BRAIDED CONNECTING ORBITS IN PARABOLIC EQUATIONS VIA COMPUTATIONAL HOMOLOGY

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Abstract. We develop and present a computational method for producing forcing theorems for stationary and periodic solutions and connecting orbits in scalar parabolic equations with periodic boundary conditions. This method is based on prior work by Van den Berg, Ghrist, and Vandervorst on a Conley index theory for solutions *braided through* a collection of known stationary solutions. Essentially, the topological structure of the stationary solutions forces the existence of additional solutions with a specified topological type. In particular, this paper studies connecting orbits and develops and implements the algorithms required to compute the index, providing sample results as illustrations.

1. Introduction. The Sturmian principle of second order parabolic equations is one of the highlights of the study of partial differential equations, because of both its elegance and its widespread applicability (e.g. in Ricci flow and other geometric evolutions). In this paper we consider scalar nonlinear parabolic equations such as

$$U_t = U_{xx} + f(U_x, U, x), \tag{1.1}$$

with *periodic* boundary conditions, i.e. $x \in S^1 = \mathbb{R}/\mathbb{Z}$ (we fix the spatial period to 1 without loss of generality).

In one spatial dimension, second order parabolic equations not only satisfy a maximum and comparison principle, but also a lap-number or intersection-number principle, see e.g. [1, 10]. When two simultaneously evolving solutions $U^1(x,t)$ and $U^2(x,t)$ develop a tangency in their graphs $\{(x, U^i(x,t)) \,|\, x \in S^1\}$ at time $t = t_0$, then this tangency is removed immediately for $t > t_0$, in such a way as to strictly *decrease* the number of intersections of the graphs (this even holds for highly degenerate tangencies [1]). When this idea is extended to the simultaneous evolution of more than two solutions $\{U^i(x,t)\}_{i=1}^n$, the natural setting turns from intersections to *braids*. In this context one obtains a *simplifying braid principle*: the braid formed by the strands $\{(x, U^i(x,t), U^i_x(x,t)) \,|\, x \in S^1\}_{i=1}^n$ can only decrease its complexity as time progresses [8]. We come back to this in full detail later.

The class of equations can be extended to cover fully nonlinear equations $U_t = F(U_{xx}, U_x, U, x)$, as long as they are uniformly parabolic and exhibit sub-quadratic growth in U_x, see [2, 9]. However, for simplicity of exposition we restrict our attention to (1.1) in this paper. Discretizing Equation (1.1) in space with discretization step size $\Delta x = 1/d$, $d \in \mathbb{N}$ and $u_i(t) = U(i\Delta x, t)$ yields the system of ordinary differential equations

$$\frac{du_i}{dt} = \frac{u_{i-1} - 2u_i + u_{i+1}}{(\Delta x)^2} + f\Big(\frac{u_{i+1} - u_{i-1}}{2\Delta x}, u_i, i\Delta x\Big), \qquad i = 0, \ldots, d. \tag{1.2}$$

The parabolic nature of the Equation (1.1) is translated into the (easily verified) property that the right hand side of (1.2) is increasing with respect to the variables $u_{i\pm1}$

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at the two neighboring discretization points, at least for small values of Δx (assuming f grows sub-quadratically in U_x).

The general discrete version of a parabolic flow (not necessarily derived from a parabolic partial differential equation) is given by

$$\frac{du_i}{dt} = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}), \tag{1.3}$$

where the *parabolic recurrence relation* $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$, with $\mathcal{R}_i \in C^1(\mathbb{R}^3; \mathbb{R})$, satisfies

(i) monotonicity: $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$ for all $i \in \mathbb{Z}$;

(ii) periodicity: for some $d \in \mathbb{N}$, $\mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$.

Here we require d-periodicity of $\mathcal{R}$ because we want to study d-periodic (or nd-periodic, $n \in \mathbb{N}$) sequences u_i. We will slightly abuse terminology and talk about the parabolic flow $\mathcal{R}$ when we really mean the local flow *generated by* the parabolic recurrence relation $\mathcal{R}$. We note that only one of the inequalities in property (i) needs to be strict.

We thus have that space (with variable x or i) is continuously or discretely periodic. The connection between the continuous and discrete versions (1.1) and (1.2) has been studied in [9], where stationary points and time-periodic orbits were considered. In Section 2 we extend this approach to (the forcing of) connecting orbits. This extension culminates in Lemma 3.7. We note that parabolic recurrence relations (1.3) also appear in the study of twist maps [16, 17], but the study of connecting orbits for (1.3) has no natural interpretation in that context. Nevertheless, the algorithms for computing the index of a single relative braid class, as explained below, are applicable in that context.

The dynamics of the flows generated by (1.1) and (1.3) have powerful topological properties. In both cases there are comparison principles, intersection number principles, and "simplifying braid" principles, see e.g. [1, 10] for Equation (1.1) and [13, 6] for Equation (1.3). Here, our goal is to combine these topological structures with Conley index techniques to derive forcing results for connecting orbits. As pioneered in [8], the natural subdivision of phase space is into *braid classes.* These form isolating neighborhoods for the flow, and we will apply Conley index arguments to study the invariant dynamics inside (collections of) braid classes.

While a full introduction to braid classes is presented in Section 2, we outline the main arguments here. In order to make the connection with the computational approach, we describe discretized braid structures used as representations of continuous braids.

In the discretized setting, we use the term *skeleton* to denote a collection $\mathbf{v}$ of stationary solutions to (1.3). A *free strand*, $\mathbf{u}$, is an initial condition for (1.3). We restrict our attention to *one* free strand in this paper. For illustration, in Figure 1.2 both the stationary solutions in the skeleton and the free strand are depicted using piecewise linear functions, where the values at the anchor or discretization points give the coordinates of the relevant objects. By considering a skeleton $\mathbf{v}$ and a free strand $\mathbf{u}$, we obtain a *relative braid* $\mathbf{u} \# \mathbf{v}$. It is the union $\mathbf{u} \cup \mathbf{v}$ of the strands in $\mathbf{u}$ and $\mathbf{v}$, but we keep track of which strands belong to $\mathbf{u}$ and which to $\mathbf{v}$. Any discretized braid can also be interpreted as a continuous braid through its piecewise linear interpolation.

Next, consider the equivalence class of relative braids whereby for a *fixed* skeleton $\mathbf{v}$ and free strand representative $\mathbf{u}$, we consider all free strands $\mathbf{u}'$ so that the collection $\mathbf{u}' \# \mathbf{v}$ is equivalent to $\mathbf{u} \# \mathbf{v}$ under the standard equivalence relation on

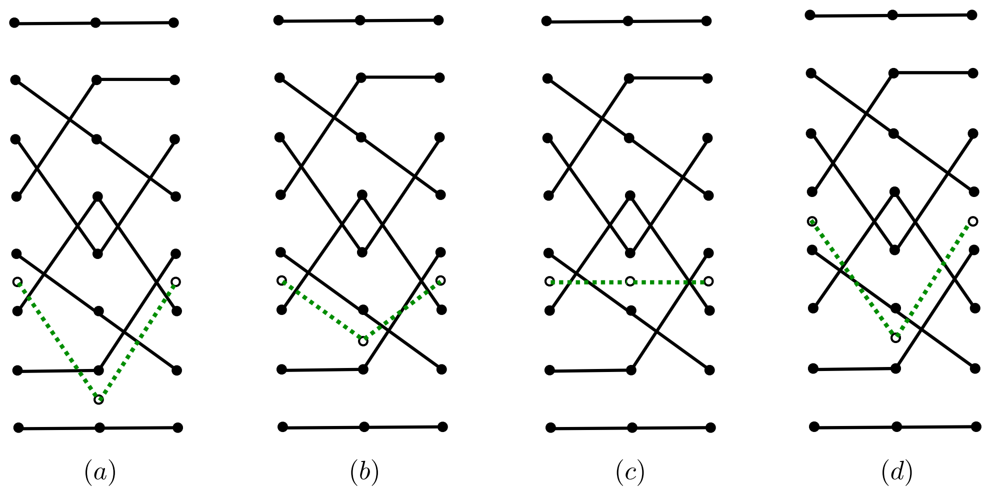


FIG. 1.1. *(a) A free strand* $\mathbf{u}$ *(dotted lines) relative to the skeleton* $\mathbf{v}$ *(solid lines). (b), (c), (d) additional free strand configurations within the same relative braid class* $\mathcal{B} = [\mathbf{u}|\mathbf{v}]$*, which is depicted in Figure 1.2.*

positive braids, see Section 2. Then we obtain the *relative braid class* $\mathcal{B} = [\mathbf{u}|\mathbf{v}]$. Relative braid classes will serve as the basic building blocks for the Conley index theory presented in Section 2.

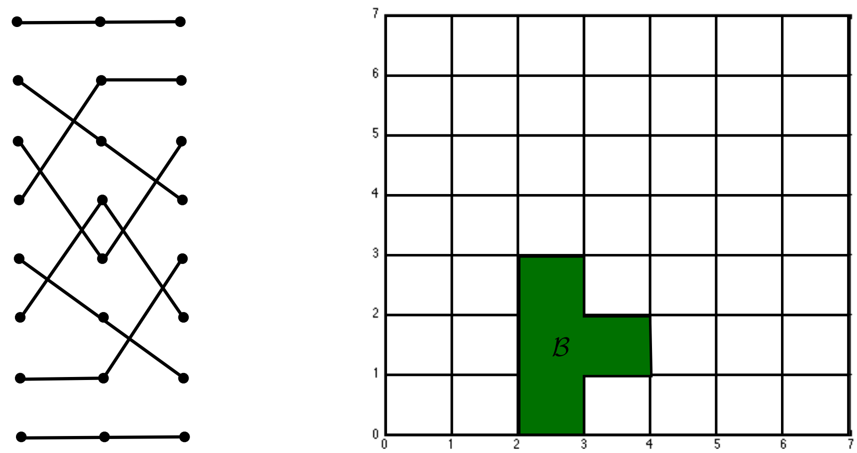
Essentially, if we are able to compute a nontrivial index for a relative braid class or collection of relative braid classes, then Conley index theory forces the existence of solutions to (1.1) and (1.3) with the topological structure prescribed by the relative braid classes.

However, we want to go beyond single braid classes and describe orbits that connect different braid classes. We thus need to identify which collections of braid classes form larger isolating neighborhoods. Those can be decomposed using Morse decompositions and information about connecting orbits is obtained by comparing index information of the individual constituents with the index of the aggregate. In Section 3 we consider these Morse decompositions and the information they encode about connecting orbits for the flows generated by (1.1) and (1.3).

To illustrate these concepts, we now consider a 2-periodic discrete skeleton $\mathbf{v}$ and one free strand $\mathbf{u}$ (with periodicity $u_3 = u_1$) as depicted in Figure 1.1(a). The four relative braids given in Figure 1.1 are all contained within the relative braid class $\mathcal{B} = [\mathbf{u}\,|\,\mathbf{v}]$. Figure 1.2(b) shows a cubical representation of $\mathcal{B}$ in the phase space for the free strand $u' = (u'_1, u'_2)$. As mentioned before, $\mathbf{u}$ and $\mathbf{v}$ can also be interpreted as representatives of continuous braids (through their piecewise linear interpolants). The following results hold for both the discrete and continuous case.

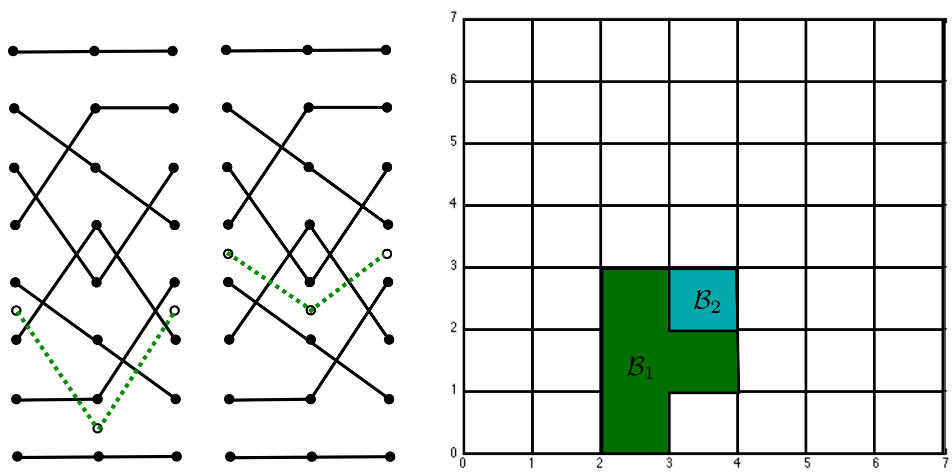
SAMPLE RESULT 1.1. *For a flow (1.1) or (1.3) with stationary solutions in the braid class depicted in discretized form by the skeleton* $\mathbf{v}$ *in Figure 1.2(a), there exists a stationary solution* $U(x)$ *or* u_i *in the relative braid class* $[\mathbf{u}|\mathbf{v}]$ *depicted (in discretized form) in Figure 1.2(b).*

As explained before, we do not want to restrict to single braid classes, but consider attractor-repeller pairs, or Morse decompositions, so that we can also obtain information about connecting orbits, in particular orbits whose α- and ω-limit sets lie in different relative braid classes.



(a) (b)

FIG. 1.2. *The skeleton* $\mathbf{v}$ *and the cubical relative braid class* $\mathcal{B}$ *corresponding the four representatives in Figure 1.1, depicted in the space of all discrete configurations of the free strand.*



(a) (b) (c)

FIG. 1.3. *Representatives* $\mathbf{u}_1$ *in (a) and* $\mathbf{u}_2$ *in (b) of braid classes* $\mathcal{B}_1$ *and* $\mathcal{B}_2$*, respectively. Both braid classes are depicted in (c).*

SAMPLE RESULT 1.2. *For a flow (1.1) or (1.3) with stationary solutions in the braid class depicted in discretized form by the skeleton* $\mathbf{v}$ *in Figure 1.2(a), there exists an orbit* $U(x,t)$ *or* $u_i(t)$ *whose* α*-limit set lies in the braid class represented by* $\mathcal{B}_1 = [\mathbf{u}_1|\mathbf{v}]$ *and whose* ω*-limit set lies in the braid class represented by* $\mathcal{B}_2 = [\mathbf{u}_2|\mathbf{v}]$*, where both these braid classes are depicted in Figure 1.3.*

The examples presented above may be computed by hand. However, as the complexity of the braided solutions increases, this approach quickly becomes impractical. In this paper we outline, implement and demonstrate a computational approach for computing the cubical braid classes and topological indices required for the forcing results, see Sections 4 and 5 for algorithms and more involved examples.

With this approach in place, one can think about bootstrapping: after locating a stationary solution by this approach, add the stationary solution to the skeleton and perform the procedure again to look for an additional solution. In this way, one may perform more expensive computations in order to increase the "topological resolution" of the computational approach.

2. Braids and the Conley-Morse index. We now introduce the required terminology, some of which may also be found in [8, 9, 15].

2.1. Topological braids. In topology, a *closed braid* on n strands is an unordered set $\{\beta^\alpha : [0,1] \to \mathbb{R}^2\}_{\alpha=1}^n$ of continuous functions with disjoint graphs, such that

$$\{\beta^\alpha(0)\}_{\alpha=1}^n = \{\beta^\alpha(1)\}_{\alpha=1}^n. \tag{2.1}$$

The graphs $\{(x, \beta^\alpha(x)) \,|\, x \in [0,1]\}$ are the *strands* of the braid. When we order the strands, there is a natural permutation τ defined by the relation (2.1) and the fact that the strands are disjoint: $\beta^\alpha(1) = \beta^{\tau(\alpha)}(0)$.

Remark 2.1 (Periodic extension). *Introducing this permutation τ allows us to extend braids periodically, i.e., we can define $\beta^\alpha(x)$ for all $x \in \mathbb{R}$ by requiring that $\beta^\alpha(x+1) = \beta^{\tau(\alpha)}(x)$ for all x. Periodic extension will be used throughout when needed.*

Having introduced the permutation τ, we slightly shift our viewpoint and introduce the following equivalent concept of closed topological braids.

Definition 2.2. *A* closed topological braid *on n strands is a pair (β, τ), where $\tau \in S_n$ is a permutation on n symbols, and $\beta = (\beta^\alpha)_{\alpha=1}^n$ is an n-tuple of functions $\beta^\alpha \in C^0([0,1];\mathbb{R}^2)$ with mutually disjoint graphs, such that $\beta^\alpha(1) = \beta^{\tau(\alpha)}(0)$.*

The space of all closed topological braids consists of all such pairs (β, τ) modulo the identification $(\beta, \tau) \cong (\tilde{\beta}, \tilde{\tau})$ if there is a permutation $\rho \in S_n$ such that $\beta^{\rho(\alpha)} = \tilde{\beta}^\alpha$ and $\rho \circ \tilde{\tau} = \tau \circ \rho$.

The above identification essentially "disorders" the strands, but it may be viewed as optional. The use of the equivalence relation is natural from a topological point of view. On the other hand, for each closed braid (β, τ) there are exactly $n! - 1$ other pairs equivalent to it (corresponding to the permutations $\rho \neq \mathrm{id}$). Hence, not much is lost if we do not use the identification, which is quite cumbersome in a computational approach. Therefore, we will not use the identification in the definitions that follow below. Since the strands of a closed topological braid are all disjoint, specifying β^α imposes τ. In the notation we will usually suppress the permutation τ, if this is not confusing.

The topology on the space of braids comes from the usual C^0-topology for each of the strands and the discrete topology with respect to the permutation τ (and then dividing out the equivalence relation if needed). The connected components of the space of closed topological braids are called topological *braid classes*. Finally, a set of strands that correspond to the same cycle of the permutation τ, is a *component* of the braid.

2.2. Braid diagrams. The specification of a topological braid class can be achieved by means of a projection onto a plane, e.g. the (x_1, x_2) plane. A braid may be perturbed slightly so that all crossings are transverse in this projection. Each crossing is then labeled "+" or "−" to indicate whether the crossing is "bottom over top" or "top over bottom", respectively.

Not all projections of a braid will look the same. The equivalence of projections can be described algebraically. Namely, braids on n strands can be described by the Artin braid group B_n generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations (see e.g. [3])

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{for } |i-j|>1, \tag{2.2a}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{for } 1 \le i \le n-2. \tag{2.2b}$$

To a projection of a braid one can associate its braid word $w = \sigma_{i_1}^{s_1}\cdots\sigma_{i_N}^{s_N}$ with $1 \le i_j \le n-1$, $s_j \in \{\pm 1\}$ and $N \in \mathbb{N}$. Equivalence of two braid words w_1 and w_2 under the relations (2.2) is denoted by $w_1 \sim w_2$. If all the crossing signs σ_j are positive, then we call the braid word *positive*. Any braid that can be described by a positive braid word is also called positive.

Closed topological braid classes can be described as the set of conjugacy classes of B_n: w_1 is conjugate to w_2 if there is a braid w_3 such that $w_3w_1 \sim w_2w_3$. The conjugacy problems deals with the problem of how to determine whether two braids are conjugate, see [3, 7].

Since we are going to use braid classes to describe solutions of Equation (1.1), the strands that we are interested in are all of the form $(x, U(x), U_x(x))$, $x \in [0,1]$. Braids consisting of such strands are called *Legendrian*. For Legendrian braids no specification of the crossing type is needed since all crossings are positive. All information is thus contained in the projection (without specifying the crossing type), or braid *diagram*. Let us concentrate on those.

Definition 2.3. *A* closed continuous braid diagram *is a pair* $(\mathbf{U}, \tau)$*, where* $\tau \in S_n$ *and* $\mathbf{U}$ *is an* n*-tuple* $\mathbf{U} = (U^\alpha)_{\alpha=1}^n$ *of* $C^0([0,1];\mathbb{R})$ *functions, the strands, that satisfy*

1. *(Periodicity)* $U^\alpha(1) = U^{\tau(\alpha)}(0)$*; and*
2. *(Transversality) for any* $\alpha \neq \alpha'$ *such that* $U^\alpha(x_0) = U^{\alpha'}(x_0)$ *for some* $x_0 \in [0,1]$*, it holds that* $U^\alpha(x) - U^{\alpha'}(x)$ *has an isolated sign change at* $x = x_0$*.*

The space of closed continuous braid diagrams on n *strands, denoted* Ω^n*, is the space of all such pairs* $(\mathbf{U}, \tau)$*.*

The topology on the space Ω^n again comes from the usual C^0-topology for each of the strands and the discrete topology with respect to the permutation τ. We usually drop the permutation τ from the notation. Nevertheless, for some braid diagrams the specification of τ is essential, since it identifies the meaning of transversality when $x_0 = 0$ or $x_0 = 1$ in Definition 2.3. It allows us to extend braids periodically, see Remark 2.1. Using periodic extension the meaning of transversality at $x_0 \in \{0,1\}$ is unambiguous.

We will often use the terminology “braid” for a closed continuous braid diagram, if this is not confusing, hence implicitly implying we are concerned with Legendrian braids. Note that we choose *not* to divide out the equivalence relation that appears in the definition of topological braids (see the discussion in Section 2.1).

Braid diagrams can be described by *positive* braid words, with the relations (2.2). To classify *closed* braid diagrams we add the following relation on positive braid words (cf. conjugacy classes):

$$\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N} \equiv \sigma_{i_2}\cdots\sigma_{i_N}\sigma_{i_1}. \tag{2.3}$$

Two positive braid words w_1 and w_2 are *positively conjugated*, denoted $w_1 \overset{+}{\sim} w_2$, if there are positive words $w_1' \sim w_1$ and $w_2' \sim w_2$ (see (2.2)), such that $w_1' \equiv w_2'$. Next, the notion of *positive conjugation* is made into an equivalence relation, again denoted by $\overset{+}{\sim}$, by taking the transitive-reflexive closure.

Definition 2.4. *A connected component of* Ω^n *is called a* closed continuous braid diagram class*, or* (continuous) braid class *for short and is denoted* $[\mathbf{U}]$*.*

Algebraically, braid classes are described by the sets of positive braid words that are positively conjugated. It will be convenient to also introduce the bigger set $\overline{\Omega^n}$, defined similarly to Definition 2.3 of Ω^n, but disregarding condition 2.

Definition 2.5 (Singular braids). *The set* $\overline{\Omega^n}$ *consists of all pairs* $(\mathbf{U},\tau)$*, where* $\tau \in S_n$ *and* $\mathbf{U}$ *is an* n*-tuple* $\mathbf{U} = (U^\alpha)_{\alpha=1}^n$ *of* $C^0([0,1];\mathbb{R})$ *functions that satisfy* $U^\alpha(1) = U^{\tau(\alpha)}(0)$*. Elements of*

$$\Sigma = \Sigma^n = \overline{\Omega^n} - \Omega^n$$

are called singular *braids.*

We interpret $\overline{\Omega^n}$ as the "full" space, including both braids and singular braids, and all closures are taken in $\overline{\Omega^n}$ unless stated otherwise. Clearly, singular braids have at least one tangency. In particular, we denote by Σ_1 those singular braids with exactly one tangency:

Definition 2.6. *The set* $\Sigma_1 = \Sigma_1^n$ *consists of those singular braids* $\mathbf{U} \in \Sigma$ *for which there is exactly one pair of strands* $\alpha \neq \alpha'$ *and exactly one point* $x_0 \in [0,1)$ *such that*

1. $U^\alpha(x_0) = U^{\alpha'}(x_0)$*; and*
2. *there is an* $\varepsilon > 0$ *such that* $U^\alpha(x) < U^{\alpha'}(x)$ *for all* $0 < |x - x_0| < \varepsilon$*.*

This definition is a bit cluttered in the topological C^0-setting; in both the C^1-setting and the discrete setting Σ_1 is the natural codimension-1 part of the boundary of Ω^n. A much more "severe" tangency occurs when two strands collapse onto each other.

Definition 2.7. *The set* $\Sigma_\infty = \Sigma_\infty^n$ *of* collapsed *singular braids consists of those* $\mathbf{U} \in \Sigma$ *for which* $U^\alpha(x) = U^{\alpha'}(x)$ *for all* $x \in \mathbb{R}$ *for some* $\alpha \neq \alpha'$*.*

A different way of expressing this is to say that a collapsed braid has two identical components.

We will also need a type of "complement" of the collapsed singular braids, namely singular braids of which the singularity is not or at least not *solely* caused by collapses of components.

Definition 2.8. *The set* $\Sigma_{\text{prop}} = \Sigma_{\text{prop}}^n$ *of* properly *singular braids consists of those* $\mathbf{U} \in \Sigma$ *for which there exist* $\alpha \neq \alpha'$ *and* $x_0, x_1 \in \mathbb{R}$ *such that*

1. $U^\alpha(x_0) = U^{\alpha'}(x_0)$*; and*
2. $U^\alpha(x) - U^{\alpha'}(x)$ *does* not *have an isolated sign change at* $x = x_0$*; and*
3. $U^\alpha(x_1) \neq U^{\alpha'}(x_1)$*, i.e. not collapsed strands.*

Let us briefly consider the issue of regularity. The projection of a Legendrian topological braid $\beta^\alpha = (U^\alpha, \frac{dU^\alpha}{dx})$ onto its first component leads to a continuous braid diagram. If one would like to have a reverse statement, one should introduce C^1 (instead of C^0) braids and require $\frac{dU^\alpha}{dx}(x_0) \neq \frac{dU^{\alpha'}}{dx}(x_0)$ as the transversality rule. In that case one can interpret a braid diagram as a Legendrian braid $\beta^\alpha = (U^\alpha, \frac{dU^\alpha}{dx})$.

However, that is not the path we want to follow, because, in view of (1.3) and our computational goals, we want to discretize, which is in some sense opposite to requiring differentiability. Here is the natural discrete version of the definition.

Definition 2.9. *A* d-periodic discrete braid diagram *is a pair* $(\mathbf{u},\tau)$*, where* $\tau \in S_n$ *and* $\mathbf{u}$ *is an* n*-tuple* $\mathbf{u} = (u^\alpha)_{\alpha=1}^n$ *of vectors* $u^\alpha = (u_0^\alpha, \ldots, u_d^\alpha) \in \mathbb{R}^{d+1}$*, the strands, that satisfy*

1. *(Periodicity)* $u_d^\alpha = u_0^{\tau(\alpha)}$*; and*

2. *(Transversality) for any $\alpha \neq \alpha'$ such that $u_i^\alpha = u_i^{\alpha'}$ for some $0 \leq i \leq d$, it holds that $\big(u_{i-1}^\alpha - u_{i-1}^{\alpha'}\big)\big(u_{i+1}^\alpha - u_{i+1}^{\alpha'}\big) < 0$.*

The space of d-periodic discrete braid diagrams on n strands, denoted $\mathcal{D}_d^n$, is the space of all such pairs $(\mathbf{u}, \tau)$.

Note that for fixed τ a d-periodic discrete braid diagram is completely determined by the nd coordinates $\{u_i^\alpha \,|\, 1 \leq \alpha \leq n, 0 \leq i \leq d-1\}$, the *anchor points* of the braid. The topology on $\mathcal{D}_d^n$ is the one induced by the standard topology on $\mathbb{R}^{nd}$ and the discrete topology on τ. Periodic extension is defined by the relation $u_{i+d}^\alpha = u_i^{\tau(\alpha)}$ for all $i \in \mathbb{Z}$. Discrete braid classes are the path components of $\mathcal{D}_d^n$, denoted by $[\mathbf{u}]$.

We may interpret a discrete braid diagram $\mathbf{u} \in \mathcal{D}_d^n$ as a continuous (piecewise linear) braid diagram $\ell\mathbf{u} \in \Omega^n$:

$$\ell u^\alpha(x) \stackrel{\text{def}}{=} u^\alpha_{\lfloor dx \rfloor} + (dx - \lfloor dx \rfloor)(u^\alpha_{\lceil dx \rceil} - u^\alpha_{\lfloor dx \rfloor}). \tag{2.4}$$

Here $\lfloor dx \rfloor$ and $\lceil dx \rceil$ denote the upper and lower integer part of dx, respectively. When drawing pictures, this piecewise linear braid is much more informative than just the anchor points $\{u_i^\alpha\}$, since it shows which points belong to the same strand.

Definition 2.10. *Given a continuous braid class $[\mathbf{U}]$ we call $\mathbf{u} \in \mathcal{D}_d^n$ a* discrete representative *of $[\mathbf{U}]$ if $\ell\mathbf{u} \in [\mathbf{U}]$.*

Due to isolation of intersections, it follows that that the discretization of a continuous braid diagram $\mathbf{U}$, given by $u_i^\alpha = U^\alpha(i/d)$, is a discrete representative of $[\mathbf{U}]$ if the number d of discretization points is sufficiently large.

Next, we introduce the set of singular discrete braid diagrams, cf. Definition 2.5. If we allow "tangencies", i.e., if we disregard condition 2 in the Definition 2.9, we obtain a closure of $\mathcal{D}_d^n$, denoted by $\overline{\mathcal{D}_d^n}$. The set $\Sigma = \Sigma_d^n \stackrel{\text{def}}{=} \overline{\mathcal{D}_d^n} - \mathcal{D}_d^n$ defines the set of (d-periodic, discrete) singular braid diagrams (on n strands). The set Σ_d^n acts as the boundary between different discrete braid classes. Notice that $\overline{\mathcal{D}_d^n}$ may be identified with $n!$ copies of $\mathbb{R}^{nd}$, one for each permutation τ. For fixed τ we will throughout identify $\overline{\mathcal{D}_d^n}$ with $\mathbb{R}^{nd}$. The sets Σ_1, Σ_∞ and Σ_{prop} in the discrete setting are defined similarly to Definitions 2.6, 2.7 and 2.8.

2.3. Relative braids. In Section 2.4 we will consider in detail how continuous and discrete braid diagrams evolve under the flows defined by (1.1) and (1.3), respectively. We are particularly interested in the idea of forcing: given a stationary braid $\mathbf{V}$, does it force special dynamics for some other braid class $\mathbf{U}$? To make this precise we need to understand how strands of $\mathbf{U}$ braid relative to those of $\mathbf{V}$.

We start by defining the set of *all* relative braid diagrams

$$\Omega^{n,m} \stackrel{\text{def}}{=} \big\{(\mathbf{U}, \mathbf{V}) : \mathbf{U} \in \Omega^n, \mathbf{V} \in \Omega^m, \mathbf{U} \cup \mathbf{V} \in \Omega^{n+m}\big\}.$$

For pairs $(\mathbf{U}, \mathbf{V})$ in $\Omega^{n,m}$ we write $\mathbf{U} \,\#\, \mathbf{V}$. If two relative braids $\mathbf{U} \,\#\, \mathbf{V}$ and $\mathbf{U}' \,\#\, \mathbf{V}'$ are in the same connected component of $\Omega^{n,m}$, they are called equivalent. Let us denote such an equivalence class (connected component) by $[\mathbf{U} \,\#\, \mathbf{V}]$, called a *relative braid class* or simply *braid class*. Clearly $[\mathbf{U} \,\#\, \mathbf{V}] \subset [\mathbf{U}] \times [\mathbf{V}]$.

Having forcing in mind, we want to fix $\mathbf{V}$ and vary $\mathbf{U}$. Associated with $\mathbf{U} \,\#\, \mathbf{V}$ we have the projection

$$\pi : \Omega^{n,m} \to \Omega^m, \qquad \mathbf{U} \,\#\, \mathbf{V} \mapsto \mathbf{V}.$$

For any $\mathbf{V}' \in [\mathbf{V}]$ we define the *fiber*, see Figure 2.1,

$$\Pi_{\mathbf{V}'}[\mathbf{U} \,\#\, \mathbf{V}] \stackrel{\text{def}}{=} \big\{\mathbf{U}' \in \Omega^n : (\mathbf{U}', \mathbf{V}') \in [\mathbf{U} \,\#\, \mathbf{V}]\big\},$$

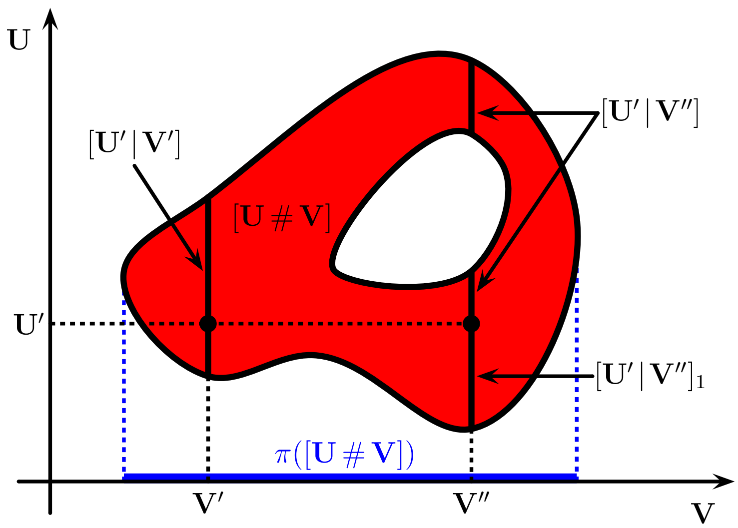


FIG. 2.1. *A relative braid class* $[\mathbf{U}\,\#\,\mathbf{V}]$ *in red and its projection* $\pi([\mathbf{U}\,\#\,\mathbf{V}])$ *in blue. Two fibers are indicated, one consisting of a single connected component, the other consisting of two components* $[\mathbf{U}'|\mathbf{V}'']_1$ *and* $[\mathbf{U}'|\mathbf{V}'']_2$.

which is nonempty if and only if $\mathbf{V}' \in \pi([\mathbf{U}\,\#\,\mathbf{V}])$. For every $\mathbf{V}' \in \pi([\mathbf{U}\,\#\,\mathbf{V}])$ and $\mathbf{U}' \in \Pi_{\mathbf{V}'}[\mathbf{U}\,\#\,\mathbf{V}]$ we have $[\mathbf{U}'\,\#\,\mathbf{V}'] = [\mathbf{U}\,\#\,\mathbf{V}]$, and we use the following alternative notation for the fiber:

$$[\mathbf{U}'\,|\,\mathbf{V}'] \stackrel{\text{def}}{=} \big\{\mathbf{U}'' \in \Omega^n : (\mathbf{U}'', \mathbf{V}') \in [\mathbf{U}'\,\#\,\mathbf{V}']\big\} = \Pi_{\mathbf{V}'}[\mathbf{U}\,\#\,\mathbf{V}].$$

In this setting $\mathbf{V}$ is called the *skeleton* (fixed under the flow), and $\mathbf{U}$ the *free* braid (free to move). A fiber $[\mathbf{U}\,|\,\mathbf{V}]$ is called a *relative braid class with fixed skeleton* $\mathbf{V}$, where the dependence on $\mathbf{V}$ is often omitted when clear from the context. A fiber $[\mathbf{U}|\mathbf{V}]$ can consist of several connected components, which we denote by $[\mathbf{U}|\mathbf{V}]_k$ for $k = 1, \ldots, K$, where we always denote the component that contains $\mathbf{U}$ by $[\mathbf{U}|\mathbf{V}]_1$.

The set of all braids relative to a fixed skeleton $\mathbf{V}$ is denoted by

$$\Omega^n\,|\,\mathbf{V} \stackrel{\text{def}}{=} \big\{\mathbf{U} \in \Omega^n : (\mathbf{U}, \mathbf{V}) \in \Omega^{n,m}\big\}.$$

This partitions Ω^n relative to $\mathbf{V}$; not only are tangencies between strands of $\mathbf{U}$ illegal, so are tangencies with the strands of $\mathbf{V}$. Any fiber $[\mathbf{U}|\mathbf{V}]$ thus consists of one or more connected components of $\Omega^n|\mathbf{V}$.

Similarly, the discrete relative braids are

$$\mathcal{D}_d^{n,m} \stackrel{\text{def}}{=} \big\{(\mathbf{u}, \mathbf{v}) : \mathbf{u} \in \mathcal{D}_d^n, \mathbf{v} \in \mathcal{D}_d^m, \mathbf{u} \cup \mathbf{v} \in \mathcal{D}_d^{n+m}\big\},$$

and elements of $\mathcal{D}_d^{n,m}$ are denoted by $\mathbf{u}\#\mathbf{v}$. Furthermore, for $\mathbf{v} \in \mathcal{D}_d^m$ let

$$\mathcal{D}_d^n\,|\,\mathbf{v} \stackrel{\text{def}}{=} \big\{\mathbf{u} \in \mathcal{D}_d^n : (\mathbf{u}, \mathbf{v}) \in \mathcal{D}_d^{n,m}\big\}.$$

To define discrete fibers we use that discrete braids represent continuous braids via piecewise linear interpolation. Consider a relative braid class $[\mathbf{U}\,\#\,\mathbf{V}]$ and a discrete representative $\mathbf{v} \in \mathcal{D}_d^m$ of $[\mathbf{V}]$, i.e., $\ell\mathbf{v} \in [\mathbf{V}]$. The natural definition of the *discrete fiber* is then

$$\Pi_{\mathbf{v}}[\mathbf{U} \# \mathbf{V}] \stackrel{\text{def}}{=} \big\{\mathbf{u} \in \mathcal{D}_d^n|\mathbf{v} : \ell\mathbf{u} \# \ell\mathbf{v} \in [\mathbf{U} \# \mathbf{V}]\big\}.$$

It may happen that this set is empty, even if $\ell\mathbf{v} \in \pi([\mathbf{U} \# \mathbf{V}])$. If there is a discrete representative $\mathbf{u}$, with d discretization points, in $\Pi_{\ell\mathbf{v}}[\mathbf{U} \# \mathbf{V}]$, then $[\ell\mathbf{u} \# \ell\mathbf{v}] = [\mathbf{U} \# \mathbf{V}]$ and the discrete fiber is given by

$$[\mathbf{u}|\mathbf{v}] \stackrel{\text{def}}{=} \big\{\mathbf{u}' \in \mathcal{D}_d^n|\mathbf{v} : \ell\mathbf{u}' \# \ell\mathbf{v} \in [\ell\mathbf{u} \# \ell\mathbf{v}]\big\} = \Pi_{\mathbf{v}}[\mathbf{U} \# \mathbf{V}].$$

The connected components of $[\mathbf{u}|\mathbf{v}]$ are again denoted by $[\mathbf{u}|\mathbf{v}]_k$ for $k = 1, \ldots, K$, where $\mathbf{u} \in [\mathbf{u}|\mathbf{v}]_1$.

Finally, we need to consider singular relative braids. For $\mathbf{V} \in \Omega^m$ let

$$\Sigma|\mathbf{V} \stackrel{\text{def}}{=} \overline{\Omega^n} - \Omega^n|\mathbf{V},$$

and

$$\Sigma_1|\mathbf{V} \stackrel{\text{def}}{=} \big\{\mathbf{U} \in \Sigma|\mathbf{V} : \mathbf{U} \cup \mathbf{V} \in \Sigma_1^{n+m}\big\},$$
$$\Sigma_\infty|\mathbf{V} \stackrel{\text{def}}{=} \big\{\mathbf{U} \in \Sigma|\mathbf{V} : \mathbf{U} \cup \mathbf{V} \in \Sigma_\infty^{n+m}\big\},$$
$$\Sigma_{\text{prop}}|\mathbf{V} \stackrel{\text{def}}{=} \big\{\mathbf{U} \in \Sigma|\mathbf{V} : \mathbf{U} \cup \mathbf{V} \in \Sigma_{\text{prop}}^{n+m}\big\}.$$

In particular, in a collapsed singular relative braid the free strands can be collapsed onto each other or onto skeletal strands. We note that in the case $n = 1$ we have $\Sigma_{\text{prop}}|\mathbf{V} = \Sigma|\mathbf{V} \setminus \Sigma_\infty|\mathbf{V}$. The above definitions extend easily to the discrete setting. For illustration, we state the definition of the codimension-1 singular relative braids (with exactly one tangency) for discrete relative braids. The definitions of $\Sigma_\infty|\mathbf{v}$ and $\Sigma_{\text{prop}}|\mathbf{v}$ follow similarly.

Definition 2.11. *For any $\mathbf{v} \in \mathcal{D}_d^m$ the set $\Sigma_1|\mathbf{v}$ consists of all singular braids $\mathbf{u} \in \Sigma|\mathbf{v}$ such that there is*

- either *exactly one $0 \le i \le d-1$ and exactly one pair $1 \le \alpha < \alpha' \le m$ such that $u_i^\alpha = u_i^{\alpha'}$;*
- or *exactly one $0 \le i \le d-1$ and exactly one $1 \le \alpha \le m$ such that $u_i^\alpha = v_i^{\alpha''}$ for some $1 \le \alpha'' \le n$.*

Notice that in this definition a free strand may be tangent to multiple skeletal strands, but in only one anchor point.

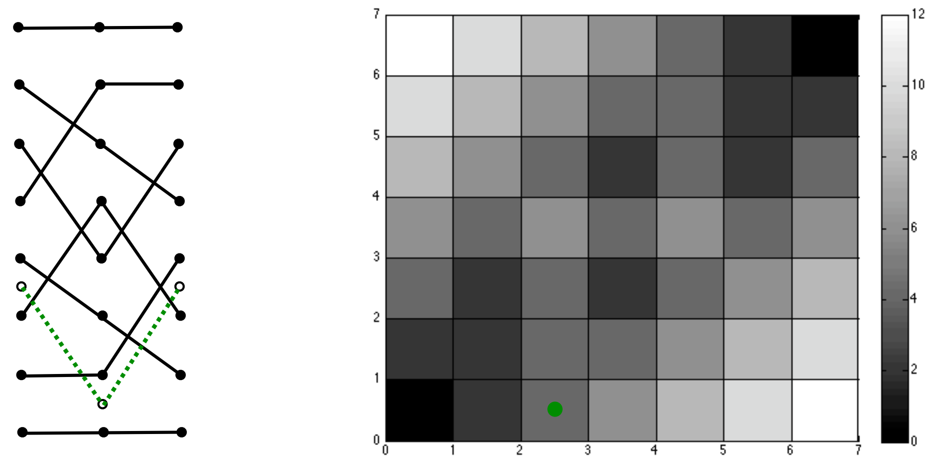
2.4. Conley Index. We consider braid diagrams evolving under the flows defined by (1.1) and (1.3). To be precise, continuous braids $\mathbf{U} = (U^\alpha(x))_{\alpha=1}^n$ evolve in time under the equation

$$U_t^\alpha = U_{xx}^\alpha + f(U_x^\alpha, U^\alpha, x), \qquad 1 \le \alpha \le n,\ x \in [0,1], \tag{2.5}$$

with identified end points $U^\alpha(1) = U^{\tau(\alpha)}(0)$. Discrete braids $\mathbf{u} = (u_i^\alpha)_{\alpha=1}^n$ evolve under the equation

$$\frac{du_i^\alpha}{dt} = \mathcal{R}_i(u_{i-1}^\alpha, u_i^\alpha, u_{i+1}^\alpha), \qquad 1 \le \alpha \le n,\ 0 \le i \le d-1, \tag{2.6}$$

with the usual periodic extension, in this case $u_d^\alpha = u_0^{\tau(\alpha)}$ and $u_{-1}^\alpha = u_{d-1}^{\tau^{-1}(\alpha)}$.



(a) (b)

FIG. 2.2. *(a) The skeleton* $\mathbf{v}$ *(solid lines) and a free strand* $\mathbf{u}$ *(dotted green lines) with intersection number* $\iota(\mathbf{u}, \mathbf{v}) = 4$. *(b) The (cubical) space of bounded configurations of free strands* $\mathbf{u}'$. *The green dot gives the location of the free stand in the left picture. The grey-scale specifies the intersection number* $\iota(\mathbf{u}', \mathbf{v})$.

The main property of these parabolic flows is that they decrease the complexity of the braid. This can be made more explicit. For *positive* braids the length of the associated braid word is an invariant of a braid class. For braid diagrams this translates into the intersection number ι.

DEFINITION 2.12. *For two strands* U^α *and* $U^{\alpha'}$ *of a continuous braid diagram* $\mathbf{U}$ *we define the* intersection number $\iota(U^\alpha, U^{\alpha'})$ *as the number of elements in the set* $\{x \in [0,1) \,|\, U^\alpha(x) = U^{\alpha'}(x)\}$. *For a closed, continuous braid* $\mathbf{U} \in \Omega^n$ *we define the intersection number* $\iota(\mathbf{U})$ *as the total number of intersections*

$$\iota(\mathbf{U}) = \sum_{1 \le \alpha < \alpha' \le n} \iota(U^\alpha, U^{\alpha'}).$$

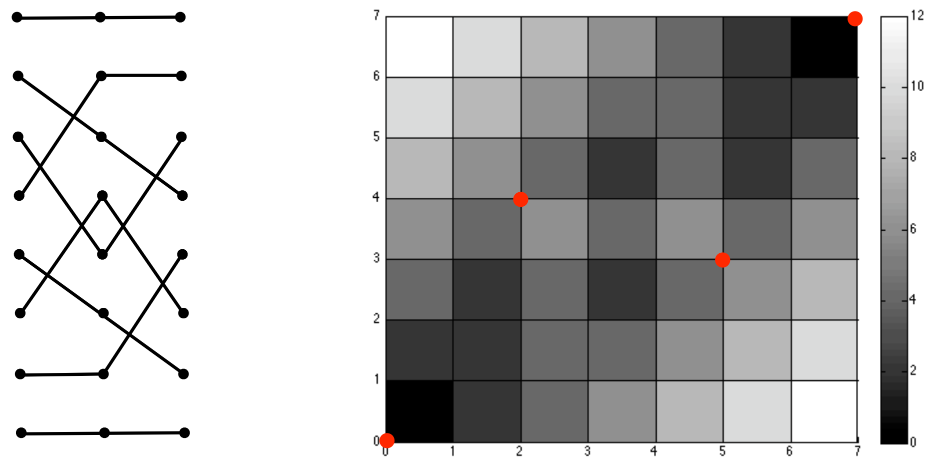
For a closed continuous relative braid $\mathbf{U} \,\#\, \mathbf{V} \in \Omega^{n,m}$, *we define the relative intersection number by*

$$\iota(\mathbf{U}, \mathbf{V}) = \sum_{\substack{1 \le \alpha \le n \\ 1 \le \alpha' \le m}} \iota(U^\alpha, V^{\alpha'}).$$

Finally, for discrete braids (and strands) $\mathbf{u} \in \mathcal{D}_d^n$ *and discrete relative braids* $\mathbf{u} \,\#\, \mathbf{v} \in \mathcal{D}_d^{n,m}$, *we extend these definitions to* $\iota(u^\alpha, u^{\alpha'})$, $\iota(\mathbf{u})$, *and* $\iota(\mathbf{u}, \mathbf{v})$ *using corresponding piecewise linear representatives (2.4).*

It follows that for $\mathbf{U} \,\#\, \mathbf{V} \in \Omega^{n,m}$ we have $\iota(\mathbf{U} \cup \mathbf{V}) = \iota(\mathbf{U}, \mathbf{V}) + \iota(\mathbf{U}) + \iota(\mathbf{V})$. The intersection number is an invariant of a (discrete or continuous) (relative) braid class. Furthermore, the intersection number of a braid equals the length of its braid word. For illustration, see Figure 2.2.

Along a parabolic flow the intersection number of a braid diagram cannot increase, see [6, 8, 13]. This property motivates our development of relative braid classes in



(a) (b)

FIG. 2.3. *(a) The skeleton* $\mathbf{v}$ *and (b) the (cubical) space of all configurations of a free strand* $\mathbf{u}$*, shaded by intersection number* $\iota(\mathbf{u}, \mathbf{v})$*. There are four* closed *skeletal strands and the red dots give the coordinates of these strands. The bounded discrete relative braid class components are those cubical regions consisting of cubes with the same crossing number which are connected by codimension-1 faces. Since a relative braid class component labeled with a red dot contains a free strand configuration that may be collapsed onto a skeletal strand, these components are not proper. The remaining depicted relative braid class components are proper and bounded.*

Section 2.3. Namely, relative braid classes are candidates for isolating neighborhoods, or even isolating blocks, for the continuous parabolic flow (2.5) and, in particular, for the discrete parabolic flow (2.6). For the Conley index to be well-defined we want the braid class to be bounded and isolating.

DEFINITION 2.13. *A relative braid class* $[\mathbf{U} \# \mathbf{V}]$ *is called* bounded *if every fiber* $\Pi_{\mathbf{V}'}[\mathbf{U} \# \mathbf{V}]$ *is bounded in* $\overline{\Omega^n}$ *(i.e.,* $[\mathbf{U}'|\mathbf{V}']$ *is bounded for any* $\mathbf{U}' \# \mathbf{V}' \in [\mathbf{U} \# \mathbf{V}]$*).*

The isolation property is summarized in the following properness definition.

DEFINITION 2.14. *A relative braid class* $[\mathbf{U} \# \mathbf{V}]$ *is called* proper *if for every fiber* $\Pi_{\mathbf{V}'}[\mathbf{U} \# \mathbf{V}] = [\mathbf{U}'|\mathbf{V}']$ *with* $\mathbf{U}' \# \mathbf{V}' \in [\mathbf{U} \# \mathbf{V}]$ *it holds that* $\mathrm{cl}([\mathbf{U}'|\mathbf{V}']) \cap (\Sigma|\mathbf{V}') \subset \Sigma_{\mathrm{prop}}|\mathbf{V}'$*. If* $[\mathbf{U} \# \mathbf{V}]$ *is not proper it is called* improper*.*

To determine properness of a relative braid in practice it is often convenient to exploit the invariants provided by intersection numbers. When $n = 1$ the condition for properness is equivalent to $\mathrm{cl}([\mathbf{U}'|\mathbf{V}']) \cap (\Sigma_\infty|\mathbf{V}') = \varnothing$. We remark that one could define bounded and proper for a *fixed* skeleton $\mathbf{V}$ as well, but we want our definitions to be invariant under perturbations of the skeleton (and independent of the coarseness of the discretization).

We will define the Conley index for *discrete* relative braid classes. Their main advantage over continuous ones is that they live in a finite dimensional setting.

DEFINITION 2.15. *A discrete relative braid class* $[\mathbf{u}|\mathbf{v}] \subset \mathcal{D}^n_d|\mathbf{v}$ *is* weakly proper *if* $\mathrm{cl}([\mathbf{u}|\mathbf{v}]) \cap (\Sigma_\infty|\mathbf{v}) = \varnothing$*.*

A discrete relative braid class in $\mathcal{D}^n_d$ can be weakly proper due to a low number of discretization points n, and it may not be proper (topologically), see Definition 2.14. On the other hand, if $[\ell\mathbf{u} \# \ell\mathbf{v}]$ is proper, then certainly $[\mathbf{u}|\mathbf{v}]$ is weakly proper. When $\mathrm{cl}([\mathbf{u}|\mathbf{v}])$ is a bounded set in $\overline{\mathcal{D}^n_d}$ and $[\mathbf{u}|\mathbf{v}]$ is weakly proper, then we associate to this relative braid class a Conley-type index for parabolic dynamics of the type (1.3), provided $\mathbf{v}$ is fixed under the flow, i.e. $\mathcal{R}(\mathbf{v}) = 0$.

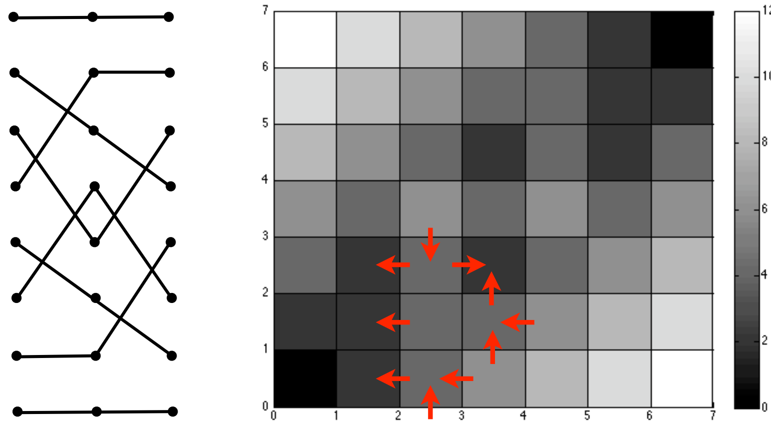


FIG. 2.4. *On the boundary of each connected component of a discrete section of a relative braid class the direction of the flow can be determined on the basis of the total intersection number indicated by the grey-scale*

.

Denote by $N = N_{[\mathbf{u}|\mathbf{v}]}$ the closure of $[\mathbf{u}|\mathbf{v}]$ in $\overline{\mathcal{D}^n_d}$. Define the exit set $N^- = N^-_{[\mathbf{u}|\mathbf{v}]}$ as those singular braids at which the intersection number ι can decrease:

$$\begin{aligned} N^-_{[\mathbf{u}|\mathbf{v}]} &= \big\{\mathbf{u}' \in \partial N : \forall \varepsilon > 0\ \exists\, \mathbf{u}'' \in B_\varepsilon(\mathbf{u}') \cap \mathcal{D}^n_d|\mathbf{v} \text{ with } \iota(\mathbf{u}'' \cup \mathbf{v}) < \iota(\mathbf{u} \cup \mathbf{v})\big\} \\ &= \mathrm{cl}\big\{\mathbf{u}' \in \partial N : \exists\, \varepsilon > 0 \text{ s.t. } \iota(\mathbf{u}'' \cup \mathbf{v}) \le \iota(\mathbf{u} \cup \mathbf{v})\ \forall \mathbf{u}'' \in B_\varepsilon(\mathbf{u}') \cap \mathcal{D}^n_d|\mathbf{v}\big\}. \quad (2.7) \end{aligned}$$

The second expression states that N^- is the closure of a subset of $\partial N \cap \Sigma_1|\mathbf{v}$, namely the subset consisting of points where the intersection number decreases when departing N at that point (of the codimension-1 boundary), see also Figure 2.4. It should be clear that the above two expressions are equivalent, and that the latter is more convenient from a computational point of view.

In the setting of Conley index theory, the sets N and N^- act as isolating block and exit set, respectively, for any discrete parabolic flow $\mathcal{R}$ that fixes $\mathbf{v}$, i.e., (N, N^-) is an index pair, see [8].

DEFINITION 2.16. *The* Conley index *of a (bounded and weakly proper) discrete (relative) braid class* $[\mathbf{u}|\mathbf{v}]$ *is defined as the pointed homotopy class of spaces*

$$h([\mathbf{u}|\mathbf{v}]) = [N/N^-] = (N/N^-, [N^-]).$$

The following proposition, which is the main result in [8], states that this Conley index is an invariant of the continuous relative braid class.

PROPOSITION 2.17 ([8]). *Let* $[\mathbf{U} \# \mathbf{V}]$ *be a bounded proper braid class. Let* $\mathbf{u} \# \mathbf{v}$ *be any discrete representative of* $[\mathbf{U} \# \mathbf{V}]$, *i.e.,* $\ell\mathbf{u} \# \ell\mathbf{v} \in [\mathbf{U} \# \mathbf{V}]$. *Then the homotopy type* $h([\mathbf{u}|\mathbf{v}])$ *is independent of the choice of discrete representative. In other words, for every* $d \in \mathbb{N}$ *and any* $\mathbf{v}' \in D^m_d$ *such that* $\ell\mathbf{v}' \in \pi([\mathbf{U} \# \mathbf{V}])$, *and any element* $\mathbf{u}'$ *in the fiber* $\Pi_{\mathbf{v}'}[\mathbf{U} \# \mathbf{V}]$, *the Conley index* $h([\mathbf{u}'|\mathbf{v}'])$ *is the same.*

Hence, we may define the invariant

$$\mathbf{H}([\mathbf{U} \# \mathbf{V}]) = \mathbf{H}([\ell\mathbf{u} \# \ell\mathbf{v}]) \stackrel{\text{def}}{=} h([\mathbf{u}|\mathbf{v}]).$$

We note that proposition 2.17 implies that if the fiber $\Pi_{\mathbf{v}}[\mathbf{U}\,\#\,\mathbf{V}]$ is empty for some discrete skeleton $\mathbf{v} \in D_d^m$ with $\ell\mathbf{v} \in [\mathbf{V}]$, then the Conley index $h([\mathbf{U}\,\#\,\mathbf{V}])$ is trivial.

As it stands, the Conley index is hard to compute. We will therefore restrict our attention to the homological index (singular or simplicial homology over $\mathbb{Z}$).

$$CH_*([\mathbf{U}\,\#\,\mathbf{V}]) \quad \text{or} \quad CH_*([\mathbf{u}|\mathbf{v}]),$$

which represents the homology of the pointed spaces $h([\mathbf{u}|\mathbf{v}])$. To encode information in $CH_*([\mathbf{u}|\mathbf{v}])$ it is convenient to use the Poincaré polynomial

$$P[\mathbf{U}\,\#\,\mathbf{V}](s) = P[\mathbf{u}|\mathbf{v}](s) = \sum_{i=0}^{nd} \beta_i\, s^i,$$

where β_i is the rank of $CH_i([\mathbf{u}|\mathbf{v}])$. Furthermore, $[\mathbf{u}|\mathbf{v}]$ in general consists of several connected components $[\mathbf{u}|\mathbf{v}]_k$, $k = 1, \ldots, K$. For each component one may define the isolating block $N_k = \mathrm{cl}([\mathbf{u}|\mathbf{v}]_k)$ and the associated exit set N_k^-. Then

$$h([\mathbf{u}|\mathbf{v}]) = \bigvee_{k=1}^{K} h([\mathbf{u}|\mathbf{v}]_k) = \bigvee_{k=1}^{K} \big(N_k/N_k^-, [N_k^-]\big),$$

where the topological wedge $\vee$ identifies all the constituent exit sets to a single point. This implies that

$$P[\mathbf{u}|\mathbf{v}](s) = \sum_{k=1}^{K} P[\mathbf{u}|\mathbf{v}]_k(s). \tag{2.8}$$

One is often only able to compute $P[\mathbf{u}|\mathbf{v}]_1(s)$, since it can be difficult (or computationally expensive) to determine the other components $[\mathbf{u}\,|\,\mathbf{v}]_k$. Hence the above decomposition is convenient.

As is usual in Conley index theory, information about the index of an isolating neighborhood can be used to draw conclusions about the invariant dynamics inside. For example, if the index is nontrivial, then any parabolic flow that fixes $\mathbf{v}$ has invariant dynamics inside. In particular, since there is a Poincaré-Bendixson type result for parabolic flows (see [5, 6]), the α- and ω-limit sets consist of stationary points, stationary points with connections between them, or periodic orbits. Moreover, for gradient type, or *exact*, systems, both periodic orbits and connections are excluded in α- and ω-limit sets.

Definition 2.18. *A parabolic recurrence relation $\mathcal{R}$ is* exact *if there exist $C^2(\mathbb{R}^2;\mathbb{R})$ functions $(S_i)_{i=0}^{d-1}$ such that (with $S_{i+d} = S_i$)*

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_{i-1}(u_{i-1}, u_i) + \partial_1 S_i(u_i, u_{i+1}).$$

For exact parabolic recurrence relations the flow becomes the gradient flow of

$$W(\mathbf{u}) = \sum_{i=1}^{d} S_i(u_i, u_{i+1}).$$

Lemma 2.19 ([8]). *Let $\mathcal{R}$ be a parabolic flow fixing $\mathbf{v}$ and let $[\mathbf{u}|\mathbf{v}]$ be a bounded and weakly proper braid class. If the Conley index $h([\mathbf{u}|\mathbf{v}])$ (or $CH_*([\mathbf{u}|\mathbf{v}])$ or $P[\mathbf{u}|\mathbf{v}](s)$) is nontrivial, then $[\mathbf{u}|\mathbf{v}]$ contains at least one stationary or periodic solution of (1.3).*

Furthermore, if the Euler characteristic $P[\mathbf{u}|\mathbf{v}](-1) \neq 0$*, then there exists at least one stationary solution in* $[\mathbf{u}|\mathbf{v}]$*. If* $\mathcal{R}$ *is exact, then the number of stationary solutions is bounded from below by the number of monomials in* $P[\mathbf{u}|\mathbf{v}](s)$*.*

In practice we usually compute the Poincaré polynomial of only one component $[\mathbf{u}|\mathbf{v}]_1$ of a fiber. The following lemma, which thus is the key to checking the assumptions in Lemma 2.19, follows directly from (2.8) and the fact that all coefficients β_i in $P[\mathbf{u}|\mathbf{v}]_k(s)$ are nonnegative.

LEMMA 2.20. *If* $P[\mathbf{u}|\mathbf{v}]_1(s)$ *is nontrivial then* $P[\mathbf{u}|\mathbf{v}](s)$ *is nontrivial. The number of monomials in* $P[\mathbf{u}|\mathbf{v}](s)$ *is bounded from below by the number of monomials in* $P[\mathbf{u}|\mathbf{v}]_1(s)$*.*

It is thus sufficient to know that $P[\mathbf{u}|\mathbf{v}]_1(s) \neq 0$ in order to conclude that $\mathbf{H}([\ell\mathbf{u} \# \ell\mathbf{v}])$ is nontrivial. On the other hand, the information that the Euler characteristic $P[\mathbf{u}|\mathbf{v}]_1(-1)$ of one component is nonzero does *not* imply that the Euler characteristic $P[\mathbf{u}|\mathbf{v}](-1)$ of the entire braid class is nonzero.

Proposition (2.17) allows a limit procedure that links the discrete setting (1.2) to the continuous case (1.1) in the limit of large d, see [9]. The information contained in the braid invariant $\mathbf{H}([\mathbf{U} \# \mathbf{V}])$ can thus be used to draw conclusions about solutions of (1.1). We need the following technical assumption:

(F) There exist constants $C > 0$ and $0 < \gamma < 2$ such that $|f(U_x, U, x)| \leq C(1+|U_x|^\gamma)$, uniformly in both $x \in S^1$ and on compact intervals in U.

LEMMA 2.21 ([9]). *Assume that Equation (1.1) with* f *satisfying hypothesis* (F) *fixes a braid* $\mathbf{V}$*. Let* $[\mathbf{U} \# \mathbf{V}]$ *be a bounded proper braid class with* $\mathbf{U}$ *a single-component braid. If* $P[\mathbf{U} \# \mathbf{V}](s)$ *is nontrivial, then* $[\mathbf{U} \# \mathbf{V}]$ *contains at least one stationary or periodic solution of (1.1). Furthermore, if* $P[\mathbf{U} \# \mathbf{V}](-1) \neq 0$*, then there exists at least one stationary solution in* $[\mathbf{U} \# \mathbf{V}]$*. If* f *in (1.1) does not depend on* U_x*, then the number of stationary solutions is bounded from below by the number of monomials in* $P[\mathbf{U} \# \mathbf{V}](s)$*.*

Slightly more general results can be found in [9].

3. Morse decompostions and connecting orbits. We have a partial order on relative braid classes.

DEFINITION 3.1. *We say that* $[\mathbf{U}' \# \mathbf{V}'] < [\mathbf{U} \# \mathbf{V}]$*, and the braid classes are called* adjacent*, if*

1. $\iota(\mathbf{U}' \cup \mathbf{V}') = \iota(\mathbf{U} \cup \mathbf{V}) - 2$*; and*
2. *there are continuous families* $\mathbf{U}_t \in \overline{\Omega^n}$*, and* $\mathbf{V}_t \in \Omega^m$*, for* $t \in [0, 2]$*, such that*
   - *(a)* $\mathbf{U}_t \# \mathbf{V}_t \in [\mathbf{U} \# \mathbf{V}]$ *for* $t \in [0, 1)$*;*
   - *(b)* $\mathbf{U}_t \# \mathbf{V}_t \in [\mathbf{U}' \# \mathbf{V}']$ *for* $t \in (1, 2]$*;*
   - *(c)* $\mathbf{U}_1 \cup \mathbf{V}_1 \in \Sigma_1 | \mathbf{V}_1$ *(exactly one tangency).*

The asymmetric relation $<$ *is made into a partial order, denoted by* $\prec$*, by taking the transitive-reflexive closure.*

It should be clear that $[\mathbf{V}] = [\mathbf{V}'] = [\mathbf{V}_t]$ in the above definition. The idea is then that $[\mathbf{U}' \# \mathbf{V}'] \prec [\mathbf{U} \# \mathbf{V}]$ if there is a path in $\overline{\Omega^n} \times \Omega^m$ from $\mathbf{U} \# \mathbf{V}$ to $\mathbf{U}' \# \mathbf{V}'$ that intersects $\overline{\Omega^{n,m}} \setminus \Omega^{n,m}$ a finite number of times in points where the free braid has exactly one, non-degenerate tangency (with itself or with the skeleton), and such that the intersection number is non-increasing along this path. The skeleton is not allowed to have tangencies with itself along the path.

The order could be defined using braid words, but this requires setting up braid words for relative braids, which is beyond the scope of the present paper. On the other hand, if one wants to define the positive conjugacy problem for relative braid diagrams, then it could be helpful to express this in terms of relative braid words.

The following sets are now (candidates for) isolating neighborhoods for the flow.

DEFINITION 3.2. *A (nonempty) collection of braids $\mathcal{C} = \{[\mathbf{U}_i \# \mathbf{V}_i]\}_{i\in I}$ is called* convex *if*

1. *$[\mathbf{V}_i]$ is independent of i; and*
2. *for any class $[\mathbf{U}' \# \mathbf{V}']$ such that $[\mathbf{U}_{i_1} \# \mathbf{V}_{i_1}] \prec [\mathbf{U}' \# \mathbf{V}'] \prec [\mathbf{U}_{i_2} \# \mathbf{V}_{i_2}]$ for some $i_1, i_2 \in I$, it holds that $[\mathbf{U}' \# \mathbf{V}'] \in \mathcal{C}$.*

The first property implies that for any convex collection $\mathcal{C}$ the skeleton braid class $[\pi\mathcal{C}]$ is well-defined. The collection $\mathcal{C}$ does not need to be fully ordered; there just cannot be any element "missing" between ordered elements. Clearly, any pair of adjacent braid classes forms a convex collection.

Let $\mathcal{C}$ be a convex collection and let $\mathbf{v} \in \mathcal{D}_d^m$ be a discrete representative of $[\pi\mathcal{C}]$, then the corresponding *discrete section* is

$$\begin{aligned}\mathcal{C}_{\mathbf{v}} &\stackrel{\text{def}}{=} \operatorname{cl}\left(\left\{\mathbf{u} \in \mathcal{D}_d^n : [\ell\mathbf{u} \# \ell\mathbf{v}] \in \mathcal{C}\right\}\right) \\ &= \operatorname{cl}\left(\bigcup\left\{[\mathbf{u}|\mathbf{v}] : [\ell\mathbf{u} \# \ell\mathbf{v}] \in \mathcal{C}\right\}\right).\end{aligned}$$

Such sections of convex collections serve as isolating neighborhoods to which we can associate a Conley index and hence draw conclusions about the invariant dynamics inside.

LEMMA 3.3. *Let $\mathcal{C}$ be a convex collection of bounded and proper braid classes, and let $\mathbf{v}$ be a discrete representative of $[\pi\mathcal{C}]$. Then the Conley index of $\mathcal{C}_{\mathbf{v}}$ is well-defined. Specifically, $N_{\mathcal{C}_{\mathbf{v}}} = \operatorname{cl}(\mathcal{C}_{\mathbf{v}})$ is an isolating block for any parabolic flow $\mathcal{R}$ fixing $\mathbf{v}$, and the exit set is (see (2.7))*

$$N_{\mathcal{C}_{\mathbf{v}}}^- = \bigcup\left\{[\mathbf{u}|\mathbf{v}] \subset \mathcal{C}_{\mathbf{v}} : N_{[\mathbf{u}|\mathbf{v}]}^- \cap \partial\mathcal{C}_{\mathbf{v}}\right\}.$$

Moreover, the Conley index of $\mathcal{C}_{\mathbf{v}}$ is independent of the choice of $\mathbf{v}$ representing $[\pi\mathcal{C}]$, i.e. $\mathbf{H}(\mathcal{C})$ is well-defined.

Proof. For a fixed discrete representative $\mathbf{v}$, isolation of the invariant set in $\mathcal{C}_{\mathbf{v}}$ follows from properness and convexity of $\mathcal{C}$ (Definition 3.2). The critical observation is that if there would be an orbit in the invariant set that touches the boundary of $\mathcal{C}_{\mathbf{v}}$ then by continuity there must be an orbit nearby that leaves $\operatorname{cl}(\mathcal{C}_{\mathbf{v}})$ and then enters $\operatorname{cl}(\mathcal{C}_{\mathbf{v}})$, which is impossible for convex collections of braid classes, since parabolic flows strictly decrease intersection numbers on boundaries of braid classes, see Theorems 11 and 15 in [8]. Independence of the choice of $\mathbf{v}$ follows from the proof of Theorem 20 in [8]. □

It follows that under the conditions stated in Lemma 3.3, the relative braid classes constituting $\mathcal{C}_{\mathbf{v}}$ form a Morse decomposition of $\mathcal{C}_{\mathbf{v}}$ (with respect to the partial order $\prec$).

Although the above construction works only if all elements of $\mathcal{C}$ are proper, suitably chosen convex collections involving improper classes may be isolating blocks as well. For example, the four tiles surrounding a red dot (stationary point) in Figure 2.3(b) are all improper, but together they form an isolating block.

The Conley index $\mathbf{H}(\mathcal{C})$ can provide information about the (forced) existence of *connecting orbits*. The following lemma describes the situation for an attractor-repeller pair.

LEMMA 3.4. *Let $\mathcal{C}$ consist of exactly two bounded and proper braid classes $[\mathbf{U}_1 \# \mathbf{V}_1]$ and $[\mathbf{U}_2 \# \mathbf{V}_2]$, with $[\mathbf{U}_2 \# \mathbf{V}_2] < [\mathbf{U}_1 \# \mathbf{V}_1]$ (adjacent). Suppose that*

$$\mathbf{H}(\mathcal{C}) \neq \mathbf{H}([\mathbf{U}_1 \# \mathbf{V}_1]) \vee \mathbf{H}([\mathbf{U}_2 \# \mathbf{V}_2]).$$

Then for any discrete representative $\mathbf{v}$ *of* $[\pi\mathcal{C}]$, *there exists at least one orbit in* $\mathcal{C}_{\mathbf{v}}$, *for any parabolic flow* $\mathcal{R}$ *fixing* $\mathbf{v}$, *with* α*-limit set in* $\Pi_{\mathbf{v}}[\mathbf{U}_1 \# \mathbf{V}_1]$ *and* ω*-limit set in* $\Pi_{\mathbf{v}}[\mathbf{U}_2 \# \mathbf{V}_2]$.

Proof. This is a consequence of well known properties of the Conley index [11, 14] and topological invariance of $\mathbf{H}(\mathcal{C})$ and $\mathbf{H}([\mathbf{U}_i \# \mathbf{V}_i])$. ∎

The following lemma is analogous to Lemma 2.21.

LEMMA 3.5. *Assume that Equation (1.1) with* f *satisfying hypothesis* (F) *fixes a braid* $\mathbf{V}$. *Let* $[\mathbf{U}_1 \# \mathbf{V}]$ *and* $[\mathbf{U}_2 \# \mathbf{V}]$ *be adjacent bounded proper braid classes, with* $\mathbf{U}_i$ *a single-component braid. Let* $\mathcal{C}$ *consists of the braid classes* $[\mathbf{U}_i \# \mathbf{V}]$, $i = 1, 2$. *If*

$$\mathbf{H}(\mathcal{C}) \neq \mathbf{H}([\mathbf{U}_1 \# \mathbf{V}]) \vee \mathbf{H}([\mathbf{U}_2 \# \mathbf{V}]),$$

then there is at least one solution of (1.1) with with α*-limit set in* $[\mathbf{U}_1|\mathbf{V}]$ *and* ω*-limit set in* $[\mathbf{U}_2|\mathbf{V}]$.

Proof. This follows from the approach and estimates in [9]. For each discrete representative, Lemma 3.4 provides an orbit going from one braid class to the other. Define $t = 0$ as the unique time the orbit is on the boundary between the braid classes. Consider, as in [9], the limit of infinitely many discretization points. Since the convergence results in [9] hold on arbitrary bounded intervals, consider time intervals $[-T, T]$, $T \in \mathbb{N}$, and use a diagonal argument to obtain an orbit for (1.1) that is in $[\mathbf{U}_1|\mathbf{V}]$ for $t < 0$ and in $[\mathbf{U}_2|\mathbf{V}]$ for $t > 0$. ∎

As discussed in Section 2.4, for computational purposes it is convenient to restrict attention to Poincaré polynomials $P\mathcal{C}(s)$ and $P[\mathbf{U}_1 \# \mathbf{V}](s)$. Let $[\mathbf{u}_i|\mathbf{v}]$ be discrete fibers, i.e. $[\ell\mathbf{u}_i \# \ell\mathbf{v}] = [\mathbf{U}_i \# \mathbf{V}]$ for $i = 1, 2$. In practice we often only have information about $[\mathbf{u}_i|\mathbf{v}]_1$, i.e. single connected components of $[\mathbf{u}_i|\mathbf{v}]$. The following lemma shows that this restricted information may be sufficient.

LEMMA 3.6. *Let* $[\mathbf{u}_i|\mathbf{v}]_1$ *be connected components of discrete fibers of adjacent proper bounded braid classes. Their union* $\mathcal{C}_{\mathbf{v},1} = \mathrm{cl}\,([\mathbf{u}_1|\mathbf{v}]_1 \cup [\mathbf{u}_2|\mathbf{v}]_1)$ *is an isolating neighborhood for any parabolic flow* $\mathcal{R}$ *fixing* $\mathbf{v}$. *We denote its Poincaré polynomial by* $P\mathcal{C}_{\mathbf{v},1}(s)$. *If*

$$P\mathcal{C}_{\mathbf{v},1}(s) \neq P[\mathbf{u}_1|\mathbf{v}]_1(s) + P[\mathbf{u}_2|\mathbf{v}]_1(s),$$

then

$$P\mathcal{C}(s) \neq P[\mathbf{u}_1|\mathbf{v}](s) + P[\mathbf{u}_2|\mathbf{v}](s).$$

Proof. First, the isolating property of $\mathcal{C}_{\mathbf{v},1}$ is analogous to Lemma 3.3.

To simplify notation we define $R_0 = \mathrm{cl}([\mathbf{u}_1|\mathbf{v}])$, $R_1 = \mathrm{cl}([\mathbf{u}_1|\mathbf{v}]_1)$ and $R_2 = \mathrm{cl}(R_0 \setminus R_1)$, and similarly $A_0 = \mathrm{cl}([\mathbf{u}_2|\mathbf{v}])$, $A_1 = \mathrm{cl}([\mathbf{u}_2|\mathbf{v}]_1)$ and $A_2 = \mathrm{cl}(A_0 \setminus A_1)$. The pairs (A_i, R_j), $i, j = 0, 1, 2$, as well as $(A_i, A_j \cup R_k)$ and $(A_k \cup R_i, R_j)$, $i \neq j = 1, 2$, $k = 0, 1, 2$, are all pairs of attracting and repelling neighborhoods. Namely, the (interior of the) union of each pair is isolating by the argument in Lemma 3.3, and orbits can only go from the repelling to the attracting neighborhood, since $\iota(\mathbf{u}_1 \cup \mathbf{v}) = \iota(\mathbf{u}_2 \cup \mathbf{v}) + 2$ and the flow strictly decreases the intersection number on the boundary of (a connected component of) a braid class. The Poincaré polynomials of the corresponding Conley indices are denoted by $P(A_i)$, $P(R_i)$, ect.

For each pair (A, R) of attracting and repelling neighborhoods it follows from the Morse relations for the Conley index [11, 14] that $P(A \cup R)(s) = P(A)(s) + P(R)(s) + Q_{A,R}(s)(1 + s)$ for some polynomial $Q_{A,R}$ with non-negative integer coefficients. For convenience, let us evaluate all Poincaré polynomials at $s = 1$ and write $P(\cdot)(1) = P(\cdot)$. In particular, the above implies that

$$P(A \cup R) \geq P(A) + P(R). \tag{3.1}$$

By assumption we have the strict inequality

$$P(A_1 \cup R_1) > P(A_1) + P(R_1). \tag{3.2}$$

Furthermore, inequality (3.1) applied to different pairs of attracting and repelling neighborhoods gives

$$P(A_2 \cup R_1) \geq P(A_2) + P(R_1) \tag{3.3}$$

$$P(A_1 \cup A_2 \cup R_1) \geq P(A_2) + P(A_1 \cup R_1) \tag{3.4}$$

$$P(A_1 \cup A_2 \cup R_1) \geq P(A_1) + P(A_2 \cup R_1), \tag{3.5}$$

and by adding the inequalities (3.2)-(3.5) we obtain

$$2P(A_1 \cup A_2 \cup R_1) > 2P(A_1) + 2P(A_2) + 2P(R_1).$$

Since $A_0 = A_1 \cup A_2$ and $P(A_0) = P(A_1) + P(A_2)$ by Equation (2.8), this implies

$$P(A_0 \cup R_1) > P(A_0) + P(R_1). \tag{3.6}$$

Next, inequality (3.1) applied to yet more pairs of attracting and repelling neighborhoods leads to

$$P(A_0 \cup R_2) \geq P(A_0) + P(R_2) \tag{3.7}$$

$$P(A_0 \cup R_1 \cup R_2) \geq P(A_0 \cup R_1) + P(R_2) \tag{3.8}$$

$$P(A_0 \cup R_1 \cup R_2) \geq P(A_0 \cup R_2) + P(R_1). \tag{3.9}$$

Again, by adding inequalities (3.6)-(3.9) we obtain

$$2P(A_0 \cup R_1 \cup R_2) > 2P(A_0) + 2P(R_1) + 2P(R_2).$$

Since $R_0 = R_1 \cup R_2$ and $P(R_0) = P(R_1) + P(R_2)$ by Equation (2.8), we conclude that

$$P(A_0 \cup R_0) > P(A_0) + P(R_0),$$

which finishes the proof. □

The following lemma leads to a computable criterion for the existence of connecting orbits between braid classes.

LEMMA 3.7. *Assume that Equation (1.1) with f satisfying hypothesis* (F) *fixes a braid* $\mathbf{V}$. *Let* $[\mathbf{U}_2 \# \mathbf{V}] < [\mathbf{U}_1 \# \mathbf{V}]$ *be adjacent bounded proper braid classes, with* $\mathbf{U}_i$ *a single-component braid. Let* $[\mathbf{u}_i | \mathbf{v}]$ *be discrete fibers, i.e.* $[\ell\mathbf{u}_i \# \ell\mathbf{v}] = [\mathbf{U}_i \# \mathbf{V}]$ *for* $i = 1, 2$. *Let* $[\mathbf{u}_i | \mathbf{v}]_1$ *be single connected components of* $[\mathbf{u}_i | \mathbf{v}]$, *and let* $\mathcal{C}_{\mathbf{v},1} = \mathrm{cl}\,([\mathbf{u}_1 | \mathbf{v}]_1 \cup [\mathbf{u}_2 | \mathbf{v}]_1)$. *If*

$$P\mathcal{C}_{\mathbf{v},1}(s) \neq P[\mathbf{u}_1 | \mathbf{v}]_1(s) + P[\mathbf{u}_2 | \mathbf{v}]_1(s),$$

then there is at least one solution of (1.1) with with α-limit set in $[\mathbf{U}_1 | \mathbf{V}]$ *and ω-limit set in* $[\mathbf{U}_2 | \mathbf{V}]$.

Proof. This follows by combining Lemmas 3.5 and 3.6. □

Hence, it suffices to find two adjacent connected components for a discrete fiber of two adjacent braid classes, such that their Poincaré polynomials do not add up to the Poincaré polynomial of their union, to conclude that there must be a "connecting orbit" between these braid classes for any parabolic flow that fixes a skeleton in $[\mathbf{V}]$. This is illustrated in Figure 3.1.

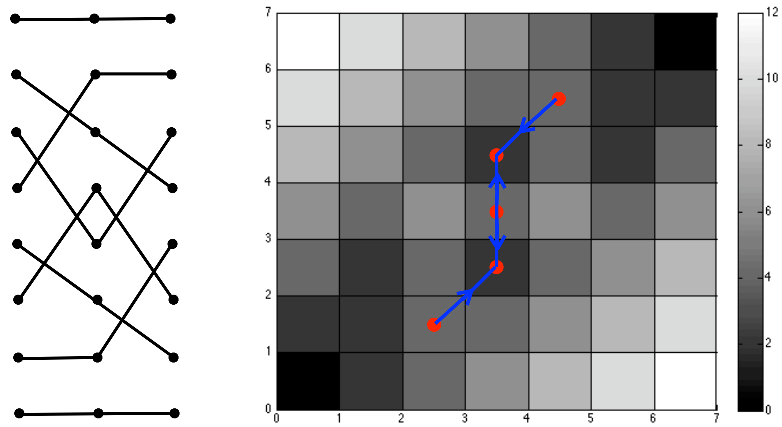


FIG. 3.1. *On the basis of the Conley indices the existence of the indicated orbit structure can be concluded for any parabolic flow (1.3) that fixes the skeleton on the left. The red dots indicate invariant sets, not necessarily stationary points.*

4. Algorithms. In this section we discuss a computational approach to determining the braid invariant $\mathbf{H}([\mathbf{U}\,\#\,\mathbf{V}])$. In fact, we "only" compute the homological Conley index $CH_*([\mathbf{u}|\mathbf{v}]_1)$, and its Poincaré polynomial $P[\mathbf{u}|\mathbf{v}]_1(s)$, of a component of a discrete section $[\mathbf{u}|\mathbf{v}]$ of $[\mathbf{U}\,\#\,\mathbf{V}]$. Due to Lemmas 2.20 and 3.7 this allows us to draw conclusions about the dynamics of (1.1) and (1.3) in $[\mathbf{U}\,\#\,\mathbf{V}]$.

We note that in low dimensional examples the Conley index can be determined by hand (see the example in the Introduction). In some high dimensional cases where the component $[\mathbf{u}|\mathbf{v}]_1$ consists of a single (high-dimensional) cube, the index can also be computed directly, see [8]. Furthermore, for a special (infinite) family of complex relative braid classes the Conley index was computed via a delicate decomposition and associated MayerVietoris sequence, see [15]. Here we consider a general approach to computing the Conley index of $[\mathbf{u}|\mathbf{v}]_1$ for relative braid classes with a *single* free strand. The case of multiple free strands is the subject of further research.

For computational purposes, we store the discretized skeleton braid $\mathbf{v} \in \mathcal{D}_d^m$ as an $m \times (d+1)$ matrix. Then $\mathbf{v}(i,j)$ gives the coordinate of the i-th strand at the j-th discretization point. A strand $\mathbf{v}(i,:)$ is closed if $\mathbf{v}(i,d+1) = \mathbf{v}(i,1)$. Note that by the choice of the fiber, we may set the entries of $\mathbf{v}$ to be consecutive integer values. More precisely, we prescribe that for each j, $\{\mathbf{v}(1,j), \ldots \mathbf{v}(m,j)\} = \{1, \ldots, m\}$. Similarly, a free strand $\mathbf{u}$ woven through $\mathbf{v}$ is given as a $1 \times (d+1)$ vector specifying the coordinates of $\mathbf{u}$ at the discretization points. In choosing a representative of the fiber $[\mathbf{u}|\mathbf{v}]$, it is convenient to choose the entries of $\mathbf{u}$ to be non-integer numbers, representing the position of the free strand relative to the fixed skeletal strands at the discretization points.

Intersection numbers may be computed in a straight-forward manner. For $i = 1, \ldots, m$,

$$\iota(\mathbf{u}, \mathbf{v}(i,:)) = \#\big\{1 \leq j \leq d : \,(\mathbf{v}(i,j) - \mathbf{u}(j))\,(\mathbf{v}(i,j+1) - \mathbf{u}(j+1)) < 0\big\}$$

is the number of intersections of the free strand $\mathbf{u}$ with the i-th skeletal strand $\mathbf{v}(i,:)$.

Then the total number of crossings between $\mathbf{u}$ and the skeleton $\mathbf{v}$ is

$$\iota(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{m} \iota(\mathbf{u},\mathbf{v}(i,:)).$$

As outlined in Section 2.4, we may use the intersection number to locate the entire relative braid component $[\mathbf{u}|\mathbf{v}]_1$ in $\mathcal{D}^1_d$. We use a cubical complex to represent these sets, where, for example, $\mathbf{u} \in B_0 \stackrel{\text{def}}{=} \prod_{j=1}^{d}[\lfloor \mathbf{u}(j) \rfloor, \lceil \mathbf{u}(j) \rceil]$. In what follows, we construct cubical sets consisting of d-dimensional cubes with vertices on the integer lattice $\mathbb{Z}^d$. More specifically, we consider cubes of the form $B = \prod_{j=1}^{d}[l_j, l_j+1]$ with $l_j \in \mathbb{Z}$. We will also restrict the set of allowed cubes to a region prescribed by $\mathbf{v}$. The result is the collection

$$\mathcal{K} \stackrel{\text{def}}{=} \Big\{ B = \prod_{j=1}^{d}[l_j, l_j+1] : l_j \in \{0,1,\ldots,n\} \Big\}.$$

For convenience in what follows, we also define the *boundary cubes in* $\mathcal{K}$ to be

$$\partial\mathcal{K} \stackrel{\text{def}}{=} \Big\{ B = \prod_{j=1}^{d}[l_j, l_j+1] \in \mathcal{K} : \text{there exists } j \text{ such that } l_j = 0 \text{ or } l_j = n-1 \Big\}.$$

The boundary layer $\partial\mathcal{K}$ is not part of $[\mathbf{u}\,|\,\mathbf{v}]_1$ if the relative braid class is bounded. Note that in Figures 2.2 and 2.3 this boundary layer is not depicted. The *center* c_B of $B = \prod_{j=1}^{d}[l_j, l_j+1] \in \mathcal{K}$ has coordinates $c_B(j) = l_j + \frac{1}{2}$ (and we set $c_B(d+1) = c_B(1)$ when needed). Finally, let $|\mathcal{S}| \stackrel{\text{def}}{=} \cup_{B\in\mathcal{S}} B$ denote the *topological realization* of $\mathcal{S} \subset \mathcal{K}$ as a subset of $\mathbb{R}^d$.

We now give a procedure for *growing* a cubical representation $\mathcal{S} \subset \mathcal{K}$ of the $[\mathbf{u}|\mathbf{v}]_1$. Begin by setting $\mathcal{S} = \{B_0\}$, where, as above, $B_0 \in \mathcal{K}$ and $\mathbf{u} \in B_0$. Now compute the *codimension-1* neighboring cubes $B \in \mathcal{K}$ with $B \cap |\mathcal{S}| \neq \emptyset$, i.e., those cubes that have a codimension-1 face in common with one of the cubes in $\mathcal{S}$. We define the intersection number for $B \in \mathcal{K}$ to be $\iota(B,\mathbf{v}) \stackrel{\text{def}}{=} \iota(c_B,\mathbf{v})$. Note that $\iota(B,\mathbf{v})$ is the intersection number for any free strand with coordinates in the interior of B with the skeleton $\mathbf{v}$. If a cube B in the codimension-1 boundary satisfies $\iota(B,\mathbf{v}) = \iota(\mathbf{u},\mathbf{v})$, then we add it to $\mathcal{S}$: $\mathcal{S} = \mathcal{S} \cup \{B\}$. We continue adding boxes to $\mathcal{S}$ that satisfy $B \notin \mathcal{S}$, $B \cap |\mathcal{S}|$ is codimension-1 and $\iota(B,\mathbf{v}) = \iota(\mathbf{u},\mathbf{v})$, until no boxes remain in $\mathcal{K}$ that satisfy these conditions. Since $\mathcal{K}$ is finite, this is a finite procedure.

Given a cubical set $\mathcal{S}$, calculated as above, we next check whether it is *bounded* and *weakly proper*. If $\mathcal{S} \subset \mathcal{K} \setminus \partial\mathcal{K}$, then $\mathcal{S}$ is bounded. In particular, all neighboring configurations of the free strand have different intersection numbers with $\mathbf{v}$ and, therefore, form a bounding layer around $\mathcal{S}$. The check that $\mathcal{S}$ is proper is an easy comparison between the centers of boxes in $\mathcal{S}$ and skeleton $\mathbf{v}$. For $B \in \mathcal{S}$, compute

$$d(c_B,\mathbf{v}) \stackrel{\text{def}}{=} \min_{\substack{1\le i\le m \\ v(i,1)=v(i,d+1)}} \left\{ \max_{1\le j\le d} |\mathbf{v}(i,j) - c_B(j)| \right\},$$

where the minimum is taken over closed strands of $\mathbf{v}$ only, since these are the strands onto which $\mathbf{u}$ could possible collapse. If $d(c_B,\mathbf{v}) < 1$ for any $B \in \mathcal{S}$, then $\mathcal{S}$ does not correspond to a proper relative braid class. In particular, a center c_B for which $d(c_B,\mathbf{v}) < 1$ corresponds to a configuration of the free strand that may be collapsed

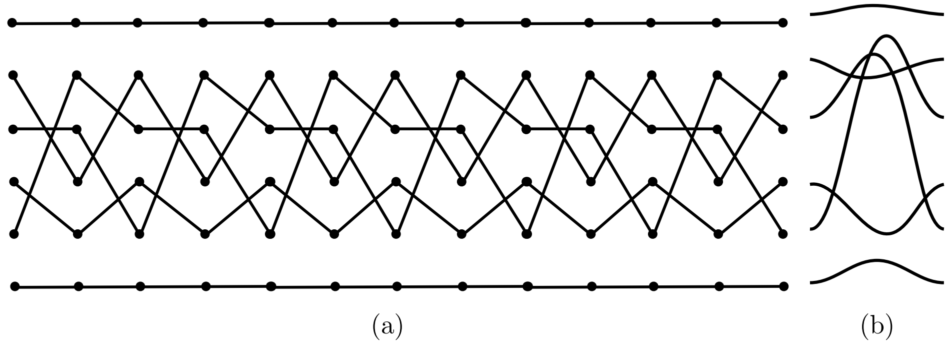


FIG. 5.1. *(a) A discretized skeleton* $\mathbf{v}$ *which is a 6-fold repetition of the continuous braid in (b).*

directly onto a strand in the skeleton $\mathbf{v}$. If, on the other hand, $d(c_B, \mathbf{v}) \geq 1$ for all $B \in \mathcal{S}$, then $|\mathcal{S}|$ is weakly proper.

If $\mathcal{S}$, as computed above, is bounded and proper then we may compute the Conley index of $|\mathcal{S}| = [\mathbf{u}|\mathbf{v}]_1$ as follows. Define the cubical exit set $\mathcal{N}^-$ as

$$\mathcal{N}^- \stackrel{\text{def}}{=} \left\{B \in \mathcal{K} : B \cap |\mathcal{S}| \neq \emptyset, \dim(B \cap |\mathcal{S}|) = d-1, \iota(B, \mathbf{v}) < \iota(\mathbf{u}, \mathbf{v})\right\},$$

where only codimension-1 neighboring cubes of $\mathcal{S}$ are considered. Note that these cubes can be identified already during the growing procedure described above. Then, by construction (N, N^-), where $N = |\mathcal{S}|$ and $N^- = |\mathcal{S}| \cap |\mathcal{N}^-|$, is an index pair for $[\mathbf{u}|\mathbf{v}]_1$. Since we have cubical representations for N and N^-, we may now use a cubical homology software program to compute the Conley index.

For the computations described in this paper, we use the binary tree structure implemented in the GAIO software package [4] with a MATLAB interface to construct the cubical complex, and the cubical homology program *homcubes* from [12] to compute the index.

5. Examples. We now present sample results based on the computational approach outlined in Section 4. Consider the discrete relative braid depicted in Figure 5.1(a). It is a discretization of a 6-fold repetition of the continuous braid in Figure 5.1(b). The configuration space of a free strand is 12-dimensional. There are many free strands that one could weave through this skeleton. We begin by studying the two free strands depicted in Figure 5.2 for which we have the following result.

SAMPLE RESULT 5.1. *Let* $K_1 = [\mathbf{u}_1 \,|\, \mathbf{v}]_1$ *and* $K_2 = [\mathbf{u}_2 \,|\, \mathbf{v}]_1$ *be the connected components of the fibers of relative braid classes given in Figure 5.2. For any parabolic flow (1.3) fixing* $\mathbf{v}$ *there are nonempty invariant sets in* K_1 *and* K_2 *as well as a connecting orbit whose* α*-limit set is in* K_1 *and whose* ω*-limit set is in* K_2*.*

Outline of proof. Using the matrix representation for the skeleton,

$$\mathbf{v} = \begin{bmatrix} 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 5 & 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 4 & 4 & 2 & 5 & 4 & 4 & 2 & 5 & 4 & 4 & 2 & 5 & 4 \\ 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\ 2 & 5 & 4 & 4 & 2 & 5 & 4 & 4 & 2 & 5 & 4 & 4 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

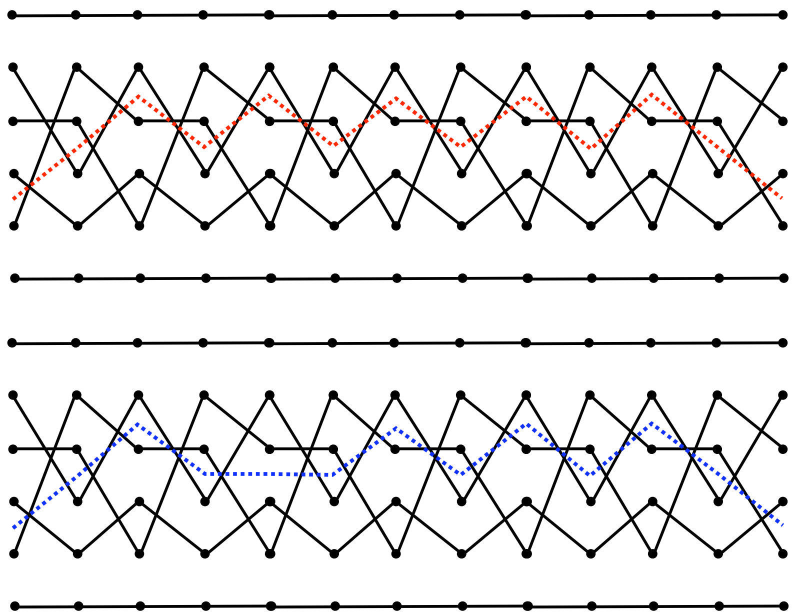


FIG. 5.2. *Two relative braids, top: (dashed) free strand $\mathbf{u}_1$, bottom: (dashed) free strand $\mathbf{u}_2$.*

and free strands

$$\mathbf{u}_1 = (2.5, 3.5, 4.5, 3.5, 4.5, 3.5, 4.5, 3.5, 4.5, 3.5, 4.5, 3.5)$$
$$\mathbf{u}_2 = (2.5, 3.5, 4.5, 3.5, 3.5, 3.5, 4.5, 3.5, 4.5, 3.5, 4.5, 3.5),$$

we first compute cubical representations for the adjacent relative braid class components $K_1 = [\mathbf{u}_1 \,|\, \mathbf{v}]_1$ and $K_2 = [\mathbf{u}_2 \,|\, \mathbf{v}]_1$. The set K_1 consists of two 12-dimensional cubes and K_2 consist of six 12-dimensional cubes. We next compute the exit faces for each class (46 for K_1 and 122 for K_2) and the corresponding relative homologies. The Poincaré polynomials for K_1 and K_2 are

$$P(K_1)(s) = s^{12} \qquad \text{and} \qquad P(K_2)(s) = s^{11}.$$

By Lemma 2.19 there is a nontrivial invariant set in each of the relative braid classes.

We now introduce the notation $\mathcal{B}_1 = [\ell\mathbf{u}_1 \# \ell\mathbf{v}]$ and $\mathcal{B}_2 = [\ell\mathbf{u}_2 \# \ell\mathbf{v}]$, and the collection $\mathcal{C} = \{\mathcal{B}_1, \mathcal{B}_2\}$, which is a pair of adjacent braid classes. In light of the definitions in Section 3 we set $\mathcal{C}_{\mathbf{v},1} = \mathrm{cl}\,(K_1 \cup K_2)$. We next compute the (164) exit faces for the union $\mathcal{C}_{\mathbf{v},1}$, and compute its index, which is trivial:

$$P\mathcal{C}_{\mathbf{v},1}(s) = 0.$$

Since the crossing number for K_1 is 36, the crossing number for K_2 is 34, and the index for their union $\mathcal{C}_{\mathbf{v},1}$ is not the sum of the indices for the individual adjacent braid classes, it follows that there is a connecting orbit with α-limit set in K_1 and ω-limit set in K_2, cf. Lemma 3.4.

Sample Result 5.2. *Given the relative braids in Figure 5.2, there are two additional relative braid classes with the same skeleton so that we have the connecting orbit structure depicted on the right for any parabolic flow (1.3) fixing the skeleton. Furthermore, the flow (1.1) also has this connecting orbit structure when considered on the spatial interval $x \in [0,6]$, provided it fixes a braid of the type depicted in Figure 5.1(b) for $x \in [0,1]$.*

$$\begin{tikzcd} & \mathcal{B}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{B}_2 \arrow[dr] & & \mathcal{B}_3 \arrow[dl] \\ & \mathcal{B}_4 & \end{tikzcd}$$

Outline of proof. Applying Lemma 3.7 to the Sample Result 5.1 yields the upper lefthand portion of the diagram, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are labeled as before. We now add the braid class components $K_3 = [\mathbf{u}_3|\mathbf{v}]_1$ and $K_4 = [\mathbf{u}_4|\mathbf{v}]_1$ where

$$\mathbf{u}_3 = (2.5, 3.5, 4.5, 3.5, 4.5, 3.5, 4.5, 3.5, 3.5, 3.5, 4.5, 3.5)$$
$$\mathbf{u}_4 = (2.5, 3.5, 4.5, 3.5, 3.5, 3.5, 4.5, 3.5, 4.5, 3.5, 4.5, 3.5).$$

The corresponding topological braid classes are $\mathcal{B}_3 = [\ell\mathbf{u}_3 \# \ell\mathbf{v}]$ and $\mathcal{B}_4 = [\ell\mathbf{u}_4 \# \ell\mathbf{v}]$.

The class K_3 consists of six 12-dimensional cubes with 122 exit faces, and K_4 consists of eighteen 12-dimensional cubes with 284 exit faces. The Poincaré polynomials of K_3 and K_4 are

$$P(K_3)(s) = s^{11} \qquad \text{and} \qquad P(K_4)(s) = s^{10}.$$

There are three more pairs of adjacent braid classes $\mathcal{C}' = \{\mathcal{B}_1, \mathcal{B}_3\}$, $\mathcal{C}'' = \{\mathcal{B}_2, \mathcal{B}_4\}$ and $\mathcal{C}''' = \{\mathcal{B}_3, \mathcal{B}_4\}$, with discretized braid class components $\mathcal{C}'_{\mathbf{v},1} = \mathrm{cl}\,(K_1 \cup K_3)$, $\mathcal{C}''_{\mathbf{v},1} = \mathrm{cl}\,(K_2 \cup K_4)$ and $\mathcal{C}'''_{\mathbf{v},1} = \mathrm{cl}\,(K_3 \cup K_4)$, which all have trivial index:

$$P\mathcal{C}'_{\mathbf{v},1}(s) = P\mathcal{C}''_{\mathbf{v},1}(s) = P\mathcal{C}'''_{\mathbf{v},1}(s) = 0.$$

Lemma 3.7 now implies the depicted connecting orbit structure.

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