Contact and noncontact type Hamiltonian systems generated by second order Lagrangians

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ABSTRACT. We show that the energy manifolds that are induced by second order Lagrangians, i.e. Lagrangrians of the form $L = L(u, u', u'')$, are in general not of contact type in $(\mathbb{R}^4, \omega)$. We also comment on the more general question whether there exist any contact forms on these energy manifolds for which the associated Reeb vector field coincides with the Hamiltionian vector field.

1. Prologue

The energy manifold $\mathrm{M} = H^{-1}(0)$[1] of a smooth Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$ is said to be of *contact type* in $(\mathbb{R}^{2n}, \omega)$ if there exists a one-form θ on M such that

$$d\theta = -j^*(\omega),$$

and

$$i_{X_H}\theta \neq 0$$

hold on M. Here $\omega = \sum dq_i \wedge dp_i$ is the usual symplectic form on $\mathbb{R}^{2n}$, $j : \mathrm{M} \to \mathbb{R}^{2n}$ is the inclusion map, and $X_H = (H_p, -H_q)$ is the Hamiltonian vector field on M.[2]

Such a contact form θ can always be written as $\theta = \theta_0 + \beta$, where $\theta_0 = \mathbf{p}d\mathbf{q} = \sum p_i dq_i$ is the *canonical form* (or *standard Liouville form*), and β is any closed one-form on M.

The contact type condition first appeared in the Weinstein conjecture: *"Every compact energy manifold* M *of contact type carries a closed orbit of the Hamiltonian vector field X_H"*, see [**16**]. This conjecture was later proved by Viterbo [**15**]. If M is of contact type then the powerful techniques involving pseudoholomorphic curves can be used to study the periodic orbits of X_H on M, and in more general settings, see [**9**]. Without the contact type condition counterexamples of energy manifolds in $(\mathbb{R}^{2n}, \omega)$ which contain no periodic orbits have been given by Ginzburg [**4, 5**] and Hermann [**7**] for dimensions $2n \geq 6$.

[1] Throughout this note we assume the energy manifolds to be regular, i.e. $dH \neq 0$ on M.
[2] The same definition can be given for arbitrary symplectic manifolds (M^{2n}, ω).

Although many Hamiltonian systems from differential geometry and classical mechanics are known to have all their energy manifolds of contact type, there does not seem to be a clear procedure for deciding if any given energy manifold $\mathrm{M} \subset \mathbb{R}^{2n}$ is or is not of contact type. In this paper we address this issue for Hamiltonian systems on $\mathbb{R}^4$ which arise from second order variational problems of the type

$$\delta \int L(u, u', u'')dt = 0. \tag{1}$$

Given any vector field X on an odd dimensional manifold M (such as X_H on M), one may also ask the more general question "is there a contact form λ on M such that the vector field X (after renormalization) is the Reeb vector field of λ?" If M is of contact type, then the answer is "yes," and one can simply put $\lambda = \theta$. The question "is M of contact type" is more restrictive since the symplectic form $d\lambda$ corresponding to the contact form one looks for is prescribed (it should be $j^*\omega$.)

1.1. Second order Lagrangians. A second order Lagrangian is a function of the form $L = L(u, u', u'')$, where $u = u(t)$ is a scalar function on $\mathbb{R}$. Such a Lagrangian is assumed to be convex in the u''-variable. The Euler-Poisson[3] equation for the variational problem (1) is a fourth order ODE, and is given by

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial u''} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0. \tag{2}$$

This ODE is equivalent to a Hamiltonian system on $(\mathbb{R}^4, \omega)$ with Hamiltonian function given by

$$H = p_u v + L^*(u, v, p_v). \tag{3}$$

Here $L^*(u, v, p_v) = \sup_{w\in\mathbb{R}} p_v w - L(u, v, w)$ is the Legendre transform of $L(u, v, w)$ with respect to third variable. The correspondence between the canonical coordinates $x = (u, v, p_u, p_v)$ and u and its derivatives is as follows: $v = u'$, $p_u = -\frac{d}{dt}\frac{\partial L}{\partial u''} + \frac{\partial L}{\partial u'}$, and $p_v = \frac{\partial L}{\partial u''}$. In §3.1 this procedure is explained in more detail for general higher order Lagrangians. See also [**1**].

Due to the $p_u v$ term in the definition of the Hamiltonian (3) the energy manifolds $\mathrm{M} = H^{-1}(0)$ are always *non-compact*.[4] The particular question we shall be concerned with is whether or not the zero energy manifold M is of contact type in general. The following result identifies a large class of second order Lagragians for this is indeed the case.

THEOREM 1.1. *If $L(u, v, w) > 0$ for all $(u, v, w) \in \mathbb{R}^3$, then* M *is of contact type and the canonical form $\theta_0 = p_u du + p_v dv$ is a contact form on* M.

This holds in much greater generality for Hamiltonian systems coming from variational problems of the type $\delta \int L(u, u', \ldots, u^{(n)})dt = 0$, where the "Lagrangian" L is strictly positive. See §3.1 for the proof.

[3] These are the equations obtained by requiring the first variation of the action to vanish along all possible variations $u \mapsto u + \epsilon\varphi$. In [**1**] these equations are called the Euler-Poisson equations rather than the Euler-Lagrange equations.

[4] We may restrict to just the zero energy manifolds since all other energy manifolds may be obtained by simply replacing L by $L + E$.

We continue now with a special class of Lagrangians which are not necessarily positive, and we show that among these there are Lagrangians for which M is not of contact type.

1.2. The Swift-Hohenberg & eFK models. As a special case one considers Lagrangians of the form

$$L(u,v,w) = \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 + F(u). \tag{4}$$

The associated Hamiltonian is then given by $H(x) = p_u v + \frac{1}{2}p_v^2 - \frac{\alpha}{2}v^2 - F(u)$, and the Euler-Poisson equation for the variational problem is fourth order;

$$\frac{d^4u}{dt^4} - \alpha\frac{d^2u}{dt^2} + F'(u) = 0.$$

For various different choices of the "potential" $F(u)$ and parameter α, this equation is known in the mathematical physics literature as extended Fisher Kolmogorov (eFK) or Swift-Hohenberg equation. See [**12**], [**14**, introduction] and the references given there.

Clearly, if the potential F is positive and the parameter $\alpha \geq 0$, then Theorem 1.1 implies that M is of contact type. However, for potentials that are not strictly positive, or for negative values of the paramter α, the question becomes more delicate, and the geometry and topology of M will come into play. The next three theorems summarize cases where M is again of contact type, but L is not necessarily strictly positive.

THEOREM 1.2. *If $\alpha \geq 0$ and $F(u)$ only has simple zeroes, then* M *is of contact type, with $\theta = \theta_0 + d(vp_v) + \beta$ as contact form, and where β is a closed form which can be chosen arbitrarily small in C^∞.*

The proof is given in §4.

In §5 we recall that for certain choices of F the flow on M admits a global Poincaré section. This additional structure also allows one to contruct contact forms compatible with ω, and is useful for the study of periodic orbits of X_H on M.

THEOREM 1.3. [5] *If $F(u) > 0$ and $F(u)/u^2 \to \infty$ as $u \to \pm\infty$, then there exists a diffeomorphism $\Lambda : \mathrm{M} \to \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ which carries trajectories of X_H to solutions of a nonautonomous planar Hamiltonian system.*

If in addition $\alpha < 0$ then the growth condition on $F(u)$ is not needed, and one can exhibit an explicit diffeomorphism Λ.

Here a nonautonomous planar Hamiltonian system refers to the case in which $\mathrm{M} = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ with coordinates $(p,q) \in \mathbb{R}^2$, $t \in (\mathbb{R}/\mathbb{Z})$ and $\lambda = pdq - H(p,q,t)dt$. Such a system is not necessarily of contact type as we show in an example in §5.3. The following is therefore not an immediate corollary of Theorem 1.3.

THEOREM 1.4. *If $F(u) > 0$ and if $F'(u)/u \to \infty$ as $u \to \pm\infty$, then* M *is of contact type.*

[5] Poincaré sections can be found in much more general settings for second order Lagrangians. In this theorem we chose to restrict the details to the Swift-Hohenberg & eFK models for simplicity. See §5 and [**13**] for a more detailed account on this subject.

1.3. Non-contact type energy manifolds. So far we have only given sufficient conditions for M to be of contact type. These conditions do not cover all cases, in particular the situation in which $F(u)$ changes sign and $\alpha < 0$ is not covered. In §8.4 we show that the energy manifold M is in fact not always of contact type in $(\mathbb{R}^4, \omega)$.

THEOREM 1.5. *There exists a potential* $F : \mathbb{R} \to \mathbb{R}$ *and an* $\alpha_* < 0$ *such that for all* $\alpha < \alpha_*$ *the energy manifold* M *is not of contact type.*

In §8.4 we describe for which shapes of F the above theorem holds. This leads to a very large class of noncontact type energy manifolds. The problem of finding energy manifold which are not of contact type has been approached from a different perspective in [**3**].

All this leaves us with a few unanswered questions of which we mention two.

Q1. For the Swift-Hohenberg equation with $F(u) = \frac{1}{4}(1-u^2)^2 + E$ (a usual choice in physics models) for $E < 0$ and $\alpha < 0$ we still do not know if M is of contact type or not, i.e. is the hypersurface in $\mathbb{R}^4$ defined by

$$p_u v + \frac{1}{2}p_v^2 - \frac{\alpha}{2}v^2 - \frac{1}{4}\left(1-u^2\right)^2 - E = 0$$

of contact type?

Q2. If one chooses $F(u)$ as in Theorem 1.5, then for sufficiently large negative α the hypersurface M is not of contact type, while for $\alpha > 0$ it is of contact type. How does the transition from contact to noncontact type take place as α decreases from $+\infty$ to $-\infty$?

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2. Representation of X_H by an arbitrary Reeb vectorfield

Any one-form θ on a $(2n-1)$ dimensional manifold defines a variational problem for closed immersed curves $\gamma \subset$ M in which one requires the *action*

$$A(\gamma) \stackrel{\text{def}}{=} \int_\gamma \theta,$$

to be stationary. If X is any vector field along γ then the variation of the action by X is

$$dA(\gamma) \cdot X = \int_\gamma i_X d\theta = \int_0^1 d\theta(X(t), \gamma'(t))dt.$$

Hence the action will be stationary at γ if and only if $i_{\gamma'(t)}d\theta = 0$ holds. See [**2**].

We will call the one-form θ *nondegenerate* if $d\theta$ has maximal rank everywhere. If θ is nondegenerate then

$$\mathfrak{L}_p = \ker d\theta \stackrel{\text{def}}{=} \{Y \in T_p\mathrm{M} \mid i_Y d\theta = 0\}\,, \tag{5}$$

is one dimensional for all $p \in \mathrm{M}$ and thus defines a linebundle (direction field) on M. In this case the stationary curves for $A(\gamma)$ are precisely the integral curves for the line bundle $\mathfrak{L}$. The nondegeneracy condition for θ is the analog for the variational problem $\delta \int_\gamma \theta = 0$ to the Legendre-Hadamard condition from the calculus of variations.

If the manifold M is oriented then $\mathfrak{L}$ has a nowhere vanishing section.[6] We will call any positively oriented section of the line bundle $\mathfrak{L}$ a *pseudo Reeb vector field* for the one-form θ. Critical points of the action $A(\gamma)$ are then closed orbits of any pseudo-Reeb vector field for θ.

If the form θ is a contact form, i.e. if $\theta \wedge (d\theta)^{n-1} \neq 0$ everywhere, then $d\theta$ clearly has maximal rank everywhere, so contact forms are nondegenerate. For a contact form θ there is a chosen section X of $\mathfrak{L}$, defined by $i_X\theta \equiv 1$. This pseudo Reeb vector field is called the *Reeb vector field* of θ. Conversely, if θ is nondegenerate and if there is a pseudo Reeb vector field X such that $i_X\theta > 0$, then θ is a contact form.

For Hamiltonian systems in $\mathbb{R}^{2n}$, i.e. if $\mathrm{M} = H^{-1}(0)$ with 0 a regular value of H, the form $\theta_0 = \mathbf{p} \cdot d\mathbf{q}$ is nondegenerate, and the Hamiltonian vector field X_H is a pseudo-Reeb vector field for θ_0 on M.

The energy surface M will be of contact type if one can find a closed form β on M such that $\theta_0 + \beta$ is a contact form. The Reeb vector field for $\theta_0 + \beta$ is then a multiple of the Hamiltonian vector field X_H. The more general question that can be asked is: *"Does there exist* ***any*** *contact form θ, such that the Hamiltonian vector field X_H is a pseudo Reeb vector field for θ?* This question leaves more freedom in choosing θ since the condition $d\theta = -j^*(\omega)$ is omitted.

Although we do not give any positive or negative results on the more general question the following observations seem to indicate that the situation in which a Hamiltonian vector field X_H is a Reeb vectorfield for a contact form λ with $d\lambda \neq cj^*\omega$, for any constant $c \neq 0$, is unusual.

LEMMA 2.1. *If X_H is a pseudo Reeb vector field for a contact form λ on* M*, then $d\lambda = fj^*\omega$ for some smooth nowhere vanishing function $f : \mathrm{M} \to \mathbb{R}$. Furthermore, this function f is a conserved quantity for X_H: $X_H(f) = 0$.*

There is a unique vectorfield Y on M *which satisfies $i_Y(d\lambda) = df$, and $i_Y\lambda = 0$.*

Let X be the Reeb vectorfield of λ (i.e. X is a multiple of X_H which satisfies $\lambda(X) = 1$). Then the vectorfields X and Y commute: $[X, Y] = 0$.

Wherever $Y \neq 0$ the vector fields X and Y are linearly independent.

[6] Indeed, at each point $p \in \mathrm{M}$ the quotient $T_p\mathrm{M}/\mathfrak{L}_p$ is oriented by the volume form $(d\theta)^{n-1}$, while the tangent space $T_p\mathrm{M}$ is assumed to have an orientation. These two orientations induce an orientation on $\mathfrak{L}_p$.

PROOF. Since both $i_{X_H} j^*\omega = 0$ and $i_{X_H} d\lambda = 0$ we have $\ker j^*\omega = \ker d\lambda$. Let $\ker \omega = \operatorname{span}(X)$, and write $\xi = \xi_1 X + \xi_2 Y + \xi_3 Z$, and $\eta = \eta_1 X + \eta_2 Y + \eta_3 Z$, where $\{X, Y, Z\}$ is a basis for TM, normalized by $\omega(Y, Z) = 1$. Write $\widetilde{\omega} = d\lambda$. Then $\widetilde{\omega}(\xi, \eta) = (\xi_2\eta_3 - \xi_3\eta_2)\widetilde{\omega}(Y, Z)$, and $\omega(\xi, \eta) = (\xi_2\eta_3 - \xi_3\eta_2)\omega(Y, Z) = \xi_2\eta_3 - \xi_3\eta_2$. Thus we find $d\lambda = f j^*\omega$, with $f = \widetilde{\omega}(Y, Z)$.

Since $f j^*\omega = d\lambda$ is closed it follows that $df \wedge j^*\omega = 0$, and thus

$$0 = i_{X_H}(df \wedge j^*\omega) = (i_{X_H} df) j^*\omega - df \wedge i_{X_H} j^*\omega = (i_{X_H} df) j^*\omega,$$

which implies that $i_{X_H} df = 0$. The function f is therefore an integral of the vector field X_H.

Since $\ker df = \ker d\lambda$, a vector Y exists such that $df(Z) = d\lambda(Y, Z)$ for all $Z \in T_p$M. Two different choices of Y differ by an element of $\ker d\lambda$ most one Y can satisfy $\lambda(Y) = 0$.

If X is the Reeb vectorfield for λ, then $\mathcal{L}_X\lambda = 0$. Hence $\lambda(\mathcal{L}_X(Y)) = \mathcal{L}_X(\lambda(Y)) = 0$ so that $[X, Y] = \mathcal{L}_X Y$ belongs to $\ker\lambda$.

One also has, using $\mathcal{L}_X\lambda = 0$ and $i_Y d\lambda = df$

$$i_{\mathcal{L}_X Y} d\lambda = \mathcal{L}_X(i_Y d\lambda) - i_Y d\mathcal{L}_X\lambda = \mathcal{L}_X(df) = d\mathcal{L}_X f = 0.$$

Since $\mathcal{L}_X Y$ belongs both to $\ker\lambda$ and to $\ker d\lambda$, we have $\mathcal{L}_X Y = 0$, and hence X and Y commute.

If $Y \neq 0$ at some $p \in$ M then at that point one has $i_Y d\lambda = df \neq 0$ and $i_X d\lambda = 0$, so that X and Y cannot be linearly dependent at p. □

It is well known that a second integral of the motion can severly restrict the possible dynamics of X_H on M. For instance, any periodic orbit of X on which $Y \neq 0$ must appear in a family of periodic orbits of X. Also, for $n = 2$ if $f : \mathrm{M} \to \mathbb{R}$ has a compact regular level surface $S = f^{-1}(c)$, then this must be a 2-torus, and the flow of X_H will be the standard linear flow. By the implicit function theorem the same will apply to $S' = f^{-1}(c')$ for c' close to c, so that an open subset of M is foliated by invariant tori with linear flow.

A typical Poincaré plot (see Fig. 1) for the Swift-Hohenberg equation shows none of these phenomena, suggesting that generally X_H will only be a pseudo-Reeb vectorfield for one particular form λ.

3. Generalities about finding contact forms

3.1. Positive Lagrangians. As explained in [**1**], the Euler-Lagrange-Poisson equations of a general n^{th} order Lagrangian variational problem of the type

$$\delta \int L(u, u', u'', \ldots, u^{(n)})\, dt = 0 \tag{6}$$

whose "Lagrangian" $L = L(u_0, u_1, \ldots, u_n)$ is strictly convex in the highest derivative u_n, can be transformed to a Hamiltonian system on the cotangent bundle of the space J^{n-1} of $(n-1)$-jets of functions $u : \mathbb{R} \to \mathbb{R}$. Alternatively, one writes

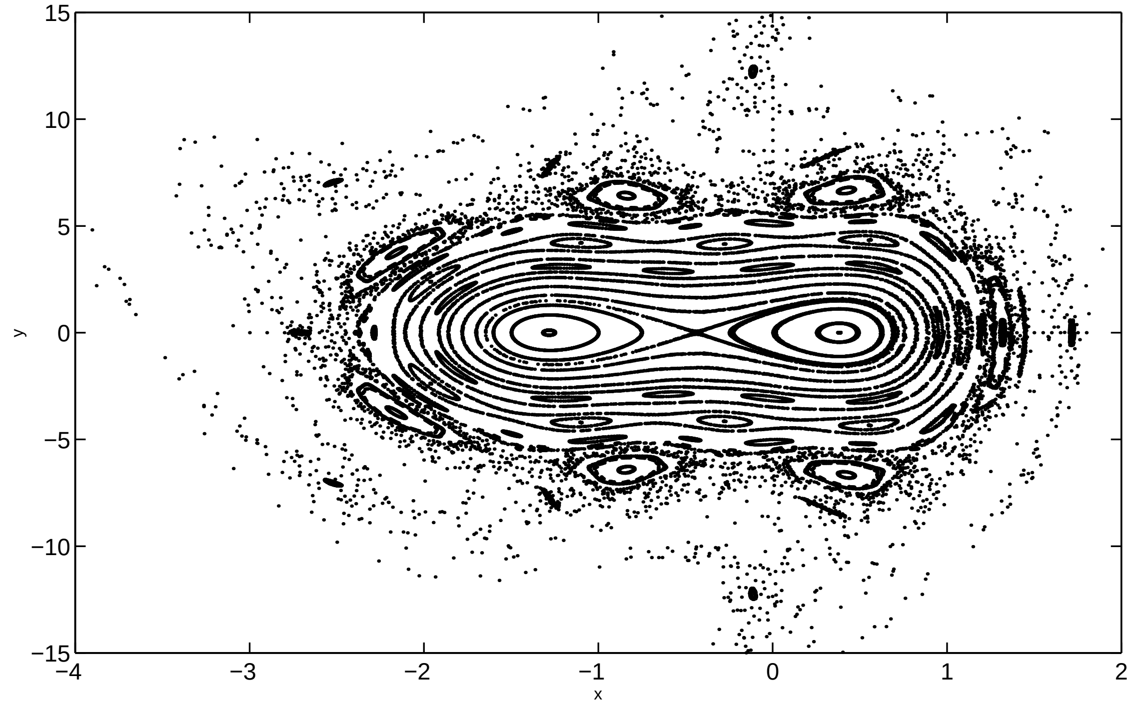


FIGURE 1. A Poincaré plot for parameter values $\alpha = -10$ and $E = 10$. Reproduced from [**14**]

$u_0, \ldots, u_n$ for the first n derivatives of u and introduces variables $p_0, \ldots, p_{n-1}$, where the last is related to $u_0, \ldots, u_n$ by

$$p_{n-1} = \frac{\partial L(u_0, \ldots, u_n)}{\partial u_n}, \tag{7}$$

due to the assumption that $\partial^2_{u_n} L > 0$. One then defines the Legendre transform by

$$L^*(u_0, u_1, \ldots, u_{n-1}, p_{n-1}) = \sup_{w \in \mathbb{R}} \{p_{n-1}w - L(u_0, u_1, \ldots, u_{n-1}, w)\}$$

and "Hamiltonian" by

$$\begin{aligned} H(u_0, \ldots, u_{n-1}, p_0, \ldots, p_{n-1}) &= p_0 u_1 + \cdots + p_{n-2} u_{n-1} \\ &\quad + L^*(u_0, u_1, \ldots, u_{n-1}, p_{n-1}). \end{aligned}$$

The Hamilton equations for H give the extremals for (6).

LEMMA 3.1. *If one defines* M *to be the zero energy manifold* $H^{-1}(0) \subset \mathbb{R}^{2n}$ *of* H, *and if the Lagrangian* L *is strictly positive, then* M *is of contact type with the standard contact form* $\theta_0 = p_0 du_0 + \cdots + p_{n-1} du_{n-1}$.

PROOF. Let p_{n-1} and u_n be related by (7), then

$$u_n = \frac{\partial L^*(u_0, \ldots, u_{n-1}, p_{n-1})}{\partial p_{n-1}}$$

and $L + L^* = p_{n-1}u_n$. One has $i_{X_H}\theta_0 = p_0 u'_0 + \cdots + p_{n-1}u'_{n-1}$, and from the Euler-Lagrange-Poisson equations one finds that $u'_i = u_{i+1}$ for $i < n-1$ and $u'_{n-1} = \frac{\partial L^*}{\partial p_{n-1}} = u_n$. Therefore

$$\begin{aligned} i_{X_H}\theta_0 &= p_0 u_1 + \cdots + p_{n-2}u_{n-1} + p_{n-1}\frac{\partial L^*}{\partial p_{n-1}} \\ &= p_0 u_1 + \cdots + p_{n-2}u_{n-1} + L + L^* \\ &= H + L. \end{aligned}$$

Since $H = 0$ on M we find $i_{X_H}(\theta_0) = L > 0$ on M, so that θ_0 is a contact form on M, whose Reeb vector field coincides with X_H. □

This Lemma, when applied to second order Lagrangians, proves Theorem 1.1.

3.2. A criterion of Hofer and Zehnder. In [**8**] an example of a hypersurface is given which is not of contact type. From this example we may distill the following criterion. Let $\mathrm{M} = H^{-1}(0)$ be an energy manifold of a Hamiltonian system on $(\mathbb{R}^{2n}, \omega)$, with ω the standard symplectic form.

LEMMA 3.2. *Given two contractable periodic orbits $\gamma_1, \gamma_2 \subset \mathrm{M}$ of X_H for which the standard contact form θ_0 yields actions of opposite signs. Then one may conclude that* M *is not of contact type.*

PROOF. Any possible contact form θ is of the form $\theta = \theta_0 + \beta$, with $d\beta = 0$.

Represent the curves γ_i as boundaries of discs, $\gamma_i = \partial\Delta_i$, $\Delta_i \subset \mathrm{M}$. Then by Stokes' theorem one has $\oint_{\gamma_i}\theta = -\int_{\Delta_i}\omega = \oint_{\gamma_i}\theta_0$. If θ were a contact form then both actions $\oint_{\gamma_i}\theta$ would have the same sign, and hence both integrals $\oint_{\gamma_i}\theta_0$ would also have the same sign, a contradiction. □

3.3. Fixing almost contact forms. In [**8**] Hofer and Zehnder show that all energy manifolds of any classical mechanical system on $\mathbb{R}^{2n}$ with Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ are of contact type. They do this by first observing that the canonical form satisfies $i_X\theta_0 \geq 0$ and then perturbing the form θ_0 to achieve strict inequality. If one replaces the explicit perturbation in [**8**] by something more abstract one arrives at the following result.

LEMMA 3.3. *Let* M *be a $(2n-1)$-dimensional manifold with a nondegenerate one-form λ. Let X be a pseudo Reeb vector field for λ. Assume that $i_X\lambda \geq 0$, and also that the set $S = \{p \in \mathrm{M} \mid i_X\lambda(p) = 0\}$ satisfies*

$$\forall p \in S\ \exists t_- < 0 < t_+ \,:\, \Phi_{t_\pm}(p) \notin S.$$

Then there exists a smooth function $f : \mathrm{M} \to \mathbb{R}$ such that $\lambda^* = \lambda + \epsilon df$ is a contact form on M *for all $\epsilon \in (0,1)$, and for which X is a pseudo Reeb vector field ($i_X d\lambda^* = 0$).*

PROOF. For any given $p \in S$ we choose a parametrized $2n-2$ ball $\sigma_0 : B^{2n-2} \to \mathrm{M}$ which is transverse to the vector field X. Then $\sigma(x_0, x_1, \ldots, x_{2n-2}) = \Phi^{x_0}(\sigma_0(x_1, \ldots, x_{2n-2}))$ is a local diffeomorphism which straightens the flow, i.e. it maps lines parallel to the x_0 axis to flow lines of the vector field X, and it maps $\frac{\partial}{\partial x_0}$ to X. It is a diffeomorphism on $B_r^{2n-2} \times [t_-, t_+]$ if $r > 0$ is small enough.

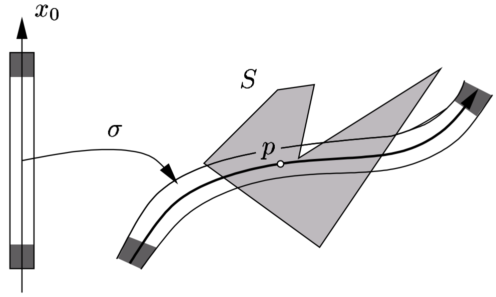


FIGURE 2. Proof of Lemma 3.3

Since $\Phi^{t_\pm}(p)$ does not belong to the closed set S one can choose $r > 0$ so small that $\sigma(x_0, \ldots, x_{2n-2})$ does not lie in S if $x_1^2 + \cdots + x_{2n-2}^2 \leq r^2$ and $x_0 \in [t_-, t_- + r] \cup [t_+ - r, t_+]$. We also choose r so small that $t_- + r < 0 < t_+ - r$. Now let $0 \leq \eta \in C^\infty(\mathbb{R})$ satisfy $\eta(x_0) = 0$ for $x_0 \leq t_-$ and $x_0 \geq t_+$, as well as $\eta(x_0) = 1$ for $x_0 \in [t_- + r, t_+ - r]$. In addition we pick a $0 \leq \zeta \in C^\infty(\mathbb{R}^{2n-2})$, which is supported in B_r^{2n-2}, and with $\zeta(0) > 0$. We then define $f : \mathrm{M} \to \mathbb{R}$ by $f \circ \sigma(x_0, \ldots, x_{2n-2}) = x_0\eta(x_0)\zeta(x_1, \ldots, x_{2n-2})$ on $\sigma(B_r^{2n-2} \times [t_-, t_+])$ and $f = 0$ elsewhere.

Using

$$i_X df = \frac{\partial x_0\eta(x_0)\zeta(x_1, \ldots, x_{2n-2})}{\partial x_0}$$

one now easily verifies that $i_X df \geq 0$ on S, and $i_X df > 0$ at p and by continuity in a neighborhood N_p of p. One also sees that $i_X df < 0$ only in $\sigma([t_-, t_- + r] \times B_r^{2n-2}) \cup \sigma([t_+ - r, t_+] \times B_r^{2n-2})$, i.e. $i_X df < 0$ outside of some neighborhood of S.

Let $\{p_k\}_{k\in\mathbb{N}}$ be a sequence of points for which the neighborhoods N_{p_i} cover S. Denote the functions obtained by the above construction by f_k. Then for each $k \in \mathbb{N}$ the quantity

$$A_k = \sup_{\mathrm{M}\setminus S} \frac{\left(-i_X df_k\right)_+}{i_X \lambda}$$

is finite, since $i_X df_k > 0$ on a neighborhood of S. We define

$$F = \sum_{k\in\mathbb{N}} \frac{2^{-k} f_k}{A_k + \|f_k\|_{C^k}}.$$

This series converges in C^∞. Its sum satisfies $i_X dF > 0$ on S and

$$\begin{aligned} -i_X dF &\leq \sum_{k\in\mathbb{N}} 2^{-k} \frac{(-i_X df_k)_+}{A_k} \\ &\leq \sum_{k\in\mathbb{N}} 2^{-k} i_X\lambda = i_X\lambda \end{aligned}$$

on $\mathrm{M} \setminus S$. Hence $i_X(\lambda + \epsilon dF) > 0$ on all of M for all $0 < \epsilon < 1$, as claimed. □

3.4. Fixing contact forms without recurrence. Our proof of Theorem 1.4 is based on the following general observation.

LEMMA 3.4. *Let* M *be* $(2n-1)$*-dimensional manifold with a nondegenerate one-form* λ*, and let* X *be a pseudo Reeb vector field for* λ*. If there is a largest compact invariant set* $K \subset \mathrm{M}$ *for the flow of* X*, and if* $i_X\lambda > 0$ *on* K*, then a function* $f \in C^\infty(\mathrm{M})$ *exists for which* $\lambda^* = \lambda + df$ *is a contact form and for which* X *is a pseudo Reeb vector field* ($i_X d\lambda^* = 0$)*.*

PROOF. Let $K = K_0 \subset K_1 \subset K_2 \cdots$ be a sequence of compact subsets of M such that K_i is in the interior of K_{i+1} and $\cup_{i\in\mathbb{N}} K_i = \mathrm{M}$. By induction we will construct a sequence of functions $f_i \in C^\infty(\mathrm{M})$ such that

$$\left\{\begin{array}{l}\text{the one-forms } \lambda_0 = \lambda,\ \lambda_i = \lambda_{i-1} + df_i\ (i \geq 1) \text{ are contact}\\ \text{forms on } K_i, \text{ and } f_i \equiv 0 \text{ on a neighborhood of } K_{i-1}.\end{array}\right. \tag{8}$$

The functions f_j with $j \geq i+1$ then all vanish on K_i, so that $f = d\Sigma_{i\in\mathbb{N}} f_i$ is well-defined, and $\lambda_* = \lambda + df$ is a contact-form.

Let $f_0, \ldots, f_i$ be constructed, so that λ_i is a contact form on K_i, i.e. $i_X(\lambda_i) > 0$ on K_i. By continuity there is an open set $K_i \subset U_i \subset K_{i+1}$ such that $i_X(\lambda_i) > 0$ on U_i.

PROPOSITION 3.5. *For each* $p \in K_{i+1} \setminus U_i$ *a function* $g_p \in C_c^\infty(\mathrm{M})$ *exists such that supp* $g_p \subset \mathrm{M} \setminus K_i$, $i_X dg_p \geq 0$ *everywhere, and* $i_X dg_p > 0$ *in* p*.*

PROOF. Let $p \in K_{i+1} \setminus U_i$ be given. The orbit $\Gamma_p = \{\Phi^t(p) \mid t \in \mathbb{R}\}$ cannot be contained in K_{i+1}, for if it were, then the closure of Γ_p would be a compact invariant set containing points outside of $K = K_0$, thereby contradicting our assumption that K is the largest compact invariant set. Hence for some $t_+ \neq 0$ one has $\Phi^{t_+}(p) \notin K_{i+1}$. Without loss of generality we assume that $t_+ > 0$. It is also impossible for the half-orbit $\Gamma_p^- = \Phi^{(-\infty,0]}(p)$ to be contained in $K_{i+1} \setminus U_i$. If this were to happen, then the α-limit of Γ_p^- would be a compact invariant subset of the flow outside of K_0. Thus a $t_- < 0$ exists with $\Phi^{t_-}(p) \in U_i$ or $\Phi^{t_-}(p) \in \mathrm{M} \setminus K_{i+1}$. In either case we can choose t_- so that $\Phi^{[t_-,0]}(p)$ is disjoint from K_i.

We straighten the flow in a neighborhood of the orbit segment $\Phi^{[t_-,t_+]}(p)$. To this end choose a smooth immersion $\sigma : \mathbb{R}^{2n-2} \to \mathrm{M}$ with $\sigma(0) = p$ and which meets the orbit Γ_p transversally at this point. Then let $\phi : [t_-, t_+] \times \mathbb{R}^{2n-2} \to \mathrm{M}$ be given by

$$\phi(x_0, x_1, \ldots, x_{2n-2}) = \Phi^{x_0}[\sigma(x_1, \ldots, x_{2n-2})].$$

For sufficiently small $r > 0$ this map is a diffeomorphism from $[t_-, t_+] \times B_r^{2n-2}$ into M.

Since $\phi(t_+, 0, \ldots, 0) = \Phi^{t_+}(p) \notin K_{i+1}$ we can choose $\epsilon, r > 0$ so small that $\phi([t_+ - \epsilon, t_+] \times B_r^{2n-2})$ is disjoint from K_{i+1}.

Choose $\eta \in C_c^\infty(t_-, t_+)$ and $\zeta \in C_c^\infty(B_r^{2n-2})$ and define g_p by requiring

$$g_p(\phi(x_0, x_1, \ldots, x_{2n-2})) = \eta(x_0)\zeta(x_1, \ldots, x_{2n-2}),$$

on the tubular neighborhood $\phi([t_-, t_+] \times B_r^{2n-2})$, and $g \equiv 0$ elsewhere. One then has $i_X dg_p = \eta'(x_0)\zeta(x_1, \ldots, x_{2n-2})$. We now choose $\zeta \geq 0$ and $\eta'(x_0) \geq 0$ for

$t_- \leq x_0 \leq t_+ - \epsilon$, so that $i_X dg_p \geq 0$ on K_{i+1}. Finally we also choose ζ and η so that $\zeta(0,\ldots,0) > 0$ and $\eta'(0) > 0$, i.e. so that $i_X dg_p > 0$ at p. □

We complete the proof of Lemma 3.4. For each $p \in K_{i+1} \setminus U_i$ a $k_p > 0$ exists such that $i_X(\lambda_i + k_p dg_p) > 0$ at p. By continuity this als holds on some small neighborhood N_p of p. A finite number of such neighbourhoods $N_{p_1}, \cdots, N_{p_m}$ cover $K_{i+1} \setminus U_i$ and the function $f_{i+1} = \Sigma_j k_{p_j} g_{p_j}$ satisfy our requirements in (8). □

4. Proof of Theorem 1.2

We try one-forms of the form

$$\begin{aligned}\lambda &= \theta_0 + \beta_1 d(up_u) + \beta_2 d(vp_v)\\ &= (\beta_1+1)p_u du + \beta_1 u dp_u + (\beta_2+1)p_v dv + \beta_2 v dp_v\end{aligned}$$

on $\mathrm{M} = H^{-1}(0)$, where $H = p_u v + L^*(u,v,p_v)$.

We then have

$$\begin{aligned}i_{X_H}\lambda &= (\beta_1+1)p_u u' + \beta_1 u p_u' + (\beta_2+1)p_v v' + \beta_2 v p_v'\\ &= (\beta_1+1)p_u v + \beta_1 u\frac{\partial L}{\partial u} + (\beta_2+1)p_v\frac{\partial L^*}{\partial p_v} + \beta_2 v\left(-p_u + \frac{\partial L}{\partial v}\right)\\ &= (\beta_1-\beta_2+1)p_u v + \beta_1 u\frac{\partial L}{\partial u} + \beta_2 v\frac{\partial L}{\partial v} + (\beta_2+1)w\frac{\partial L}{\partial w}\\ &= -(\beta_1-\beta_2+1)L^* + \beta_1 u\frac{\partial L}{\partial u} + \beta_2 v\frac{\partial L}{\partial v} + (\beta_2+1)w\frac{\partial L}{\partial w}\\ &= (\beta_1-\beta_2+1)L + \beta_1 u\frac{\partial L}{\partial u} + \beta_2 v\frac{\partial L}{\partial v} + (-\beta_1+2\beta_2)w\frac{\partial L}{\partial w}.\end{aligned}$$

If $(\beta_1-\beta_2+1)L + \beta_1 u L_u + \beta_2 v L_v + (-\beta_1+2\beta_2)wL_w \geq 0$, for some constants β_1 and β_2, then we see that $i_X\lambda \geq 0$ everywhere. This generalizes the calculation of §3.1, where $\beta_1 = \beta_2 = 0$. Different choices of β_i lead to different "starshapedness-like" conditions on L. For instance, if one puts $\beta_1 = \beta_2 = \gamma^{-1} > 0$, then one finds that

$$\gamma L + uL_u + vL_v + wL_w > 0$$

for all (u,v,w), implies that λ is a contact form and that M is of contact type.

We will apply this more general criterion now to the Lagrangians defined by (4). Taking $\beta_1 = 0$ and $\beta_2 = 1$, it follows that $2wL_w + vL_v = 2w^2 + \alpha v^2 \geq 0$ for all $\alpha \geq 0$. In order to find a contact form we need to examine the zero set $\{p \in \mathrm{M} \mid i_{X_H}\lambda(p) = 0\}$ and apply Lemma 3.3.

When $\alpha > 0$ we have that $i_X\lambda = 0$ on the set $S = \{(u,0,p_u,0) \mid L(u,0,0) = 0\}$. At any point on $(u,0,p_u,0) \in S$, with $p_u \neq 0$, one has $p_v' = -p_u \neq 0$ so that all trajectories of the flow immediately leave S in forward and backward time. If $p_u = 0$ then one has

$$p_v'' = -p_u' + \alpha v' = -F_u(u) + \alpha p_v = -F_u(u).$$

At $(u,0,0,0)$ we have $p_v'' = -F_u(u) \neq 0$ and it follows again that orbits of the flow through $(u,0,0,0)$ leave S immediately in both time directions. By Lemma 3.3 we see that M is of contact type, and that one can take $\lambda + \epsilon df$ to be the contact form, for some $f \in C^\infty(\mathrm{M})$ and any $\epsilon \in (0,1)$.

If $\alpha = 0$, then the set S is larger,

$$S = \{(u, v, p_u, 0) \mid p_u v = F(u)\}.$$

On S one has $p_v' = -p_u$ so that all orbits through points on S with $p_u \neq 0$ leave S in forward and backward time. Points on S with $p_u = 0$ are of the form $(\bar{u}, v, 0, 0)$ with $F(\bar{u}) = 0$. Since F only has simple zeroes, this implies $p_u' = F_u(\bar{u}) \neq 0$, so that trajectories through $(\bar{u}, v, 0, 0)$ also leave S in forward and backward time. Again we can apply Lemma 3.3 and conclude that M is of contact type, concluding the proof of Theorem 1.2.

5. When there is a global Poincaré section

5.1. Poincaré sections and contact forms. A surface $\Sigma \subset \mathrm{M} = H^{-1}(0) \subset (\mathbb{R}^4, \omega_0)$ is called a global Poincaré section if (i) it is a closed subset of M, (ii) the Hamiltonian vector field is transverse to Σ, and (iii) for every $p \in \mathrm{M}$ there exist $t_- < 0 < t_+$ such that $\Phi^{t_\pm}(p) \in \Sigma$ (here Φ^t denotes the Hamiltonian flow on M as defined before.)

Given a global Poincaré section one defines the return time $T : \Sigma \to \mathbb{R}_+$ by $T(p) = \inf\{t > 0 \mid \Phi^t(p) \in \Sigma\}$, and the return map $\Psi : \Sigma \to \Sigma$ by $\Psi(p) = \Phi^{T(p)}(p)$. Both T and Ψ are smooth. The suspension of the return map Ψ is the space $\mathcal{B} = (\Sigma \times \mathbb{R})/\sim$, where the equivalence is given by $(\Psi(p), t) \sim (p, t+1)$. The manifolds $\mathcal{B}$ and M are diffeomorphic. The map $\psi : (p, t) \mapsto \Phi^{tT(p)}(p)$ induces a homeomorphism from $\mathcal{B}$ to M. To construct a diffeomorphism one must choose a C^∞-function $\tau : [0,1] \times (0, \infty) \to \mathbb{R}$ with $\frac{\partial \tau(t,T)}{\partial t} > 0$ for all (t, T), $\frac{\partial \tau(t,T)}{\partial t} = 1$ for t close to 0 or 1, and $\tau(0, T) \equiv 0$, $\tau(1, T) \equiv T$. The map $\psi : \Sigma \times [0, 1] \to \mathrm{M}$ given by $\psi(p, t) = \Phi^{\tau(t,T(p))}(p)$ then induces a diffeomorphism from $\mathcal{B}$ to M.

LEMMA 5.1. *If* M *is compact and if* M *has a global section, then* M *is of contact type.*

PROOF. The form dt on $\mathcal{B}$ is closed (but not exact, since $t \in \mathbb{R}/\mathbb{Z}$ is multi-valued). Let β be the pushforward of dt under the diffeomorphism $\psi : \mathcal{B} \to \mathrm{M}$. From $\frac{\partial \psi}{\partial t} = \frac{\partial \tau}{\partial t} X_H(\Phi^{\tau(t,T(p))}(p))$ it follows that $i_{X_H}\beta = \tau_t(t, T(p))^{-1} > 0$ holds at $\Phi^{\tau(t,T(p))}(p)$. Since M is compact, $i_{X_H}\theta_0$ is bounded from below on M, where θ_0 is the standard Liouville form with $d\theta_0 = \omega_0$. For sufficiently large $k \in \mathbb{R}$ one then has $i_{X_H}(\theta_0 + k\beta) > 0$ everywhere, so that $\theta_0 + k\beta$ is a contact form for M. □

It is clear from the proof that one can get away with less than compactness. If M is non-compact then one must verify that

$$\tau_t(t, T(p)) \cdot i_{X_H}(\theta_0)\big|_{\Phi^{\tau(t,T(p))}(p)} \geq -\delta$$

for some positive constant δ independent of $(p, t) \in \Sigma \times [0, 1]$.

5.2. Representation by planar Hamiltonian systems. A planar Hamiltonian system is a system defined by a smooth Hamiltonian $H(p,q,t)$ on $\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})$. Its orbits satisfy the variational problem $\delta\int pdq - H(p,q,t)dt = 0$, i.e. they satisfy $\delta\int\theta = 0$, where θ is the one-form $\theta = pdq - Hdt$. We continue the discussion of §5.1, assuming from here on that the section Σ is diffeomorphic to $\mathbb{R}^2$.

Any isotopy $\{\Psi^s : 0\le s\le 1\}$ of the return map Ψ to the identity, induces a diffeomorphism of $\mathcal{B}$ (and hence M) with $\Sigma\times(\mathbb{R}/\mathbb{Z})$. By reparametrizing the s variable we may assume that $\Psi^s(p)$ does not depend on s for $0\le s\le\frac{1}{3}$ and for $\frac{2}{3}\le s\le 1$. We can then extend the isotopy Ψ^s to all $s\in\mathbb{R}$ by requiring $\Psi^{s+1} = \Psi^s\circ\Psi$. The map $\eta:\Sigma\times\mathbb{R}\to\Sigma\times\mathbb{R}$, $\eta(p,t) = (\Psi^t(p),t)$, which sends $(\Psi(p),t)$ to $(\Psi^{t+1}(p),t)$, and $(p,t+1)$ to $(\Psi^{t+1}(p),t+1)$, induces a diffeomorphism from $\mathcal{B}$ to $\Sigma\times(\mathbb{R}/\mathbb{Z})$.

Since any orientation preserving diffeomorphism $\Psi:\mathbb{R}^2\to\mathbb{R}^2$ is isotopic to the identity it follows that M is diffeomorphic with $\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})$. (To find an isotopy of Ψ to the identity first translate so that the origin becomes a fixed point and then consider $\Psi^s(p) = s^{-1}\Psi(sp)$; this is an isotopy of Ψ to a linear map, namely $D\Psi(0)$; finally, all orientation preserving linear maps are isotopic.)

Because the Hamiltonian vector field is transverse to Σ the two-form $d\theta_0$ restricted to Σ is nondegenerate.

PROPOSITION 5.2. *If $\int_\Sigma d\theta_0$ diverges then there exist coordinates $(x,y,t):\mathrm{M}\to\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})$, a smooth function $K:\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})\to\mathbb{R}$ and a closed one-form β such that $\theta_0 = -ydx + K(x,y,t)dt + \beta$. In particular the trajectories of the Hamiltonian flow on* M *are mapped to those of the planar Hamiltonian system $\dot{x} = K_y$, $\dot{y} = -K_x$.*

PROOF. Identify M with $\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})$, and assume that we have chosen coordinates (u,v) on $\mathbb{R}^2$ so that $d\theta_0 = f(u,v,t)du\wedge dv$ with $f(u,v,t) > 0$. We now construct "action angle variables." Let $I(u,v,t)$ be the area measured with $f(u,v,t)du\wedge dv$ of the disc in $\mathbb{R}^2$ centered at the origin and with radius $\sqrt{u^2+v^2}$. Then I is a smooth function on $\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})$. For each fixed $t\in\mathbb{R}/\mathbb{Z}$ the orbits of the Hamiltonian vector field of $I(\cdot,\cdot,t)$ with respect to the symplectic form $d\theta_0$ are circles centered at the origin. With this choice of I all orbits have period 1.

Let $\phi(u,v,t)$ be the time it takes to reach the point (u,v,t) along the Hamiltonian vector field of $I(\cdot,\cdot,t)$ starting from the positive u axis. Then $\phi(u,v,t)$ is a smooth function on the universal cover of $(\mathbb{R}^2\setminus 0)\times(\mathbb{R}/\mathbb{Z})$, and that $\phi \mod \mathbb{Z}$ is smooth on $(\mathbb{R}^2\setminus 0)\times(\mathbb{R}/\mathbb{Z})$ itself. By direct computation one verifies that in the coordinates $x = x(u,v,t) = \sqrt{2I}\cos 2\pi\phi$ and $y = y(u,v,t) = \sqrt{2I}\sin 2\pi\phi$, the pullback of $d\theta_0$ to $\mathbb{R}^2\times\{t\}$ is $dx\wedge dy$. On the entire manifold $\mathbb{R}^2\times(\mathbb{R}/\mathbb{Z})$ one therefore has $d\theta_0 = dx\wedge dy + A(x,y,t)dx\wedge dt + B(x,y,t)dy\wedge dt$.

From $0 = dd\theta_0 = (B_x - A_y)\,dx\wedge dy\wedge dt$ it follows that there is a smooth fuction $K(x,y,t)$ such that $A = K_x$ and $B = K_y$. Hence $d\theta_0 = dx\wedge dy + dK\wedge dt$, and $\beta = \theta_0 + ydx - Kdt$ is closed, as claimed. ☐

5.3. Example of a planar Hamiltonian system which is not of contact type. As pointed out at the beginning of this section, for non-compact Poincaré sections Σ, a contact form is not immediately found without certain additional requirements.

Consider an autonomous Hamiltonian of the form $H(p,q,t) = h(I)$ where $I = \frac{1}{2}(p^2 + q^2)$, and $h(I)$ is some smooth increasing function of I.

The manifold M on which our flow is defined is $\mathrm{M} = \mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ and the Hamiltonian vector field is a pseudo Reeb vector field for the one-form $\theta = pdq - h(I)dt$.

Then the periodic orbits in M of the Hamiltonian system are exactly those periodic solutions of $\dot{q} = H_p$, $\dot{p} = -H_q$, whose minimal period is a fraction m/n (after n oscillations the time variable increases by an integer m, and the (q,p) variables return to their original positions). In this section we only consider periodic solutions for which $m = 1$.

All solutions of the Hamiltonian system are given by

$$p(t) = \sqrt{2I}\sin(\phi - \Omega t), \qquad q(t) = \sqrt{2I}\cos(\phi - \Omega t).$$

where the angular frequency is given by $\Omega = h'(I)$. A solution has minimal period $1/n$ if $h'(I) = 2\pi n$.

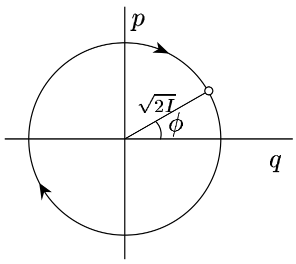


FIGURE 3. Symplectic polar coordinates

Introduce polar coordinates $I = (q^2 + p^2)/2$, $\phi = \arctan(p/q)$. Then along our periodic orbit the angle ϕ decreases by $2\pi n$. Using that $pdq = -Id\phi + d(pq/2)$, we compute the action of such an orbit:

$$\begin{aligned} A(I) &= \int_{t=0}^{1} pdq - h(I)dt \\ &= \int_{t=0}^{1} -Id\phi - h(I)dt \\ &= 2n\pi I - h(I) \\ &= h'(I)I - h(I). \end{aligned}$$

If we choose $h(I) = 4\pi(I - \frac{1}{2}\sin I)$, then we get

$$A(I) = 2\pi(-I\cos I + \sin I).$$

A sequence of periodic solutions occurs at $I = k\pi$, $k \in \mathbb{N}$ and their action is $A(k\pi) = 2k\pi^2(-1)^{k+1}$. For large k the action attains arbitrarily large positive and negative values. The Hamiltonian system with $H(p,q,t) = I + \alpha \sin I$ is therefore not of contact type ($H^1(\mathrm{M})$ is generated by dt).

Although the energy manifold is not of contact type, the Hamiltonian vector field is a Reeb vector field (we are in the special situation that there is a (second) conservation law on M). We now construct a contact form μ whose Reeb vector field is a multiple of the vector field $X_H = \partial_t + H_p\partial_q - H_q\partial_p$. Our Hamiltonian flow on $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ is determined by the one form $\lambda = Id\phi + h(I)dt$. If there is any other one-form μ which determines the same foliation, then for some function $f(p,q,t)$ one has $d\lambda = f(p,q,t)d\mu$ (Lemma 2.1). This function f must be constant on orbits of X_H, from which it is not hard to see that f must be a function of I alone. Thus we assume $d\mu = f(I)d\lambda$, i.e.

$$d\mu = f(I)dI \wedge d\phi + f(I)h'(I)dI \wedge dt = d\big(Jd\phi + g(J)dt\big)$$

where

$$J(I) = \int_0^I f(\hat{I})d\hat{I}, \qquad g(J) = \int_0^J h'(I(\hat{J}))d\hat{J} + C,$$

where $I(J)$ is the inverse of $J(I)$. We set $\mu = Jd\phi + g(J)dt$. The form μ will be a contact form provided $\mu \wedge d\mu = -f(Jg'(J) - g(J))dI \wedge d\phi \wedge dt \neq 0$, i.e. provided $Jg'(J) - g(J) \neq 0$ for all $J \in \mathbb{R}$. Using $g'(J) = h'(I)$ we compute

$$\begin{aligned}
Jg'(J) - g(J) &= -C + \int Jg''(J)dJ \\
&= -C + \int J\frac{dI}{dJ}h''(I)dJ \\
&= -C + \int J(I)h''(I)dI \\
&= -C + 2\pi \int J(I)\sin I dI.
\end{aligned}$$

Thus if we let $J(I) = 1 - e^{-I}$, and if we choose the constant $-C$ sufficiently large, then μ will be a contact form with the same flow as X_H, up to reparametrization.

5.4. Additional remarks. The example we gave above has an additional integral (namely, I) which allows us to write down a large class of one-forms with the same flows. A small non-autonomous perturbation of the Hamiltonian $H = h(I) + \epsilon h_1(I,\phi,t)$ will in general destroy the integral I and make it impossible to find the form μ. It thus seems reasonable to conjecture that an arbitrary small pertubation of λ exists for whose flow is not that of a Reeb vectorfield on $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$.

The example is also different from the other classes of Hamiltonian systems we consider in that M is not embedded as a hypersurface of $\mathbb{R}^4$. However, it is easy to produce such an embedding. Namely, given any strictly positive $h(I)$, we consider the Hamiltonian function

$$H(p,q,x,y) = \pi h\left(\tfrac{1}{2}(p^2+q^2)\right) - \pi(x^2+y^2)$$

on $\mathbb{R}^4$ with symplectic form $dq \wedge dp + dx \wedge dy$. This Hamiltonian truly has a second integral, namely $\pi(x^2 + y^2)$. Under the Hamiltonian vectorfield of H the x and y variables undergo harmonic oscillations with period 1.

The zero energy surface $\mathrm{M} = H^{-1}(0)$ is then diffeomorphic to $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ via $(p, q, t) \mapsto (p, q, R\cos 2\pi t, R\sin 2\pi t)$ with $R = \sqrt{\frac{1}{2}h(p^2 + q^2)}$. On M one has $\frac{1}{2}(ydx - xdy) = -R^2 dt$, so that

$$pdq + \tfrac{1}{2}(ydx - xdy) = pdq - h(I)dt,$$

as required.

6. Poincaré sections for second order Lagrangians

Following [**13**] we can define a transverse slice to the flow on M provided the Lagrangian $L(u, v, w)$ is a strictly convex function of w which satisfies $L(u, 0, 0) > 0$ for all $u \in \mathbb{R}$, and $\lim_{w\to\pm\infty} \frac{L(u,0,w)}{w} = \pm\infty$. Under this hypothesis the Legendre transform $L^*(u, 0, p_v) = \sup_{w\in\mathbb{R}} p_v w - L(u, 0, w)$ satisfies $L^*(u, 0, 0) \le -L(u, 0, 0) < 0$. Since $L^*(u, 0, p_v)$ is a strictly convex and proper function of p_v there are for each $u \in \mathbb{R}$ precisely two solutions $p_v^-(u) < 0 < p_v^+(u)$ of $L^*(u, 0, p_v) = 0$. At the positive solution one has $\frac{\partial L^*}{\partial p_v} > 0$.

On M the equation $v = 0$ implies $L^*(u, 0, p_v) = 0$. Hence the set $\mathrm{M} \cap \{v = 0\}$ consists of two smooth surfaces

$$\Sigma^\pm = \{(u, 0, p_u, p_v) \mid u, p_u \in \mathbb{R}, p_v = p_v^\pm(u)\}.$$

Both $\Sigma^\pm$ are transverse slices for the flow since, e.g., on Σ^+ one has $\dot{v} = \frac{\partial L^*}{\partial p_v} > 0$.

To verify that Σ^+ is a Poincaré section we must show that all orbits return to Σ^+ in both time directions. We do this in the special case where $L(u, v, w) = \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 = F(u)$.

6.1. A section when $F(u) > 0$.

Assume that $F(u) > 0$. Then the section Σ^+ is given by

$$\Sigma = \left\{(u, 0, p_u, \sqrt{2F(u)}) \mid u, p_u \in \mathbb{R}\right\}.$$

If we also assume that for large u the potential $F(u)$ grows superquadratically (to be precise, $F(u)/u^2 \to \infty$ as $u \to \pm\infty$) all solutions oscillate, i.e. for any solution $u(t)$ of the Euler-Lagrange-Poisson equations and any t_0 there exist $t_- < t_0 < t_+$ at which the solution has local minima. Thus any orbit of the Hamiltonian flow on M returns to Σ both in forward and backward time. This has been shown in [**11**] for $F(u) = \frac{1}{4}(1 - u^2)^2$ and with a minor modification the proof can be generalised to superquadratic potentials.

So Σ is a global Poincaré section for M and clearly Σ is diffeomorphic with the plane, while $\int_\Sigma d\theta_0 = \infty$. Therefore the Hamiltonian flow on M is conjugate to a planar Hamiltonian system, as was explained in Proposition 5.2.

6.2. Proof of Theorem 1.4. We claim that for some $M < \infty$ any bounded solution $u(t)$ of the Euler-Poisson equations is actually bounded by $|u(t)| \le M$. This then immediately implies that $|u'''' - \alpha u''| \le M'$, and hence, using an interpolation inequality, that all derivatives $u^{(j)}$ of order $j \le 4$ are also uniformly bounded. It follows that the Hamiltonian flow on the level set M has a largest compact invariant set $K \subset$ M.

Although this may be well known to some, let us prove our claim. Multiply the equation with $u(t)$ and $h(t) = \frac{1}{1+(t-a)^2}$ $(a \in \mathbb{R})$, and integrate by parts to get

$$\int_{\mathbb{R}} \left[h(uF'(u) + (u'')^2) + (2h'' - \alpha h)uu'' - \tfrac{1}{2}h''''u^2\right]dt = 0.$$

Use $|h''|, |h''''| \le Ch$ and also $uu'' \le \frac{1}{\epsilon}u^2 + \epsilon(u'')^2$ to obtain

$$\int_{\mathbb{R}} h[uF'(u) - C'u^2 + \tfrac{1}{2}(u'')^2]dt \le 0.$$

Superlinearity of $F'(u)$ implies $uF'(u) - C'u^2 \ge u^2 - C''$. One gets $\int h(u^2 + (u'')^2)dt \le C'''$ independent of u and a. This leads to the asserted L^∞-bound.

Using the Poincaré section for the flow we obtain a closed one-form β on M with $i_X\beta > 0$ everywhere. Since K is compact $i_X\theta_0$ is bounded from below on K by some positive constant. We can therefore choose $k > 0$ large enough so that $i_X(\theta_0 + k\beta) > 0$ on K. Lemma 3.4 then provides us with a smooth function f on M for which $\theta_0 + k\beta + df$ is a contact form.

Assuming also that $\alpha < 0$, we write $\alpha = -a^2$ and define

$$I = \frac{1}{2}(av^2 + p_v^2/a), \quad \varphi = \arctan\frac{p_v}{av},$$

i.e.

$$p_v = \sqrt{2aI}\sin\varphi, \quad v = \sqrt{2I/a}\cos\varphi.$$

Thus φ is a smooth function on the universal cover of M. A short calculation reveals that

$$-Id\varphi = p_v dv - d\left(\frac{vp_v}{2}\right)$$

so that the canonical form on M is given by $\theta_0 = p_u du - Id\varphi + d(vp_v/2)$. The Hamiltonian as a function of (u, p_u, I, φ) is $H = aI + p_u\sqrt{2I/a}\cos\varphi - F(u)$. On its zeroset one therefore has

$$I = \left\{-\frac{p_u}{\sqrt{2a^3}}\cos\varphi + \sqrt{\frac{p_u^2}{2a^3}\cos^2\varphi + \frac{F(u)}{a}}\right\}^2. \tag{9}$$

The trajectories of the Hamiltonian vector field on M are determined by the "principle of least action" [**2**, section 45C] $\delta\int\theta_0 = 0$, i.e. by $\delta\int p_u du - I(u, p_u, \varphi)d\varphi = 0$, where we regard I as the smooth function of (u, p_u, φ) specified in (9). Consequently they are integral curves of the Hamilton equations of $I(u, p_u, \varphi)$, i.e.

$$\frac{du}{d\varphi} = \frac{\partial I(u, p_u, \varphi)}{\partial p_u}, \quad \frac{dp_u}{d\varphi} = -\frac{\partial I(u, p_u, \varphi)}{\partial u}.$$

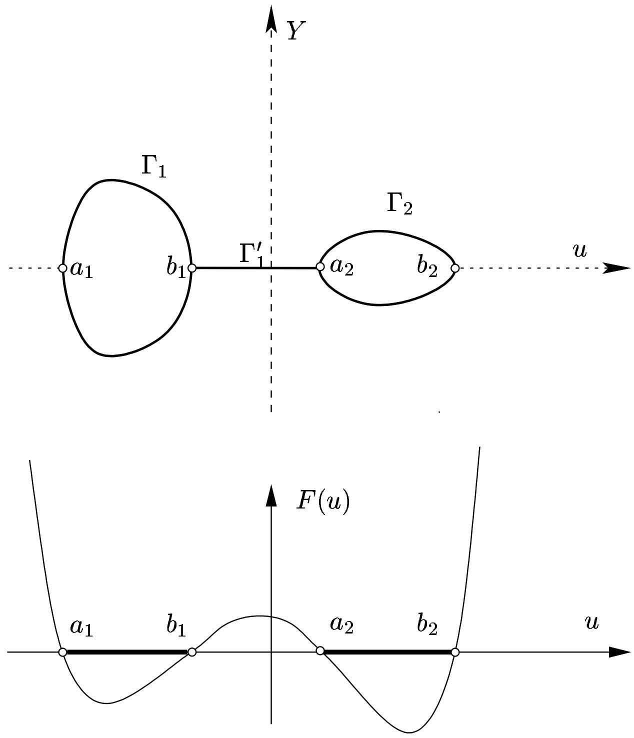


FIGURE 4. The (u, Y) plane.

7. Topology of M

Instead of analysing the topology of energy manifolds of second order Lagrangians in general we restrict ourselves here to the Swift-Hohenberg and eFK Lagrangians. In that case an energy manifold is given by the equation $H = \frac{1}{2}p_v^2 + p_u v - \frac{\alpha}{2}v^2 - F(u) = 0$. By rewriting this as

$$H = \frac{1}{2}p_v^2 + \frac{1}{4}\left\{(p_u + (1-\alpha/2)v)^2 - (p_u - (1+\alpha/2)v)^2\right\} - F(u) = 0$$

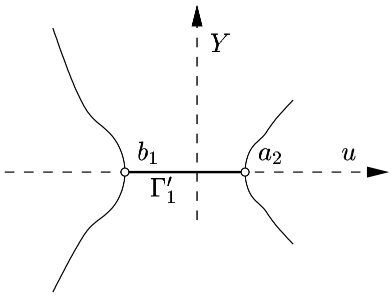
and introducing the new coordinates $X = \frac{1}{2}\{p_u + (1-\alpha/2)v\}$ and $Y = \frac{1}{2}\{p_u - (1+\alpha/2)v\}$, we see that M is given by

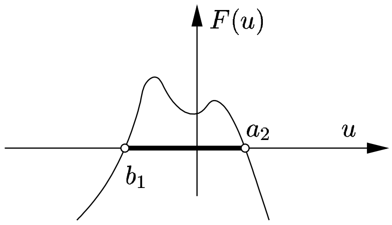
$$\frac{1}{2}p_v^2 + X^2 = Y^2 + F(u). \tag{10}$$

It is immediately clear that all hypersurfaces obtained by varying $\alpha \in \mathbb{R}$ are diffeomorphic.

If $F(u) \geq c > 0$ for some constant c, then (10) also shows that M is diffeomorphic to $S^1 \times \mathbb{R}^2$, since one can parametrize M by $p_v = \sqrt{2Y^2 + 2F(u)}\cos\theta$, $Y = \sqrt{Y^2 + F(u)}\sin\theta$, where $Y, u \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ are the parameters.

We can also easily compute the homotopy type of M for general F. Let us assume that $F(u) > 0$ for large enough $|u|$, and that F only has simple zeroes. Then the set $\{u \in \mathbb{R} \mid F(u) \leq 0\}$ is the union of a finite number of intervals, say





$[a_1, b_1] \cup \cdots \cup [a_k, b_k]$. The set in the (u, Y)-plane where $Y^2 + F(u)$ vanishes is the union of k closed curves Γ_j, given by $Y = \pm\sqrt{F(u)}$, with $u \in [a_j, b_j]$, $1 \le j \le k$.

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the projection in (u, X, Y, p_v) space onto the (u, Y) plane. Then $\pi(\mathrm{M})$ is the region outside the curves Γ_j, with the curves Γ_j included. Over each (u, Y) in the interior of $\pi(\mathrm{M})$ the preimage $\pi^{-1}(u, Y)$ is a circle whose radius shrinks to zero as the point (u, Y) approaches one of the curves Γ_j.

One can deform the projection $\pi(\mathrm{M})$ onto the union of the circles Γ_j and the $k-1$ line segments $\Gamma'_j = \{(u, 0) \mid b_j \le a_{j+1}\}$. This deformation can be lifted to a deformation retraction of M onto $\pi^{-1}(\Gamma_1 \cup \cdots \cup \Gamma_k \cup \Gamma'_1 \cup \cdots \cup \Gamma'_{k-1})$. Since each $\pi^{-1}(\Gamma_j)$ is an S^1 and each $\pi^{-1}(\Gamma'_j)$ is an S^2 one finds that M has the homotopy type of a bouquet of k circles and $k-1$ two-spheres. The first singular homology group is generated by the $\pi^{-1}\Gamma_j$, the second homology group is generated by $\pi^{-1}(\Gamma'_j)$.

For instance, when $k = 1$, one finds that M the homotopy type of a circle. In this case one actually finds that M is diffeomorphic with $S^1 \times \mathbb{R}^2$.

If one applies a similar analysis to the case in which $F(u) > 0$ in some bounded interval (b_1, a_2) and $F(u) < 0$ on $(-\infty, b_1) \cup (a_2, \infty)$, then one finds that M has the homotopy type of S^2 and in particular is simply connected. One finds that M is diffeomorphic with $\mathbb{R} \times S^2$.

8. $F(u)$ arbitrary, $\alpha \ll 0$

8.1. A very singular perturbation problem. In this section we present the example promised in Theorem 1.5. Consider the Swift-Hohenberg Lagrangian $L(u, v, w) = \frac{1}{2}w^2 + \frac{\alpha}{2}v^2 + F(u)$, $\alpha < 0$. Rescale time by $t \to \sqrt{\epsilon}t$, and set $\alpha = -1/\epsilon$. This yields a more convenient formulation. The Lagrangian now becomes

$$L(u, v, w) = \frac{\epsilon^2}{2}w^2 - \frac{1}{2}v^2 + F(u) \tag{11}$$

in which ϵ is a small positive constant. The variational equation for this Lagrangian is

$$\epsilon^2 u'''' + u'' + F_u(u) = 0, \tag{12}$$

As $\epsilon \searrow 0$ equation (12) formally reduces to a second order equation

$$u'' + F_u(u) = 0. \tag{13}$$

We will verify that for small enough ϵ solutions of (12) are essentially solutions of the second order equation (13), with a small rapid oscillation superimposed. This will allow us to construct many periodic orbits on the corresponding zero-energy manifold M and compute their actions.

We again pass to the Hamiltonian formulation of (12) and introduce new variables p_u and p_v, where now

$$p_v = \frac{\partial L}{\partial w} = \epsilon^2 w.$$

Thus u is a solution of (12) if and only if (u, v, p_u, p_v) is a solution of the Hamiltonian system with Hamiltonian function $H(u, v, p_u, p_v) = \frac{p_v^2}{2\epsilon^2} + p_u v + \frac{1}{2}v^2 - F(u)$. We rewrite this as

$$H = \frac{p_v^2}{2\epsilon^2} + \frac{1}{2}(v + p_u)^2 - \left\{\frac{1}{2}p_u^2 + F(u)\right\},$$

and apply the following coordinate change

$$U = u + p_v, \quad p_U = p_u, \quad V = \sqrt{\epsilon}(v + p_u) \quad p_V = \frac{p_v}{\sqrt{\epsilon}} \tag{14}$$

It follows from $p_U dU + p_V dV = p_u du + p_v dv + d(p_u p_v)$ that this is a symplectic coordinate change.

The Hamiltonian in these new coordinates is

$$H = \frac{1}{2\epsilon}\left(p_V^2 + V^2\right) - \left\{\frac{1}{2}p_U^2 + F(U - \sqrt{\epsilon}p_V)\right\}.$$

The Hamiltonian equations are

$$\begin{cases} \dot U = -p_U, \quad \dot p_U = F'(U - \sqrt{\epsilon}p_V), \\ \dot V = \epsilon^{-1}p_V + \sqrt{\epsilon}F'(U - \sqrt{\epsilon}p_V), \quad \dot p_V = -\epsilon^{-1}V \end{cases} \tag{15}$$

We will consider solutions which lie on the zero energy manifold $\mathrm{M} = H^{-1}(0)$. In particular, we will assume ϵ is small and that both sides in the identity

$$\frac{1}{2\epsilon}\left(p_V^2 + V^2\right) = \frac{1}{2}p_U^2 + F(U - \sqrt{\epsilon}p_V) \tag{16}$$

which defines M are bounded. Then V and p_V are of order $\sqrt{\epsilon}$, and the Hamiltonian equations are approximately given by two uncoupled two-dimensional systems,

$$\dot U = -p_U \qquad \dot p_U = F'(U) \tag{17}$$

and

$$\dot V = \epsilon^{-1}p_V \qquad \dot p_V = -\epsilon^{-1}V. \tag{18}$$

The first of these is the Hamiltonian system corresponding to the second order ODE (13). The second is a simple harmonic oscillator with angular frequency ϵ^{-1}. If we assume that (V, p_V) is given by

$$V = \sqrt{2I\epsilon}\sin(t/\epsilon - \phi), \qquad p_V = \sqrt{2I\epsilon}\cos(t/\epsilon - \phi)$$

then the zero total energy condition (16) forces (U, p_U) to be a solution of (17) with

$$\frac{1}{2}p_U^2 + F(U) = I \tag{19}$$

Since $\sqrt{2I\epsilon}$ is the amplitude of the (V, p_V) oscillation we must always have $I \geq 0$. Formally one would expect solutions of the Swift-Hohenberg equation (12) to be approximated by

$$u(t) = U(t) - \epsilon\sqrt{2I}\cos(t/\epsilon - \gamma) + o(\epsilon) \tag{20}$$

where $(U(t), p_U(t))$ is a solution of (17) with energy $I \geq 0$, and $\gamma \in [0, 2\pi)$ is some phase angle.

8.2. Action angle variables. We replace (V, p_V) by new coordinates (I, ϕ) given by

$$I = \frac{1}{2}\left(p_V^2 + V^2\right), \quad \phi = \arctan\frac{p_V}{V}$$

and thus $V = \sqrt{2I}\cos\phi$, $\quad p_V = \sqrt{2I}\sin\phi$. One has $Id\phi = \frac{1}{2}(Vdp_V - p_V dV) = -p_V dV + \frac{1}{2}d(Vp_v)$, so that

$$p_U dU + p_V dV = p_U dU - Id\phi + \frac{1}{2}d(Vp_V). \tag{21}$$

It follows that $dU \wedge dp_U + dV \wedge dp_V = dU \wedge dp_U + dI \wedge d\phi$, so that (U, p_U, I, ϕ) are symplectic coordinates. The Hamiltonian H appears in these variables as

$$H = \frac{I}{\epsilon} - \left\{\frac{1}{2}p_U^2 + F(U - \sqrt{2I\epsilon}\cos\phi)\right\}.$$

Define

$$\Omega_\epsilon = \left\{(U, p_U) \in \mathbb{R}^2 \mid c \leq \frac{1}{2}p_U^2 + F(U) \leq C, |U| + |p_U| \leq \bar{C}\right\}$$

for certain $0 < c < C, \bar{C} < \infty$ with c small and C and $\bar{C}$ large. We also define M_ϵ to be the portion of M for which (U, p_U) lies in Ω_ϵ. After rewriting the equation $H = 0$ as

$$\frac{I}{\epsilon} = \frac{1}{2}p_U^2 + F(U - \epsilon\sqrt{2\frac{I}{\epsilon}}\cos\phi)$$

one concludes from the implicit function theorem that for small enough ϵ there is a unique solution $I = \mathcal{I}(U, p_U, \phi, \epsilon)$ which is smooth in ϵ and whose Taylor series begins with

$$\mathcal{I}(U, p_U, \phi, \epsilon) = \epsilon I_0(U, p_U) + \epsilon^2 I_1(U, p_U, \phi) + \mathcal{O}(\epsilon^3) \tag{22}$$

where $I_0 = \frac{1}{2}p_U^2 + F(U)$, $\qquad I_1 = -F'(U)\sqrt{p_U^2 + 2F(U)}\cos\phi$. In particular we see that for small ϵ the portion M_ϵ of the zero energy manifold is diffeomorphic with the product space $\Omega_\epsilon \times S^1$.

On M_ϵ we have

$$\frac{d\phi}{dt} = -\frac{\partial H}{\partial I} = -\frac{1}{\epsilon} - F'(U - \sqrt{2I\epsilon}\cos\phi)\sqrt{\frac{\epsilon}{2I}}\cos\phi$$

while for small enough ϵ one has $c' < I/\epsilon < C'$ for any $0 < c' < c < C < C' < \infty$. Therefore, for small ϵ one has

$$\frac{d\phi}{dt} = -\frac{1}{\epsilon} + \mathcal{O}(1) < 0,$$

so we may parametrize orbits of X_H by the angle variable ϕ instead of time t. Orbits are then determined by specifying (U, p_U) as functions of ϕ; given the (U, p_U) component of an orbit, one recovers the action from $I = \mathcal{I}(U, p_U, \phi; \epsilon)$.

On M_ϵ orbits satisfy the least action principle, i.e.

$$\delta \int p_U dU + p_V dV = 0, \text{ i.e. } \delta \int p_U dU - \mathcal{I}(U, p_U, \phi, \epsilon) d\phi = 0$$

where we have used (21). Hence (U, p_U) as function of ϕ satisfies the Hamiltonian equations with Hamiltonian $\mathcal{I}(U, p_U, \phi, \epsilon)$. Moreover, (21) shows us that if $(U(\phi), p_U(\phi))$ is a $2N\pi$ periodic solution of

$$\frac{dU}{d\phi} = \frac{\partial \mathcal{I}}{\partial p_U}, \quad \frac{dp_U}{d\phi} = -\frac{\partial \mathcal{I}}{\partial U}, \tag{23}$$

then the corresponding periodic solution of (15) has action

$$\mathcal{A} = \oint p_U dU + p_V dV = \int_0^{2N\pi} p_U dU - \mathcal{I}(U, p_U, \phi, \epsilon)\, d\phi.$$

We write (23) out to get

$$\begin{aligned} \frac{1}{\epsilon}\frac{dU}{d\phi} &= p_U + \epsilon \frac{\partial I_1(U, p_U, \phi)}{\partial p_U} + \mathcal{O}(\epsilon^2), \\ \frac{1}{\epsilon}\frac{dp_U}{d\phi} &= -F'(U) - \epsilon \frac{\partial I_1(U, p_U, \phi)}{\partial U} + \mathcal{O}(\epsilon^2). \end{aligned}$$

Introducing a new time variable $\tau = \epsilon\phi$ we get

$$\begin{cases} \dfrac{dU}{d\tau} = p_U + \epsilon \dfrac{\partial I_1(U, p_U, \frac{\tau}{\epsilon})}{\partial p_U} + \mathcal{O}(\epsilon^2), \\ \dfrac{dp_U}{d\tau} = -F'(U) - \epsilon \dfrac{\partial I_1(U, p_U, \frac{\tau}{\epsilon})}{\partial U} + \mathcal{O}(\epsilon^2) \end{cases} \tag{24}$$

where the $\mathcal{O}(\epsilon^2)$ terms are smooth functions of $(U, p_U, \frac{\tau}{\epsilon})$ and all terms are 2π-periodic in $\tau/\epsilon = \phi$.

Denote the flow determined by these ODEs by Φ_ϵ^τ. Since the perturbation terms in (24) are uniformly smooth in (U, p_U) the flow Φ_ϵ^τ is an $\mathcal{O}(\epsilon)$ perturbation of Φ_ϵ^τ in the C^∞ topology. [7] Here Φ_0^τ is the flow corresponding to the autonomous equations

$$\frac{dU}{d\tau} = p_U, \quad \frac{dp_U}{d\tau} = -F'(U). \tag{25}$$

[7] In fact, by the method of single frequency averaging [**2**, section 52] the returnmap Φ_ϵ^τ with $\tau = 2N\pi\epsilon$, $N \in \mathbb{N}$, is an $\mathcal{O}(\epsilon^2)$ perturbation of the return map of the averaged Hamiltonian. Since first order term in (22) has zero time average, Φ_ϵ^τ is $\mathcal{O}(\epsilon^2)$ close to Φ_0^τ in the C^∞ topology.

These are the Hamiltonian equations for $H_0 = \frac{1}{2}p_U^2 + F(U)$, and thus periodic orbits of (25) come in families parametrized by their energy $E = H_0(U, p_U)$. On any such family the period T depends smoothly on the energy E. We shall call a periodic orbit *nondegenerate* if at this orbit one has $\frac{dT}{dE} \neq 0$.

Assume that $\big(\bar{U}(\tau), \bar{p}_U(\tau)\big)$ is a T periodic nondegenerate solution of (25). It traces out a closed curve γ in the (U, p_U) plane. This curve consists of fixed points for the map Φ_0^T. The nondegeneracy condition implies that for small $\epsilon > 0$ the map Φ_ϵ^T will have at least one fixed point in an ϵ neighborhood of γ. In general there will be many fixed points. If $T = 2N\pi\epsilon$ for some $N \in \mathbb{N}$ then these fixed points correspond to periodic solutions of (24) which are ϵ close to $\big(\bar{U}(\tau - \tau_0), \bar{p}_U(\tau - \tau_0)\big)$ for some phase τ_0. The action of this solution is

$$\begin{aligned} \mathcal{A} &= \int p_U dU - \mathcal{I}(U, p_U, \phi; \epsilon) d\phi \\ &= \int p_U dU - \frac{\mathcal{I}}{\epsilon} d\tau. \end{aligned}$$

Now use (24) and (22) to replace $dU/d\tau$ and $\mathcal{I}/\epsilon$ respectively. One obtains

$$\begin{aligned} \mathcal{A} &= \int_0^T \left\{ \frac{1}{2}\bar{p}_U^2 - F(\bar{U}) + \mathcal{O}(\epsilon) \right\} d\tau \\ &= \int_0^T \left\{ \frac{1}{2}\bar{p}_U^2 - F(\bar{U}) \right\} d\tau + \mathcal{O}(\epsilon). \end{aligned}$$

Thus we have proved: near any nondegenerate $T = 2\pi N\epsilon$ periodic orbit $\big(\bar{U}, \bar{p}_U\big)$ of (25) there is a T periodic solution of (24) whose action is within $\mathcal{O}(\epsilon)$ equal to the Lagrangian action of $\big(\bar{U}, \bar{p}_U\big)$.

8.3. Simple mechanical systems. Consider the mechnical system with Lagrangian $L(u, \dot{u}) = \frac{1}{2}\dot{u}^2 - F(u)$. The energy of a periodic orbit is $E = \frac{1}{2}\dot{u}^2 + F(u)$. Using $dt = du/\dot{u}$ one finds that the action of such an orbit is

$$S = \oint L dt = \oint \frac{\frac{1}{2}\dot{u}^2 - F(u)}{\sqrt{2(E - F(u))}} du = \oint \frac{E - 2F(u)}{\sqrt{2(E - F(u))}} du.$$

For the special case of the harmonic oscillator, $L = \frac{1}{2}(\dot{u}^2 - a^2u^2)$ all periodic orbits are of the form $u(t) = A \sin a(t - t_0)$, and by direct substitution one finds

$$S = \int_0^{2\pi/a} \frac{1}{2} \left(A^2a^2\cos^2 a(t - t_0) - A^2a^2\sin^2 a(t - t_0)\right) dt = 0.$$

In other words *the action of any orbit of the harmonic oscillator vanishes.*

LEMMA 8.1. *There exists a potential $F : \mathbb{R} \to \mathbb{R}$ with $F(u) > 0$ on $-L < u < L$ and $F(u) < 0$ when $|u| > L$ such that $\ddot{u} + F_u(u) = 0$ has one periodic orbit with positive Lagrangian action, and one with negative Lagrangian action.*

PROOF. We construct the potential by perturbing $F(u) = \frac{1}{2}u^2$. Let $0 \leq f \in C^\infty(\mathbb{R})$ be supported in $1 < u < 2$. Then for small λ the potential $F_\lambda(q) =$

$F(u) - \lambda f(u)$ has a periodic orbit $u_\lambda(t)$ which oscillates between -2 and 2, and hence has energy $E = 2$. The action S_λ of this orbit satisfies

$$\frac{dS_\lambda}{d\lambda}\bigg|_{\lambda=0} = \sqrt{2}\int_{-2}^{2} \frac{\frac{3}{2}E - F(u)}{(E - F(u))^{3/2}} f(u)\, du = 2\int_{-2}^{2} \frac{6-u^2}{(4-u^2)^{3/2}} f(u)\, du > 0.$$

For sufficiently small $\lambda > 0$ the amplitude 2 orbit of the potential F_λ will have positive action. Since F_λ coincides with the quadratic potential $u^2/2$ in the interval $|u| \le 1$, the potential F_λ still has an amplitude 1 orbit ($u(t) = \cos t$) with zero action. We now perturb F_λ to $F_{\lambda,\mu} = F_\lambda + \mu g(q)$, where $0 \le g \in C^\infty(\mathbb{R})$ is supported in $|u| < 1$. Reasoning as above we find that for sufficiently small $\mu > 0$ the amplitude 1 orbit of the potential $F_{\lambda,\mu}$ will have negative action. By first choosing $\lambda > 0$ small but fixed, so that the amplitude 2 orbit has positive action and then choosing $\mu > 0$ small enough we can guarantee that the amplitude 2 orbit of $F_{\lambda,\mu}$ still has positive action, while the amplitude 1 orbit has negative action.

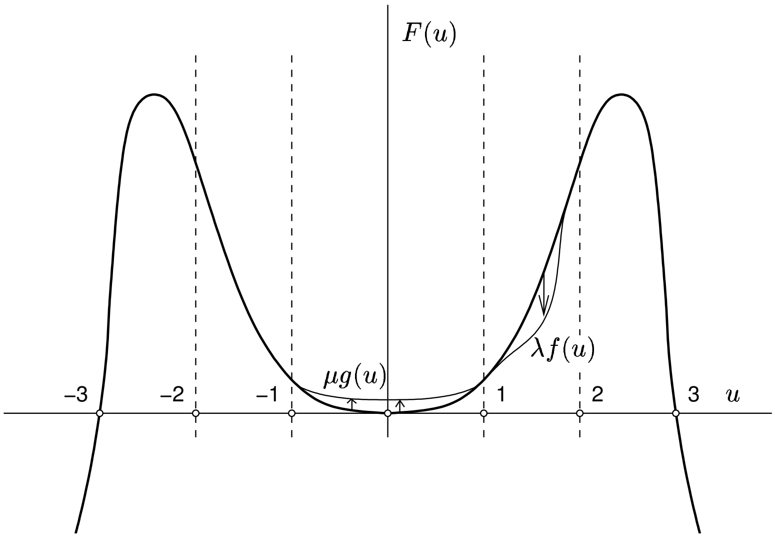


FIGURE 5. The potential $F(u)$

Since the amplitude 1 and 2 orbits are unaffected by changes in the potential outside the interval $|u| \le 2$, we may define $F(u)$ as we like for $|u| > 2$, and in particular we could make it vanish at $u = \pm 3$, and also be negative for $|u| > 3$. $\square$

8.4. The example. We choose $F(u)$ as in Lemma 8.1. Then for sufficiently small $\epsilon > 0$ the zero-energy manifold $\mathrm{M} = H^{-1}(0)$, $H = \frac{p_v^2}{\epsilon} + p_u v + \frac{1}{2}v^2 - F(u)$ contains two periodic orbits with actions of opposite signs. Furthermore, since $F(u)$ is negative outside of the interval $|u| \le 3$, the manifold M is homeomorphic to $S^2 \times \mathbb{R}$, i.e. M is simply connected. By the Hofer-Zehnder criterion, Lemma 3.2, M cannot be of contact type.

A different example for a potential of globally the same shape could be found via the methods used in [**13**], where simple closed periodic orbits are found with estimates on their actions. This also yields periodic solutions with both negative and positive action, but with an explicit estimate for the range of the parameter α for which M_α is not of contact type.

8.5. More examples. It is very easy to extend the example in §8.4 to other examples. Namely, one can modify the potential $F(u)$ outside the interval $-3 \leq u \leq 3$ any way one likes, and the resulting hypersurface M will still not be of contact type. Indeed, the new M contains the two periodic orbits with opposite actions, and they are still contractible (since the contraction takes place in the region $\mathrm{M} \cap \{|u| \leq 3\}$.)

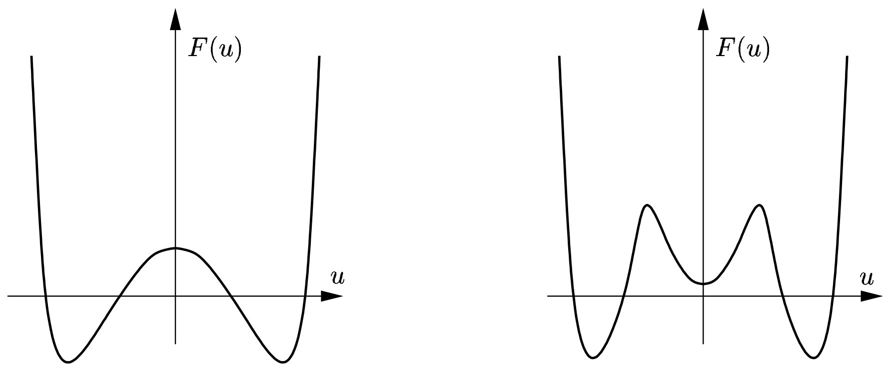


FIGURE 6. Swift-Hohenberg and Modified Swift-Hohenberg potentials

Thus the potential on the right in figure 6 contains our previous example and hence yields an energy manifold M which is not of contact type. It can be deformed into the potential $F(u) = \frac{1}{4}(1-u^2)^2+E$ (with $-\frac{1}{4} < E < 0$) of the Swift-Hohenberg equation without changing the number of zeros of the potential$F$, and without changing the topology of the hypersurface M. As we observed in the introduction, we do not know if M with the Swift-Hohenberg potential is of contact type or not.

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