Closed characteristics on non-compact hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

Viterbo demonstrated that any $(2n-1)$-dimensional compact hypersurface $M \subset (\mathbb{R}^{2n}, \omega)$ of contact type has at least one closed characteristic. This result proved the Weinstein conjecture for the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. Various extensions of this theorem have been obtained since, all for compact hypersurfaces. In this paper we consider *non-compact* hypersurfaces $M \subset (\mathbb{R}^{2n}, \omega)$ coming from mechanical Hamiltonians, and prove an analogue of Viterbo's result. The main result provides a strong connection between the top half homology groups $H_i(M)$, $i = n, \ldots, 2n-1$, and the existence of closed characteristics in the non-compact case (including the compact case).

1 Introduction

It was proven by Rabinowitz [20] that any convex and compact hypersurface in $\mathbb{R}^{2n}$, i.e., a hypersurface that occurs as a regular energy surface of the Hamilton equations, contains at least one periodic orbit for the Hamilton equations, also called a *closed characteristic*. Still under the assumption of compactness this problem was formulated in more geometric terms by Weinstein, generalizing the convexity hypothesis. Weinstein [29] conjectured that compact smooth hypersurfaces $M \subset \mathbb{R}^{2n}$ (in fact an arbitrary symplectic manifold) with $H_1(M) = 0$, that satisfy a specific geometric property, always contain a closed characteristic for the (normalized) Hamilton equations

$$x' = J\mathbf{n}_M.$$

Here $\mathbf{n}_M$ is the outward pointing normal on M and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the standard symplectic matrix, i.e. $\omega(\cdot, J\cdot) = \langle \cdot, \cdot \rangle$, with ω the standard symplectic form on $\mathbb{R}^{2n}$, and $\langle \cdot, \cdot \rangle$ the standard inner product. Viterbo [26] proved Weinstein's conjecture in $\mathbb{R}^{2n}$ without the condition on the first homology group. The geometric condition in Weinstein's conjecture, known as the *contact type* condition, can be explained as follows. A hypersurface $M \subset \mathbb{R}^{2n}$ is of contact type if there exists a so-called *Liouville* vector field Y (i.e. a vector field Y such that $\mathcal{L}_Y \omega = \omega$) defined on a neighborhood of M, which is transverse

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to M. Given such a Liouville vector field Y, the associated 1-form $\alpha = i_Y\omega$ is a *contact* form on M. There are examples by Ginzburg of compact hypersurfaces which are not of contact type and contain no closed characteristics [8, 9].

To give some more background, the problem posed by Weinstein can also be phrased in purely geometric terms. The characteristic line bundle of M is defined by

$$\ell_M = \big\{\xi \in TM \;\big|\; i_\xi\omega = 0 \text{ on } TM\big\}.$$

A *closed characteristic* of M is an embedded circle $\gamma : S^1 \to M$ such that $T\gamma = \ell_M$ on γ. For a hypersurface $M \subset (\mathbb{R}^{2n}, \omega)$ the contact type condition is equivalent to the existence of a 1-form α on M such that $d\alpha = \omega|_M$, and α is non-vanishing on $\ell_M\backslash\{0\}$. As we mentioned before, Viterbo proved that any compact hypersurface $M \subset (\mathbb{R}^{2n}, \omega)$ of contact type has a closed characteristic for the characteristic line bundle. We note that regular compact and convex hypersurfaces, as considered by Rabinowitz, are automatically of contact type [13]. In that sense the results on compact hypersurfaces of contact type are an extension of the results by Rabinowitz.

The objective of this paper is to investigate this result in the case of *non-compact* hypersurfaces. The complications encountered in dealing with the non-compactness of a hypersurface lead to formidable difficulties. Therefore, in this paper, we choose to consider the class of hypersurfaces that occur as energy surfaces of a classical mechanical Hamiltonians. For *compact* hypersurfaces coming from mechanical Hamiltonians existence of closed characteristics was proven by Weinstein [28]. To be precise about this definition, let (p, q) be the standard symplectic coordinates on $\mathbb{R}^{2n}$, and consider a hypersurface $M \subset \mathbb{R}^{2n}$ given as 0-level set of a *Hamiltonian* function $H(p, q) = \frac{1}{2}|p|^2 + V(q)$, i.e.

$$M = H^{-1}(0) = \big\{(p, q) \in \mathbb{R}^{2n} \;\big|\; \tfrac{1}{2}|p|^2 + V(q) = 0\big\},$$

where the *potential* V is a $C^2(\mathbb{R}^n; \mathbb{R})$ function (in particular, it is not singular). From now on we will restrict our attention to hypersurfaces of the above type, which we refer to as *mechanical* hypersurfaces. There is some freedom in the choice of the potential. Let N be the projection of M onto the q-coordinate:

$$N = \pi(M) = \{q \in \mathbb{R}^n \,|\, V(q) \le 0\},$$

where π is the projection $(p, q) \mapsto q$. The shape of M only fixes the function V on $N \subset \mathbb{R}^n$, hence on $\mathbb{R}^n - N$ the potential can be suitably altered.

It is worth mentioning that regular energy surfaces of mechanical systems are always of contact type, also in the non-compact case, cf. [1]. Some simple counterexamples show that non-compact hypersurfaces of contact type need not contain any closed characteristics. Consider $M_1 = \{|p|^2 - |q|^2 - 1 = 0\} \cong S^{n-1} \times \mathbb{R}^n$, which is of contact type by virtue of the contact form $\alpha = \frac{1}{2}(pdq - qdp)$, but clearly contains no closed characteristics. The nonzero homology groups in this cases are $H_0(M_1) = H_{n-1}(M_1) \cong \mathbb{Z}$. The topologically different example $M_2 = \{|p|^2 + \sum_{i=1}^{n-1} q_i^2 + \frac{2}{\pi}\arctan q_n = 1\} \cong S^{2n-2}\times\mathbb{R}$ also contains no closed characteristics, and its homology is given by $H_0(M_2) = H_{2n-2}(M_2) \cong \mathbb{Z}$, and zero elsewhere.

For compact hypersurfaces Poincaré duality reveals that the first n Betti numbers are equal to the last n Betti numbers: $\beta_i = \beta_{2n-1-i}$, or more precisely $H^i(M) \cong$

$H_{2n-1-i}(M)$. In the non-compact case this result is not true; since M is orientable it holds that M is non-compact if and only if $H_{2n-1}(M) = 0$. Our main theorem states that the *latter* n homology groups give information about the existence of closed characteristics. In the above examples, the manifold M_1 has nontrivial homology for $i < n$ only, while M_2 has nontrivial homology for $i = 2n - 2$. Nevertheless, both examples have no closed characteristics. Topology is thus not the only requirement for existence. An additional geometric condition is needed in the non-compact case. The topological information about M will be used to construct critical values of an appropriate action functional and therefore construct closed characteristics. In Viterbo's proof of the Weinstein Conjecture compactness is used analytically for the convergence of Palais-Smale sequences, and topologically to construct critical points of the action functional. We want to replace compactness by a geometric condition that still ensures Palais-Smale sequences to convergence, yet allows for hypersurfaces to be non-compact.

Let us fix some notation: DV denotes the gradient of V, while D^2V is the matrix of second derivatives of V. As usual, the hypersurfaces under consideration should be regular (i.e. not containing any critical points, or equivalently $DV \neq 0$ on ∂N). In addition, hypersurfaces are assumed to satisfy the *asymptotic regularity* condition

$$|DV(q)| \geq c > 0 \quad \text{and} \quad \frac{\|D^2V(q)\|}{|DV(q)|} \to 0 \qquad \text{as } |q| \to \infty.$$

Here the constant c is a q-independent positive constant. Intuitively, the former assumption excludes large (near-) critical points (which obviously would lead to difficulties in Palais-Smale sequences). The latter, slightly more technical, assumption gives us, asymptotically, some control over the rate of change of DV. Note that many polynomial potentials satisfy these conditions. In Section 7 we discuss a set of slightly different sufficient assumptions (also including the possibility of exponential growth of V). Under this *geometric* assumption on the asymptotic behavior of the potential, which ensures the necessary compactness properties for our problem, a *topological* condition implies the existence of a closed characteristic on M.

Theorem 1. *Let M be a regular mechanical hypersurface of dimension $2n-1$ which is asymptotically regular. If $H_i(M) \neq 0$ for some $i \geq n$, then M contains a closed characteristic.*

Notice that this topological condition means that we need one nonzero homology group among the top half, which implies that compact hypersurfaces always contain a closed characteristic since $H_{2n-1}(M) \cong \mathbb{Z}$. The example M_1 given above shows that Theorem 1 is sharp in its setting with respect to the topological condition. On the other hand, the example M_2 shows that an additional geometric condition is indeed necessary. This theorem deals with general *non-compact* hypersurfaces. As opposed to compact hypersurfaces very few results are known about the non-compact case. A special case, namely when the complement of N is disconnected, was studied by Offin [17], where some rather complicated additional conditions were needed. Other examples of closed characteristics on non-compact hypersurfaces occur in singular potentials, see e.g. [22, 23]. In this paper we consider a very general topological property (see also Theorem 2 below) that leads to the existence of closed characteristics. Furthermore, the asymptotic regularity condition is not too restrictive, very concrete and easily checked for examples.

The proof of Theorem 1 hinges on two ideas. The analytical part is to formulate a variational setting for finding closed characteristics as critical points, establishing a version of the Palais-Smale condition along the way. We allow variations in both the profile and the period in order to be able to determine a priori the energy level in which the closed orbit is found. We introduce a penalizing function for the Lagrangian action. In essence, this allows us to reduce Palais-Smale sequences to ones consisting of periodic solutions for approximating problems (on nearby or not-so-nearby energy levels) in which we then take the appropriate limit (Section 3). Another essential step in this analysis is to obtain the right function space bounds, for which we employ geometric properties of V and thus of M. The fact that the contact type condition always holds for mechanical hypersurfaces also plays an important role. The asymptotic regularity condition introduced above enables us to carry out these analytical steps. We emphasize again that the asymptotic regularity is a very mild condition. As examples, any asymptotically quadratic potential, i.e. $D^2V(q)$ tends to some invertible matrix as $|q| \to \infty$, is asymptotically regular. More generally, if the spectrum of the matrices $D^2V(q)$ is bounded away from zero and infinity for large q, then V is asymptotically regular. Also, the class of compact hypersurfaces forms a special case of Theorem 1 (the potential outside the compact projection N can easily be chosen to be asymptotically regular). Another family of examples covered by Theorem 1 are potentials of the form $V(q) = P_k(q) + \sum_{j=1}^{n} C_j q_j^{k+1}$, where $C_j \neq 0$, and $P_k(q)$ is any k-th order polynomial. In this case Theorem 1 immediately applies. We should point out that this argument also hold for potentials which are obtained by compactly supported perturbations of such polynomials. In Section 7 we present some alternatives for the asymptotic regularity condition. In that context we also discuss the relation with the contact type condition. This is best postponed until after the proof of Theorem 1, where the main analytical steps are explained.

Concerning the topological part we employ a variational linking principle that leads to Theorem 1. This linking principle is completely homological in nature (see Section 4). Since we allow variations in the periodic profiles as well as in the period, the linking sets have to be chosen in a rather subtle way (see Section 5) using the homological characterization of the topology of M provided in Theorem 2 below. Let us give the intuitive idea behind the linking principle. Two sets A_0 and S_0 in $\mathbb{R}^n$ are said to *homologically link* if the inclusion-induced homomorphism

$$j_i : H_i(A_0) \longrightarrow H_i(\mathbb{R}^n - S_0),$$

is nontrivial for some $0 \leq i \leq n$, i.e., j_i does not map the whole of $H_i(A_0)$ to the zero element in $H_i(\mathbb{R}^n - S_0)$. This means that there exists a nontrivial homology class $[a_0] \in H_i(A_0)$ which is also a nontrivial class in $H_i(\mathbb{R}^n - S_0)$, and since $H_*(\mathbb{R}^n) = 0$, the representative a_0 can be 'filled' so that S_0 intersects any such filled set. From linking sets A_0 and S_0 in $\mathbb{R}^n$ we can 'grow' a link in $\mathbb{R}^{n+1}$, see Figure 1 for an example from $\mathbb{R}^2$ to $\mathbb{R}^3$. Since our variational setup is not in $\mathbb{R}^n$ but in an infinite dimensional function space $(H^1(S^1) \times \mathbb{R})$ we need to grow, or lift, a link (A_0, S_0) in $\mathbb{R}^n$ to a link (A, S) in the function space. The minimax principle in e.g. [5] or [19] then states that for any link (A, S) one can minimax a functional over the link as follows: maximize over a 'fill' of A, and minimize over all admissible 'fills', see Section 4. The difficulty is twofold: find an appropriate "initial" link (A_0, S_0) in $\mathbb{R}^n$, and then construct a lift to

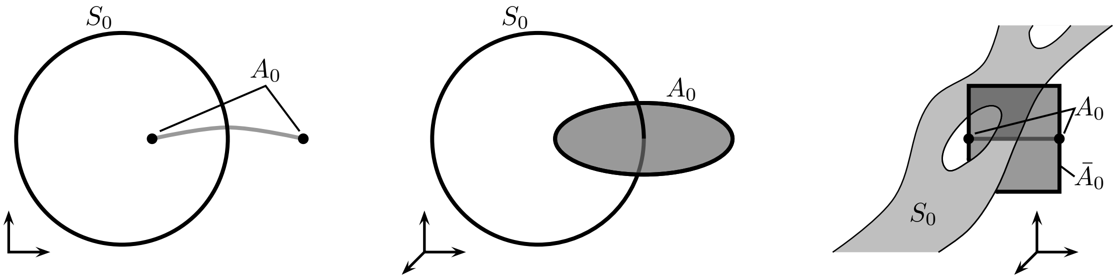

Figure 1: The left and middle picture show linking sets in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. The 'filled sets' are indicated in each one. The figure on the right depicts how a link (A_0, S_0) in $\mathbb{R}^2$ can be grown into a link $(\bar{A}_0, S_0)$ in $\mathbb{R}^3$.

a link (A, S) in the function space. In Section 5.3 all details are explained. As for the initial link, we invoke the topology of M. This topology is characterized in terms of the 0-sublevel set of the potential V as follows. Recalling that N is the projection of M onto the q-coordinate, we have that

$$M \cong \Big(S^{n-1} \times N\Big) \cup_{S^{n-1}\times\partial N} \Big(D^n \times \partial N\Big).$$

The topological information from $H_i(M)$ is closely related to the topology of the projection N, and, in particular, its complement.

Theorem 2. *Let M be a regular mechanical hypersurface of dimension $2n-1$, then*

$$H_{i+n}(M) \cong \widetilde{H}_i(\mathbb{R}^n - N), \quad 0 \le i \le n-1.$$

As usual, $\widetilde{H}_*$ denotes reduced homology. The nontrivial topology of the complement of the projection of M is used in an essential way to construct an initial link (A_0, S_0). We take $S_0 = N$, and since $H_i(M) \neq 0$ for some $i \geq n$ implies that $\widetilde{H}_{i-n}(\mathbb{R}^n - N) \neq 0$, one can, roughly speaking, find a set $A_0 \subset \mathbb{R}^n$ such that $[A_0]$ is a nonzero element of $\widetilde{H}_{i-n}(\mathbb{R}^n - N)$. Crucial for the minimax construction is that $V|_N \le 0$, and $V|_{A_0} > 0$. The second part of the argument in Section 5 (and appendix A) is to lift the link (A_0, S_0) to the function space. Here one also has to take into account that the period is a variable. In the end we find a nontrivial relative homology class in the function space, based on the homological data in $H_i(M)$, $i \geq n$. Reformulated in terms of the topology of sublevel sets, we show that there are nontrivial homomorphisms

$$\begin{aligned} h &: H_{i-n+2}(\mathcal{X}, \mathcal{A}^{\bar{a}}) \longrightarrow H_i(M) \qquad \text{for some } n \le i \le 2n-1, \\ \hat{h} &: H_{i-n+2}(\mathcal{A}^{\hat{a}}, \mathcal{A}^{\bar{a}}) \longrightarrow H_i(M) \qquad \text{for some } n \le i \le 2n-1. \end{aligned}$$

Here $\mathcal{X}$ denotes the space of periodic functions, $\mathcal{A}$ is the Lagrangian action functional, and $\mathcal{A}^{\bar{a}}$ its $\bar{a}$-sublevel set. The level $\bar{a}$ is chosen such that $\sup_A \mathcal{A} < \bar{a} < \inf_S \mathcal{A}$, and $\hat{a}$ is some suitable level above $\bar{a}$. Nontriviality of $H_{i-n+2}(\mathcal{A}^{\hat{a}}, \mathcal{A}^{\bar{a}})$ leads, in view of the established Palais-Smale property, to a critical value between $\bar{a}$ and $\hat{a}$, and the critical point corresponds to a closed characteristic on M.

The theorem that we prove in this paper says nothing about multiplicity of solutions. In some special cases however, the homological information can also provide multiplicity results, e.g. [14]. Recent results by Long [15] show that such statements are extremely hard to prove in general. In Section 7 we will elaborate some more on the question of multiplicity, and possible future directions. Furthermore, we obviously did not choose the most general class of Hamiltonians in this paper, and in Section 7 we also discuss some generalizations that can easily be made. These include Hamiltonians of the form $H(p,q) = \frac{1}{2}\langle A(q)p, p\rangle + V(q)$, Hamiltonians defined on an underlying configuration space different from $\mathbb{R}^n$, and Hamiltonians stemming from higher order Lagrangians.

The outline of the paper is as follows. In Section 2 some preliminary observations are made, which are subsequently used in Section 3 to establish the Palais-Smale property. The linking (or minimax) characterization of existence of a closed characteristic is presented in Section 4, while the linking sets are constructed in Section 5. Theorem 2 is proved in Section 6. As already mentioned, Section 7 deals with variations on asymptotic regularity and other generalizations, as well as a view towards the future. Appendix A presents the construction of an important auxiliary function needed in the construction of the linking sets. In appendix B we collect the results that lead to nontriviality of $H_{i-n+2}(\mathcal{A}^{\hat{a}}, \mathcal{A}^{\bar{a}})$ as expressed above. Finally, appendix C provides the proofs of some more technical lemmas.

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2 Mechanical Lagrangian systems

2.1 Notation and preliminary observations

In the introduction we defined a hypersurface M to be the 0-energy surface of a Hamiltonian $H(p,q) = \frac{1}{2}|p|^2 + V(q)$. Closed characteristics on such hypersurfaces can be regarded as critical points of a suitable action functional. For this purpose we define the Lagrangian function

$$L(q,q') = \frac{1}{2}|q'|^2 - V(q).$$

The variational principle for finding closed characteristics on M can be formulated as follows:

$$\delta_{q,T} \int_0^T L\big(q,q'\big)dt = 0, \tag{1}$$

where the variations are with respect to T-periodic functions $q : [0,T] \to \mathbb{R}^n$ and periods $T > 0$. Indeed, extremals of the variational problem (1) are related to closed characteristics on hypersurfaces due to the 'conservation of energy'; extremals $q(t)$ of (1) satisfy a conservation law $\frac{1}{2}|q'(t)|^2 + V(q(t)) = E = \text{constant}$. Extremals satisfy the differential equation $q'' + DV(q) = 0$. In canonical coordinates this yields the first order (Hamiltonian) system: $q' = p$, and $p' = -DV(q)$, where the right-hand side is the Hamiltonian vector field X_H defined by the relation $i_{X_H}\omega = -dH$. From the equation it is immediately clear that solutions lie on level sets of H. Therefore, X_H restricted to $H^{-1}(E)$ satisfies $i_{X_H}\omega = 0$, which implies that X_H is a section in the characteristic line bundle of $H^{-1}(E)$. The variational principle in (1) produces characteristics in the specific level set $H^{-1}(0)$ due to variations in both q and T, see Lemma 3 below.

Let us start with a functional analytic framework for the variational principle.

Define the set

$$\mathcal{X} = \big\{(q,T) \;\big|\; q \in H^1(\mathbb{R}/T\mathbb{Z};\mathbb{R}^n),\, T \in \mathbb{R}^+\big\},$$

which can be given the structure of a Hilbert manifold over $H^1(\mathbb{R}/\mathbb{Z};\mathbb{R}^n) \times \mathbb{R}$. This can be done via a *global* coordinate transformation:

$$(q(t),T) \mapsto \big(q(sT), \log(T)\big) = \big(u(s), \tau\big).$$

We will denote the inverse of this coordinate transformation by ξ. For the action $\mathcal{A}(q,T) = \int_0^T L(q,q')dt$ this yields

$$\begin{aligned}\mathcal{B}(u,\tau) = (\mathcal{A}\circ\xi)(u,\tau) = e^{\tau}\int_0^1 L(u, e^{-\tau}u')ds &\stackrel{\text{def}}{=} \int_0^1 e^{\tau}\widehat{L}(u,u',\tau)ds \\ &= \frac{e^{-\tau}}{2}\int_0^1 |u'|^2 ds - e^{\tau}\int_0^1 V(u)ds,\end{aligned} \tag{2}$$

where $\widehat{L}(u,u',\tau) = e^{-2\tau}|u'|^2 - V(u)$. We now show that extremals of $\mathcal{A}$ (or $\mathcal{B}$) are in the energy level $H = 0$. This may also be compared with the general variation formula (4).

Lemma 3. *Extremals of* (1) *satisfy* $H = 0$.

Proof. With respect to variations δu and $\delta\tau$, the integral in (1), using its reformulation in (2) via the coordinate transformation ξ, yields (with periodic boundary conditions)

$$\begin{aligned}\delta_{u,\tau}\int_0^1 e^{\tau}\widehat{L}(u,u',\tau)ds = &\int_0^1 e^{\tau}\Big\{\frac{\partial\widehat{L}}{\partial u} - \frac{d}{ds}\frac{\partial\widehat{L}}{\partial u'}\Big\}\delta u\, ds \\ &+ \int_0^1 e^{\tau}\Big\{\widehat{L}(u,u',\tau) - \frac{\partial\widehat{L}}{\partial\tau}\Big\}\delta\tau\, ds.\end{aligned}$$

For extremals the Euler-Lagrange equations are satisfied, so that

$$\int_0^1 e^{\tau}\Big\{\widehat{L} - \frac{\partial\widehat{L}}{\partial\tau}\Big\}ds = \int_0^T \Big\{L(q,q') - \frac{\partial L}{\partial q'}q'\Big\}dt = \int_0^T H(p,q)dt = 0,$$

which proves that extremals lie in the level set $H = 0$. ☐

On the Sobolev space $H^1(\mathbb{R}/\mathbb{Z};\mathbb{R}^n)$ we will use two equivalent norms:

$$\begin{aligned}\|u\|_{H^1}^2 &= \int_0^1 |u'(s)|^2 + |u(s)|^2 ds,\\ \|u\|_1^2 &= \int_0^1 |u'(s)|^2 ds + \Big|\int_0^1 u(s)ds\Big|^2.\end{aligned}$$

The interpretation is that a function u is split into its average $u^0 = \int_0^1 u(s)ds$ and its oscillatory part $u^+ = u - u^0$ (which has zero average). We will use the notation $u = u^0 + u^+$ throughout, as well as the decomposition $H^1(\mathbb{R}/\mathbb{Z};\mathbb{R}^n) \cong E^0 \oplus E^+$, where $E^0 \cong \mathbb{R}^n$ and $E^+ = \{u \in H^1 \mid \int_0^1 u(s)ds = 0\}$. A straightforward estimate shows that the deviation of u from its average is controlled by the norm of the oscillatory part:

$$|u(t) - u^0| \le \|u^+\|_1 \qquad \text{for all } t. \tag{3}$$

The next lemma states how asymptotic regularity of V controls the variability of DV.

Lemma 4. *If* $\|D^2V\| \leq C|DV|$ *on the line segment joining* u *and* u_0*, then*

$$|DV(u) - DV(u_0)| \leq |DV(u_0)|\,(e^{C|u-u_0|} - 1).$$

Proof. The proof, provided in appendix C, is analogous to that of Gronwall's inequality. □

2.2 The variation formula

In order to explain the variational principle we start with a general variation formula. Although the problem will be put in a purely functional analytic framework later, we first use some notation from the Lie group setting (cf. [18]), mainly to illustrate the ideas and difficulties and to motivate our subsequent choices. Let $\tau \in \mathbb{R}$, and let u be a 1-periodic function of s. We consider variations as 1-parameter transformations: $\tau \mapsto \hat{\tau}(\epsilon, \tau)$, $s \mapsto \hat{s}(\epsilon, s)$, $u \mapsto \hat{u}(\epsilon, u, s)$. We could choose more general classes of variations, but these suffice for our purposes. In first order approximation this yields

$$\tau \mapsto \tau + \epsilon\delta\tau, \qquad s \mapsto s + \epsilon\delta s, \qquad u \mapsto u + \epsilon\delta u,$$

where $\delta\tau = \frac{d}{d\epsilon}\hat{\tau}|_{\epsilon=0}$, $\delta s = \frac{d}{d\epsilon}\hat{s}|_{\epsilon=0}$, and $\delta u = \frac{d}{d\epsilon}\hat{u}|_{\epsilon=0}$. Infinitesimal variations are described by the vector field

$$X = \delta\tau\frac{\partial}{\partial\tau} + \delta s\frac{\partial}{\partial s} + \delta u\frac{\partial}{\partial u}.$$

Since τ and s are independent variables, and u is an s-dependent variable, we need a prolonged vector field to describe variations of u and its derivative (cf. Olver [18]):

$$X^1 = \delta\tau\frac{\partial}{\partial\tau} + \delta s\frac{\partial}{\partial s} + \delta u\frac{\partial}{\partial u} + \delta u'\frac{\partial}{\partial u'},$$

with $\delta u' = (\delta u)' - u'(\delta s)'$. Here primes denote total derivatives with respect to the variable s, e.g. $(\delta u)' = \frac{\partial(\delta u)}{\partial s} + \frac{\partial(\delta u)}{\partial u}u'$. We apply these variations to the 1-form

$$\lambda = \frac{1}{2}e^{-\tau}|u'|^2 ds - e^{\tau}V(u)ds.$$

Recall the Lie-derivative of λ along the vector field X^1: $\mathcal{L}_{X^1}\lambda = di_{X^1}\lambda + i_{X^1}d\lambda$. When we evaluate $\mathcal{L}_{X^1}\lambda$ we obtain:

$$\mathcal{L}_{X^1}\lambda = -\big(\delta\tau + (\delta s)'\big)\Big[\tfrac{1}{2}e^{-\tau}|u'|^2 + e^{\tau}V(u)\Big]ds + e^{-\tau}u'(\delta u)'ds - e^{\tau}DV(u)\delta u\,ds.$$

The associated variation for the functional $\mathcal{B}(u,\tau) = \int_0^1 \lambda$ can be derived from the above formula:

$$\begin{aligned}
\mathcal{B}'(u,\tau)(\delta u, \delta s, \delta\tau) &= \int_0^1 \mathcal{L}_{X^1}\lambda \\
&= \int_0^1 e^{-\tau}u'(\delta u)'ds - \int_0^1 e^{\tau}DV(u)\delta u ds && \text{(4a)}\\
&\quad - \int_0^1 \Big[\frac{1}{2}e^{-\tau}|u'|^2 + e^{\tau}V(u)\Big]\delta\tau ds && \text{(4b)}\\
&\quad + \int_0^1 \Big[\frac{1}{2}e^{-\tau}|u'|^2 + e^{\tau}V(u)\Big]'\delta s ds, && \text{(4c)}
\end{aligned}$$

where we used 1-periodicity of u and δs. We have split the right-hand side in three integrals. The variational formula suggests that if $\{u_n\}$ is a (Palais-Smale) sequence of functions converging to a critical point of $\mathcal{B}$ with respect to these variations, then the functions $u_n(t)$ not only almost satisfy the Euler-Lagrange equation due to (4a), but are also everywhere near the energy level $H = 0$ due to (4c). This means that the functions 'live' in a neighborhood of the energy level, giving useful analytic estimates. However, variations in s are not in accordance with the functional analytic setting. Therefore, due to (4b), we only have control over the average energy along a Palais-Smale sequence. To enhance analytic control, we need to introduce a penalizing function as is done in the next section. To summarize, we work with variations $\delta\tau$ and δu only. Additionally, we will only use variations δu that are independent of s, further reducing the variational formula:

$$\begin{aligned}\mathcal{B}'(u,\tau)(\delta u,\delta\tau) \quad &= \quad \int_0^1 \left\{ e^{-\tau} \left\langle u', \frac{d(\delta u)}{du} u' \right\rangle - e^{\tau} \langle DV(u), \delta u \rangle \right\} ds \\ &\quad - \int_0^1 \left\{ \left[\frac{1}{2} e^{-\tau} |u'|^2 + e^{\tau} V(u) \right] \delta\tau \right\} ds. \qquad (5)\end{aligned}$$

3 The Palais-Smale condition

We start by introducing a penalizing function. For $\varepsilon > 0$ consider the functional

$$\mathcal{B}_\varepsilon(u,\tau) = \mathcal{B}(u,\tau) + \varepsilon\big(e^{-\tau} + e^{\tau/2}\big). \qquad (6)$$

A sequence $(u_n, \tau_n) \in H^1 \times \mathbb{R}$ is called a Palais-Smale sequence if

$$\mathcal{B}_\varepsilon'(u_n,\tau_n) \to 0, \quad 0 < c_1 \le \mathcal{B}_\varepsilon(u_n,\tau_n) \le c_2 < \infty, \qquad \text{as } n \to \infty.$$

The following proposition states that the functionals $\mathcal{B}_\varepsilon$, $\varepsilon > 0$ satisfy the Palais-Smale condition. We will use this in Section 4 to find critical points of $\mathcal{B}_\varepsilon$.

Proposition 5. *Let (u_n, τ_n) be a Palais-Smale sequence for $\mathcal{B}_\varepsilon$, then there exists a convergent subsequence $(u_{n'}, \tau_{n'}) \to (u_\varepsilon, \tau_\varepsilon)$ in $H^1 \times \mathbb{R}$, $n' \to \infty$. The limit function satisfies $\mathcal{B}_\varepsilon'(u_\varepsilon, \tau_\varepsilon) = 0$, and $0 < c_1 \le \mathcal{B}_\varepsilon(u_\varepsilon, \tau_\varepsilon) = c_\varepsilon \le c_2$.*

The next proposition states that the critical points of the penalized functional $\mathcal{B}_\varepsilon$ converge to a critical point of $\mathcal{B}$ as $\varepsilon \to 0$. The latter critical point corresponds to a closed characteristic on the energy surface M.

Proposition 6. *Let $(u_\varepsilon, \tau_\varepsilon)$, $\varepsilon \to 0$ be a sequence satisfying $\mathcal{B}_\varepsilon'(u_\varepsilon, \tau_\varepsilon) = 0$, and $0 < c_1 \le \mathcal{B}_\varepsilon(u_\varepsilon, \tau_\varepsilon) \le c_2$. Then there exists a convergent subsequence $(u_{\varepsilon'}, \tau_{\varepsilon'}) \to (u, \tau)$ in $H^1 \times \mathbb{R}$, $\varepsilon' \to 0$. The limit function satisfies $\mathcal{B}'(u,\tau) = 0$, and $0 < c_1 \le \mathcal{B}(u,\tau) \le c_2$.*

Using the transformation ξ from Section 2 and Lemma 3 we see that the limit function from Proposition 6 leads to a closed characteristic $q(t) = u(e^{-\tau} t)$ on M of period $T = e^{\tau}$. Combining the two propositions above thus implies that existence of Palais-Smale sequences for $\mathcal{B}_\varepsilon$, for all sufficiently small $\varepsilon > 0$, leads to a proof of Theorem 1. Those Palais-Smale sequences will be obtained in Sections 4 and 5 using homological linking arguments.

The proof of these propositions is based on several auxiliary lemmas. The total variation of $\mathcal{B}_\varepsilon$ with respect to variations $(\delta u, \delta\tau) \in H^1 \times \mathbb{R}$ is given by (cf. (5)):

$$\begin{aligned}
\mathcal{B}'_\varepsilon(u,\tau)(\delta u, \delta\tau) &= \mathcal{B}'(u,\tau)(\delta u,\delta\tau) + \varepsilon\big(-e^{-\tau} + \frac{1}{2}e^{\tau/2}\big)\delta\tau \\
&= \int_0^1 \Big\{ e^{-\tau}\Big\langle u', \frac{d(\delta u)}{du}u'\Big\rangle - e^{\tau}\left\langle DV(u), \delta u\right\rangle \Big\} ds \\
&\qquad - \int_0^1 \Big\{ \Big[\frac{1}{2}e^{-\tau}|u'|^2 + e^{\tau}V(u)\Big]\delta\tau \Big\} ds + \varepsilon\big(-e^{-\tau} + \frac{1}{2}e^{\tau/2}\big)\delta\tau.
\end{aligned}$$

For fixed $\varepsilon > 0$, let (u_n, τ_n) be a Palais-Smale sequence for $\mathcal{B}_\varepsilon$. Extracting a subsequence we may assume that $\mathcal{B}_\varepsilon(u_n,\tau_n) \to c_\varepsilon$ for some $c_\varepsilon \in [c_1, c_2]$. The derivative of $\mathcal{B}_\varepsilon(u_n,\tau_n)$ going to zero is equivalent to

$$\mathcal{B}'_\varepsilon(u_n,\tau_n)(\delta u,\delta\tau) = o(1)\big(\|\delta u\|_{H^1} + |\delta\tau|\big), \qquad \text{as } n\to\infty, \tag{7}$$

uniformly for all variations $(\delta u, \delta\tau) \in H^1 \times \mathbb{R}$. The first step is to obtain estimates on the integrals $\int_0^1 e^{-\tau_n}|u_n'|^2 ds$ and $\int_0^1 e^{\tau_n}V(u_n)ds$.

Lemma 7. *A Palais-Smale sequence (u_n,τ_n) satisfies*

$$\int_0^1 e^{-\tau_n}|u_n'|^2 ds + \varepsilon\big(2e^{-\tau_n} + \frac{1}{2}e^{\tau_n/2}\big) = c_\varepsilon + o(1), \qquad \textit{as } n\to\infty; \tag{8}$$

$$\int_0^1 e^{\tau_n}V(u_n)ds - \varepsilon\frac{3}{4}e^{\tau_n/2} = -\frac{c_\varepsilon}{2} + o(1), \qquad \textit{as } n\to\infty. \tag{9}$$

Proof. Consider variations of the form $(\delta u, \delta\tau) = (0,1)$. From the variation formula and (7) we then derive that

$$\frac{1}{2}\int_0^1 e^{-\tau_n}|u_n'|^2 ds + \int_0^1 e^{\tau_n}V(u_n)ds = -\varepsilon\big(e^{-\tau_n} - \frac{1}{2}e^{\tau_n/2}\big) + o(1),$$

as $n\to\infty$. On the other hand, $\mathcal{B}_\varepsilon(u_n,\tau_n)\to c_\varepsilon$ means that

$$\frac{1}{2}\int_0^1 e^{-\tau_n}|u_n'|^2 ds - \int_0^1 e^{\tau_n}V(u_n)ds = -\varepsilon\big(e^{-\tau_n} + e^{\tau_n/2}\big) + c_\varepsilon + o(1).$$

Combining these two estimates completes the proof. □

This leads to bounds from below and above on τ_n.

Lemma 8. *Let (u_n,τ_n) be a Palais-Smale sequence. There are constants $T_0 < T_1$ (depending on ε) such that $T_0 \le \tau_n \le T_1$ for sufficiently large n.*

Proof. Equation (8) implies that $\varepsilon(2e^{-\tau_n} + \frac{1}{2}e^{\tau_n/2}) \le c_\varepsilon + 1$ for sufficiently large n. The assertion follows immediately from this inequality. □

With these bounds on τ_n we obtain a bound on u_n.

Lemma 9. *Let (u_n,τ_n) be a Palais-Smale sequence. There is a constant C (depending on ϵ) such that $\|u_n\|_1 \le C$ for sufficiently large n.*

Proof. Equation (8), combined with the bounds on τ_n from Lemma 8, implies that $\|u_n^+\|_1 = (\int_0^1 |u_n'|^2 ds)^{1/2}$ is bounded, say by C_0. It remains to estimate $u_n^0 = \int_0^1 u_n ds$. Let us argue by contradiction, and assume that $|u_n^0| \to \infty$ as $n \to \infty$. Note that $\|u_n - u_n^0\|_\infty \leq C_0$ by (3). It now follows from asymptotic regularity and Lemma 4 that $DV(u_n(t)) \neq 0$ and

$$\frac{\|D^2V(u_n)\|}{|DV(u_n)|^2} \to 0 \quad \text{uniformly} \qquad \text{as } n \to \infty. \tag{10}$$

Consider variations of the form

$$\delta u = -\frac{DV(u_n)}{|DV(u_n)|^2} \qquad \text{and} \qquad \delta\tau = 0.$$

The variation formula gives

$$\begin{aligned}
&\mathcal{B}_\varepsilon'(u_n, \tau_n)(\delta u, \delta\tau) = \\
&\quad -\int_0^1 e^{-\tau_n}\left(\frac{\langle u_n', D^2V(u_n)u_n'\rangle}{|DV(u_n)|^2} - 2\frac{\langle u_n', DV(u_n)\rangle\,\langle DV(u_n), D^2V(u_n)u_n'\rangle}{|DV(u_n)|^4}\right)ds \\
&\qquad + \int_0^1 e^{\tau_n}\frac{\langle DV(u_n), DV(u_n)\rangle}{|DV(u_n)|^2}ds. \\
&\quad = e^{\tau_n} + o(1) \qquad \text{as } n \to \infty,
\end{aligned} \tag{11}$$

where we have used (10) and the bounds on $\int_0^1 |u_n'|^2 ds$ and τ_n. On the other hand, (u_n, τ_n) is a Palais-Smale sequence, hence, again using asymptotic regularity,

$$\mathcal{B}_\varepsilon'(u_n, \tau_n)(\delta u, \delta\tau) = o(1)\|\delta u\|_{H^1} = o(1)(c^{-1} + C_0 o(1)) = o(1) \qquad \text{as } n \to \infty, \tag{12}$$

where the bound on $\|\delta u\|_{H^1}$ is obtained as follows. Clearly, $|\delta u| \leq c^{-1}$, and

$$\begin{aligned}
|(\delta u)'| &= \left| -\frac{D^2V(u)u'}{|DV(u)|^2} + 2\frac{\langle DV(u), D^2V(u)u'\rangle}{|DV(u)|^4}DV(u) \right| \\
&\leq \left| \frac{D^2V(u)u'}{|DV(u)|^2} \right| + 2\left| \frac{\langle DV(u), D^2V(u)u'\rangle}{|DV(u)|^3} \right| \\
&\leq 3\,\frac{\|D^2V(u)\|}{|DV(u)|^2}\,|u'| = o(1)|u'|,
\end{aligned}$$

with $(\int_0^1 |u'|^2 ds)^{1/2} \leq C_0$. Since τ_n is bounded below by Lemma 8, estimate (12) contradicts (11). □

We now finish the proof of the Palais-Smale property for the (penalized) functional $\mathcal{B}_\varepsilon$.

Proof of Proposition 5. The sequence τ_n is bounded by Lemma 8, hence it has a convergent subsequence, say $\tau_n \to \tau_\varepsilon \in \mathbb{R}$. Let $\partial_u \mathcal{B}_\varepsilon(u, \tau) = \mathcal{B}_\varepsilon'(u, \tau)(\cdot, 0)$, then $\partial_u \mathcal{B}_\varepsilon(\cdot, \tau_n)$ is of the form $e^{-\tau_\varepsilon}\mathrm{id} + K + R_n$, where K is compact and $R_n \to 0$ as $n \to \infty$. Since $\mathcal{B}_\varepsilon' \to 0$ as $n \to \infty$, the boundedness of u_n implies that there exists a convergent subsequence $u_{n'} \to u_\varepsilon \in H^1$. Since $\mathcal{B}_\varepsilon$ is continuously differentiable, this establishes the Palais-Smale property. □

We can characterize the critical points of $\mathcal{B}_\varepsilon$ as follows.

Lemma 10. *A critical point $(u_\varepsilon, \tau_\varepsilon)$ of $\mathcal{B}_\varepsilon$ solves the Euler-Lagrange equation*

$$e^{-\tau_\varepsilon} u_\varepsilon'' + e^{\tau_\varepsilon} DV(u_\varepsilon) = 0,$$

and satisfies the energy identity

$$E_\varepsilon \stackrel{\text{def}}{=} \frac{e^{-2\tau_\varepsilon}}{2}|u_\varepsilon'|^2 + V(u_\varepsilon) = \varepsilon\Big(-e^{-2\tau_\varepsilon} + \frac{1}{2}e^{-\tau_\varepsilon/2}\Big). \tag{13}$$

Proof. Since $\mathcal{B}_\varepsilon'(u_\varepsilon, \tau_\varepsilon) = 0$, taking variations $(\delta u, 0)$ leads to the first statement, while variations $(0, \delta\tau)$ then prove the energy identity. □

A consequence of Lemma 10 is that $q_\varepsilon(t) = u_\varepsilon(e^{-\tau_\varepsilon} t)$ is a closed characteristic on $H^{-1}(E_\varepsilon)$ with period $T_\varepsilon = e^{\tau_\varepsilon}$. To prove Proposition 6 we need to show that τ_ε is bounded, and in turn the same for u_ε. We start with an upper bound on τ_ε. In Section 7 we come back to the proof of this lemma and put it into the context of contact forms.

Lemma 11. *Let $(u_\varepsilon, \tau_\varepsilon)$ be critical points of $\mathcal{B}_\varepsilon$ with $0 < c_1 \le \mathcal{B}_\varepsilon(u_\varepsilon, \tau_\varepsilon) \le c_2$. Then there is a constant T_2, independent of ε, such that $\tau_\varepsilon \le T_2$ for sufficiently small ε.*

Proof. Let $\tau_\varepsilon \ge 0$, then from (13) we see that $0 \le |E_\varepsilon| \le \varepsilon$. We are going to use variations

$$\delta u = -\kappa \frac{DV(u_\varepsilon)}{1 + |DV(u_\varepsilon)|^2} \qquad \text{and} \qquad \delta\tau = -1, \tag{14}$$

for some small $\kappa > 0$ to be chosen shortly.

The first claim is that (similar to Lemma 9), for some $C_1 > 0$,

$$\begin{aligned} |(\delta u)'| &= \kappa \left| \frac{D^2V(u_\varepsilon)u_\varepsilon'}{1 + |DV(u_\varepsilon)|^2} + 2\frac{DV(u_\varepsilon)\left\langle DV(u_\varepsilon), D^2V(u_\varepsilon)u_\varepsilon'\right\rangle}{(1 + |DV(u_\varepsilon)|^2)^2} \right| \\ &\le C_1 \kappa |u_\varepsilon'|. \end{aligned} \tag{15}$$

For u_ε sufficiently large, say $|u_\varepsilon| > R$, this follows from asymptotic regularity. On the other hand, inside the ball $B_R(0)$, since V is a C^2 function, the derivatives D^2V and DV are uniformly bounded. This proves the claim $|(\delta u)'| \le C_1\kappa|u_\varepsilon'|$.

Next we choose $\kappa = \frac{1}{2C_1}$ and claim there is a constant $C_2 > 0$ such that

$$\begin{aligned} e^{-2\tau_\varepsilon}\Big(|u_\varepsilon'|^2 + \left\langle u_\varepsilon', (\delta u)'\right\rangle\Big) - \langle DV(u_\varepsilon), \delta u\rangle &\ge \frac{e^{-2\tau_\varepsilon}}{2}|u_\varepsilon'|^2 + \kappa\frac{|DV(u_\varepsilon)|^2}{1 + |DV(u_\varepsilon)|^2} \\ &\ge C_2. \end{aligned} \tag{16}$$

The first inequality follows from (14) and (15). To prove the second inequality we again start by exploiting asymptotic regularity to infer that it holds for $u_\varepsilon(s)$ outside some large ball $B_R(0)$ with $C_2 \le \kappa\frac{c^2}{1+c^2}$.

When $u_\varepsilon(s)$ is inside the ball the argument is more subtle. Since $M = H^{-1}(0)$ is a regular energy level, we have $DV(u) \ne 0$ at the level set $V(u) = 0$. By continuity this also holds for nearby level sets of V, at least when restricted to the ball $B_R(0)$. We conclude that $|DV(u)| \ge C_3 > 0$ for all $u \in B_R(0)$ with $V(u)$ sufficiently small. If

$\frac{1}{2}e^{-2\tau_\varepsilon}|u_\varepsilon'|^2 \le C_4$, then it follows from (13) that $|V(u_\varepsilon)| \le |E_\varepsilon| + C_4 \le \varepsilon + C_4$. Hence for C_4 and ε sufficiently small, $|DV(u_\varepsilon)| \ge C_3$ whenever $\frac{1}{2}e^{-2\tau_\varepsilon}|u_\varepsilon'|^2 \le C_4$. Taking $C_2 \le \min\{\kappa\frac{C_3^2}{1+C_3^2}, C_4\}$ we see that (16) also holds in $B_R(0)$.

We are now suitably prepared to use the variations (14):

$$\begin{aligned}
c_\varepsilon &= \mathcal{B}_\varepsilon(u_\varepsilon,\tau_\varepsilon) + \mathcal{B}_\varepsilon'(u_\varepsilon,\tau_\varepsilon)(\delta u,\delta\tau)\\
&= e^{\tau_\varepsilon}\int_0^1 \Big\{e^{-2\tau_\varepsilon}\left(|u_\varepsilon'|^2 + \langle u_\varepsilon', (\delta u)'\rangle\right) - \langle DV(u_\varepsilon), \delta u\rangle\Big\}ds + \varepsilon(2e^{-2\tau_\varepsilon} + \frac{1}{2}e^{-\tau_\varepsilon/2})\\
&\ge C_2 e^{\tau_\varepsilon}.
\end{aligned}$$

Since $c_\varepsilon \le c_2$, we find the upper bound $\tau_\varepsilon \le \max\{\log(c_2/C_2), 0\}$. ☐

Next we establish a lower bound on τ_ε, corresponding to a lower bound on the period T_ε.

Lemma 12. *Let $(u_\varepsilon,\tau_\varepsilon)$ be critical points of $\mathcal{B}_\varepsilon$ with $0 < c_1 \le \mathcal{B}_\varepsilon(u_\varepsilon,\tau_\varepsilon) \le c_2$. Then there is a constant T_3, independent of ε, such that $\tau_\varepsilon \ge T_3$ for sufficiently small ε.*

Proof. We argue by contradiction and assume that $\tau_\varepsilon \to -\infty$ as $\varepsilon \to 0$. From (8) we have that

$$\int_0^1 |u_\varepsilon'|^2 ds = c_\varepsilon e^{\tau_\varepsilon} - 2\varepsilon - \frac{\varepsilon}{2}e^{3\tau_\varepsilon/2} \to 0, \qquad \text{as } \varepsilon \to 0.$$

We decompose u_ε in its average and its oscillatory part: $u_\varepsilon = u_\varepsilon^0 + u_\varepsilon^+$. It follows from (3) that $u_\varepsilon^+ \to 0$ uniformly. If u_ε^0 is bounded as $\varepsilon \to 0$, then $\int_0^1 e^{\tau_\varepsilon} V(u_\varepsilon)ds \to 0$. On the other hand, Equation (9) implies that $\int_0^1 e^{\tau_\varepsilon} V(u_\varepsilon)ds = -\frac{c_\varepsilon}{2} + o(1)$ as $\varepsilon \to 0$, and $c_\varepsilon \ge c_1 > 0$, contradicting the assumption that u_ε^0 is bounded as $\varepsilon \to 0$.

It remains to show that $|u_\varepsilon^0| \to \infty$ also leads to a contradiction. We now use what is in essence a flow box argument. The periodic functions u_ε satisfy the Euler-Lagrange equation $e^{-2\tau_\varepsilon}u_\varepsilon'' + DV(u_\varepsilon) = 0$, and therefore (component-wise)

$$\int_0^1 DV(u_\varepsilon(s))\, ds = 0. \tag{17}$$

Since $|u_\varepsilon^0| \to \infty$ it follows from asymptotic regularity that $|DV(u_\varepsilon^0)| \ge c > 0$ for small ε, so there is a component i_ε of the vector $DV(u_\varepsilon^0)$, denoted by $D_{i_\varepsilon}V(u_\varepsilon^0)$, such that $|D_{i_\varepsilon}V(u_\varepsilon^0)| \ge |DV(u_\varepsilon^0)|/n > 0$. From Lemma 4, asymptotic regularity, and the fact that $\|u_\varepsilon - u_\varepsilon^0\|_\infty \to 0$, it follows that $|DV(u_\varepsilon(t)) - DV(u_\varepsilon^0)| \le |DV(u_\varepsilon^0)|/2n$ for all t, provided ε is sufficiently small. In particular, $|D_{i_\varepsilon}V(u_\varepsilon(t))| \ge |DV(u_\varepsilon^0)|/2n > 0$ for all t, which contradicts (17). ☐

Finally, we prove that the critical points of $\mathcal{B}_\varepsilon$ converge to a critical point of $\mathcal{B}$.

Proof of Proposition 6. Lemmas 11 and 12 provide a uniform bound on τ_ε, We then observe that the arguments in the proof of Lemma 9 lead to an ε-independent bound on $\|u_\varepsilon\|_1$. An argument analogous to the one in the proof of Proposition 5 shows that $(u_\varepsilon,\tau_\varepsilon)$ converges along a subsequence to a critical point of $\mathcal{B}$. ☐

4 Minimax characterizations

In this section we will link the topology of M to minimax values of the functionals $\mathcal{B}$ and $\mathcal{B}_\varepsilon$. Here we follow the general setup of [19]. Consider disjoint sets A and S in $H^1 \times \mathbb{R}$. The sets A and S are said to (homologically) link if the inclusion induced homomorphism

$$i_q : \widetilde{H}_q(A) \longrightarrow \widetilde{H}_q(H^1 \times \mathbb{R} - S)$$

is nontrivial for some $q \geq 0$. We will drop the tilde from our notation to prevent cluttered symbols, but we always silently assume that for $q = 0$ we are considering reduced homology. In order to use the linking sets A and S for finding a critical value we assume that the functional $\mathcal{B}$ satisfies the following conditions with respect to A and S:

(i) $\mathcal{B}|_S \geq a > 0$,

(ii) $\mathcal{B}|_A \leq b < a$.

Lemma 13. *Let A and S be linking subsets of $H^1 \times \mathbb{R}$. If $\mathcal{B}$ satisfies* (i) *and* (ii)*, and A is bounded, then $\mathcal{B}$ has a critical value $c_{A,S}$ with $0 < \frac{a}{2} \leq c_{A,S} < \infty$.*

Proof. Let q be the dimension for which the homomorphism i_q is nontrivial. Choose auxiliary values $\bar{a} \geq \frac{a}{2}$ and $\bar{b}$ such that $b < \bar{b} < \bar{a} < a$. For the penalized functional $\mathcal{B}_\varepsilon$ defined by (6) we have:

$$\begin{aligned} &\mathcal{B}_\varepsilon\big|_S \geq \mathcal{B}\big|_S \geq a > \bar{a}, \\ &\mathcal{B}_\varepsilon\big|_A \leq \mathcal{B}\big|_A + \varepsilon(e^{-\tau} + e^{\tau/2})\big|_A \leq b + \varepsilon(e^{-\tau} + e^{\tau/2})\big|_A \leq \bar{b}, \end{aligned}$$

for all $\varepsilon \leq \varepsilon^*$, when $\varepsilon^* > 0$ is chosen sufficiently small. Here we have used that A is bounded (and ε^* depends on A). For any d, let $\mathcal{B}_\varepsilon^d = \{(u,\tau) \in H^1 \times \mathbb{R} \,|\, \mathcal{B}_\varepsilon(u,\tau) \leq d\}$ be the sublevel set of $\mathcal{B}_\varepsilon$. Then we have, for all $\varepsilon \leq \varepsilon^*$, the following inclusions:

$$A \subset \mathcal{B}_\varepsilon^{\bar{b}} \subset \mathcal{B}_\varepsilon^{\bar{a}} \subset H^1 \times \mathbb{R} - S,$$

and i_q factors as

$$H_q(A) \longrightarrow H_q(\mathcal{B}_\varepsilon^{\bar{b}}) \longrightarrow H_q(\mathcal{B}_\varepsilon^{\bar{a}}) \longrightarrow H_q(H^1 \times \mathbb{R} - S).$$

Since i_q is nontrivial by assumption, $H_q(\mathcal{B}_\varepsilon^{\bar{b}}) \neq 0$ and $H_q(\mathcal{B}_\varepsilon^{\bar{a}}) \neq 0$ for all $\varepsilon \leq \varepsilon^*$.

Notice that for $\varepsilon \leq \varepsilon^*$, the corresponding functionals satisfy $\mathcal{B}_\varepsilon \leq \mathcal{B}_{\varepsilon^*}$, so we also have inclusions

$$\mathcal{B}_{\varepsilon^*}^{\bar{a}} \subset \mathcal{B}_\varepsilon^{\bar{a}}, \qquad \mathcal{B}_{\varepsilon^*}^{\bar{b}} \subset \mathcal{B}_\varepsilon^{\bar{b}},$$

with induced maps in homology.

Consider the long exact homology sequence of the pair $(H^1 \times \mathbb{R}, \mathcal{B}^{\bar{a}})$:

$$H_{q+1}(H^1 \times \mathbb{R}) \longrightarrow H_{q+1}(H^1 \times \mathbb{R}, \mathcal{B}_\varepsilon^{\bar{a}}) \xrightarrow{\partial_{q+1}} H_q(\mathcal{B}_\varepsilon^{\bar{a}}) \longrightarrow H_q(H^1 \times \mathbb{R}). \quad (18)$$

Since $H_q(H^1 \times \mathbb{R}) \cong 0$ for all $q \geq 0$ (reduced homology for $q = 0$), the connecting morphism ∂_{q+1} is an isomorphism between $H_{q+1}(H^1 \times \mathbb{R}, \mathcal{B}_\varepsilon^{\bar{a}})$ and $H_q(\mathcal{B}_\varepsilon^{\bar{a}}) \neq 0$, hence $H_{q+1}(H^1 \times \mathbb{R}, \mathcal{B}_\varepsilon^{\bar{a}}) \neq 0$ for all $\varepsilon \leq \varepsilon^*$.

In exactly the same way one shows that $H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_\varepsilon)\neq 0$ for all $\varepsilon\le\varepsilon^*$. In particular, $H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_{\varepsilon^*})\neq 0$, so we can choose a class $\theta\in H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_{\varepsilon^*})$, $\theta\neq 0$. More precisely, using the fact that i_q is nontrivial, let $x\in H_q(A)$ be such that $i_qx\neq 0$ in $H_q(H^1\times\mathbb{R}-S)$, and denote by y the image of x under the map $H_q(A)\longrightarrow H_q(\mathcal{B}^{\bar b}_{\varepsilon^*})$. Since y gets mapped to $i_qx\neq 0$ (hence $y\neq 0$) by $H_q(\mathcal{B}^{\bar b}_{\varepsilon^*})\longrightarrow H_q(H^1\times\mathbb{R}-S)$, and since ∂_{q+1} is an isomorphism, there exists a $\theta\neq 0$ such that $\partial_{q+1}\theta=y$.

The inclusions $\mathcal{B}^{\bar b}_{\varepsilon^*}\subset\mathcal{B}^{\bar b}_\varepsilon$ induce maps in homology

$$\begin{aligned} H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_{\varepsilon^*})&\longrightarrow H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_\varepsilon), &&\theta\to\theta_\varepsilon,\\ H_q(\mathcal{B}^{\bar b}_{\varepsilon^*})&\longrightarrow H_q(\mathcal{B}^{\bar b}_\varepsilon), && y\to y_\varepsilon,\end{aligned}$$

which fit into the commutative diagram

$$\begin{array}{ccccccc} H_q(A) & \longrightarrow & H_q(\mathcal{B}^{\bar b}_{\varepsilon^*}) & \longrightarrow & H_q(\mathcal{B}^{\bar b}_\varepsilon) & \longrightarrow & H_q(H^1\times\mathbb{R}-S)\\ & & {\scriptstyle\partial_{q+1}}\uparrow\cong & & {\scriptstyle\partial_{q+1}}\uparrow\cong & & \\ & & H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_{\varepsilon^*}) & \longrightarrow & H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_\varepsilon) & & \end{array}$$

where the horizontal maps are induced by inclusions. This shows that $\partial_{q+1}\theta_\varepsilon=y_\varepsilon\neq 0$.

On the other hand, we also have an inclusion $\mathcal{B}^{\bar b}_{\varepsilon^*}\subset\mathcal{B}^{\bar a}_{\varepsilon^*}$, which induces

$$\begin{aligned} H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_{\varepsilon^*})&\longrightarrow H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar a}_{\varepsilon^*}), &&\theta\to\hat\theta,\\ H_q(\mathcal{B}^{\bar b}_{\varepsilon^*})&\longrightarrow H_q(\mathcal{B}^{\bar a}_{\varepsilon^*}), && y\to\hat y.\end{aligned}\tag{19}$$

By commutativity of

$$\begin{array}{ccccccc} H_q(A) & \longrightarrow & H_q(\mathcal{B}^{\bar b}_{\varepsilon^*}) & \longrightarrow & H_q(\mathcal{B}^{\bar a}_{\varepsilon^*}) & \longrightarrow & H_q(H^1\times\mathbb{R}-S)\\ & & {\scriptstyle\partial_{q+1}}\uparrow\cong & & {\scriptstyle\partial_{q+1}}\uparrow\cong & & \\ & & H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_{\varepsilon^*}) & \longrightarrow & H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar a}_{\varepsilon^*}) & & \end{array}$$

we have $\partial_{q+1}\hat\theta=\hat y\neq 0$. The class $\hat\theta$ descends to $\hat\theta_\varepsilon\in H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar a}_\varepsilon)$ such that $\partial_{q+1}\hat\theta_\varepsilon=\hat y_\varepsilon\neq 0$ for all $\varepsilon\le\varepsilon^*$ (exactly as above for θ_ε and y_ε).

Now let

$$c_\varepsilon=\inf_{\theta'_\varepsilon\in\theta_\varepsilon}\max_{|\theta'_\varepsilon|}\mathcal{B}_\varepsilon,$$

where the infimum is over all relative cycles θ'_ε in $C_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar b}_\varepsilon)$ that represent the class θ_ε, and $|\theta'_\varepsilon|$ denotes the support of such a cycle. By definition $\partial\theta'_\varepsilon$ is a q-cycle in $\mathcal{B}^{\bar b}_\varepsilon$, so

$$\sup_{\theta'_\varepsilon\in\theta_\varepsilon}\max_{|\partial\theta'_\varepsilon|}\mathcal{B}_\varepsilon\le\bar b<\bar a.$$

By (19) we may regard θ'_ε as a relative cycle $\hat\theta'_\varepsilon\in C_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar a}_\varepsilon)$, which represents the class $\hat\theta_\varepsilon\in H_{q+1}(H^1\times\mathbb{R},\mathcal{B}^{\bar a}_\varepsilon)$. Since the support does not change we get

$$\max_{|\theta'_\varepsilon|}\mathcal{B}_\varepsilon=\max_{|\hat\theta'_\varepsilon|}\mathcal{B}_\varepsilon>\bar a,$$

otherwise we would have $|\hat\theta'_\varepsilon|\subset\mathcal{B}^{\bar a}_\varepsilon$ and hence $\hat\theta_\varepsilon=[\hat\theta'_\varepsilon]=0$, which is a contradiction. It follows that

$$\inf_{\theta'_\varepsilon\in\theta_\varepsilon}\max_{|\theta'_\varepsilon|}\mathcal{B}_\varepsilon\ge\bar a>\bar b.$$

Since $\mathcal{B}_\varepsilon$ satisfies the Palais-Smale condition (see Proposition 5), the Linking Theorem (e.g. [5]) now implies that c_ε is a critical value for $\mathcal{B}_\varepsilon$ for all $\varepsilon \le \varepsilon^*$.

Clearly $c_\varepsilon \ge c_1 = \bar{a}$ for all $\varepsilon \le \varepsilon^*$, and choosing a particular representative $\tilde{\theta}' \in [\theta]$ we have $c_\varepsilon \le c_2 = \max_{|\tilde{\theta}'|} \mathcal{B}_{\varepsilon^*}$ for all $\varepsilon \le \varepsilon^*$. Due to these a priori bounds on c_ε, Proposition 6 produces a critical value $c_{A,S} \ge \bar{a} \ge \frac{a}{2}$ for $\mathcal{B}$. □

In order to conclude that $\mathcal{B}$ has a critical value (and thus prove Theorem 1) we need to find linking sets A and S satisfying the conditions (i) and (ii) given above. In the forthcoming section we specify such sets A and S using the topology of M.

5 The construction of the linking sets

5.1 Preliminaries: a link in $\mathbb{R}^n$

We need to find linking sets A and S in $H^1 \times \mathbb{R}$ satisfying the conditions (i) and (ii) in Section 4. We start by constructing linking sets in $\mathbb{R}^n$. By assumption, $H_{n+i}(M) \neq 0$ for some $i = 0, \ldots, n-1$, and from Theorem 2 we see that $H_{n+i}(M) \cong \widetilde{H}_i(\mathbb{R}^n - N) \neq 0$. We infer that one of the homology groups of the complement $\mathbb{R}^n - N$ of the projection of M is nonzero. It is more convenient to use the index $k = i+1$, hence let $k \in \{1, \ldots, n\}$ be such that $\widetilde{H}_{k-1}(\mathbb{R}^n - N) \neq 0$.

Notice that the case $k = n$ corresponds to the compact case, namely M closed and N compact (with boundary). In this case, in order to find a link it suffices to take a ball in $\mathbb{R}^n$, large enough to contain N: then the boundary of this ball links with any point in the interior of N. We focus on the non-compact case, but the case of compact M is included in this construction as well. Furthermore, note that in the case $k = 1$ we obtain

$$\widetilde{H}_0(\mathbb{R}^n - N) \neq 0,$$

and hence we conclude that $H_0(\mathbb{R}^n - N)$ is at least 2-dimensional, that is, the complement of N consists of at least two connected components. In this situation a link is also easy to find, namely between N and a pair of points in different connected components of $\mathbb{R}^n - N$. For this special case, the reader might want to compare the present work with [17].

We want to find a non-vanishing homology class in the complement of N, and from that a link between the support W of a representative of this class and a subset of N (cf. Figure 2). In order to get a clear (although simplified) picture of the situation, consider first the case where $\mathbb{R}^n - N$ is simply connected and the highest non-vanishing homology group $H_{k-1}(\mathbb{R}^n - N)$ has $k \le n/2$. Let χ be a nonzero element of $H_{k-1}(\mathbb{R}^n - N)$. By Hurewicz's theorem, $H_{k-1}(\mathbb{R}^n - N) \cong \pi_{k-1}(\mathbb{R}^n - N)$, so we find a $(k-1)$-dimensional sphere representing the class χ. Since $k - 1 < n/2$, we may assume such a sphere to be embedded [11]. By the assumption on its co-dimension, this embedding, as a map into $\mathbb{R}^n$, is isotopic to the standard embedding of the $(k-1)$-sphere. In fact, by compactness the two embeddings are also ambient isotopic, so our sphere bounds an embedded k-dimensional ball B^k (cf. [11]). Obviously, $B^k \cap N \neq \varnothing$.

More generally, since $\widetilde{H}_{k-1}(\mathbb{R}^n - N) \neq 0$, there exists a cycle σ_{k-1} that represents a non-trivial homology class in $\widetilde{H}_{k-1}(\mathbb{R}^n - N)$. We set $W = |\sigma_{k-1}|$, the support of σ_{k-1}. Since $\widetilde{H}_*(\mathbb{R}^n) = 0$ it follows that in $\mathbb{R}^n$ any cycle is a boundary and therefore there

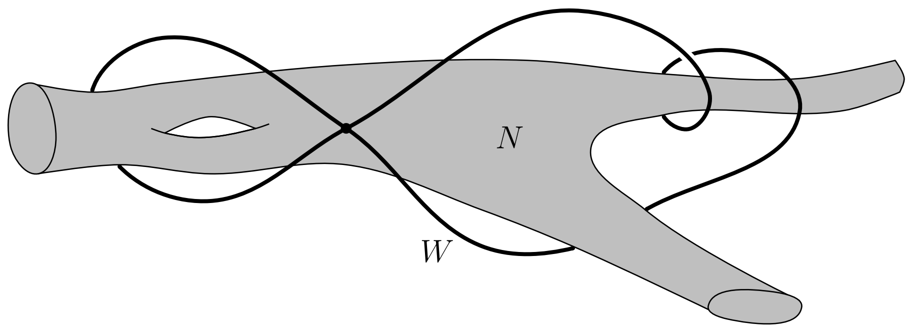


Figure 2: An example of a cycle W linking with N in $E^0 \cong \mathbb{R}^n$, with $n = 3$. The set N extends to infinity in this example.

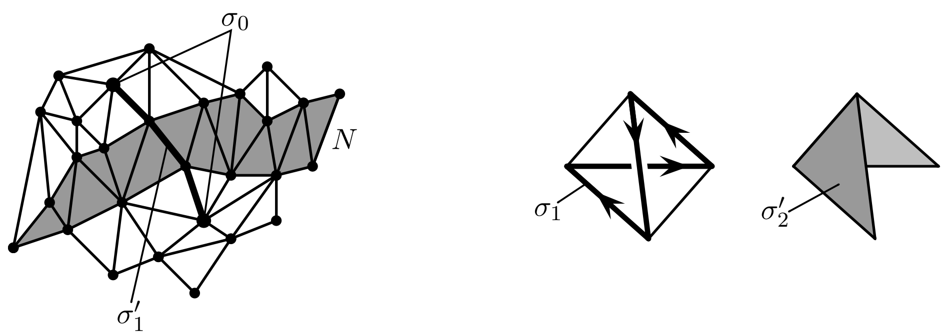


Figure 3: The left part of the figure shows a simultaneous triangulation of N (extending to infinity) and $\mathbb{R}^2$, together with a linking cycle σ_0 and its fill σ'_1. The right part of the figure depicts a cycle and its fill in $\mathbb{R}^3$.

is a chain $\sigma'_k \in C_k(\mathbb{R}^n)$ such that $\partial \sigma'_k = \sigma_{k-1}$. The support $U = |\sigma'_k|$ "fills" W, and $U \cap N \neq \varnothing$.

Since in the case of triangulated spaces the singular and simplicial chain complexes give rise to isomorphic homology groups, we will allow ourselves the freedom to deal at times with singular and at times with simplicial chains, depending on what fits more properly with a certain argument. To make the construction a little easier, in the beginning, for example, we choose to exploit *simplicial* homology, so that σ_{k-1} and σ'_k are simplicial chains (cf. Figure 3). Here we use that there exists a triangulation $\mathcal{T}$ of $\mathbb{R}^n$ such that $\mathcal{T}_N \subset \mathcal{T}$ and $\mathcal{T}_{\mathbb{R}^n - N} \subset \mathcal{T}$ are triangulations of N and $\mathbb{R}^n - N$ respectively (i.e. $\mathcal{T}$ is a simultaneous triangulation of N and $\mathbb{R}^n$, see e.g. [16]). For the bounded sets $W = |\sigma_{k-1}|$ and $U = |\sigma'_k|$ we have that

$$\partial U \subset W. \tag{20}$$

The sets W and N link: the map $\widetilde{H}_{k-1}(W) \to \widetilde{H}_{k-1}(\mathbb{R}^n - N)$ induced by inclusion, is nontrivial. We will now use W and U to construct A and S. In other words, we will grow a link (A, S) of homological dimension k from the link (W, N) of homological dimension $k - 1$.

As before, we have the decomposition $H^1(\mathbb{R}/\mathbb{Z}) = E^+ \oplus E^0$, where $E^0 \cong \mathbb{R}^n$, and

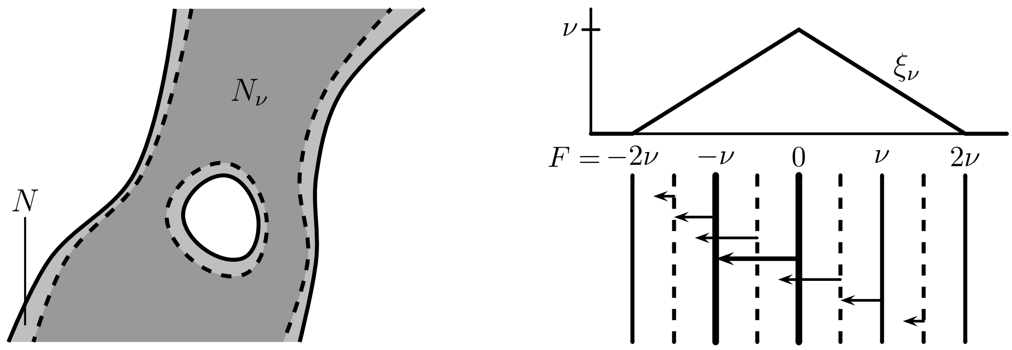


Figure 4: The (sublevel) sets $N = \{F \leq 0\}$ and $N_\nu = \{F \leq -\nu\}$ are homeomorphic for small ν. A cut-off function ξ_ν is used to extend the local gradient flow ψ_t to the whole of $\mathbb{R}^n$. The way the rescaled flow ϕ_s acts on the level curves of F is depicted at the bottom right.

$E^+ = \{u \in H^1 \mid \int_0^1 u(s)ds = 0\}$. Throughout this section we will identify $u^0 \in E^0 \cong \mathbb{R}^n$ with the constant function $u^0 \in H^1$. For analytic reasons to be clarified later, we need W to link with a subset of N. For ν sufficiently small we define the sets N_ν as follows:

$$N_\nu = \left\{u^0 \in E^0 \;\middle|\; V(u^0) \leq -\nu\sqrt{1 + |DV(u^0)|^2}\right\},$$

so that $N_\nu \subsetneq N = N_0$ for $\nu > 0$.

Lemma 14. *For ν sufficiently small the sets N_ν (resp. $\mathbb{R}^n - N_\nu$) are homeomorphic to N (resp. $\mathbb{R}^n - N$), and W links with N_ν. Moreover, for each sufficiently small $\nu > 0$ there exists a $\rho_\nu > 0$ such that $V|_{B_{\rho_\nu}(u^0)} \leq -\nu/2$ for all $u^0 \in N_\nu$.*

Proof. We start by proving that for ν sufficiently small the sets N_ν and N are homeomorphic. This is illustrated in Figure 4. To construct the homeomorphisms we shall use a gradient flow of the function

$$F(u^0) = \frac{V(u^0)}{\sqrt{1 + |DV(u^0)|^2}}.$$

The gradient of F is given by

$$\nabla F(u^0) = \frac{DV(u^0)}{\sqrt{1 + |DV(u^0)|^2}} - F(u^0)\,\frac{D^2V(u^0)DV(u^0)}{1 + |DV(u^0)|^2}.$$

We only use the gradient flow on the strip $T_{2\nu} = \{u^0 \mid -2\nu \leq F(u^0) \leq 2\nu\}$ around the boundary ∂N. Provided ν is small, on $T_{2\nu}$ we have the bounds $|DV(u^0)| \geq C_1$ and $\|D^2V(u^0)\| \leq C_2|DV(u^0)|$, for some (small) $C_1 > 0$ and (large) $C_2 > 0$. Namely, for large u^0, say $|u^0| > R$, this follows from asymptotic regularity. For u^0 in the ball $B_R(0)$ the first inequality follows from regularity of the energy surface: $DV \neq 0$ on $\partial N = \{V = 0\} = T_0$, which extends to $T_{2\nu} \cap B_R(0)$, provided ν is sufficiently small. The second inequality then follows from boundedness of D^2V on $B_R(0)$.

For $u^0 \in T_{2\nu}$ we thus have the estimate

$$|\nabla F(u^0)| \geq \frac{|DV(u^0)|}{\sqrt{1+|DV(u^0)|^2}} - |F|\frac{|D^2V(u^0)DV(u^0)|}{1+|DV(u^0)|^2}$$
$$\geq \frac{C_1}{\sqrt{1+C_1^2}} - 2\nu C_2 \geq \frac{C_1}{2\sqrt{1+C_1^2}} > 0,$$

provided ν is sufficiently small. This estimate shows that on $T_{2\nu}$ the initial value problem for the differential equation

$$\frac{du^0}{dt} = -\frac{\nabla F(u^0)}{|\nabla F(u^0)|^2}, \qquad \text{for } u^0 \in T_{2\nu},$$

is well-posed, and that solutions exist for all time, as long as they stay in $T_{2\nu}$. Denote this gradient flow by $\psi_t(u^0)$, where u^0 is the initial value at $t = 0$. An easy calculation shows that $\frac{d}{dt}F(\psi_t(u^0)) = -1$. To extend the flow to $\mathbb{R}^n$ we introduce a cut-off function $\xi_\nu \in C^0(\mathbb{R})$, see also Figure 4,

$$\xi_\nu(x) = \begin{cases} \nu - \frac{1}{2}|x| & |x| \leq 2\nu, \\ 0 & |x| > 2\nu. \end{cases}$$

The properties of ξ_ν that are needed in the following are: ξ_ν is continuous with support in $[-2\nu, 2\nu]$, $\xi_\nu(0) = \nu$, and $\frac{d\xi_\nu}{dx} < 1$. We now use the flow ψ_t and the cut-off function ξ_ν to construct an isotopy between N (resp. $\mathbb{R}^n - N$) and N_ν (resp. $\mathbb{R}^n - N_\nu$), for ν sufficiently small, namely

$$\phi_s(u^0) = \psi_{s\xi_\nu(F(u_0))}(u^0) \qquad \text{for all } u^0 \in \mathbb{R}^n,$$

with $0 \leq s \leq 1$. It follows from the choice of the cut-off function ξ_ν, that the family ϕ_s, $s \in [0,1]$ are homeomorphisms on $\mathbb{R}^n$. Indeed, one may also interpret ϕ_s as a flow acting on level curves of F: it sends the level $\{F = F_0\}$ to $\{F = F_0 - s\xi_\nu(F_0)\}$, i.e. $F(\phi_s(u^0)) = F(u^0) - s\xi_\nu(F(u^0))$. The property $\frac{d\xi_\nu}{dx} < 1$ therefore guarantees that as a map on level curves ϕ_s is bijective for all $s \in [0,1]$. Finally, by construction $\phi_1(N) = N_\nu$ and $\phi_1(\mathbb{R}^n - N) = \mathbb{R}^n - N_\nu$. This proves that $N \cong N_\nu$ and $\mathbb{R}^n - N \cong \mathbb{R}^n - N_\nu$.

Since W is a bounded set in $\mathbb{R}^n - N$, we have that $W \cap T_{2\nu} = \varnothing$ for ν sufficiently small. By construction, $\phi_s = \mathrm{id}$ on $T_{2\nu}$ for $s \in [0,1]$, hence W links with N_ν for sufficiently small ν.

Finally, let $u^0 \in N_\nu$ and consider points of the form $u = u^0 + v$ with $|v| \leq \rho_\nu$, for some $0 < \rho_\nu \leq 1$ to be determined. Recall that by asymptotic regularity, we have $|DV| \geq c$ outside some large ball $B_R(0)$. We now consider two cases: u^0 inside the slightly larger ball $B_{R+1}(0)$, and u^0 outside this ball.

In the latter case, i.e. $|u^0| > R+1$, we use asymptotic regularity and Lemma 4 to conclude that $|DV(u^0 + \tilde{v})| \leq 2|DV(u^0)|$ for all $|\tilde{v}| \leq \rho_\nu$, provided ρ_ν is sufficiently small. We then estimate

$$V(u^0 + v) = V(u^0) + \langle DV(u^0 + \theta v), v \rangle \qquad \text{for some } \theta \in [0,1],$$
$$\leq -\nu\sqrt{1+|DV(u^0)|^2} + 2|DV(u^0)|\,|v|$$
$$\leq -\frac{\nu}{\sqrt{2}} - \frac{\nu}{\sqrt{2}}|DV(u^0)| + 2|DV(u^0)|\,\rho_\nu,$$

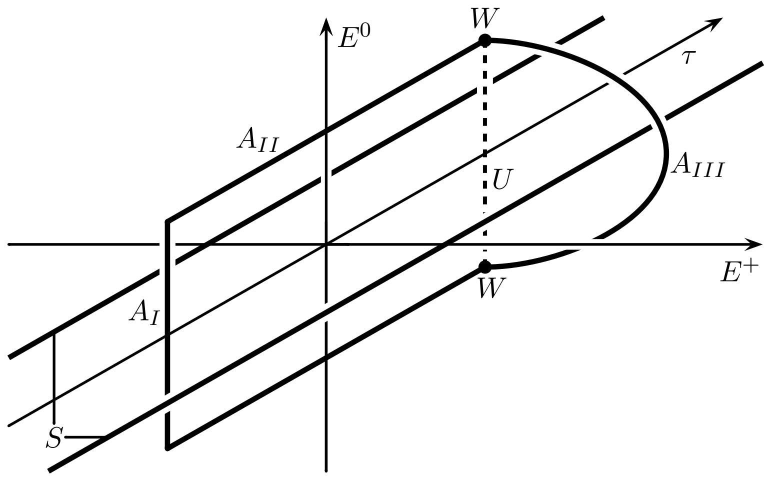


Figure 5: A schematic view of the sets A and S. The set A consists of three pieces marked I, II and III. The belly A_{III} goes around S.

and if we choose $\rho_\nu \leq 2^{-3/2}\nu$, it follows that $V(u^0 + v) \leq -\nu/2$ for $|\tilde{v}| \leq \rho_\nu$.

In the former case, i.e. $u_0 \in B_{R+1}(0)$, since $|DV|$ is uniformly bounded in the slightly larger ball $B_{R+2}(0)$, an estimate similar to the one above shows that $V(u^0+v) \leq -\nu/2$ for $|v| \leq \rho_\nu$, if ρ_ν is sufficiently small. This finishes the proof. □

5.2 Definition of the linking set in $H^1 \times \mathbb{R}$

The above lemma implies that the sets W and N_ν also form a link for all admissible ν. For small ν and $\rho \leq \rho_\nu$ we define

$$S = \{(u,\tau) \mid \tau \in \mathbb{R},\, u^0 \in N_\nu,\, \|u^+\|_1 = \rho\} \subset H^1 \times \mathbb{R}.$$

The set A is defined as follows. Let $A = A_I \cup A_{II} \cup A_{III}$, with

$$\begin{aligned} A_I &= \{(u,\tau) \mid u = u^0 \in U \subset E^0,\, \tau = R_1\} \\ A_{II} &= \{(u,\tau) \mid u = u^0 \in W \subset E^0,\, R_1 \leq \tau \leq R_2\}, \\ A_{III} &= \{(u,\tau) \mid u = g(u^0),\, u^0 \in U,\, \tau = R_2\}, \end{aligned}$$

where the parameters are $R_1 < R_2$, and $g : U \to H^1(\mathbb{R}/\mathbb{Z})$ is a continuous map, with the properties

1. $g(u^0) \equiv u^0$ for all $u^0 \in W$;
2. $g(u^0)^+ \equiv 0$ if and only if $u^0 \in W$;
3. $\int_0^1 V(g(u^0)(s))ds > 0$ for all $u^0 \in U$.

In the "ideal" case described at the beginning of this section, namely that the sets U and W are an embedded ball and its boundary, respectively, a map g satisfying the properties listed above is easily defined. It can be helpful to (over)simplify even further and hypothesize that g is even defined in such a way that its image is the graph of a function $\tilde{g} : U \to E^+$, that is, $g(u^0) = (u^0, \tilde{g}(u^0))$. This leads to simple pictures and the reader may keep this case in mind through the more general (and technical) arguments and use it to interpret them, because it already contains the essential ideas of the proof.

The existence of such a continuous map g in the general case is established in the appendix (where we will use the fact that W and U consists of simplices, so that $\partial U \subset W$, cf. (20)). Property 1 guarantees that the set A is connected. Property 2 will be used in Lemma 16 to establish that A and S link (for sufficiently small ρ). The idea is that the "belly" in Figure 5, i.e. the set A_{III} defined by g, goes around S. Property 3 is needed in Lemma 15 to prove estimates on $\mathcal{B}|_A$. Here the idea is to choose $g(u^0)(s) \in W$ for almost all s, since by construction $W \subset \mathbb{R}^n - N = \{V > 0\}$.

Figure 5 gives a schematic account of the sets A and S. The above definition of A and S yields a possible link in $H^1 \times \mathbb{R}$ grown out of the (W, N). If the parameters ν, ρ, R_1 and R_2 are chosen properly, and if g satisfies the three properties listed above, A and S indeed form a (homological) link. In fact, for linking only the parameters ρ and ν matter; for ν so small that the assertion in Lemma 14 holds, the sets A and S link for all $\rho \leq \rho_g$, where $\rho_g > 0$ is some g-dependent constant. This will be established a little later in Lemma 16. First we show in Lemma 15 below that with the remaining parameters the sets A and S can be tuned in such a way that the estimates (i) and (ii) from Section 4 on $\mathcal{B}|_S$ and $\mathcal{B}|_A$ are satisfied.

Lemma 15. *If ν and ρ are sufficiently small, then there exist constants $R_1 < R_2 \in \mathbb{R}$ and $a > b > 0$ such that A and S satisfy $\mathcal{B}|_S \geq a$ and $\mathcal{B}|_A \leq b$.*

Proof. Let us start with S. From (3) we have that $\|u^+\|_{L^\infty} \leq \|u^+\|_1$, and if follows from Lemma 14 that $V(u) \leq -\nu/2$ for all $u \in S$ if $\rho \leq \rho_\nu$. For the functional $\mathcal{B}$ we obtain:

$$\mathcal{B}|_S = \frac{e^{-\tau}}{2} \int_0^1 |u'(s)|^2 ds - e^{\tau} \int_0^1 V(u(s)) ds \geq \frac{e^{-\tau}\rho^2}{2} + \frac{e^{\tau}\nu}{2} \geq \rho\sqrt{\nu}.$$

Therefore we choose $\rho \leq \rho_\nu$ and set $a = \rho\sqrt{\nu} > 0$.

As for the set A, a more detailed analysis is required. As Figure 5 indicates, A consists of three parts, hence let us estimate $\mathcal{B}$ on the successive parts of A. We start with the boundaries A_I and A_{II} which are contained in E^0. For A_I we have:

$$\mathcal{B}|_{A_I} = -e^{R_1} \int_0^1 V(u(s)) ds \leq e^{R_1} M,$$

where $M = \max_U(-V)$ (the set U is bounded). Now choose $R_1 \leq \log \frac{a}{2M}$, then $\mathcal{B}_{A_I} \leq \frac{a}{2} = b$. The section A_{II} is characterized as $W \times [R_1, R_2]$ and consequently, independent of the choice of R_1 and R_2,

$$\mathcal{B}|_{A_{II}} = -e^{\tau} \int_0^1 V(u(s)) ds < 0,$$

for $\tau \in [R_1, R_2]$, since $V > 0$ on $W \subset \mathbb{R}^n - N$. Finally, to estimate $\mathcal{B}$ on A_{III}, recall that g is a continuous map from U to H^1, so that $C = \max_U \int_0^1 |g(u)'(s)|^2 ds < \infty$. Then

$$\mathcal{B}|_{A_{III}} = \frac{e^{-R_2}}{2} \int_0^1 \left|g(u)'(s)\right|^2 ds - e^{R_2} \int_0^1 V\big(g(u)(s)\big) ds \leq \frac{1}{2} e^{-R_2} C,$$

using property 3 of the map g. Choosing $R_2 \geq \log \frac{C}{a}$, we obtain $\mathcal{B}|_{A_{III}} \leq \frac{a}{2} = b$. Combining the estimates on the three pieces of A, we infer that $\mathcal{B}|_A \leq b = \frac{a}{2}$. □

5.3 Proof of the linking property

In order the find a minimax we need to show that the sets A and S link. We again take ν so small that the assertion in Lemma 14 holds.

Lemma 16. *If ν and ρ are chosen sufficiently small then the sets A and S link, i.e. the map $H_k(A) \to H_k(H^1 \times \mathbb{R} - S)$ is non-trivial. The choice of ρ depends only on the function g, i.e. $\rho \le \rho_g$.*

Proof. We start with some preliminary observations and the introduction of some notation. As explained by Lemma 14, for linking it is irrelevant whether we consider N or N_ν, and hence, to relieve notation, we write N instead of N_ν throughout this proof. To reduce confusion between Sobolev spaces and homology, we denote the Sobolev space $H^1 = H^1(\mathbb{R}/\mathbb{Z}; \mathbb{R}^n)$ by E. Let Z be the union of U and $g(U)$ in the function space E:

$$Z = U \cup g(U) \ \subset E.$$

Consider the projection

$$\begin{aligned} \widehat{\pi} : E = E^0 \times E^+ &\longrightarrow \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}, \\ (u^0, u^+) &\longmapsto (u^0, \rho - \|u^+\|_1), \end{aligned}$$

where ρ is as in the definition of the set S. Our proof is a generalization to homology of a well-known degree argument that uses a similar projection (cf. [2, 21]). Under this projection, U is mapped homeomorphically onto $\widehat{\pi}(U) = U \times \{\rho\}$, and S (or rather S_u defined below) is mapped to $N \times \{0\}$. The set S is of the form $S_u \times \mathbb{R}$, where

$$S_u = \{u \in E \,|\, u^0 \in N, \|u^+\|_1 = \rho\}.$$

It has the property that $\widehat{\pi}^{-1}\widehat{\pi}(S_u) = S_u$. We observe that since N is closed and g is continuous, the set $G \overset{\text{def}}{=} \{u \in U \,|\, g(u)^0 \in N\}$ is closed. Since $G \cap W = \varnothing$ by property 1, it follows from property 2 that

$$\rho_g \overset{\text{def}}{=} \tfrac{1}{2} \min_G \|g(u)^+\|_1 > 0.$$

This implies that $A_{III} \cap S = \varnothing$, provided that $\rho \le \rho_g$. Since $\widehat{\pi}^{-1}\widehat{\pi}(S_u) = S_u$, this is equivalent to $\widehat{\pi}(S_u) \cap \widehat{\pi}(g(U)) = (N \times \{0\}) \cap \widehat{\pi}(g(U)) = \varnothing$. The arrangement of $U \cup g(U)$ and S_u in E, as well as the projection $\widehat{\pi}$ to $\mathbb{R}^{n+1}$, are depicted in Figure 6.

The proof of Lemma 16 proceeds in three steps. The first one lifts the link from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$, the second from $\mathbb{R}^{n+1}$ to E, and the third from E to $E \times \mathbb{R}$.

Step 1. W $(k-1)$-links with N in $\mathbb{R}^n \implies \widehat{\pi}(Z)$ k-links with $N \times \{0\}$ in $\mathbb{R}^{n+1}$.

Starting from the nontrivial homomorphism $i_{k-1} : H_{k-1}(W) \to H_{k-1}(\mathbb{R}^n - N)$, we are going to show that $i_k : H_k(\widehat{\pi}(Z)) \to H_k(\mathbb{R}^{n+1} - N \times \{0\})$ cannot be trivial either. From property 2 it follows that $\widehat{\pi}(Z) = \widehat{\pi}(U) \cup \widehat{\pi}(g(U))$ and $\widehat{\pi}(U) \cap \widehat{\pi}(g(U)) = \widehat{\pi}(W)$.

First of all, we introduce a new set $\widetilde{Z}$, which is obtained by gluing a copy of $W \times I$ between $\widehat{\pi}(U) = U \times \{\rho\}$ and $\widehat{\pi}(g(U))$, with I the interval $[0, \rho]$, cf. Figure 7. More precisely, if $\Sigma_{-\rho}$ denotes translation by ρ in the negative x_{n+1}-direction, then

$$\widetilde{Z} = \widehat{\pi}(U) \cup (W \times I) \cup \Sigma_{-\rho}\widehat{\pi}(g(U)).$$

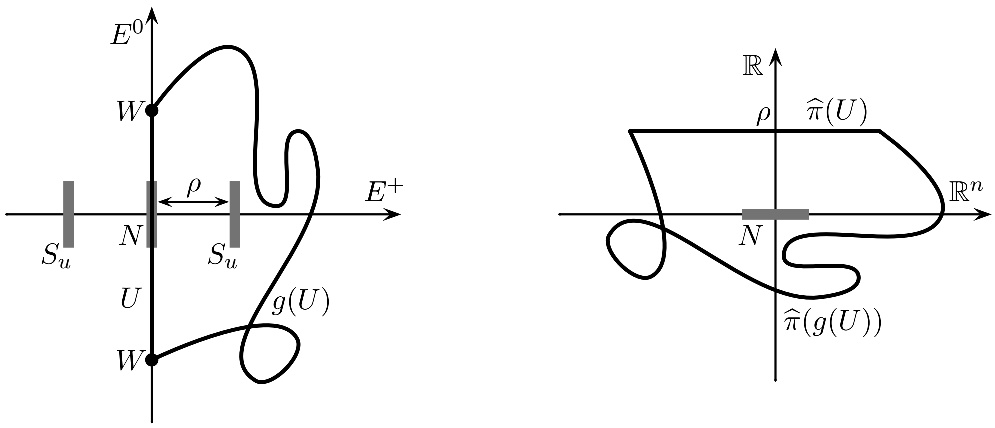


Figure 6: On the left a sketch of the situation in the infinite dimensional space $E = E^0 \times E^+$. The belly $g(U)$ goes around S_u for $\rho \leq \rho_g$. Note that $U \cap g(U) = W$. On the right the projections $\widehat{\pi}(Z) = \widehat{\pi}(U) \cup \widehat{\pi}(g(U))$ and $\widehat{\pi}(S_u) = N \times \{0\}$ are shown in the finite dimensional space $\mathbb{R}^{n+1}$.

We claim that $\widehat{\pi}(Z)$ links N if and only if $\widetilde{Z}$ links N. We define a homotopy $h_t : \widetilde{Z} \to \mathbb{R}^{n+1} - N$ by

$$h_t(x) = \begin{cases} x & x \in U \times \{\rho\}, \\ (x_1, \dots x_n, t\rho + (1-t)x_{n+1}) & (x_1, \dots, x_n, x_{n+1}) \in W \times I, \\ x + t\rho(0, \dots, 0, 1) & x \in \Sigma_{-\rho}\widehat{\pi}(g(U)). \end{cases}$$

Using that $\rho \leq \rho_g$, we see that the inclusion $\tilde{i}_k$ of $\widetilde{Z}$ in $\mathbb{R}^{n+1} - N$ is thus homotopic (as a map from $\widetilde{Z}$ to $\mathbb{R}^{n+1} - N$) to the map h_1 followed by the inclusion i_k of $\widehat{\pi}(Z)$. This leads to the following commutative diagram on the level of homology groups:

$$\begin{array}{ccc} H_k(\widetilde{Z}) & \xrightarrow{\tilde{i}_k} & H_k(\mathbb{R}^{n+1} - N \times \{0\}) \\ {\scriptstyle (h_1)_*}\downarrow & & \| \\ H_k(\widehat{\pi}(Z)) & \xrightarrow{i_k} & H_k(\mathbb{R}^{n+1} - N \times \{0\}) \end{array}$$

In fact h_1 has a homotopy inverse: its construction is based on the observation that $\Sigma_\rho \widetilde{Z}$ and $\widehat{\pi}(Z)$ are both deformation retracts of $(U \times [\rho, 2\rho]) \cup \widehat{\pi}(g(U))$. A homotopy inverse of h_1 is produced as follows: take the inclusion of $\widehat{\pi}(Z)$ into $(U \times [\rho, 2\rho]) \cup \widehat{\pi}(g(U))$, followed by a map which is the identity on $\widehat{\pi}(g(U))$ and retracts $U \times [\rho, 2\rho]$ onto $(U \times \{2\rho\}) \cup (W \times [\rho, 2\rho])$; the pair (U, W) is a CW-complex pair, so $U \cup (W \times I)$ is a deformation retract of $U \times I$. In the end perform a translation by ρ in the negative x_{n+1}-direction. This yields the desired map from $\widehat{\pi}(Z)$ to $\widetilde{Z}$. Thus $(h_1)_*$ is an isomorphism, which shows that $\tilde{i}_k$ and i_k can only be simultaneously trivial or nontrivial.

The new set $\widetilde{Z}$ can be seen as the union of the two sets

$$U_1 = U \times \{\rho\} \cup W \times [0, \rho] \qquad \text{and} \qquad U_2 = \Sigma_{-\rho}(\widehat{\pi}(g(U))),$$

which intersect in $W \times \{0\}$, as illustrated in Figure 7. Then $\widehat{\pi}(U)$ is a deformation retract of U_1, whereas U_2 is isomorphic to $\widehat{\pi}(g(U))$. The Mayer-Vietoris sequence for

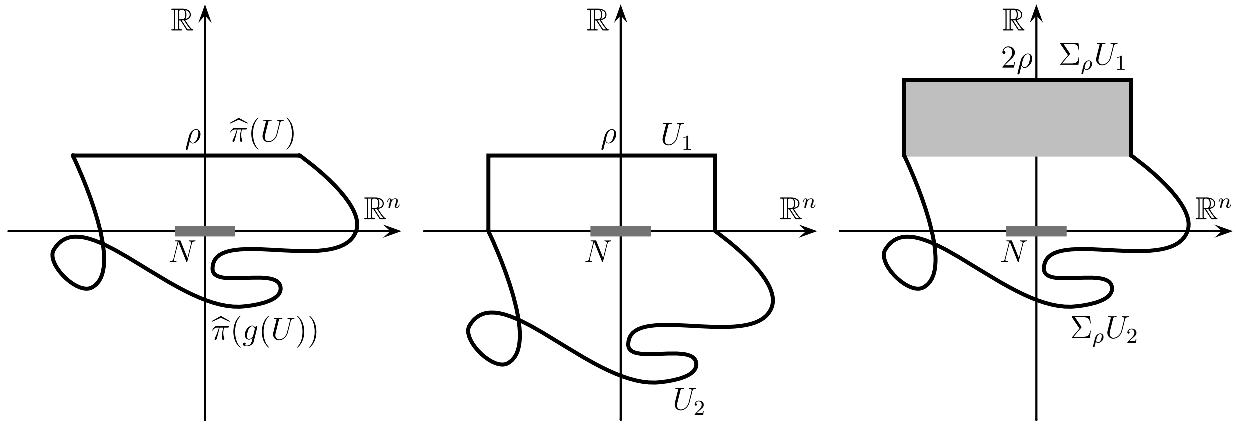


Figure 7: The sets $\widehat{\pi}(Z) = \widehat{\pi}(U) \cup \widehat{\pi}(g(U))$ on the left and $\widetilde{Z} = U_1 \cup U_2$ in the middle are homotopic. The right picture illustrates that $\Sigma_\rho \widetilde{Z}$ and $\widehat{\pi}(Z)$ are both deformation retracts of $(U \times [\rho, 2\rho]) \cup \widehat{\pi}(g(U))$.

the triad $(\widetilde{Z}, U_1, U_2)$ looks as follows:

$$H_k(U_1) \oplus H_k(U_2) \longrightarrow H_k(\widetilde{Z}) \xrightarrow{\delta} H_{k-1}(W \times \{0\}).$$

Let $[w] \in H_{k-1}(W)$ be the class such that $i_k([w]) \neq 0 \in H_{k-1}(\mathbb{R}^n - N)$. Identify the following isomorphic sets: W with $W \times \{0\}$, U with $\widehat{\pi}(U)$ and U_2 with $\widehat{\pi}(g(U))$, the latter isomorphism being induced by h_1. We know that the cycle w is equal to ∂v, with v a chain of U with boundary on W. Then it is also the boundary of a chain $v' = v + w \times I$ of U_1, which satisfies $(h_1)_*(v') = v$. Moreover, v induces a singular chain $\widehat{\pi}_* g_*(v)$ in U_2, where g_* and $\widehat{\pi}_*$ are the chain maps associated to g and $\widehat{\pi}$. We have: $\partial \widehat{\pi}_* g_*(v) = \widehat{\pi}_* g_* \partial(v) = \widehat{\pi}_* g_*(w) = w$. This implies that $y = v' - \widehat{\pi}_* g_*(v)$ is a closed chain in $\widetilde{Z}$. By construction, and by definition of the connecting morphism δ in the Mayer-Vietoris sequence, $\delta[y] = [w]$.

Consider another triad, namely $(\mathbb{R}^{n+1} - N \times \{0\}, \mathbb{R}^{n+1}_+ - N \times \{0\}, \mathbb{R}^{n+1}_- - N \times \{0\})$, where $\mathbb{R}^{n+1}_+ = \{x = (x_1, \ldots, x_{n+1}) | x_{n+1} \geq 0\}$ is the upper half-space, and $\mathbb{R}^{n+1}_-$ is the analogously defined lower half-space. Notice that $(\mathbb{R}^{n+1}_+ - N \times \{0\}) \cap (\mathbb{R}^{n+1}_- - N \times \{0\}) = (\mathbb{R}^n - N) \times \{0\}$. By naturality of Mayer-Vietoris sequences, and by inclusion of the triads, we get the commutative diagram

$$\begin{array}{ccc}
H_k(\widetilde{Z}) & \xrightarrow{\delta} & H_{k-1}(W) \\
{\scriptstyle i_k}\big\downarrow & & \big\downarrow{\scriptstyle i_{k-1}} \\
H_k(\mathbb{R}^{n+1} - N \times \{0\}) & \xrightarrow{\delta} & H_{k-1}(\mathbb{R}^n - N)
\end{array}$$

Let y be the chain constructed earlier. Then

$$\delta i_k[y] = i_{k-1}\delta[y] = i_{k-1}[w] \neq 0 \in H_{k-1}(\mathbb{R}^n - N),$$

which implies in particular that $i_k[y]$ cannot be zero and therefore the morphism i_k is not trivial.

Step 2. $\widehat{\pi}(Z)$ k-links with $N \times \{0\}$ in $\mathbb{R}^{n+1}$ $\implies$ Z k-links with S_u in E.

We start with the observation that the pre-image of $N \times \{0\}$ under $\widehat{\pi}$ is exactly S_u, so that $\widehat{\pi}$ maps $E - S_u$ to $\mathbb{R}^{n+1} - N \times \{0\}$. Hence we may consider the following diagram, where the inclusions commute with the (restrictions of the) projection:

$$\begin{array}{ccc} Z & \xrightarrow{\widehat{\pi}|_Z} & \widehat{\pi}(Z) \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle i_k} \\ E - S_u & \xrightarrow{\widehat{\pi}|_{E-S_u}} & \mathbb{R}^{n+1} - N \times \{0\} \end{array}$$

In turn this induces a commutative diagram on the level of homology groups, namely

$$\begin{array}{ccc} H_k(Z) & \xrightarrow{(\widehat{\pi}|_Z)_*} & H_k(\widehat{\pi}(Z)) \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle i_k} \\ H_k(E - S_u) & \xrightarrow{(\widehat{\pi}|_{E-S_u})_*} & H_k(\mathbb{R}^{n+1} - N \times \{0\}) \end{array}$$

Just as in the arguments in Step 1, denote by g_* the map induced by g on the level of singular chains and notice that $\partial g_* v = g_* \partial v = w$, where w and v are as in Step 1, so we may define a closed chain in Z by $z = v - g_*(v)$. This satisfies $\widehat{\pi}_*(z) = (h_1)_*(v' - \widehat{\pi}_* g_*(v)) = (h_1)_*(y)$, using the identification of $\widehat{\pi}(g(U))$ with U_2. Therefore it represents a class $[z] \in H_k(Z)$, which is mapped to $(h_1)_*[y]$ under $\widehat{\pi}_*$. Since $(h_1)_*$ is an isomorphism, we have

$$\widehat{\pi}_* i_k[z] = i_k \widehat{\pi}_*[z] = i_k (h_1)_*[y] \neq 0.$$

In particular, $i_k[z] \neq 0$, which implies that $i_k : H_k(Z) \to H_k(E - S_u)$ is not trivial.

Step 3. Z k-links with S_u in E $\implies$ A k-links with S in $E \times \mathbb{R}$.

Identify Z with $Z \times \{R_2\} \subset E \times \mathbb{R}$. We define a homotopy $r_t : A \longrightarrow (E \times \mathbb{R}) - S = (E - S_u) \times \mathbb{R}$, which is the identity along A_{III} and sends points (u, τ) on either A_I or A_{II} to $(u, (1-t)\tau + tR_2)$. Then $r_1(A) = Z$ and the inclusion of A in $(E - S_u) \times \mathbb{R}$, followed by projection $\widetilde{\pi}$ onto $E - S_u$ is homotopic to the composition of r_1 with the inclusion in $E - S_u$ (a homotopy being given by $\widetilde{\pi} \circ r_t$), thus giving rise to the following commutative diagram of homology groups:

$$\begin{array}{ccc} H_k(A) & \xrightarrow{i_k} & H_k((E - S_u) \times \mathbb{R}) \\ {\scriptstyle (r_1)_*}\downarrow & & \downarrow{\scriptstyle \widetilde{\pi}_*} \\ H_k(Z) & \xrightarrow{i_k} & H_k(E - S_u) \end{array}$$

Following the same line of arguments as in Step 1, we observe that A and Z are both deformation retracts of the set $\big\{(u, \tau) \;\big|\; u = u^0 \in U \subset E^0,\, R_1 \leq \tau \leq R_2\big\} \cup g(U)$, so we may construct a suitable homotopy inverse of r_1. We then have that both the vertical maps in the above diagram are isomorphisms, and we conclude that the horizontal ones can only be either both trivial or both nontrivial. This completes the proof of Lemma 16. $\square$

This concludes the construction of the linking sets A and S. Lemma 16 proves that they link for ν and ρ sufficiently small, while Lemma 15 establishes that $\mathcal{B}|_S$ and $\mathcal{B}|_A$ satisfy the minimax estimates for appropriate choices of R_1 and R_2 (and in particular we take $\rho \leq \min\{\rho_\nu, \rho_g\}$). In turn, Sections 3 and 4 show that this implies the existence of a critical point of $\mathcal{B}$, corresponding to a closed characteristic on M. To finish the argument though, we still need to prove Theorem 2, which is the subject of the next section.

6 The homology of M

In this section we prove Theorem 2. The theorem is obviously true when $N = \mathbb{R}^n$. In the remainder of this section we consider the case $N \neq \mathbb{R}^n$. For notational purposes it is easier to work with indices $k = i + 1$. We will prove that

$$H_{k+n-1}(M) \cong H_k(N, \partial N) \cong \widetilde{H}_{k-1}(\mathbb{R}^n - N), \tag{21}$$

for $k = 1, \ldots, n$. The second isomorphism is fairly straightforward. Indeed, by Poincaré duality for non-compact manifolds (cf. [10]) we have $H_k(N, \partial N) \cong H_c^{n-k}(N)$, where H_c^* denotes compactly supported cohomology. On the other hand, from Alexander duality (cf. [6, VIII.8.15]) we get $H_c^{n-k}(N) \cong \widetilde{H}_{k-1}(\mathbb{R}^n - N)$, so that

$$H_k(N, \partial N) \cong \widetilde{H}_{k-1}(\mathbb{R}^n - N).$$

To establish Theorem 2, it remains to prove the first isomorphism in (21).

Lemma 17. *For every $k = 1, \ldots, n$ there is an isomorphism ($N \neq \mathbb{R}^n$)*

$$H_{k+n-1}(M) \cong H_k(N, \partial N).$$

Proof. Recall that we have from the introduction the following description of the topology of M:

$$M \cong \Big(S^{n-1} \times N\Big) \cup_{S^{n-1} \times \partial N} \Big(D^n \times \partial N\Big),$$

and since our arguments will be of purely topological nature, we will from now on identify M with the above boundary sum. Notice that the restriction of the projection $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$ to M is a proper map (i.e. pre-images of compact sets are compact). In fact, if $K \subset N$ is a compact subset, then $(\pi|_M)^{-1}(K) = K \times S^{n-1}$ if $K \cap \partial N = \emptyset$, whereas if $K \cap \partial N \neq \emptyset$, then the spheres over points of $K \cap \partial N$ are collapsed to points in the pre-image (or, using the above identification, disks are glued in). Since the functor Ω_c^* (i.e. taking compactly supported forms) is contravariant with respect to proper maps [3], the pullback $\pi^* : H_c^*(N) \to H_c^*(M)$ is well defined.

Let $\pi_! : H_k(N, \partial N) \to H_{k+n-1}(M)$ be the transfer map (cf. [4, VI.11.2]) defined by the commutative diagram

$$\begin{array}{ccc} H_k(N, \partial N) & \xrightarrow[\cong]{\text{PD}} & H_c^{n-k}(N) \\ \Big\downarrow \pi_! & & \Big\downarrow \pi^* \\ H_{k+n-1}(M) & \xrightarrow[\cong]{\text{PD}} & H_c^{n-k}(M) \end{array}$$

where PD denotes the Poincaré isomorphism for non-compact manifolds. In other words, $\pi_! = PD_M^{-1}\pi^* PD_N$. We are going to show that the pull-back map $\pi^* : H_c^{n-k}(N) \to H_c^{n-k}(M)$ is an isomorphism, hence in turn $\pi_!$ also is. Notice first of all that it has a left inverse $j^* : H_c^{n-k}(M) \to H_c^{n-k}(N)$, induced by the inclusion $j : N \to M$ given by $q \mapsto [(q, x_0)]$ for some fixed $x_0 \in S^{n-1}$. Again, we remark that this induced morphism is well defined because the inclusion map of N in M is proper: since $j(N)$ is closed in M, the pre-image of a compact set K is compact (j is a homeomorphism between N and $j(N)$, and $j(j^{-1}(K)) = K \cap j(N)$ is compact).

With inclusions denoted by $i_1 : \partial N \times S^{n-1} \to N \times S^{n-1}$, $i_2 : \partial N \times S^{n-1} \to \partial N \times D^n$, $j_1 : N \times S^{n-1} \to M$ and $j_2 : \partial N \times D^n \to M$, the Mayer-Vietoris sequence for compactly supported cohomology of the triad $(M, N \times S^{n-1}, \partial N \times D^n)$ looks as follows:

$$H_c^{n-k}(M) \xrightarrow{(j_1^*, -j_2^*)} H_c^{n-k}(N \times S^{n-1}) \oplus H_c^{n-k}(\partial N \times D^n) \xrightarrow{i_1^* + i_2^*} H_c^{n-k}(\partial N \times S^{n-1}), \tag{22}$$

and from it we would like to show that the map j^* is in fact an isomorphism. Using the Künneth formula for compactly supported cohomology (cf. [6, VIII.8.20]), for $k > 1$ we may rewrite the sequence as follows:

$$\longrightarrow H_c^{n-k}(M) \xrightarrow{\alpha} H_c^{n-k}(N) \oplus H_c^{n-k}(\partial N) \xrightarrow{\beta} H_c^{n-k}(\partial N) \longrightarrow$$

Since the Künneth isomorphism $H_c^{n-k}(N \times S^{n-1}) \cong H_c^{n-k}(N)$ coincides in this case with the pullback map induced by the inclusion of N in $N \times S^{n-1}$ [3], we see that the first component of α is in fact j^*. In turn, β is of the form $i_1^* + \mathrm{id}$, where by a slight abuse of notation i_1 is also taken to denote the inclusion $\partial N \to N$. Therefore β is surjective and its kernel is isomorphic to $H_c^{n-k}(N)$. Surjectivity implies that the maps at both outer ends of the sequence are trivial, hence α is an isomorphism onto its image $H_c^{n-k}(N) \oplus 0$. In other words, $j^* : H_c^{n-k}(M) \to H_c^{n-k}(N)$ is bijective. By uniqueness of the inverse, π^* also is an isomorphism, as is $\pi_!$, proving the assertion for the case $k > 1$.

If $k = 1$, the Künneth formula yields $H_c^{n-1}(N \times S^{n-1}) \cong H_c^{n-1}(N) \oplus H_c^0(N)$, and the sequence (22) may be rewritten as

$$\xrightarrow{\cdot 0} H_c^{n-1}(M) \xrightarrow{\alpha'} H_c^{n-1}(N) \oplus H_c^0(N) \oplus H_c^{n-1}(\partial N) \xrightarrow{\beta'} H_c^{n-1}(\partial N) \oplus H_c^0(\partial N),$$

with the map on the far left hand side being trivial because of the previous step. By naturality of the Künneth isomorphism with respect to maps between spaces, α' is of the form $(j^*, \gamma, i_1^{n-1} j^*)$ and β' of the form $(i_1^{n-1} + 0 + \mathrm{id}, 0 + i_1^0 + 0)$, where i_1 is again taken to denote the inclusion of ∂N in N and we have indicated the degree of the induced maps in cohomology. Notice that H_c^0 consists of constant functions with compact support; in particular, it is trivial in the case of a (connected) non-compact space, and in general $H_c^0(N) \to H_c^0(\partial N)$ is an injective morphism. Because of this, i_1^0 is injective. This shows that $\ker \beta'$ consists of elements of the form $(a, 0, i_1^{n-1} a)$, with $a \in H_c^{n-1}(N)$ and hence that $j^* : H_c^{n-1}(M) \to H_c^{n-1}(N)$ is an isomorphism because $H_c^{n-1}(M) \cong \mathrm{im}\, \alpha' \cong \ker \beta' \cong H_c^{n-1}(N)$. Thus π^* and $\pi_!$ are isomorphisms, finishing the proof for the remaining case $k = 1$. $\square$

7 Further extensions and generalizations

For the mechanical Hamiltonians $H(p,q) = \frac{1}{2}|p|^2 + V(q)$ we have chosen the asymptotic regularity conditions as given in the introduction. Certain variations on these conditions lead to the same results. For example, if we consider

$$|DV(q)| \to \infty \quad \text{and} \quad \frac{\|D^2V(q)\|}{|DV(q)|} \leq \bar{c} \qquad \text{as } |q| \to \infty, \tag{23}$$

for some $\bar{c} > 0$, all the arguments still work with only very minor adjustments. Hence these conditions also guarantee the existence of a closed characteristic, provided the topological condition of Theorem 1 is met. The geometric conditions on the potential are used at four different stages, namely in Lemmas 9, 11, 12 and 14. Of those, Lemma 9 stands out. For the (proofs of the) other three lemmas slightly weaker conditions, such as

$$|DV(q)| \geq c > 0 \quad \text{and} \quad \frac{\|D^2V(q)\|}{|DV(q)|} \leq \bar{c} \qquad \text{as } |q| \to \infty, \tag{24}$$

suffice. Alternatively, for these latter three lemmas a different set of sufficient conditions is

$$|DV(q)| \geq c > 0 \quad \text{and} \quad \left\langle p, D^2V(q)p \right\rangle \leq \tilde{c}|p|^2 \text{ for all } p \in \mathbb{R}^n \qquad \text{as } |q| \to \infty, \tag{25}$$

for some $\tilde{c} > 0$, with thus only a *one-sided* bound on the quadratic form D^2V. Note that conditions (25) require significant alterations to the proofs of the above-mentioned lemmas (and to the definition of N_ν). We postpone these alternative proofs to appendix C.

We stress that neither (24) nor (25) suffices to prove Lemma 9. We would thus need an additional condition to establish the existence of a closed characteristic. One such condition is (cf. (10))

$$\frac{\|D^2V(q)\|}{|DV(q)|^2} \to 0 \qquad \text{as } |q| \to \infty, \tag{26}$$

which, due to the square in the denominator, is a very weak condition. That this condition indeed suffices in the proof of Lemma 9 is easily checked. This implies that Theorem 1 holds under the pair of conditions (24) *and* (26), or under the pair (25) *and* (26), of course always under the constraint that the topological condition is met. If some specific potential $V(u)$ does not satisfy any of these sufficient conditions, one could still try to make the proofs of those lemmas work, but we shall not pursue that here.

The above generalizations all apply to the standard mechanical systems. There are several classes of systems to which one could attempt to extend these results.

1. Hamiltonian functions $H(p,q)$ which are the sum of a potential energy $V(q)$ and a "kinetic energy" which is quadratic in p, i.e. $H(p,q) = \frac{1}{2}\langle A(q)p, p\rangle + V(q)$. Here the matrices $A(q)$ are symmetric and positive definite, with both $\|A(q)\|$ and $\|A^{-1}(q)\|$ uniformly bounded in q. One may view the kinetic energy as a metric $g(\cdot,\cdot) = \langle A(q)\cdot,\cdot\rangle$ on $\mathbb{R}^n$ and therefore the topological characterization in Theorem 2 remains unchanged. The Lagrangian in the case is $L(q,q') = \frac{1}{2}\left\langle A^{-1}(q)q', q'\right\rangle - V(q)$, which reveals that in order to extend the proof from this paper one will need some appropriate growth condition on $A(q)$ as $q \to \infty$.

2. More generally, when the Hamiltonian $H(p,q)$ is *convex* in p, then we may employ the Legendre transform to convert the problem to the Lagrangian setting. One example in which the calculations remain surveyable is when we add a linear term of the form $\langle B(q)p, q\rangle$, i.e. $H(p,q) = \frac{1}{2}\langle A(q)p, p\rangle + \langle B(q)p, q\rangle + V(q)$. The Legendre transform now leads to the explicit relation $p = A^{-1}(q)[q' - B^*(q)q]$, and a straightforward calculation shows that the Lagrangian becomes

$$\begin{aligned} L(q,q') &= \langle p, q'\rangle - H(p,q) \\ &= \frac{1}{2}\langle A^{-1}q', q'\rangle - \langle A^{-1}q', B^*q\rangle - V(q) + \frac{1}{2}\langle A^{-1}B^*q, B^*q\rangle . \end{aligned}$$

   Only minor changes are needed to establish a topological characterization of $M = H^{-1}(0)$, namely replacing $V(q)$ by $V(q) - \frac{1}{2}\langle A^{-1}(q)B^*(q)q, B^*(q)q\rangle$ in the definition of N. One may then proceed along the same lines as in the present paper to establish an existence theorem in the spirit of Theorem 1.

3. Another possible extension is to generalize the underlying configuration space $\mathbb{R}^n$. Let P be any smooth n-dimensional Riemannian manifold (without boundary), and consider the cotangent space T^*P with its canonical symplectic structure. By considering mechanical Hamiltonians $H : T^*P \to \mathbb{R}$ we obtain the generalization of our problem for cotangent bundles. To prove the analogue of Theorem 1 requires additional thought, although we believe that the result still holds under a suitable geometric conditions. In the compact case results like these were obtained by Hofer and Viterbo [12].

4. An interesting class of Hamiltonian systems that does not fall into any of the above categories are the so-called *higher order* Lagrangian systems. To illustrate this let us consider *second* order Lagrangians of the form $L(q,q',q'') = \frac{1}{2}|q''|^2 + \frac{\beta}{2}|q'|^2 + V(q)$, where q is scalar. The associated Hamiltonian then is $H(p_1,p_2,q_1,q_2) = p_1q_2 + \frac{1}{2}p_2^2 - \frac{\beta}{2}q_2^2 - V(q_1)$. A topological characterization of the energy manifolds is not hard to obtain, see e.g. [1]. Let us illustrate the important issues by specializing further to the specific example $V_{\pm}(q) = \pm(\frac{1}{4}q^4 - \frac{1}{2}q^2 + E)$ with $E \in (0, \frac{1}{4})$. For these choices we have $H_2(M_+) \cong \mathbb{Z}$, while $H_2(M_-) \cong \mathbb{Z}^2$ (and $H_1(M_+) \cong \mathbb{Z}^2$ and $H_1(M_-) \cong \mathbb{Z}$). For $\beta \geq 0$ the energy manifold is of contact type, whereas for sufficiently large $\beta < 0$ it is not [1]. The results in [14, 25] show that for either sign of β there are closed characteristics. Moreover, on M_- (whose second homology group has rank 2) there are at least *two* different closed characteristics [14, 25]. On the other hand, varying E outside $[0, \frac{1}{4}]$ one easily arrives at situations where $H_2(M_+) = 0$ (but $H_1(M_+) \cong \mathbb{Z}$). In such a case there are values of β (for example $\beta > 0$ sufficiently large) for which no closed characteristics exists on M_+ (cf. [24]).

The above generalizations are all in the setting where the Hamiltonian problem can be converted into a Lagrangian one. The challenge is to obtain a similar result on non-compact energy surfaces in the general Hamiltonian setting. The above considerations, as well as those put forward in the introduction, seem to indicate that the existence of closed characteristics on non-compact hypersurfaces depends on two main ingredients, namely on the topology of the hypersurface, and on some well-chosen geometrical condition. The *topological* condition that we propose in this paper is that at least one of the homology groups of M *in the top half* be nontrivial.

As already mentioned, even for compact hypersurfaces a *geometric* condition is needed to ascertain the existence of closed characteristics: one usually requires M to be of contact type. The contact type condition can also be recognized in the proof of the Palais-Smale property in Section 3. Consider variations of the form $X = \mu \frac{\partial}{\partial \tau} + a(u)\frac{\partial}{\partial u}$, which are all the variations that are useful for the Lagrangian variational principle. A straightforward calculation shows that we have the identity (with $i_{X_H}\omega = -dH$ and the notation introduced in Section 2.2)

$$e^{-\tau} i_{\frac{\partial}{\partial s}}(\lambda + \mathcal{L}_{X^1}\lambda) = i_{X_H}\alpha - (\mu+1)H, \tag{27}$$

where $\alpha = pdq + d(pa(q))$ with, as usual, $q(t) = u(e^{-\tau}t)$ and $p(t) = e^{-\tau}u'(e^{-\tau}t)$. In Section 3 we chose $\mu = -1$ to simplify the above relation. Note that if the manifold does not satisfy the asymptotic regularity condition required for Theorem 1, then it could in particular cases be useful to consider other values of μ in the variations used in the proofs of Lemmas 9 and 11. But let us come back to the relation between the Palais-Smale condition and the contact type condition, which is equivalent to the existence of a 1-form α such that $d\alpha = \omega$ and $i_{X_H}\alpha > 0$. If a 1-form can be found which satisfies the slightly stronger inequality $i_{X_H}\alpha \geq C > 0$, and which, in addition, is of the special form $\alpha = pdq + d(pa(q))$, then via (27) with $\mu = -1$ this leads to an upper bound on the periods in the Palais-Smale sequence. This is precisely what is made explicit in Lemma 11.

In the general Hamiltonian setting the situation is similar. Let $M = H^{-1}(0)$ for some Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. Writing $x = (p,q)$, the Hamiltonian action of some parametrized closed curve γ is given by

$$\mathcal{A}(x,T) = \int_0^T \big[\langle p, q'\rangle - H(x)\big]dt = \oint_\gamma \alpha_0 - \int_0^T H(x)\,dt,$$

where $\alpha_0 = \frac{1}{2}(pdq - qdp) = i_{Y_0}\omega$, with $Y_0 = \frac{1}{2}x\frac{\partial}{\partial x}$. The vectorfield Y_0 is a Liouville vectorfield, i.e. $\mathcal{L}_{Y_0}\omega = \omega$. Recalling Cartan's identity $\mathcal{L}_{Y_0}\omega = di_{Y_0}\omega + i_{Y_0}d\omega$, we conclude from $d\omega = 0$ that α_0 satisfies $d\alpha_0 = \omega$. When we rescale time by $T = e^\tau$, and set $x(e^{-\tau}s) = u(s)$, then the action becomes

$$\mathcal{A}(x,T) = \mathcal{B}(u,\tau) = \oint_\gamma \alpha_0 - e^\tau \int_0^1 H(u)\,ds.$$

We define the 1-form $\bar\lambda = \alpha_0 - e^\tau H(u)ds$. Variations of the form $X = \delta\tau\frac{\partial}{\partial \tau} + \delta u\frac{\partial}{\partial u}$ lead to

$$\mathcal{L}_X\bar\lambda = i_{\delta u\frac{\partial}{\partial u}}\omega + di_{\delta u\frac{\partial}{\partial u}}\alpha_0 - e^\tau\langle DH(u), \delta u\rangle\,ds - e^\tau H(u)\delta\tau ds.$$

We now consider a more restrictive class of variations, namely $\delta\tau = \mu$ and $\delta u\frac{\partial}{\partial u} = -Y$, where Y is a Liouville vectorfield. Writing $\alpha = i_Y\omega$, we obtain

$$\bar\lambda + \mathcal{L}_X\bar\lambda = \alpha_0 - \alpha - di_Y\alpha_0 + e^\tau\langle DH(u), Y\rangle\,ds - e^\tau(\mu+1)H(u)ds. \tag{28}$$

Since $d(\alpha_0 - \alpha) = 0$ there is a function f such that $\alpha_0 - \alpha = df(u)$, which corresponds to $Y = Y_0 - JDf(u)$. Using the identity $\langle DH, Y\rangle = \omega(Y, JDH) = \omega(Y, X_H) = i_{X_H} i_Y \omega = i_{X_H}\alpha$, we see that (28) becomes, analogous to (27),

$$\bar{\lambda} + \mathcal{L}_X \bar{\lambda} = d(f - i_Y \alpha_0) + e^\tau \big[i_{X_H}\alpha - (\mu + 1)H(u)\big] ds.$$

Finally, choosing $\mu = -1$ implies that, with $\alpha = \alpha_0 - df$,

$$\mathcal{B}(u, \tau) + \mathcal{B}'(u, \tau)(-Y_0 + JDf(u), -1) = e^\tau \int_0^1 i_{X_H}\alpha \, ds.$$

In particular, for critical points we have $\mathcal{A}(x, T) = \int_0^T i_{X_H}\alpha$. Therefore, if for some function $f(u)$ the inequality $i_{X_H}\alpha \geq C > 0$ holds on M, where $\alpha = \frac{1}{2}(pdq - qdp) - df(x)$, then this leads to a bound on the period of closed characteristics that have bounded action $\mathcal{A}$. For compact manifolds one can deduce (after some rather classical estimates) that the set of closed characteristics is compact. Moreover, the results by Viterbo [26] show that on compact manifolds of contact type there is at least one closed characteristic.

For non-compact hypersurfaces we do not expect the contact type condition to be a sufficient condition for the existence of closed characteristics, even when the topology is such that one of the homology groups in the top half is nontrivial, see the examples in the introduction. The non-compactness makes it particularly difficult to obtain (a priori) bounds on the set of closed characteristics. In the Lagrangian context this is overcome in Lemma 12. Our impression is that, in addition to the topological constraint and the contact type condition, some well-chosen geometric condition on the asymptotic behavior of the Hamiltonian (and thus of the energy surface M) is needed. This condition should at least ensure an a priori bound on the set of closed characteristics on M that satisfy an action bound, since this is an essential requirement for a Palais-Smale property to hold. In this context, one may for example think of the existence of a 1-form α' with $d\alpha' = \omega$ such that $i_{X_H}\alpha'$ asymptotically grows "sufficiently" fast for large $x \in M$. This should perhaps be complemented by an asymptotic regularity condition on $H(x)$ (instead of one on $V(q)$ for the Lagrangian setting). Obviously, at this point it is unclear what would be a suitable condition that is correct, sufficiently general, as well as aesthetically pleasing. On the other hand, since our topological condition is well-defined, these considerations lead us to the following conjecture.

Conjecture. *A regular hypersurface M of dimension $2n-1$ with $H_i(M) \neq 0$ for some $i \geq n$, and which satisfies an appropriate geometric/asymptotic condition, contains at least one closed characteristic.*

The vagueness of the "appropriate" geometric/asymptotic condition highlights that we not just would like to prove this conjecture, but that we also need to find a suitable condition that ensures its correctness.

Finally, our result concerns the existence of at least one closed characteristic. In most situations the topology implies the existence of many different closed characteristics. This problem has sparked, especially in the compact case, many interesting results in Hamiltonian mechanics, and still is a mostly uncharted field of research, although in the past decades various multiplicity result were obtained by Ekeland and Hofer [7], Viterbo [27], and more recently by Long [15]. To get a flavor of how the topological information in the top half homology groups is related to multiplicity let us consider $M \cong S^{2n-1}$ given by a quadratic Hamiltonian: $H = \sum_{i=1}^n \frac{1}{2}\left(p_i^2 + \zeta_i^2 q_i^2\right) - 1$, with $\zeta_i > 0$. For the associated linear Hamiltonian system we know all the closed characteristics at $M = H^{-1}(0)$. We find exactly n periodic orbits if the ζ_i-s are independent over $\mathbb{Z}$. The conjecture is that n is a lower bound in the case $M \cong S^{2n-1}$, where the only nontrivial top half homology group is $H_{2n-1}(M) \cong \mathbb{Z}$. This shows that the Betti numbers should probably be counted with a weight. To further substantiate this, recall from the discussion of second order Lagrangians above that if $H_2(M) \cong \mathbb{Z}^k$, then M has at least k closed characteristics. Although this has only been proved so far for second order Lagrangians, it suggests that a Betti number $\beta_n = k$ implies at least k closed characteristics. Summarizing, multiplicity is an extremely interesting but difficult direction for further research, and (non-compact) mechanical hypersurfaces could serve as a useful initial step towards an understanding of general non-compact energy surfaces.

Appendix A: Filling a simplex

The objective of this appendix is to prove the existence of the map g used in the construction of the linking sets in Section 5.2. The properties of the map g are repeated below for convenience. Recall that σ_{k-1} is a simplicial $(k-1)$-cycle, and σ'_k is a simplicial k-chain such that $\partial\sigma'_k = \sigma_{k-1}$. Their (bounded) supports $W = |\sigma_{k-1}|$ and $U = |\sigma'_k|$ satisfy $W \subset \{V > 0\} = \mathbb{R}^n - N$ and $\partial U \subset W$. The triangulation of U is denoted by $\mathcal{T}_U$, and its restriction to W by $\mathcal{T}_W$. We use the notation u (rather than u^0) for points in U throughout. We construct a continuous map $g : U \to H^1(\mathbb{R}/\mathbb{Z}; \mathbb{R}^n)$ satisfying

1. $g(u) \equiv u \quad$ for all $u \in W$;
2. $g(u)^+ \equiv 0 \quad$ if and only if $u \in W$;
3. $\int_0^1 V(g(u)(s))ds > 0 \quad$ for all $u \in U$.

As before, we identify points in $\mathbb{R}^n$ with constant functions in H^1.

We proceed by first constructing a continuous map $\bar{g}$ from U to $L^2(\mathbb{R}/\mathbb{Z}; \mathbb{R}^n)$, and subsequently smoothing it. For any $u \in U$ the function $\bar{g}(u)$ will be piecewise constant with values in $\{V > 0\}$, so that it satisfies property 3. In view of property 1, let $\bar{g} : W \to L^2$ be given by $\bar{g}(u) = u$ for all $u \in W$. We will extend the domain of definition gradually to all of U.

We start the construction by explaining an interpolation procedure for individual simplices. Define the standard m-simplex by

$$\Delta^m = \Big\{\eta = (\eta_1, \cdots, \eta_m) \;\Big|\; \sum_{i=1}^m \eta_i \leq 1,\, \eta_i \geq 0\Big\}.$$

Given a continuous map h on the boundary $\partial\Delta^m$, we describe a way to extend/interpolate h to a continuous map on all of Δ^m, denoted by $I_{\Delta^m}(h)$. Define the base of the simplex by $\Delta^m_B = \{\eta \mid \sum_{i=1}^m \eta_i = 1,\, \eta_i \geq 0\}$, and its normal by $\mathbf{1} = \sum_{i=1}^m e_i$, where e_i are the standard unit vectors in $\mathbb{R}^m$. On Δ^m we can give alternative coordinates $(\bar{\eta}, \lambda)$

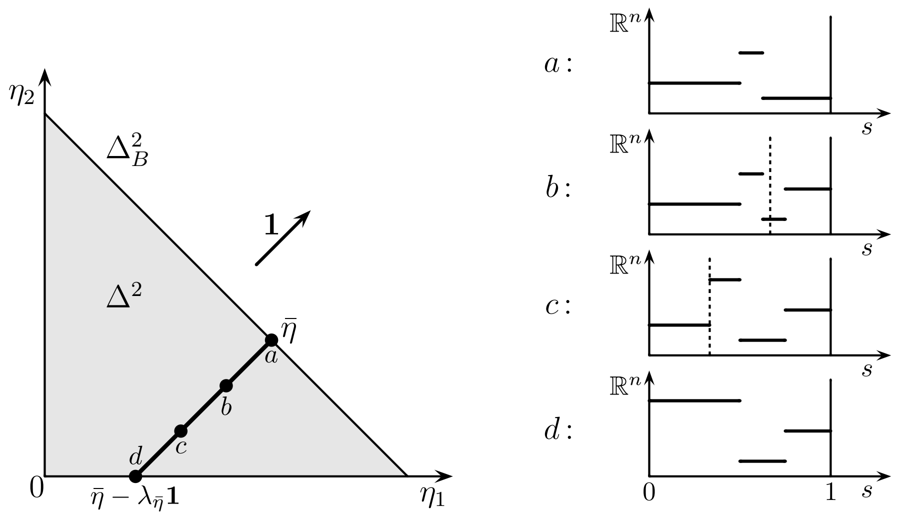


Figure 8: A map h defined on the boundary of an m-simplex Δ^m is interpolated to a map $I_{\Delta^m}(h)$ on all of Δ^m. On the left a picture of the standard 2-simplex with an interpolation line. Interpolated profiles along this line are shown on the right. The dotted lines indicate the value $s = 1 - \frac{\lambda}{\lambda_{\bar\eta}}$.

so that $\eta = \bar\eta - \lambda\mathbf{1}$, with $\bar\eta \in \Delta^m_B$ and $\lambda \in [0, \lambda_{\bar\eta}]$, where $\lambda_{\bar\eta} = \min_{1\le i\le m} \bar\eta_i$ depends on $\bar\eta$. Let h be a continuous map from the boundary $\partial\Delta^m$ to L^2. We now define an interpolation of h on Δ^m (see Figure 8):

$$I_{\Delta^m}(h(\bar\eta - \lambda\mathbf{1}))(s) = \begin{cases} h(\bar\eta)(s), & s \in \big[0, 1 - \frac{\lambda}{\lambda_{\bar\eta}}\big), \\ h(\bar\eta - \lambda_{\bar\eta}\mathbf{1})(s), & s \in \big(1 - \frac{\lambda}{\lambda_{\bar\eta}}, 1\big]. \end{cases}$$

This map is then defined on all of Δ^m, it is a continuous map from Δ^m to L^2, and it coincides with h on the boundary. The interpolation operator I_{Δ^m} thus sends a map $h \in C(\partial\Delta^m; L^2)$ to a map $I_{\Delta^m}(h) \in C(\Delta^m; L^2)$.

As in singular homology, we consider linear (affine) maps

$$\ell^m_j : \Delta^m \longrightarrow \mathcal{T}_U,$$

which map the standard m-simplex to an m-dimensional "triangle" in $\mathcal{T}_U$. The image $\ell^m_j(\Delta^m)$ is again called an m-simplex, denoted by L^m_j. When h is a map defined on ∂L^m_j, then $I_{L^m_j}(h) \stackrel{\text{def}}{=} \ell^m_j \circ I_{\Delta^m} \circ (\ell^m_j)^{-1}(h)$ is its interpolation to L^m_j.

We need to do some careful bookkeeping. We order the 0-simplices in $\mathcal{T}_U$ and denote them by $\{L^0_i\}^p_{i=1}$. For any m-simplex L^m_j in $\mathcal{T}_U$, we denote the 0-simplices that form the corners of L^m_j by $\{L^0_{i(l)}\}^m_{l=0}$, ordered in such a way that $i(0) < i(1) < \cdots < i(m)$. Here the $i(l)$ depend on the simplex L^m_j under consideration, and which simplex is meant should be clear from the context. We choose *all* the maps ℓ^m_j so that $\ell^m_j(0, 0, \ldots, 0) = L^0_{i(0)}$, i.e., they map the origin to the "first" corner point. The importance of this choice will become clear later. For now, notice that the role of the origin in Δ^m in the interpolation construction is geometrically different from that of the other corner points in Δ^m_B.

Having defined $\bar{g}(u) = u$ on all simplices in $\mathcal{T}_W$, we use the interpolation operator to extend its domain of definition to $\mathcal{T}_U$. We begin with defining $\bar{g}$ on the 0-simplices in $\mathcal{T}_U - \mathcal{T}_W$, and then inductively/recursively deal with the m-simplices, $0 < m \leq k$. Let $\{v_i^{\pm}\}_{i=1}^p$ be a set of $2p$ *distinct* points in $\mathbb{R}^n - (N \cup W) \subset \{V > 0\}$. On all 0-simplices $L_j^0 \in \mathcal{T}_U - \mathcal{T}_W$ we define $\bar{g}$ to be piecewise constant, but not uniformly constant:

$$\bar{g}\big(L_j^0(\Delta^0)\big)(s) = \begin{cases} v_j^-, & s \in [0, \frac{1}{2}), \\ v_j^+, & s \in (\frac{1}{2}, 1]. \end{cases}$$

On $\bigcup_j L_j^0$ we now have that $\bar{g}$ satisfies the three required properties.

Next, we consider the 1-simplices. On all $L_j^1 \in \mathcal{T}_U - \mathcal{T}_W$ we define $\bar{g}$ via the interpolation operator. Namely, since $\bar{g}$ is already defined on the 0-simplices that form the boundary ∂L_j^1, we may define $\bar{g} = I_{L_j^1}\bar{g}$ on L_j^1. On $\bigcup_j L_j^1$ we have that $\bar{g}$ is continuous and satisfies the three required properties. First, $\bar{g}(u) = u$ for $u \in W$ by definition. Second, $\bar{g}(u)$ is not a constant for $\bigcup_j L_j^1 - W$, since the points $v_i^{\pm} \notin W$ and they are all different. Third, $\bar{g}(u)(s) \in W \cup \{v_i^{\pm}\}_{i=1}^p$ a.e. by construction, and $W \cup \{v_i^{\pm}\}_{i=1}^p \subset \{V > 0\}$, hence $\int_0^1 V(\bar{g}(u)(s))ds > 0$.

We now proceed recursively. Let $2 \leq m \leq k$ and let $\bar{g}$ be defined on all $L_j^{m-1} \in \mathcal{T}_U$, where it satisfies the three required properties, and $\bar{g}(u)(s) \in W \cup \{v_i^{\pm}\}_{i=1}^p$ a.e.. On the m-simplices $L_j^m \in \mathcal{T}_U - \mathcal{T}_W$ we define $\bar{g}$ again via the interpolation operator: since $\bar{g}$ is defined in the boundary ∂L_j^m, we may define $\bar{g} = I_{L_j^m}\bar{g}$ on L_j^m. On $\bigcup_j L_j^m$ we have that $\bar{g}$ is continuous and we claim that it satisfies the three properties. Properties 1 and 3 are straightforward, but property 2 requires a more detailed investigation. Let $u \in \text{int}(L_j^m)$ with $L_j^m \in \mathcal{T}_U - \mathcal{T}_W$. We need to show that $\bar{g}(u)$ is not a constant function. In particular, we assert that $\bar{g}(u)(0) \neq \bar{g}(u)(1)$.

Since ℓ_j^m is a bijection between Δ^m and L_j^m, let us identify them, and write $u = (\bar{\eta}, \lambda)$. The function $\bar{g}(u)$ is an interpolation between $\bar{g}(\bar{\eta})$ and $\bar{g}(u_1)$, where $u_1 = \bar{\eta} - \lambda_{\bar{\eta}}\mathbf{1}$. In particular, $\bar{g}(u)(0) = \bar{g}(\bar{\eta})(0)$ and $\bar{g}(u)(1) = \bar{g}(u_1)(1)$. Let us single out the special corner point $L^* = L_{i(0)}^0 = (0, 0, \ldots, 0)$, and let $v^* = v_{i(0)}^+$. Since $\bar{\eta} \in \Delta_B^m$, and the corners of Δ_B^m are $\{L_{i(l)}^0\}_{l=1}^m$, we see that $\bar{g}(\bar{\eta})$ takes values in the set $\Lambda \stackrel{\text{def}}{=} (W \cap \Delta_B^m) \cup \{v_{i(l)}^{\pm}\}_{l=1}^m$. In particular, $\bar{g}(u)(0) = \bar{g}(\bar{\eta})(0) \in \Lambda$. Note that $u_1 \notin \Delta_B^m$, and if $u_1 \in W$ then $u_1 \notin \Lambda$, see also Figure 9.

To prove our assertion, we show that $\bar{g}(u)(1) = \bar{g}(u_1)(1) \notin \Lambda$. This is true if $u_1 \in W$, since then $\bar{g}(u_1) = u_1 \notin \Lambda$. It is also true if $u_1 \notin W$ is a corner point, since then $u_1 = L^*$, and hence $\bar{g}(u_1)(1) = v^* = v_{i(0)}^+ \notin \Lambda$. The final possibility is that $u_1 \notin W$ is not a corner point. This means that u_1 is in some lower dimensional simplex, i.e. $u_1 \in \text{int}(L_j^{m_1}) \in \mathcal{T}_U - \mathcal{T}_W$ for some $1 \leq m_1 < m$. Notice that the corner points of $L_j^{m_1}$ are a subset of those of L_j^m, and in particular, by our ordering of the corner points, $L^* = L_{i(0)}^0$ also for this m_1-simplex. Furthermore, since we have chosen $\ell_j^{m_1}(0, \ldots, 0) = L_{i(0)}^0 = \ell_j^m(0, \ldots, 0)$, we see that $\ell_j^{m_1}(\Delta_B^{m_1}) \subset \ell_j^m(\Delta_B^m)$.

Identifying $L_j^{m_1}$ with Δ^{m_1}, we write $u_1 = (\bar{\eta}_1, \lambda_1)$. The function $\bar{g}(u_1)$ is an interpolation between $\bar{g}(\bar{\eta}_1)$ and $\bar{g}(u_2)$, where $\bar{\eta}_1 \in \Delta_B^{m_1}$ and $u_2 = \bar{\eta}_1 - \lambda_{\bar{\eta}_1}\mathbf{1}$. As before, $\bar{g}(u_1)(1) = \bar{g}(u_2)(1)$, and the same arguments apply to u_2 as to u_1 above. If $u_2 \in W$ then $\bar{g}(u)(1) = \bar{g}(u_1)(1) = \bar{g}(u_2)(1) \notin \Lambda$. If $u_2 \notin W$ is a corner point, then $u_2 = L_*$, and hence $\bar{g}(u)(1) = \bar{g}(u_2)(1) = v^* \notin \Lambda$. Finally, if $u_2 \notin W \cup L^*$, then $u_2 \in \text{int}(L_j^{m_2}) \in \mathcal{T}_U - \mathcal{T}_W$ for some $1 \leq m_2 < m_1$, and we repeat the argument. Since

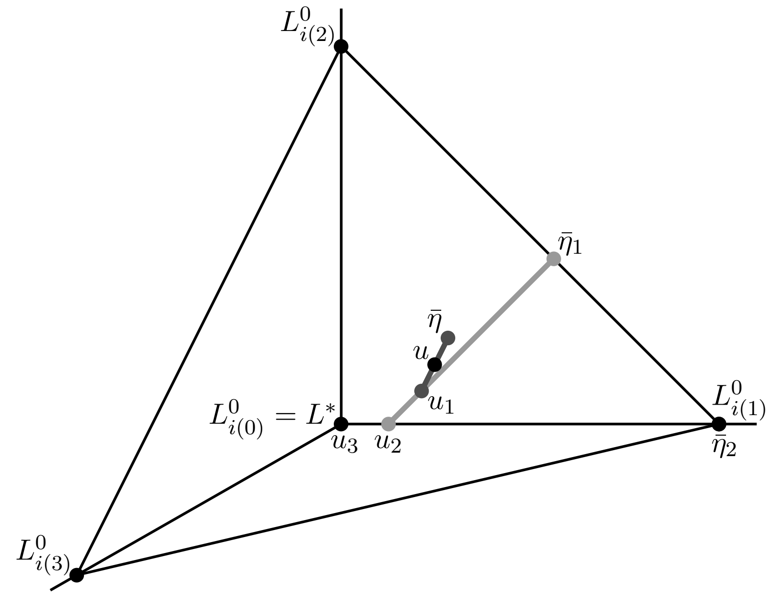


Figure 9: We depict the argument why $\bar{g}(u)^+ \not\equiv 0$ for $u \notin W$ for a 3-simplex in $\mathcal{T}_U - \mathcal{T}_W$, where we assume for simplicity that $\{L_{i(l)}\}_{l=0}^3 \cap W = \varnothing$. For any $u = (\bar{\eta}, \lambda)$ in the interior of the 3-simplex it follows that $\bar{\eta} \in \Delta^3_B$ and hence $\bar{g}(u)(0) = \bar{g}(\bar{\eta})(0) \in \Lambda = \{v^\pm_{i(1)}, v^\pm_{i(2)}, v^\pm_{i(3)}\}$. On the other hand, $\bar{g}(u)(1) = \bar{g}(u_1)(1)$. Repeating this argument twice we see that $\bar{g}(u)(1) = \bar{g}(u_1)(1) = \bar{g}(u_2)(1) = \bar{g}(u_3)(1) = v^* = v^+_{i(0)} \notin \Lambda$. In particular, we conclude that $\bar{g}(u)(0) \neq \bar{g}(u)(1)$, so that $\bar{g}(u)$ is not identically constant.

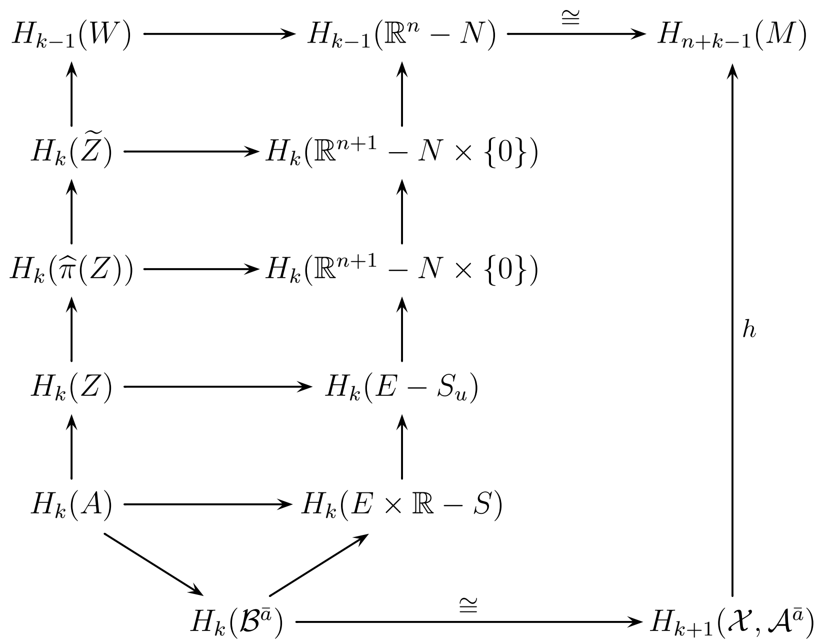
$1 \leq \cdots < m_{i+1} < m_i < \cdots < m_1 < m$, this construction breaks off after at most m steps, and we conclude that $\bar{g}(u)(1) \notin \Lambda$, and hence $\bar{g}(u)(0) \neq \bar{g}(u)(1)$.

This finishes our proof of the assertion that $\bar{g}$ satisfies property 2. We have thus found a continuous map $\bar{g} : U \to L^2$ that satisfies the three required properties, and $\bar{g}(u)(s) \in W \cup \{v^\pm_i\}_{i=1}^p \subset \{V > 0\}$ a.e.. Furthermore, it follows from compactness of U that $\int_0^1 V(\bar{g}(u)(s))ds \geq C$ for some u-independent positive constant C. The final step is to smoothen $\bar{g}$. Using a standard mollifier φ_ϵ, let $\bar{g}_\epsilon$ be the convolution $\varphi_\epsilon \star \bar{g}$. It is not difficult to derive that $\bar{g}_\epsilon$ is a continuous map from U to H^1, and that it satisfies the three required properties for small ϵ. Choosing a sufficiently small $\bar{\epsilon}$, this completes the construction of the map $g = \bar{g}_{\bar{\epsilon}}$.

Appendix B: Homology of the sublevel set

This appendix explains why the arguments in this paper imply that the relative homologies $H_*(\mathcal{X}, \mathcal{A}^{\bar{a}})$ are $H_*(\mathcal{A}^{\hat{a}}, \mathcal{A}^{\bar{a}})$ are nontrivial, as stated at the end of the introduction. In a Morse theoretical context this can be interpreted as follows: $\mathcal{A}^{\hat{a}}$ (or $\mathcal{X}$) and $\mathcal{A}^{\bar{a}}$ are sublevel sets of the action functional $\mathcal{A}$ corresponding to the values $\hat{a}$ (or ∞) and $\bar{a}$. Nontriviality of the relative homology of these sublevel sets implies, in view of the Palais-Smale property, that there is a critical value for $\mathcal{A}$ which is larger than $\bar{a}$, and less than $\hat{a}$. We start with investigating $H_*(\mathcal{X}, \mathcal{A}^{\bar{a}})$. We note that our coordinate changes (see Section 2.1) imply that $H_*(\mathcal{X}, \mathcal{A}^{\bar{a}}) \cong H_*(E \times \mathbb{R}, \mathcal{B}^{\bar{a}})$, where $E = H^1(\mathbb{R}/\mathbb{Z})$.

The topological arguments in our paper may be summarized by the following diagram:



The "ladder" on the left-hand side of the diagram consists of the various steps in the proof of Lemma 16, each one giving rise to a commutative square. The commutative triangle at the bottom is induced by the inclusions $A \subset \mathcal{B}^{\bar{a}} \subset E \times \mathbb{R} - S$ and appears in the proof of Lemma 13. In the same proof, in (18), the reader has come across the *isomorphism* δ_{k+1} as the connecting map in the long exact homology sequence of the pair $(E \times \mathbb{R}, \mathcal{B}^{\bar{a}})$. Recalling that the coordinate transformations imply that $H_{k+1}(E \times \mathbb{R}, \mathcal{B}^{\bar{a}}) \cong H_{k+1}(\mathcal{X}, \mathcal{A}^{\bar{a}})$, this leads to the isomorphism at the bottom of the diagram. The isomorphism in the top right-hand corner is the statement of Theorem 2. The map h is defined by commutativity of the right side of the diagram. We still need to argue why the homomorphism h is nontrivial. What we have shown in the paper is that, given an element w of $H_{k-1}(W)$ with nontrivial image y in $H_{k-1}(\mathbb{R}^n - N)$, we can lift it to an element, say $\tilde{w}$, of $H_k(A)$ with nontrivial image in $H_k(E \times \mathbb{R} - S)$. If we denote by x the image of $\tilde{w}$ in $H_k(\mathcal{B}^{\bar{a}})$, by commutativity we get that h maps the element in $H_{k+1}(\mathcal{X}, \mathcal{A}^{\bar{a}})$ that corresponds to x, to the nonzero element of $H_{n+k-1}(M)$ corresponding to y. This shows that the map h and the homology group $H_{i-n+2}(\mathcal{X}, \mathcal{A}^{\bar{a}})$ are nontrivial when $H_i(M)$ is nontrivial for some $n \leq i \leq 2n-1$.

As for the non-triviality of $H_*(\mathcal{A}^{\hat{a}}, \mathcal{A}^{\bar{a}})$ we argue as follows. Again, the coordinate transformations imply that $H_*(\mathcal{A}^{\hat{a}}, \mathcal{A}^{\bar{a}}) \cong H_*(\mathcal{B}^{\hat{a}}, \mathcal{B}^{\bar{a}})$. Let $\bar{x} = (\delta_{k+1})^{-1}(x)$ be the element in $H_{k+1}(E \times \mathbb{R}, \mathcal{B}^{\bar{a}})$ corresponding to the nontrivial class $x \in H_k(\mathcal{B}^{\bar{a}})$. Choose a k-cycle $\hat{x}$ representing $\bar{x}$, and define $\hat{a}$ such that $\hat{a} > \max_{|\hat{x}|} \mathcal{B}$. Consider the triple $(E \times \mathbb{R}, \mathcal{B}^{\hat{a}}, \mathcal{B}^{\bar{a}})$ and its long exact homology sequence

$$H_{k+1}(\mathcal{B}^{\hat{a}}, \mathcal{B}^{\bar{a}}) \xrightarrow{\gamma_1} H_{k+1}(E \times \mathbb{R}, \mathcal{B}^{\bar{a}}) \xrightarrow{\gamma_2} H_{k+1}(E \times \mathbb{R}, \mathcal{B}^{\hat{a}}) \longrightarrow H_k(\mathcal{B}^{\hat{a}}, \mathcal{B}^{\bar{a}}).$$

By the choice of $\hat{a}$, we have that γ_2 maps $\bar{x}$ to $0 \in H_{k+1}(E \times \mathbb{R}, \mathcal{B}^{\hat{a}})$. This implies that the image of γ_1 is non-trivial and, in particular, $\bar{x} \in \mathrm{im}(\gamma_1) = \ker(\gamma_2)$, hence there is an element $\tilde{x} \neq 0$ such that $\gamma_1(\tilde{x}) = \bar{x}$. Therefore $\hat{h} = h \circ \gamma_1$ is a nontrivial homomorphism from $H_{k+1}(\mathcal{B}^{\hat{a}}, \mathcal{B}^{\bar{a}})$ to $H_{n+k-1}(M)$.

Appendix C: Additional proofs

In this appendix we provide the proof of the technical lemma 4, and we give substituting proofs of several lemmas when asymptotic regularity is replaced by (25).

Lemma 4. *If* $\|D^2V\| \le C|DV|$ *on the line segment joining* u *and* u_0*, then*

$$|DV(u) - DV(u_0)| \le |DV(u_0)|\,(e^{C|u-u_0|} - 1).$$

Proof of Lemma 4. The proof is analogous to that of Gronwall's lemma. First, we set $\mu = C|u - u_0|$, and let $f(t) = DV(tu + (1-t)u_0)$, so that $f(0) = DV(u_0)$ and $f(1) = DV(u)$. Since $f'(t) = D^2V(tu+(1-t)u_0)[u-u_0]$, it follows from the hypotheses that $|f'(t)| \le \mu|f(t)|$. We thus get the estimate

$$\begin{aligned} |f(t) - f(0)| &= \left|\int_0^t f'(s)ds\right| \le \mu \int_0^t |f(s)| \le \mu \int_0^t \big[|f(s) - f(0)| + |f(0)|\big]ds \\ &\le \mu|f(0)|t + \mu \int_0^t |f(s) - f(0)|ds. \end{aligned} \tag{29}$$

Let $h(t) = \int_0^t |f(s) - f(0)|ds$. Then $h(t) \le \tilde{h}(t)$, where $\tilde{h}$ satisfies the differential *equation* that corresponds to the differential *inequality* for h:

$$\begin{cases} h'(t) \le \mu|f(0)|t + \mu h(t), \\ h(0) = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \tilde{h}'(t) = \mu|f(0)|t + \mu\tilde{h}(t), \\ \tilde{h}(0) = 0. \end{cases}$$

It is straightforward to deduce that

$$h(t) \le \tilde{h}(t) = \mu^{-1}|f(0)|(e^{\mu t} - \mu t - 1).$$

We infer from the definition of h and (29) that

$$|f(t) - f(0)| \le \mu|f(0)|t + |f(0)|(e^{\mu t} - \mu t - 1) = |f(0)|(e^{\mu t} - 1).$$

Substituting $t = 1$ and using the definitions of f and μ we arrive at the asserted estimate. □

We now present alternative proofs of Lemmas 11, 12 and 14/15 assuming, instead of the usual asymptotic regularity,

$$|DV(q)| \ge c > 0 \quad \text{and} \quad \big\langle p, D^2V(q)p\big\rangle \le \tilde{c}|p|^2 \text{ for all } p \in \mathbb{R}^n \qquad \text{as } |q| \to \infty, \tag{30}$$

for some $\tilde{c} > 0$. The adjustments to the proof of Lemma 11 are minimal.

Lemma 11*. *Let* $(u_\varepsilon, \tau_\varepsilon)$ *be critical points of* $\mathcal{B}_\varepsilon$ *with* $0 < c_1 \le \mathcal{B}_\varepsilon(u_\varepsilon, \tau_\varepsilon) \le c_2$*. There is a constant* T_2*, independent of* ε*, such that* $\tau_\varepsilon \le T_2$ *for sufficiently small* ε*.*

Proof. We follow the proof of Lemma 11, but use variations

$$\delta u = -\kappa DV(u_\varepsilon) \qquad \text{and} \qquad \delta\tau = -1.$$

Since on bounded sets D^2V is bounded, it follows from (30) that there exists a $C_1 \ge \tilde{c}$ such that

$$\big\langle u_\varepsilon'(s), D^2V(u_\varepsilon(s))u_\varepsilon'(s)\big\rangle \le C_1|u_\varepsilon'(s)|^2.$$

We choose $\kappa = \frac{1}{2C_1}$, so that estimate (16) holds with $C_2 \le \kappa c^2$ for u_ε outside some large ball. For $u_\varepsilon(s)$ inside the ball one argues in the same way as in the proof of Lemma 11, and also the rest of the argument remains unchanged. □

The adjustments to the proof of Lemma 12 are substantial.

Lemma 12*. *Let $(u_\varepsilon, \tau_\varepsilon)$ be critical points of $\mathcal{B}_\varepsilon$ with $0 < c_1 \leq \mathcal{B}_\varepsilon(u_\varepsilon, \tau_\varepsilon) \leq c_2$. There is a constant T_3, independent of ε, such that $\tau_\varepsilon \geq T_3$ for sufficiently small ε.*

Proof. We follow the first paragraph of the proof of Lemma 12 without alterations. We thus know that, arguing by contradiction, $u_\varepsilon \to u_\varepsilon^0$ uniformly, and it remains to show that $|u_\varepsilon^0| \to \infty$ (also) leads to a contradiction. We reach this contradiction as follows. We use coordinates $q_\varepsilon(t) = u_\varepsilon(e^{\tau_\varepsilon} t)$ and $T_\varepsilon = e^{\tau_\varepsilon}$, so that q_ε is a T_ε-periodic solution of

$$q_\varepsilon'' = -DV(q_\varepsilon).$$

Clearly $q_\varepsilon^0 = T_\varepsilon^{-1} \int_0^{T_\varepsilon} q_\varepsilon(s) ds = u_\varepsilon^0$, so that $|q_\varepsilon^0| \to \infty$ and $\|q_\varepsilon - q_\varepsilon^0\|_{L^\infty} \to 0$ as $\varepsilon \to 0$.

We observe that

$$\frac{d^2 V(q_\varepsilon(t))}{dt^2} = \left\langle q_\varepsilon', D^2V(q_\varepsilon) q_\varepsilon' \right\rangle + \left\langle DV(q_\varepsilon), q_\varepsilon'' \right\rangle = \left\langle q_\varepsilon', D^2V(q_\varepsilon) q_\varepsilon' \right\rangle - |DV(q_\varepsilon)|^2.$$

Since q_ε is T_ε periodic and hence $\int_0^{T_\varepsilon} \frac{d^2V(q_\varepsilon(t))}{dt^2} = 0$, we infer that

$$\int_0^{T_\varepsilon} |DV(q_\varepsilon(s))|^2 ds = \int_0^{T_\varepsilon} \left\langle q_\varepsilon'(s), D^2V(q_\varepsilon(s)) q_\varepsilon'(s) \right\rangle ds \leq \tilde{c} \int_0^{T_\varepsilon} |q_\varepsilon'(s)|^2 ds, \tag{31}$$

where the inequality follows from (30). We can also estimate

$$\begin{aligned}
\int_0^{T_\varepsilon} |q_\varepsilon'(s)|^2 &= -\int_0^{T_\varepsilon} \left\langle q_\varepsilon''(s), q_\varepsilon(s) - q_\varepsilon^0 \right\rangle ds \\
&\leq \left(\int_0^{T_\varepsilon} |q_\varepsilon''(s)|^2 ds \right)^{1/2} \left(\int_0^{T_\varepsilon} |q_\varepsilon(s) - q_\varepsilon^0|^2 ds \right)^{1/2} \\
&\leq \left(\int_0^{T_\varepsilon} |DV(q_\varepsilon(s))|^2 ds \right)^{1/2} T_\varepsilon^{1/2} \, \|q_\varepsilon - q_\varepsilon^0\|_{L^\infty}.
\end{aligned} \tag{32}$$

Combining (31) and (32) we obtain

$$\left(\int_0^{T_\varepsilon} |DV(q_\varepsilon(s))|^2 ds \right)^{1/2} \leq \tilde{c}\, T_\varepsilon^{1/2} \, \|q_\varepsilon - q_\varepsilon^0\|_{L^\infty}. \tag{33}$$

On the other hand, it follows from the first inequality in (30) that for small ε

$$\left(\int_0^{T_\varepsilon} |DV(q_\varepsilon(s))|^2 ds \right)^{1/2} \geq c\, T_\varepsilon^{1/2}. \tag{34}$$

Since $\|q_\varepsilon - q_\varepsilon^0\|_{L^\infty} \to 0$ as $\varepsilon \to 0$, estimates (33) and (34) are contradictory. □

The arguments in Sections 5.1 and 5.2 need to be adjusted to fit the assumptions (30) as follows. Right above Lemma 14 we change the definition of N_ν to simply

$$N_\nu = \left\{ u^0 \in E^0 \;\middle|\; V(u^0) \leq -\nu \right\}.$$

The first claim in Lemma 14 about N ($\mathbb{R}^n - N$) and N_ν ($\mathbb{R}^n - N_\nu$) being homeomorphic for small ν remains true, but we drop the second claim about B_{ρ_ν} (it is not necessarily true). In the proof we need to take $F(u^0) = V(u^0)$. Then $\nabla F(u^0) = DV(u^0)$, and the remainder of the proof (of the first claim of the lemma) is analogous (we only need to use the first inequality in (30)).

The proof of Lemma 15 now changes, since the estimate on the ball B_{ρ_ν} is no longer available. Instead we argue as follows to obtain the estimate on $\mathcal{B}|_S$. Let $k(t) = V(u^0+t(u-u^0))$, then a Taylor expansion shows that $k(1) = k(0)+k'(0)+\frac{1}{2}k''(\theta)$ for some $\theta \in (0,1)$. If the second inequality in (30) holds for large q, then, at the expense of choosing a larger value for $\tilde{c}$, it holds for all $q \in \mathbb{R}^n$. Hence, substituting the definition of k into its Taylor expansion,

$$\begin{aligned} V(u) &= V(u^0) + DV(u^0)(u-u^0) + \frac{1}{2}\left\langle u-u^0, D^2V(u^0+\theta(u-u^0))(u-u^0)\right\rangle \\ &\le V(u^0) + DV(u^0)(u-u^0) + \frac{1}{2}\tilde{c}\,|u-u^0|^2. \end{aligned}$$

Using that $u^0 = \int_0^1 u(s)ds$, we obtain

$$\begin{aligned} \int_0^1 V(u(s))ds &\le V(u^0) + DV(u^0)\int_0^1 (u(s)-u^0)ds + \frac{1}{2}\tilde{c}\int_0^1 |u(s)-u^0|^2 ds. \\ &\le V(u^0) + \frac{1}{2}\tilde{c}\,\|u^+\|_\infty^2. \end{aligned}$$

Since for $u \in S$ we have $\|u^+\|_{L^\infty} \le \|u^+\|_1 = \rho$ by (3), it follows that for $\rho \le \sqrt{\nu/\tilde{c}}$

$$\mathcal{B}|_S = \frac{e^{-\tau}}{2}\int_0^1 |u'(s)|^2 ds - e^\tau \int_0^1 V(u(s))ds \ge \frac{e^{-\tau}\rho^2}{2} + e^\tau\Big(\nu - \frac{1}{2}\tilde{c}\,\rho^2\Big) \ge \rho\sqrt{\nu}.$$

We therefore choose $\rho \le \sqrt{\nu/\tilde{c}}$, which in some sense replaces ρ_ν, and the remainder of the proof of Lemma 15 is identical. This establishes that the sets A and S lead to a minimax under the assumption (30).

References

[1] S.B. Angenent, J.B. van den Berg, and R.C. Vandervorst. Contact and non-contact type Hamiltonian systems generated by second order Lagrangians, 2005. Preprint.

[2] V. Benci and P.H. Rabinowitz. Critical point theorems for indefinite functionals. *Invent. Math.*, 52:241–273, 1979.

[3] R. Bott and L.W. Tu. *Differential forms in algebraic topology.* Springer-Verlag, New York, 1982.

[4] G.E. Bredon. *Topology and geometry.* Springer-Verlag, New York, 1997.

[5] K.-C. Chang. *Infinite-dimensional Morse theory and multiple solution problems.* Birkhäuser Boston Inc., Boston, MA, 1993.

[6] A. Dold. *Lectures on algebraic topology.* Springer-Verlag, New York, 1972.

[7] I. Ekeland and H. Hofer. Convex Hamiltonian energy surfaces and their periodic trajectories. *Comm. Math. Phys.*, 113:419–469, 1987.

[8] V.L. Ginzburg. An embedding $S^{2n-1} \to \mathbf{R}^{2n}$, $2n-1 \geq 7$, whose Hamiltonian flow has no periodic trajectories. *Internat. Math. Res. Notices*, 2:83–97 (electronic), 1995.

[9] V.L. Ginzburg. A smooth counterexample to the Hamiltonian Seifert conjecture in $\mathbf{R}^6$. *Internat. Math. Res. Notices*, 13:641–650, 1997.

[10] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[11] M.W. Hirsch. *Differential topology.* Springer-Verlag, New York, 1994.

[12] H. Hofer and C. Viterbo. The Weinstein conjecture in cotangent bundles and related results. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 15:411–445, 1988.

[13] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics.* Birkhäuser Verlag, Basel, 1994.

[14] W.D. Kalies and R.C. Vandervorst. Closed characteristics of second-order Lagrangians. *Proc. Roy. Soc. Edinburgh Sect. A*, 134:143–158, 2004.

[15] Y. Long. Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Adv. Math.*, 154:76–131, 2000.

[16] J.R. Munkres. *Elementary differential topology.* Princeton University Press, Princeton, N.J., 1966.

[17] D. Offin. A class of periodic orbits in classical mechanics. *J. Diff. Eq.*, 66:90–117, 1987.

[18] P.J. Olver. *Applications of Lie groups to differential equations.* Springer-Verlag, New York, second edition, 1993.

[19] K. Perera. Critical groups of pairs of critical points produced by linking subsets. *J. Diff. Eq.*, 140:142–160, 1997.

[20] P.H. Rabinowitz. Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.*, 31:157–184, 1978.

[21] M. Struwe. *Variational methods.* Springer-Verlag, Berlin, third edition, 2000. Applications to nonlinear partial differential equations and Hamiltonian systems.

[22] K. Tanaka. Periodic solutions of first order singular Hamiltonian systems. *Nonlinear Anal.*, 26:691–706, 1996.

[23] K. Tanaka. Periodic solutions for singular Hamiltonian systems and closed geodesics on non-compact Riemannian manifolds. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17:1–33, 2000.

[24] J.B. van den Berg. The phase-plane picture for a class of fourth-order conservative differential equations. *J. Diff. Eq.*, 161:110–153, 2000.

[25] J.B. van den Berg and R.C. Vandervorst. Second order Lagrangian twist systems: simple closed characteristics. *Transactions AMS*, 354:1393–1420, 2002.

[26] C. Viterbo. A proof of Weinstein's conjecture in $\mathbf{R}^{2n}$. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 4:337–356, 1987.

[27] C. Viterbo. Equivariant Morse theory for starshaped Hamiltonian systems. *Trans. Amer. Math. Soc.*, 311:621–655, 1989.

[28] A. Weinstein. Periodic orbits for convex Hamiltonian systems. *Ann. of Math. (2)*, 108:507–518, 1978.

[29] A. Weinstein. On the hypotheses of Rabinowitz' periodic orbit theorems. *J. Diff. Eq.*, 33:353–358, 1979.