} - \mathrm{Id} = (\exp(A_c t) + G_{c,t}) \circ k_c \circ \exp(-A_c t) + G_{c,t} \circ \exp(-A_c t)$$

instead of k_c in Theorem 1.1. We note that we need to check that $\tilde{k}_c$ satisfies the conditions of Theorem 1.1, i.e. $\tilde{k}_c$ lies in C_b^n and its derivative is bounded by L_c. We will then use Lemma 3.10 to show that K_t is the conjugacy obtained by Theorem 1.1, and hence we can apply Lemma 3.11. Indeed, we have to show that

1. The non-linearity $\tilde{k}_c$ lies in C_b^n and its derivative is bounded by L_c.
2. The (un)stable parts of K_t are bounded.
3. The linear part of R_t is $\exp(A_c\tau)$, its non-linearity is bounded, and R_t is a homeomorphism.

1) We have

$$\tilde{k}_c = (\exp(A_c t) + G_{c,t}) \circ k_c \circ \exp(-A_c t) + G_{c,t} \circ \exp(-A_c t).$$

Since all terms of the right hand side are C^n, we have that $\tilde{k}_c$ is C^n. Furthermore, $\exp(A_c t)$ is a bounded linear operator, k_c and $G_{c,t}$ are bounded functions in C^n, hence $\tilde{k}_c$ is bounded in C^n norm. Finally, its derivative is given by

$$\begin{aligned} D\tilde{k}_c(x) = &\left(\exp(A_c t) + DG_{c,t}(k_c(\exp(-A_c t)x))\right) Dk_c(\exp(-A_c t)x) \exp(-A_c t) \\ &+ DG_{c,t}(\exp(-A_c t)x) \exp(-A_c t). \end{aligned}$$

We obtain the estimate

$$\begin{aligned} \|D\tilde{k}_c\|_0 &\leq (\|\exp(A_c t)\|_{\mathrm{op}} + \|DG_t\|_0) \|Dk_c\|_0 \|\exp(-A_c t)\|_{\mathrm{op}} + \|DG_t\|_0 \|\exp(-A_c t)\|_{\mathrm{op}} \\ &\leq (\|\exp(A_c t)\|_{\mathrm{op}} + L_G(L_g, t)) L_c \|\exp(-A_c t)\|_{\mathrm{op}} + L_G(L_g, t) \|\exp(-A_c t)\|_{\mathrm{op}}. \end{aligned}$$

We want to prove that the right hand side is bounded by L_c. From the definition of L_G in Proposition 3.3, it follows that $s \mapsto L_G(L_g, s)$ is continuous and $L_G(L_g, 0) = 0$. Furthermore, $s \mapsto \|\exp(A_c s)\|_{\mathrm{op}}$ is continuous for all $s \in \mathbb{R}$ since A_c is bounded. In particular, we have $\|\exp(A_c 0)\|_{\mathrm{op}} = 1$. Thus for $t = 0$ we have $\|D\tilde{k}_c\|_0 < L_c$ and by continuity there exists an interval $[0, T)$ such that $\|D\tilde{k}_c\|_0 < L_c$ for all $t \in [0, T)$.

2) We note that k_u is bounded and continuous, hence also

$$K_{u,t} = \exp(A_u t) k_u \circ \exp(-A_c t) + G_{u,t} \circ k_u \circ \exp(-A_c t)$$

is bounded and continuous, as $\exp(A_u t)$ is a bounded operator and $G_{u,t}$ is bounded. Similar, we can show that $K_{s,t}$ is bounded and continuous.

3) From Lemma 3.8 it follows that $\Psi_\tau = \exp(A_c \tau) + \psi_\tau$ is a homeomorphism, thus $R_t = \exp(A_c t) \Psi_\tau \circ \exp(-A_c t)$ is also a homeomorphism. Furthermore, $\exp(\pm A_c t)$ and $\exp(A_c \tau)$ commute, thus we have

$$\begin{aligned} R_t &= \exp(A_c t) \exp(A_c \tau) \exp(-A_c t) + \exp(A_c t) \psi_\tau \circ \exp(-A_c t) \\ &= \exp(A_c \tau) + \exp(A_c t) \psi_\tau \circ \exp(-A_c t). \end{aligned}$$

Thus the linear part of R_t is $\exp(A_c \tau)$, and its non-linearity is bounded, since $\exp(A_c t)$ is a bounded linear operator and ψ_τ is bounded.

Now that we have proven the conditions for both Lemmas 3.10 and 3.11, we apply Lemmas 3.10 and 3.11 to K_t for all $t \in [0, T)$. That is, for $t \in [0, T)$ there exists a diffeomorphism $\zeta_t = \mathrm{Id} + \psi_t \circ \exp(-A_c t)$ such that

$$\varphi_t \circ K \circ \exp(-A_c t) = K_t = K \circ \zeta_t = K \circ (\exp(A_c t) + \psi_t) \circ \exp(-A_c t).$$

Therefore, K is the conjugacy between the time t-map φ_t on X and the dynamical system $\Psi_t := \exp(A_c t) + \psi_t$ on the center subspace for any $t \in [0, T)$. $\square$

3.3 Obtaining the vector field

We now have a collection $(\Psi_t)_{t \in [0,T)}$ of dynamical systems on the center subspace, where each dynamical system Ψ_t is conjugate with φ_t. We want to show that there exists a vector field $\dot{x} = A_c x + r(x)$ on the center subspace such that Ψ_t is its time t-map for all $t \in [0, T)$. Hence we first want to show that $t \mapsto \Psi_t$ is differentiable.

Lemma 3.12. *Under the conditions of Theorem 3.4, the function* $(t,x) \mapsto \psi_t(x)$ *is a fixed point of*

$$\Xi : \Phi \mapsto \exp(A_c t)k_c(x) + G_{c,t}(K(x)) - k_c(\exp(A_c t)x + \Phi_t(x)). \tag{3.3.1}$$

Furthermore, if Φ *is differentiable with respect to time and* $x \mapsto \frac{\partial \Phi}{\partial t}(t,x)$ *is* C^n*, then* $\Xi(\Phi)$ *has the same smoothness.*

Proof. From Lemma 3.9 it follows that ψ_t satisfies

$$\begin{pmatrix} \exp(A_c t) + G_{c,t} \\ \exp(A_u t) + G_{u,t} \\ \exp(A_s t) + G_{s,t} \end{pmatrix} \circ \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} = \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} \circ (\exp(A_c t) + \psi_t).$$

In particular, from the center part it follows that ψ_t satisfies

$$\psi_t(x) = \exp(A_c t)k_c(x) + G_{c,t}(K(x)) - k_c(\exp(A_c t)x + \psi_t(x)) \tag{3.3.2}$$

for $t \in [0,T)$. Hence $\psi : [0,T) \times X_c \to X_c$ is a fixed point of Ξ.

Assume that Φ is differentiable with respect to time and $x \mapsto \frac{\partial \Phi}{\partial t}(x)$ is C^n, then we want to show that all three terms in (3.3.1), namely $\exp(A_c t)k_c(x)$, $G_{c,t}(K(x))$ and $k_c(\exp(A_c t)x + \Phi_t(x))$ have the same smoothness as Φ.

Term 1) Since A_c is bounded, we have that $t \mapsto \exp(A_c t)$ is differentiable at all $t \in \mathbb{R}$ with derivative $A_c \exp(A_c t)$. Hence $t \mapsto \exp(A_c t)$ is differentiable with respect to time and its time derivative is C^n as function on X.

Term 2) For the center part of G_t, we have

$$G_{c,t} = \exp(A_c t) \int_0^t \exp(-A_c s) g_c(\exp(As)x + G_s(x)) \mathrm{d}s.$$

From Proposition 3.1 it follows that $G_{c,t}$ is differentiable at $t > 0$ with derivative

$$\left. \frac{\partial G_{c,s}}{\partial s} \right|_{s=t} = A_c G_{c,t} + g_c \circ \varphi_t,$$

which is a C^n function X. For the derivative at 0 we derive, where we denote $H(s,x) = \exp(-A_c s) g_c(\exp(As)x + G_s(x))$,

$$\begin{aligned} \frac{G_{c,t}(x) - G_{c,0}(x) - t g_c(x)}{t} &= \frac{1}{t} \int_0^t \exp(A_c t) H(s,x) - g_c(x) \mathrm{d}s \\ &= \frac{\exp(A_c t) - \mathrm{Id}}{t} \int_0^t H(s,x) \mathrm{d}s \qquad (3.3.3) \\ &\quad + \frac{1}{t} \int_0^t \exp(-A_c s) g_c(\exp(As)x + G_s(x)) - g_c(x) \mathrm{d}s. \qquad (3.3.4) \end{aligned}$$

We want to show that (3.3.3) and (3.3.4) both go to 0 when $t \downarrow 0$. This shows that $G_{c,t}$ is differentiable at $t = 0$ with derivative g_c, which is C^n. We start with (3.3.3). We note that $H(s,x)$ is uniformly bounded on $[0,T]\times X_c$, and we denote its bound by $\mathcal{C}$. Then we obtain

$$\begin{aligned}\lim_{t\downarrow 0}\left\|\frac{\exp(A_c t)-\mathrm{Id}}{t}\int_0^t H(s,x)\mathrm{d}s\right\|_{X_c} &\le \lim_{t\downarrow 0}\frac{1}{t}\|\exp(A_c t)-\mathrm{Id}\,\|_{\mathrm{op}}\int_0^t \|H(s,x)\|_{X_c}\,\mathrm{d}s\\ &\le \lim_{t\downarrow 0}\frac{1}{t}\|\exp(A_c t)-\mathrm{Id}\,\|_{\mathrm{op}}\int_0^t \mathcal{C}\mathrm{d}s\\ &\le \lim_{t\downarrow 0}\mathcal{C}\|\exp(A_c t)-\mathrm{Id}\,\|_{\mathrm{op}}\\ &=0.\end{aligned}$$

Thus (3.3.3) goes to 0 if $t\downarrow 0$.

To analyze (3.3.4), fix $x\in X_c$ and let $\varepsilon>0$. Then there exists a $\delta>0$ such that $\|\exp(-A_c s)g_c(y)-g_c(x)\|_{X_c}<\varepsilon$ if $s<\delta$ and $y\in B_\delta(x)$. Furthermore, there exists a $\gamma\in(0,\delta)$ such that $\exp(As)x+G_s(x)\in B_\delta(x)$ for all $s\in[0,\gamma)$ by continuity of $s\mapsto \exp(As)x+G_s(x)$. Hence for $t\in(0,\gamma)$ we have

$$\left\|\frac{1}{t}\int_0^t \exp(-A_c s)g_c(\exp(As)x+G_s(x))-g_c(x)\mathrm{d}s\right\|_{X_c}\le\frac{1}{t}\int_0^t\varepsilon\mathrm{d}s=\varepsilon.$$

Thus also (3.3.4) goes to 0 if $t\downarrow 0$. Hence $G_{c,t}$ is also differentiable at $t=0$, and its time derivative g_c at $t=0$ is C^n as function on X_c.

Term 3) Finally, we have assumed that $k_c\in C^{n+1}(X_c,X_c)$, hence

$$\frac{\partial}{\partial t}k_c(\exp(A_c t)x+\Phi_t(x))=Dk_c(\exp(A_c t)x+\Phi_t(x))\left(A_c\exp(A_c t)x+\frac{\partial\Phi_t}{\partial t}(x)\right)$$

is C^n as function on X.

Thus if Φ is differentiable with respect to time and $x\mapsto\frac{\partial\Phi}{\partial t}(t,x)$ is C^n, then so is $\Xi(\Phi)$. □

Lemma 3.13. *Under the conditions of Theorem 3.4, the function Ξ defined in* (3.3.1) *is a contraction on the set*

$$\left\{\Phi\in C_b^0([0,T)\times X_c,X_c)\;\middle|\;\Phi_t(0)=0 \textit{ for all } t\in[0,T)\right\}.$$

Proof. Recall that $\|Dk_c\|_0<L_c$, which is small enough in the sense of Theorem 1.1. In particular, this means that $L_c<1$. The result then follows from the estimate $\|\Xi(\Phi)-\Xi(\tilde\Phi)\|_0\le\|Dk_c\|_0\|\Phi-\tilde\Phi\|_0$. □

Lemma 3.14. *Under the conditions of Theorem 3.4, the fixed point Φ_Ξ of the contraction Ξ is differentiable with respect to time and $x\mapsto\frac{\partial\Phi}{\partial t}(t,x)$ is C^n.*

Proof. The argument for the first part of the lemma is analogous to the argument used in the proof of Lemma 3.2 i) in [14]. □

In particular, we have shown in Lemma 3.12 that $(t,x)\mapsto\psi_t(x)$ is a fixed point of Ξ, and hence from Lemma 3.13 it follows that it is the unique fixed point of Ξ. Therefore, it follows from Lemma 3.14 that $t\mapsto\psi_t$ is differentiable and $x\mapsto\frac{\partial\psi_t}{\partial t}(x)$ is C^n. In particular, we can now prove that the time t-map of $\dot x=\frac{\partial\Psi_t}{\partial t}\big|_{t=0}(x)$ is given by Ψ_t for all $t\in[0,T)$.

Lemma 3.15. *Under the conditions of Theorem 3.4, the time t-map of the vector field $\dot{x} = R(x) := \frac{\partial \Psi_t}{\partial t}\big|_{t=0}(x)$ on the center manifold is the map Ψ_t defined in Lemma 3.9 for all $t \in [0,T)$. Moreover, for all $t \geq 0$, we have $\varphi_t \circ K = K \circ \Psi_t$, where Ψ_t is the time t-map of R.*

Proof. It follows from Lemmas 3.12 to 3.14 that $x \mapsto \frac{\partial \varphi_t}{\partial t}(x)$ is well-defined and C^n for all $t \in [0,T)$. Recall that $\Psi_t(x) = \exp(A_c t)x + \psi_t(x)$ and A_c is bounded, hence $x \mapsto \frac{\partial \Psi_t}{\partial t}(x)$ is well-defined and C^n for all $t \in [0,T)$. Thus we can define the C^n map

$$R(x) := \frac{\partial \Psi_t}{\partial t}\bigg|_{t=0}(x) = A_c x + \frac{\partial \psi_t}{\partial t}(x)\bigg|_{t=0}(x).$$

For $x \in X_c$, we want to show that $t \mapsto \Psi_t(x)$ is the orbit of x under $\dot{x} = R(x)$ for $t \in [0,T)$. That is, we have to show that

$$\frac{\partial}{\partial t}\Psi_t = R(\Psi_t) = \frac{\partial \Psi_t}{\partial t}\bigg|_{t=0}(\Psi_t) \qquad \text{for all } t \in [0,T).$$

Equivalently, we want to show that Ψ_t has the flow property on $[0,T)$, i.e. $\Psi_s \circ \Psi_t = \Psi_{t+s}$ for all $t, s \in [0,T)$ such that $t+s \in [0,T)$. Let $t, s \in [0,T)$ such that $t+s \in [0,T)$, then from Lemmas 3.12 and 3.13 it follows that ψ_{t+s} is the unique fixed point of Ξ, and hence $\Psi_{t+s} = \exp(A_c(t+s)) + \psi_{t+s}$ is the unique solution of $\varphi_{t+s} \circ K = K \circ \Psi_{t+s}$. On the other hand, from the flow property $\varphi_s \circ \varphi_t = \varphi_{t+s}$ we obtain

$$K \circ \Psi_{t+s} = \varphi_{t+s} \circ K = \varphi_s \circ \varphi_t \circ K = \varphi_s \circ K \circ \Psi_t = K \circ \Psi_s \circ \Psi_t.$$

Therefore, we obtain $\Psi_{t+s} = \Psi_s \circ \Psi_t$. Hence, the time t-map of $\dot{x} = R(x)$ is given by Ψ_t defined in Lemma 3.9 for all $t \in [0,T)$.

We still have to show that all orbits in X_c given by $\dot{x} = R(x)$ exist for all $t \geq 0$ and that $\varphi_t \circ K = K \circ \Psi_t$ for all $t \geq 0$, where Ψ_t is the time t-map of R. Since $\Psi_t : X_c \to X_c$ is well-defined for all $t \in [0,T)$, we can use the flow property of the time t-maps of R to see that time t-map of R exists for all $t \geq 0$. Furthermore, if $t \geq 0$, then there exists an $N \in \mathbb{N}$ such that $s = t - TN/2 \in [0, T/2)$, hence we have

$$\begin{aligned}\varphi_t \circ K &= \varphi_s \circ \underbrace{\varphi_{T/2} \circ \cdots \circ \varphi_{T/2}}_{N \text{ times}} \circ K = \varphi_s \circ \underbrace{\varphi_{T/2} \circ \cdots \circ \varphi_{T/2}}_{N-1 \text{ times}} \circ K \circ \Psi_{T/2} \\ &= \cdots = K \circ \Psi_s \circ \underbrace{\Psi_{T/2} \circ \cdots \circ \psi_{T/2}}_{N \text{ times}} = K \circ \Psi_t.\end{aligned}$$

□

3.4 Proof of Theorem 3.4

We can now finish the proof of Theorem 3.4

Proof of Theorem 3.4. It follows from Lemma 3.15 that $K : X_c \to X$ maps the flow of $R : X_c \to X_c$ to the flow of $f : X \to X$. Furthermore, it follows from Lemma 3.8 that K has the desired properties. It follows from Lemma 3.14 and (3.3.2) that r is a fixed point of the contraction

$$\Theta : \xi \mapsto A_c k_c(x) + A_c g_c(K(x)) - Dk_c(x)(A_c x + \xi(x)).$$

Hence r is bounded, and the set $\{\xi \mid \xi(0) = 0 \text{ and } D\xi(0) = 0\}$ is invariant under Θ, thus both $r(0) = 0$ and $Dr(0) = 0$. Since A_c is also bounded, we see that $\dot{x} = R(x) = A_c x + r(x)$ has forward and backward flow. $\square$

Since the conjugate vector field R has backward flow, we can show that the vector field f has backward flow on the center manifold.

Corollary 3.16. *Under the conditions of Theorem 3.4, there exists a forward and backward flow for $\dot{x} = Ax + g(x)$ on $K(X_c)$.*

Proof. If $x_0 \in K(X_c)$, there exists an $y_0 \in X_c$ such that $x_0 = K(y_0)$. The forward flow satisfies

$$\varphi_t(x_0) = \varphi_t(K(y_0)) = K(\Psi_t(y_0)),$$

thus $K(X_c)$ in invariant under the forward flow of f. Furthermore, we define the backward flow on $K(X_c)$ by

$$\varphi_{-t}(x_0) = \varphi_{-t}(K(y_0)) := K(\Psi_{-t}(y_0)).$$

In particular, we have for all $t, s \in \mathbb{R}$

$$\varphi_t \circ \varphi_s = \varphi_t \circ K \circ \Psi_s = K \circ \Psi_t \circ \Psi_s = K \circ \Psi_{t+s} = \varphi_{t+s}.$$

Thus φ_t has the flow property on the center manifold for all $t \in \mathbb{R}$, and in particular we have that φ_t is the flow of f for $t \geq 0$, hence φ_t is the flow of f for all $t \in \mathbb{R}$. $\square$

4 An application

As an illustration of the parameterization method, we will prove the existence of a period doubling bifurcation in a two dimensional discrete dynamical system. Furthermore, we will give explicit regions in the phase space in which the period orbit will lie for small parameter values after the bifurcation. Finally, we also show the existence of heteroclinic orbits from the periodic orbit to the stationary point that the periodic orbit bifurcated from.

4.1 A bifurcation in a in reaction-diffusion system

We consider a reaction-diffusion equation with transport on the integer lattice given by

$$\dot{u}_n = \underbrace{u_{n-1} - 2u_n + u_{n+1}}_{\text{Diffusion}} + \underbrace{(6+\lambda)u_n + 3u_n^2 - u_n^3}_{\text{Reaction}} + \underbrace{u_{n-1} - u_n}_{\text{Transport}}. \tag{4.1.1}$$

Here $n \in \mathbb{Z}$ and λ is a bifurcation parameter. The coefficient $6+\lambda$ in the reaction term is chosen so that a bifurcation happens at $\lambda = 0$. A stationary solution to (4.1.1) is an orbit of the discrete dynamical system

$$\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \mapsto \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} u_n \\ -(3+\lambda)u_n - 2u_{n-1} - 3u_n^2 + u_n^3 \end{pmatrix}. \tag{4.1.2}$$

We will prove that at $\lambda = 0$ and $u = 0$ there occurs a period doubling bifurcation in this map. Furthermore, we want to find computationally the parameter range for which we can prove that there exists a period 2-orbit, and show that for these parameter values, there also exists a heteroclinic orbit between the origin and the period 2-orbit. Hence we find a family of stationary solutions $(u_n)_{n\in\mathbb{Z}}$ of (4.1.1) which limits to 0 as $n \to -\infty$ and limits to a 2-periodic profile as $n \to \infty$, or vice versa.

For $\lambda = 0$, the linearization of (4.1.2) at the origin is given by

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

which has eigenvalues -1 and -2. We apply the coordinate transformation $(u_{n-1}, u_n) \mapsto (x, y) = (-2u_{n-1} - u_n, 2u_{n-1} + 2u_n)$ such that the linearization becomes diagonal, and (4.1.2) is equivalent to

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_c(\lambda, x, y) \\ g_u(\lambda, x, y) \end{pmatrix}, \tag{4.1.3}$$

where we define

$$\begin{aligned} g_c(\lambda, x, y) &= -\left((x+y)^3 - 3(x+y)^2 - \lambda(x+y)\right), \\ g_u(\lambda, x, y) &= 2\left((x+y)^3 - 3(x+y)^2 - \lambda(x+y)\right). \end{aligned}$$

If we want to apply Theorem 2.1, we need to replace g_c and g_u with bounded C^n functions. Assume for now that we replaced g_c by h_c such that h_c is a uniformly bounded C^n function with $h_c \equiv g_c$ on a neighborhood around 0. Likewise, assume that g_u is replaced by h_u such that h_u is uniformly bounded in C^n and $h_u \equiv g_u$ on the same neighborhood around 0. Let us denote this neighborhood by $\mathcal{B}$. Then we consider the dynamical system

$$\tilde{F} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_c(\lambda, x, y) \\ h_u(\lambda, x, y) \end{pmatrix}. \tag{4.1.4}$$

If we can apply Theorem 2.1 to (4.1.4) for a suitable choice of k_c and parameter set $W \subset \mathbb{R}$, then the center manifold of (4.1.4) will be a local center manifold for (4.1.3). In fact, the intersection of the center manifold of (4.1.4) and $\mathcal{B}$ is a local center manifold of (4.1.3).

4.2 Conditions on the linearization

To apply Theorem 2.1 to (4.1.4), we have to choose k_c and a parameter set $W \subset \mathbb{R}$ for which the six conditions of the theorem hold. Conditions 4 and 6 depend not only on our choice of k_c and W, but also on our choice of the functions h_c and h_u, whereas the other conditions do not depend on the choices we make. Therefore, we will check conditions 1, 2 and 3a of Theorem 2.1 first.

Condition 1: From the definition of $\tilde{F}$ in (4.1.4), it follows that

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Hence we have the invariant subspaces $X_c = \langle e_1 \rangle$ and $X_u = \langle e_2 \rangle$.

Condition 2: We have $A_c = -1$ and $A_u = -2$, which are both invertible.
Condition 3a: We have for $n \in \mathbb{N}$

$$\|A_u^{-1}\|_{\mathrm{op}} \max\{1, \|A_c\|_{\mathrm{op}}\}^n = \frac{1^n}{2} < 1.$$

4.3 Normal forms

Before we choose k_c, W, h_c and h_u and apply Theorem 2.1 to (4.1.4), we want to prove that there is a period doubling bifurcation, and find where the periodic orbit is on the center subspace. This allows us to make an informed choice for k_c, W, h_u and h_c. Assume for now that we can apply Theorem 2.1, i.e. we have chosen k_c, W, h_c and h_u all sufficiently small. Then we obtain a conjugate dynamical system

$$R : W \times X_c \to X_c, x \mapsto -x + r_\lambda(x),$$

such that $r(0,0) = 0$ and $\frac{\partial r}{\partial x}(0,0) = \frac{\partial r}{\partial \lambda}(0,0) = 0$. The conjugacy we obtain is given by

$$K : W \times X_c \to X, x \mapsto \begin{pmatrix} x + k_c(x) \\ k_{u,\lambda}(x) \end{pmatrix},$$

such that $k_u(0,0) = 0$ and $\frac{\partial k_u}{\partial x}(0,0) = \frac{\partial k_u}{\partial \lambda}(0,0) = 0$.

It is clear from the definition of $\tilde{F}$ that the origin is always a fixed point for any $\lambda \in W$. Since we obtain r and k_u as fixed points of a contraction operator Θ, they will lie in any invariant set of Θ. In particular, we have that the set $\{h : \mathbb{R} \times X_c \to \mathbb{R} \times X \mid h(\lambda, 0) = 0 \text{ for } \lambda \in W\}$ is invariant under Θ as the origin is a fixed point of $\tilde{F}$. Therefore, we have $r(\lambda, 0) = 0$ and $k_u(\lambda, 0) = 0$ for $\lambda \in W$. To prove the existence of the period-doubling bifurcation, we must have that R is at least C^3, and satisfies the conditions

$$\frac{\partial}{\partial \lambda}\frac{\partial R}{\partial x}(0,0) \neq 0 \qquad \text{and} \qquad 2\frac{\partial^3 R}{\partial x^3}(0,0) + 3\left(\frac{\partial^2 R}{\partial x^2}(0,0)\right)^2 \neq 0,$$

see for instance Theorem 4.3 in [11].

Since K and R satisfy the conjugacy equation

$$\tilde{F}_\lambda \circ K_\lambda = K_\lambda \circ R_\lambda \qquad \text{for all } \lambda \in W,$$

we can compute the higher order derivatives of r and k_u once we have chosen k_c. That is, we can solve the conjugacy equation order by order given the Taylor expansion of k_c. In particular, when we take k_c independent of λ, we obtain an explicit formula for the partial derivative of R with respect to x at $(\lambda, 0)$. We compute

$$\frac{\partial R}{\partial x}(\lambda, 0) = -1 + \lambda + \frac{1}{2}\left(-3\lambda - 1 + \sqrt{\lambda^2 + 6\lambda + 1}\right).$$

We use Taylor's Theorem to write

$$\sqrt{\lambda^2 + 6\lambda + 1} = 1 + 3\lambda - o(\lambda^2).$$

Thus we obtain

$$\frac{\partial^2 R}{\partial\lambda\partial x}(0,0) = \frac{\partial}{\partial\lambda}(-1+\lambda+o(\lambda^2)) = 1 \neq 0. \tag{4.3.1}$$

For the second and third derivative with respect to x of R at $(0,0)$, we write $k_c(x) = c_2x^2 + o(x^3)$ and compute

$$\begin{aligned} \frac{\partial^2 R}{\partial x^2}(0,0) &= 6 - 4c_2, \\ \frac{\partial^3 R}{\partial x^3}(0,0) &= -78 + 72c_2 - 24c_2^2. \end{aligned} \tag{4.3.2}$$

For any choice of c_2 we have $2\frac{\partial^3 R}{\partial x^3}(0,0) + 3\left(\frac{\partial^2 R}{\partial x^2}(0,0)\right)^2 = -48 \neq 0$. Therefore, (4.1.4) has a period-doubling bifurcation at $\lambda = 0$. However, this gives us no a priori information about the interval of existence or the location of the periodic orbit of (4.1.4). To obtain this information of the 2-periodic orbit or to prove the existence of heteroclinic orbits, we need an approximation of both R and K as well as error bounds on our approximations.

4.4 Taylor approximations

As we see in (4.3.2), our choice of k_c has influence on both the second and third derivative of R with respect to x. The normal form of a period doubling bifurcation is $x \mapsto (-1+\lambda)x - cx^3 + o(x^4)$, so we take k_c such that the second derivative of R with respect to x vanishes in the origin. To make the second derivative of R vanish, i.e. $c_2 = \frac{3}{2}$, we take

$$k_c \equiv \frac{3}{2}x^2$$

locally near 0. From (4.3.1) and (4.3.2) we obtain the Taylor polynomial P_R of R around $(0,0)$:

$$P_R(\lambda, x) = (-1+\lambda)x - 4x^3.$$

We see that P_R has the 2-periodic orbit $x = \pm\sqrt{\lambda}/2$ for $\lambda > 0$. On the other hand, we could also have chosen $k_c = 0$, in which case the Taylor polynomial of R would be $P(\lambda, x) = (-1+\lambda)x + 3x^2 - 13x^3$. This means that we cannot find an analytic expression for the 2-periodic orbit near 0 of P. As a consequence, our analysis of the conjugate dynamics becomes harder when we choose $k_c = 0$. Furthermore, we also compute the Taylor Polynomial P_K of k_u around $(0,0)$ for our choice of $k_c \equiv \frac{3}{2}x^2$ locally around 0. We obtain

$$P_K(\lambda, x) = -2\lambda x - 2x^2 + 8x^3.$$

4.5 Error bounds on the conjugate dynamics

As P_R is a good approximation for the conjugate dynamics R, the periodic orbit of R should lie close to the periodic orbit of P_R. In fact, if we have a bound on the derivative of $R - P_R$, we can explicitly find an interval which contains the periodic orbit of R. This is the content of the following proposition.

Proposition 4.1. *Let $n = 3$ and assume that k_c, W, h_c and h_u are chosen such that Theorem 2.1 holds for (4.1.4). Let $0 \le \mathcal{E}_R < 1$ and denote $\mathcal{I}_\lambda = [-\sqrt{1+\mathcal{E}_R}\sqrt{\lambda}/2, \sqrt{1+\mathcal{E}_R}\sqrt{\lambda}/2]$.*

i) If for all $x \in \mathcal{I}_\lambda$ we have the estimate $\|R(\lambda, x) - P_R(\lambda, x)\|_X \le \mathcal{E}_R\lambda|x|$, then R has a 2-periodic orbit alternating between W_- and W_+, where we define

$$\begin{aligned} W_- &:= [-\sqrt{1+\mathcal{E}_R}\sqrt{\lambda}/2, -\sqrt{1-\mathcal{E}_R}\sqrt{\lambda}/2], \\ W_+ &:= [\sqrt{1-\mathcal{E}_R}\sqrt{\lambda}/2, \sqrt{1+\mathcal{E}_R}\sqrt{\lambda}/2]. \end{aligned}$$

ii) If we have the stronger estimates $0 \le \mathcal{E}_R < 1/2$ and $\|D_x[R - P_R](\lambda, x)]\|_{\mathrm{op}} \le \mathcal{E}_R\lambda$ for all $x \in \mathcal{I}_\lambda$, then the periodic orbit alternating between W_- and W_+ is unique for $0 \le \lambda < 4/3$. Let us denote the periodic orbit by $x_\pm \in W_\pm$. Then all points $x \in (x_-, x_+)$ converge to the periodic orbit as $n \to -\infty$ and to the origin as $n \to \infty$.

Remark 4.2. If we choose $k_c \equiv \frac{3}{2}x^2$ on $\mathcal{I}_\lambda$, then the Taylor Series of R starts with P_R, i.e. $R(\lambda, x) - P_R(\lambda, x) = O(\lambda^2, \lambda x^2, x^4)$. Furthermore, if $x \in \mathcal{I}_\lambda$, we have that x is of order $\sqrt{\lambda}$. Thus we have that $D_x[R - P_R](\lambda, x) = o(\lambda x, x^3) = o(\lambda^{3/2})$ for $x \in \mathcal{I}_\lambda$. Hence for λ sufficiently small we may expect that $\mathcal{E}_R$ exists and is less than $1/2$. That is, we expect that the conditions of Proposition 4.1 ii) are satisfied for λ sufficiently small.

Proof. i) We want to show that $R(W_\pm) \supset W_\mp$, as the existence of a periodic orbit then follows from the Intermediate Value Theorem. We define $\lambda_\pm := \sqrt{1 \pm \mathcal{E}_R}\sqrt{\lambda}/2$.

1. First, consider the extreme case $R(\lambda, x) = P_R(\lambda, x) - \mathcal{E}_R\lambda x$. Then the periodic orbit of R is $\pm\lambda_-$. This means that the conjugate dynamics R maps λ_- to, with $\gamma \in [-\mathcal{E}_R, \mathcal{E}_R]$,

$$\begin{aligned} R(\lambda, \lambda_-) &= P_R(\lambda, \lambda_-) + \gamma\lambda\lambda_- \\ &= P_R(\lambda, \lambda_-) - \mathcal{E}_R\lambda\lambda_- + (\gamma + \mathcal{E}_R)\lambda\lambda_- \\ &= -\lambda_- + (\gamma + \mathcal{E}_R)\lambda\lambda_-. \end{aligned}$$

   As $\gamma + \mathcal{E}_R \ge 0$, we see that $R(\lambda, \lambda_-) \ge -\lambda_-$. Similarly, we have that $R(\lambda, \lambda_+) \le -\lambda_+$. Hence we have $W_- \subset R(\lambda, W_+)$.

2. As a consequence, there exists an interval $U_+ \subset W_+$ such that $R(\lambda, U_+) = W_-$.

3. Similarly to steps 1 and 2, we can find an interval $U_- \subset W_-$ such that $R(\lambda, U_-) = W_+$. Therefore, we can find an interval $V_+ \subset U_+$ such that $R(\lambda, V_+) = U_-$.

Hence, we find an interval $V_+ \subseteq W_+$ such that $R^2(\lambda, V_+) = W_+$. Since $0 \notin W_+$, R^2 has a non-trivial fixed point in W_+. Therefore, we see that the 2-periodic orbit of R alternates between W_- and W_+.

ii) For the second part of the proposition, we denote a 2-periodic orbit with $x_\pm \in W_\pm$, i.e. $R(\lambda, x_\pm) = x_\mp$. We want to show that for all $x \in (x_-, x_+)$ the orbit of x goes to the origin. First, we will show for all $x \in (-\lambda_-, \lambda_-)$ that the orbits of x goes to the origin. In particular, we will show the stronger result that $|R(\lambda,x)| < |x|$ for any non-zero $x \in (-\lambda_-, \lambda_-)$. Let $\gamma = R(\lambda,x) - P_R(\lambda,x) \in [-\mathcal{E}_R\lambda, \mathcal{E}_R\lambda]$, then we write out what it means that $|R(\lambda,x)| < |x|$, where we distinguish four different cases:

$$\begin{cases} R(\lambda,x) > -x \iff \lambda x - 4x^3 + \gamma x > 0 \iff \lambda + \gamma > 4x^2 \\ R(\lambda,x) < x \iff \lambda x - 4x^3 + \gamma x < 2x \iff \lambda + \gamma < 2 \end{cases} \qquad \text{if } x \in (0, \lambda_-),$$

$$\begin{cases} R(\lambda,x) < -x \iff \lambda x - 4x^3 + \gamma x < 0 \iff \lambda + \gamma > 4x^2 \\ R(\lambda,x) > x \iff \lambda x - 4x^3 + \gamma x > 2x \iff \lambda + \gamma < 2 \end{cases} \qquad \text{if } x \in (-\lambda_-, 0).$$

We have $\lambda + \gamma \geq \lambda - \mathcal{E}_R\lambda = 4\lambda_-^2 > 4x^2$ and with $\mathcal{E}_R < 1/2$ and $\lambda < 4/3$ we also have $\lambda + \gamma \leq \lambda + \mathcal{E}_R\lambda < 3/2\lambda < 2$. Therefore, if $x \in (-\lambda_-, \lambda_-)$ we have that the orbit of x goes to the origin.

Now, let $x \in (x_-, -\lambda_-]$. It follows from Theorem 2.1 that $R(\lambda,\cdot)$ is invertible and continuous, hence we have $x_+ = R(\lambda, x_-) > R(\lambda,x) > R(\lambda,0) = 0$. We want to show that x moves away from the periodic orbit under one iteration of $R(\lambda,\cdot)$, i.e. we want to show that $x_+ - R(\lambda,x) > x - x_-$. We can enclose

$$D_xR(\lambda,y) \in [-1 - (2 + 4\mathcal{E}_R)\lambda, -1 - (2 - 4\mathcal{E}_R)\lambda].$$

Since $\mathcal{E}_R < 1/2$, we have $D_xR(\lambda,y) < -1$ for all $y \in (x_-, x)$. We obtain from the mean value theorem, with $y \in (x_-, x)$, that

$$x_+ - R(\lambda,x) = R(\lambda,x_-) - R(\lambda,x) = D_xR(\lambda,y)(x_- - x) > x - x_-.$$

Therefore, we have that x moves away from the periodic orbit $x_\pm$ after one iteration of $R(\lambda,\cdot)$. Analogous, we can show that for $x \in [\lambda_-, x_+)$ we obtain that $R(\lambda,x) \in (x_-, 0)$ as well as that x moves away from the periodic orbit $x_\pm$ under one iteration of $R(\lambda,\cdot)$. Together with the fact that all orbits starting in $(-\lambda_-, \lambda_-)$ go towards the origin, we conclude that orbits starting in $(x_-, -\lambda_-]$ and $[\lambda_-, x_+)$ also go towards the origin. As we already mentioned, the dynamical system $R(\lambda,\cdot)$ is invertible and continuous, thus the orbit of $x \in (x_-, x_+)$ emerges from the periodic orbit. Hence we have shown that all points $x \in (x_-, x_+)$ converge to the periodic orbit as $n \to -\infty$ and to the origin as $n \to \infty$.

We still have to show that the periodic orbit in $W_\pm$ is unique, so assume that there is another periodic orbit $x'_\pm \in W_\pm$. Then either $x'_- \in (x_-, x_+)$ or $x_- \in (x'_-, x'_+)$. Thus the orbit of either x'_- or x_- goes to the origin, which contradicts that both $x_\pm$ and $x'_\pm$ are 2-periodic. □

4.6 Error bounds on the conjugacy

Similarly to how we found explicit error bounds on the 2-periodic orbit of R given explicit errors on its Taylor polynomial, we want to find an explicit rectangle in $\mathbb{R}^2$ in which the image of K lies for all $x \in \mathcal{I}_\lambda$ given the Taylor polynomial P_K of k_u. These are given by the following proposition.

Proposition 4.3. *Let $n = 3$ and assume that k_c, W, h_c and h_u are chosen such that Theorem 2.1 holds for* (4.1.4). *Furthermore, let $\mathcal{E}_R < 1/2$ and $\lambda < 4/3$ such that Proposition 4.1 i) holds. Let $0 \leq \mathcal{E}_K \leq 9/2$ and $0 \leq \lambda < 1/43 < 4/3$. If for* *all $x \in \mathcal{I}_\lambda$ we have the pointwise estimate $\|k_u(\lambda, x) - P_K(\lambda, x)\|_X \leq 2\mathcal{E}_K\lambda|x|$, then we have*

$$k_u(\lambda, x) \leq \frac{\left(\sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda} - 1\right)^2 \left(2\sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda} + 1\right)}{216}, \tag{4.6.1}$$

$$k_u(\lambda, x) \geq -\frac{\lambda}{2}\left(1 + \mathcal{E}_R + 2\mathcal{E}_R\sqrt{(1 + \mathcal{E}_R)\lambda} + 2\mathcal{E}_K\sqrt{(1 + \mathcal{E}_R)\lambda}\right), \tag{4.6.2}$$

for all $x \in \mathcal{I}_\lambda$.

Proof. The computations in this proof are checked symbolically in the Mathematica Notebook available at [15]. We have for $x \in \mathcal{I}_\lambda$

$$\begin{aligned} P_K(\lambda, x) - 2\mathcal{E}_K\lambda x \leq k_u(\lambda, x) \leq P_K(\lambda, x) + 2\mathcal{E}_K\lambda x \qquad & \text{if } x \geq 0, \\ P_K(\lambda, x) + 2\mathcal{E}_K\lambda x \leq k_u(\lambda, x) \leq P_K(\lambda, x) - 2\mathcal{E}_K\lambda x \qquad & \text{if } x \leq 0, \end{aligned}$$

with $P_K(\lambda, x) = -2\lambda x - 2x^2 + 8x^3$. Hence we can bound the minimum of $k_u(\lambda, x)$ on $\mathcal{I}_\lambda$ by

$$\min\left\{\min_{x\in[0,\lambda_+]} P_K(\lambda, x) - 2\mathcal{E}_K\lambda x, \min_{x\in[-\lambda_+,0]} P_K(\lambda, x) + 2\mathcal{E}_K\lambda x\right\}. \tag{4.6.3}$$

Under the constraints $0 \leq \mathcal{E}_R < 1/2$, $0 \leq \mathcal{E}_K \leq 9/2$ and $0 \leq \lambda < 1/43$, it turns out that the the minimum in (4.6.3) is obtained at the boundary $x = -\lambda_+$, hence we have the following lower bound on $k_u(\lambda, x)$ for all $x \in \mathcal{I}_\lambda$:

$$\begin{aligned} k_u(\lambda, x) &\geq P_K(\lambda, -\sqrt{1 + \mathcal{E}_R}\sqrt{\lambda}/2) - 2\mathcal{E}_K\lambda\sqrt{1 + \mathcal{E}_R}\sqrt{\lambda}/2 \\ &= -\frac{\lambda}{2}\left(1 + \mathcal{E}_R + 2\mathcal{E}_R\sqrt{(1 + \mathcal{E}_R)\lambda} + 2\mathcal{E}_K\sqrt{(1 + \mathcal{E}_R)\lambda}\right). \end{aligned}$$

Likewise, we can bound the maximum of $k_u(\lambda, x)$ on $\mathcal{I}_\lambda$ by

$$\max\left\{\max_{x\in[0,\lambda_+]} P_K(\lambda, x) + 2\mathcal{E}_K\lambda x, \max_{x\in[-\lambda_+,0]} P_K(\lambda, x) - 2\mathcal{E}_K\lambda x\right\}. \tag{4.6.4}$$

The maximum of (4.6.4) is obtained in $x = 1/12 - \sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda}/12$, hence for $x \in \mathcal{I}_\lambda$ we have the upper bound

$$\begin{aligned} k_u(\lambda, x) &\leq P_K(\lambda, 1/12 - \sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda}/12) \\ &\quad - 2\mathcal{E}_K\lambda\left(1/12 - \sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda}/12)\right) \\ &= \frac{\left(\sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda} - 1\right)^2 \left(2\sqrt{1 + 12\lambda + 12\mathcal{E}_K\lambda} + 1\right)}{216}. \end{aligned}$$

□

Corollary 4.4. *Under the assumptions of Proposition 4.3 and the extra assumption that $k_c \equiv \frac{3}{2}x^2$ on $\mathcal{I}_\lambda$, the dynamical system (4.1.4) for parameter value λ has a 2-periodic orbit inside*

$$\mathcal{B}_\lambda = [\lambda_{c,-}, \lambda_{c,+}] \times [\lambda_{u,-}, \lambda_{u,+}],$$

where

$$\begin{aligned}
\lambda_{c,-} &:= -\sqrt{1+\mathcal{E}_R}\sqrt{\lambda}/2 + \frac{3(1+\mathcal{E}_R)\lambda}{8},\\
\lambda_{c,+} &:= \sqrt{1+\mathcal{E}_R}\sqrt{\lambda}/2 + \frac{3(1+\mathcal{E}_R)\lambda}{8},\\
\lambda_{u,-} &:= -\frac{\lambda}{2}\left(1+\mathcal{E}_R + 2\mathcal{E}_R\sqrt{(1+\mathcal{E}_R)\lambda} + 2\mathcal{E}_K\sqrt{(1+\mathcal{E}_R)\lambda}\right),\\
\lambda_{u,+} &:= \frac{\left(\sqrt{1+12\lambda+12\mathcal{E}_K\lambda}-1\right)^2\left(2\sqrt{1+12\lambda+12\mathcal{E}_K\lambda}+1\right)}{216}.
\end{aligned}$$

Proof. The first interval of $\mathcal{B}_\lambda$ is the image of $\mathcal{I}_\lambda$ under $\mathrm{Id}+k_c \equiv x + \frac{3}{2}x^2$. The second interval of $\mathcal{B}_\lambda$ follows from Proposition 4.3. ◻

We note that we can obtain the location of the periodic orbit more precisely inside $\mathcal{B}_\lambda$. Since we know that the periodic orbit of R lies inside $W_\pm$, the periodic orbit of (4.1.4) is contained in the image of $W_\pm$ under K. We are however not only interested in the periodic orbit, but also in heteroclinic connections between the origin and the periodic orbit, which is why we consider the image of $\mathcal{I}_\lambda$ instead.

4.7 Periodic orbits and connections

From Corollary 4.4 we find the box where the 2-periodic orbits of (4.1.4) are. To prove the same periodic orbits for (4.1.3), we want that $h_c \equiv g_c$ and $h_u \equiv g_u$ on the boxes $\mathcal{B}_\lambda$ for some parameter interval $[0, \lambda_{\max}]$. Furthermore, we have to check that we can indeed find $\mathcal{E}_R$ and $\mathcal{E}_K$ such that Propositions 4.1 and 4.3 are both satisfied for all $\lambda \in [0, \lambda_{\max}]$.

Theorem 4.5. *Consider the dynamical system given by* (4.1.3)*.*

i) The dynamical system undergoes a period doubling bifurcation at $(\lambda, x) = (0, 0)$*.*

ii) For $0 < \lambda \leq 7.6 \cdot 10^{-5}$ *the* 2*-periodic orbit of* (4.1.3) *for parameter value* λ *lies inside the box* $\mathcal{B}_\lambda$ *from Corollary 4.4, where we take* $\mathcal{E}_R = 57.1\sqrt{\lambda}$ *and* $\mathcal{E}_K = 61.9\sqrt{\lambda}$*. Furthermore, there exists a 1D manifold inside* $\mathcal{B}_\lambda$ *consisting of heteroclinic connections between the origin and the periodic orbit.*

Remark 4.6. As we already mentioned, we have that the periodic orbit of (4.1.3) lies inside the image of $W_\pm$. As $W_\pm \sim \sqrt{\lambda}$ and $k_c = x + \frac{3}{2}x^2$ on $\mathcal{I}_\lambda$, the distance between the origin and the periodic orbit has a magnitude of 10^{-3}.

Proof. We used the Mathematica Notebook available at [15] to check several inequalities in the proof below. We will replace g_c and g_u with C^3 bounded functions h_c and h_u. To do this, we want to consider $h_c = g_c \circ \Phi$, where $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a bounded C^3 function. Furthermore, we want to construct Φ such that $\Phi \equiv \mathrm{Id}$ in a neighborhood around the origin. Then h_c is bounded in C^3 and $h_c \equiv g_c$ in the neighborhood where $\Phi \equiv \mathrm{Id}$. To define Φ, we define

$\varphi_{\alpha_1,\Delta_1}^{\alpha_2,\Delta_2} : \mathbb{R} \to \mathbb{R}$ as

$$\varphi_{\alpha_1,\Delta_1}^{\alpha_2,\Delta_2}(x) = \begin{cases} \alpha_1 - \frac{1}{2}\Delta_1 & x \leq \alpha_1 - \Delta_1 \\ x + \frac{(x-\alpha_1)^6}{\Delta_1^5} + \frac{3(x-\alpha_1)^5}{\Delta_1^4} + \frac{5(x-\alpha_1)^4}{2\Delta_1^3} & \alpha_1 - \Delta_1 \leq x \leq \alpha_1 \\ x & \alpha_1 \leq x \leq \alpha_2 \\ x - \frac{(x-\alpha_2)^6}{\Delta_2^5} + \frac{3(x-\alpha_2)^5}{\Delta_2^4} - \frac{5(x-\alpha_2)^4}{2\Delta_2^3} & \alpha_2 \leq x \leq \alpha_2 + \Delta_2 \\ \alpha_2 + \frac{1}{2}\Delta_2 & \alpha_2 + \Delta_2 \leq x \end{cases}$$

Here we assume that Δ_1 and Δ_2 are both positive. We find that $\varphi_{\alpha_1,\Delta_1}^{\alpha_2,\Delta_2}$ is C^3, it image is the interval $[a_1 - \Delta_1/2, a_2 + \Delta_2/2]$ and its derivative lies in the interval $[0, 1]$ for all $x \in \mathbb{R}$.

For Theorem 2.1 we consider the parameter set $W = [-10^{-6}, 7.61 \cdot 10^{-5}]$. We have to take negative λ-values in W for Theorem 2.1 if we want to conclude something for $\lambda = 0$.

Let $\lambda_{\max} = 7.6 \cdot 10^{-5}$, $\mathcal{E}_R = 57.1\sqrt{\lambda_{\max}}$, $\mathcal{E}_K = 61.9\sqrt{\lambda_{\max}}$ and consider the set $\tilde{W} = [-10^{-7}, \lambda_{\max}] \subset W$. Recall the definition of the interval $\mathcal{I}_\lambda \subset \mathbb{R}$ and λ_+ from Proposition 4.1, and the definition of the box $\mathcal{B}_\lambda \subset \mathbb{R}^2$ from Corollary 4.4. Then we use $a_1, a_2, b_1, b_2, c_1, c_2$ to denote $\mathcal{I}_{\lambda_{\max}} = [a_1, a_2]$ and $\mathcal{B}_{\lambda_{\max}} = [b_1, b_2] \times [c_1, c_2]$. Furthermore, let $\Delta = 10^{-6}$, which we can choose to be arbitrary small, but is chosen to be 10^{-6} so we can compute explicit bounds in (4.7.2) below. We define h_c and h_u as

$$h_c(x, y) := g_c\left(\varphi_{b_1,\Delta}^{b_2,\Delta}(x), \varphi_{c_1,\Delta}^{c_2,\Delta}(y)\right) \quad \text{and} \quad h_u(x, y) := g_u\left(\varphi_{b_1,\Delta}^{b_2,\Delta}(x), \varphi_{c_1,\Delta}^{c_2,\Delta}(y)\right). \tag{4.7.1}$$

Finally, we define $d_1 = -\lambda_{\max,+} - 2\mathcal{E}_R\lambda_{\max}\lambda_{\max,+}$ and $d_2 = \lambda_{\max,+} - 2\mathcal{E}_R\lambda_{\max}\lambda_{\max,+}$. Then we choose $k_c(x) = \frac{3}{2}\varphi_{d_1,\Delta}^{d_2,\Delta}(x)^2$, which is also a C^3 bounded function. For all $\mu \in W$ we compute, where we use Mathematica to check the inequalities,

$$\begin{aligned} \|Dh\|_0 &\leq \sup_{\substack{b_1-\Delta/2 \leq x \leq b_2+\Delta/2 \\ c_1-\Delta/2 \leq y \leq c_2+\Delta/2}} |12(x+y)^2 - 24(x+y) - 4\mu| < 0.13, \\ \|Dk_c\|_0 &\leq \sup_{d_1-\Delta/2 \leq x \leq d_2+\Delta/2} 3|x| < 0.017. \end{aligned} \tag{4.7.2}$$

Hence we take $L_g = 0.13$ and $L_c = 0.017$, in which case we find with Mathematica that condition 4 of Theorem 1.1 holds for $n = 3$. In particular, the conditions of Theorem 2.1 are satisfied for $n = 3$ for the system (4.1.4), which means that there exists a C^3 conjugacy $K : \tilde{W} \times \mathbb{R}$ and C^3 dynamical system $R : \tilde{W} \times \mathbb{R} \to \mathbb{R}$ such that locally for all $\mu \in \tilde{W}$

$$F_\mu \circ K_\mu = K_\mu \circ R_\mu.$$

We will now prove the two statements of the theorem.

i) From Section 4.3 it follows that (4.1.3) undergoes a period doubling bifurcation at the origin at $\lambda = 0$.

ii) Fix $0 < \lambda \leq \lambda_{\max}$. We want to use Proposition 2.4 for the dynamical system $F_\lambda := F(\lambda, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ to find explicit error bounds on the Taylor approximations of $R_\lambda := R(\lambda, \cdot)$ and $K_\lambda := K(\lambda, \cdot)$ on the neighborhood $0 \in \mathcal{I}_\lambda = [a_1, a_2]$. In particular, we want to find error bounds on the first derivative, thus we use Remark 2.8 in combination with the proof of Proposition 2.4. Furthermore, we estimate both $DR(x)$ and $DK(x)$ using the Taylor approximation to obtain better bounds, as we explained in Remark 2.9. Finally, since we have used the Ansatz that

$$\begin{aligned}\|DR_\lambda(x) - DP_R(\lambda, x)\| &\leq \mathcal{E}_R\lambda,\\ \|Dk_{u,\lambda}(x) - DP_K(\lambda, x)\| &\leq 2\mathcal{E}_K\lambda,\end{aligned}$$

we check that the bound we obtain is consistent with both these inequalities.

We use the Ansatz on R_λ in order to guarantee that $R_\lambda(\mathcal{I}_\lambda) \subset [d_1, d_2]$, i.e. we have that $k_c = \frac{3}{2}x^2$ on the image of $\mathcal{I}_\lambda$ under R. Furthermore, the Ansatz on K_λ allows us to use Proposition 4.3, which tells us that $K_\lambda(\mathcal{I}_\lambda) \subset \mathcal{B}_\lambda \subset \mathcal{B}_{\lambda_{\max}}$, i.e. the cut-off functions we used in (4.7.1) are the identity on the image of K_λ. We can now apply Proposition 2.4 in combination with Remark 2.7, where we choose the cut-off function $\varphi_{\mathcal{I}_\lambda}$ for (2.0.23). We refer to our Mathematica supplement for the computation of the (generalization of the) system (2.0.25), as well as checking that the bounds we obtain from solving this system are indeed at most $\mathcal{E}_R\lambda$ and $2\mathcal{E}_K\lambda$.

Finally, the assertion follows from Corollary 4.4 since we check with Mathematica that $\mathcal{E}_R < \frac{1}{2}$, $\mathcal{E}_K < \frac{9}{2}$ and $\lambda < \frac{1}{43}$ and thus the conditions of Proposition 4.3, and hence the conditions of Corollary 4.4, are satisfied. □

References

[1] Inmaculada Baldomá, Ernest Fontich, and Pau Martín, *Invariant manifolds of parabolic fixed points (i). existence and dependence on parameters*, ArXiv e-prints (2016), available at `https://arxiv.org/pdf/1603.02533.pdf`.

[2] ———, *Invariant manifolds of parabolic fixed points (ii). approximations by sums of homogeneous functions*, ArXiv e-prints (2016), available at `https://arxiv.org/pdf/1603.02535.pdf`.

[3] Jan Bouwe van den Berg, Christian Reinhardt, and J.D. Mireles James, *Computing (un)stable manifolds with validated error bounds: Non-resonant and resonant spectra*, Journal of Nonlinear Science **26** (2015), no. 4, 1055–1095.

[4] Xavier Cabré, Ernest Fontich, and Rafael de la Llave, *The parameterization method for invariant manifolds i: manifolds associated to non-resonant subspaces*, Indiana University mathematics journal (2003), 283–328.

[5] ———, *The parameterization method for invariant manifolds ii: regularity with respect to parameters*, Indiana University mathematics journal (2003), 329–360.

[6] ———, *The parameterization method for invariant manifolds iii: overview and applications*, Journal of Differential Equations **218** (2005), no. 2, 444–515.

[7] Roberto Castelli, Jean-Philippe Lessard, and J.D. Mireles James, *Parameterization of invariant manifolds for periodic orbits (ii): A posteriori analysis and computer assisted error bounds*, Journal of Dynamics and Differential Equations **30** (2018), no. 4, 1525–1581.

[8] Shui-Nee Chow, Chengzhi Li, and Duo Wang, *Normal forms and bifurcation of planar vector fields*, Cambridge University Press, 1994.

[9] Chris Groothedde and J.D. Mireles James, *Parameterization method for unstable manifolds of delay differential equations*, Journal of Computational Dynamics **4** (2017), no. 1&2, 21–70.

[10] Dan Henry, *Geometric theory of semilinear parabolic equations*, Springer Berlin Heidelberg, 1993.

[11] Yuri A. Kuznetsov, *Elements of applied bifurcation theory*, 2nd ed., Springer-Verlag New York, 1998.

[12] Jean-Philippe Lessard, J.D. Mireles James, and Christian Reinhardt, *Computer assisted proof of transverse saddle-to-saddle connecting orbits for first order vector fields*, Journal of Dynamics and Differential Equations **26** (2014), no. 2, 267–313.

[13] Christian Reinhardt and J.D. Mireles James, *Fouriertaylor parameterization of unstable manifolds for parabolic partial differential equations: Formalism, implementation and rigorous validation*, Indagationes Mathematicae **30** (2019), no. 1, 39 –80.

[14] Jan Bouwe van den Berg, Wouter Hetebrij, and Bob Rink, *The parameterization method for center manifolds*, Journal of Differential Equations (2020). To appear.

[15] ———, *Mathematica code for "generalizations of the parameterization method for center manifolds"*, 2020. `https://www.math.vu.nl/~janbouwe/code/center`.

[16] Jan Bouwe van den Berg, J.D. Mireles James, Jean-Philippe Lessard, and Konstantin Mischaikow, *Rigorous numerics for symmetric connecting orbits: even homoclinics of the gray-scott equation*, SIAM Journal on Mathematical Analysis **43** (2011), no. 4, 1557–1594.