A domain-wall between single-mode and bimodal states

G. J. B. van den Berg and R. C. A. M. van der Vorst

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Abstract

We examine a model equation describing spatial patterns in a class of physical systems where instabilities to travelling waves occur. The spatial patterns are modelled by a system of two second order ordinary differential equations, in which the cross-coupling coefficient is spatially dependent. The system has two clearly distinct types of stationary states, of which the stability depends on the cross-coupling coefficient. Under mild assumptions on the cross-coupling coefficient, we apply a variational method to prove the existence of a heteroclinic orbit between both types of states, corresponding to a domain-wall in the physical picture. This solution is found as a minimiser of a Lyapunov functional and the variational structure is exploited to obtain detailed information about the shape of the solution. In the case of a constant cross-coupling coefficient we find heteroclinic solutions connecting stationary states of the same type.

1 Introduction

In the description of near-threshold pattern formation in physics *amplitude equations* play a central role. The patterns typically occur via a bifurcation of a homogeneous state or a traveling wave. A large number of such pattern forming phenomena can be analysed using amplitude equations, that describe the dynamics on long time and length scales. The form of these amplitude equations is quite universal; the details of the underlying physical system are merely reflected in the coefficients [CH, HHS]. Just above the threshold the amplitude equations yield an almost complete description of the dynamical behaviour of the system. Both qualitative and quantitative predictions have been successfully confronted with experiments [CH].

If the patterns occur via a bifurcation of a traveling wave, the amplitude equations for the near-threshold behaviour in a one-dimensional system are [CH, HHS]

$$\left\{\begin{array}{rcl} \partial_t A + v_0 \partial_x A &=& \varepsilon A + (1+ic_1)\partial_x^2 A - (1-ic_3)|A|^2 A - g_2(1-ic_2)|B|^2 A, \\ \partial_t B - v_0 \partial_x B &=& \varepsilon B + (1+ic_1)\partial_x^2 B - (1-ic_3)|B|^2 B - g_2(1-ic_2)|A|^2 B. \end{array}\right.$$

Here A and B are the complex amplitudes of the right and left travelling waves, v_0 denotes the group velocity of the waves, g_2 is the coupling coefficient and ε is the bifurcation parameter which is positive. The constants c_i are real and can in principle be calculated from the equations of motion of the underlying physical phenomenon. The coefficients in both equations are equal on grounds of space reflection symmetry.

In this paper we take both the group velocity v_0 and the constants c_i equal to zero. We consider the case where g_2 varies spatially, i.e., $g_2 = g(x)$ (cf. [HM, He]). After scaling we then obtain, for real-valued amplitudes, the system

$$\text{(I)} \left\{ \begin{array}{rcl} \partial_t A & = & A + \partial_x^2 A - \left\{A^2 + g(x)B^2\right\}A, \\ \partial_t B & = & B + \partial_x^2 B - \left\{B^2 + g(x)A^2\right\}B. \end{array} \right.$$

Although these equations have no direct physical interpretation in terms of travelling waves, they can serve as a starting point for investigation of the effect of the group velocity term [HM, He]. We remark that a related system is obtained in the study of a class of physical problems in which there are two interacting instability mechanisms [DR].

System (I) can (at least formally) be derived from the Lyapunov functional

$$L \stackrel{\text{def}}{=} \int \left\{ \tfrac{1}{2}(\partial_x A)^2 + \tfrac{1}{2}(\partial_x B)^2 - \tfrac{1}{2}A^2 - \tfrac{1}{2}B^2 + \tfrac{1}{4}A^4 + \tfrac{1}{4}B^4 + \tfrac{1}{2}g(x)A^2B^2 \right\} dx, \qquad (1)$$

by setting $\partial_t A = -\frac{\delta L}{\delta A}$ and $\partial_t B = -\frac{\delta L}{\delta B}$. We find that $L(t)$ is non-increasing:

$$\frac{dL(t)}{dt} = \int \frac{\delta L}{\delta A}\frac{\partial A}{\partial t} + \frac{\delta L}{\delta B}\frac{\partial B}{\partial t}\, dx = -\int \left(\frac{\delta L}{\delta A}\right)^2 + \left(\frac{\delta L}{\delta B}\right)^2 dx \le 0.$$

Therefore it is expected that generically the system will tend to a final state that corresponds to a local minimum of L.

The coupling coefficient $g(x)$ has a critical value of 1. When $g(x)$ is constant the situation is the following:

i) For $g \neq 1,\ g > -1$ system (I) has *nine* homogeneous stationary states:

- First of all there is the zero-solution $(0,0)$; it is a local maximum of L for all $g \in \mathbb{R}$.
- Secondly there are four *single-mode* states $(\pm 1, 0)$ and $(0, \pm 1)$. For $g > 1$ the single-mode states minimise L, whereas for $g < 1$ they are saddle points.
- Thirdly there are four *bimodal* states $\left(\pm\frac{1}{\sqrt{1+g}}, \pm\frac{1}{\sqrt{1+g}}\right)$ and $\left(\pm\frac{1}{\sqrt{1+g}}, \mp\frac{1}{\sqrt{1+g}}\right)$. For $g > 1$ these bimodal mode states are saddle points of L, while for $-1 < g < 1$ they are global minimisers.

ii) At the critical value $g = 1$ the set of equilibrium points consist of the zero-solution and

$$\left\{(A,B) \in \mathbb{R}^2 \;\middle|\; A^2 + B^2 = 1\right\}.$$

All points on the unit circle minimise L.

iii) Finally, for $g \le -1$ the integrand in (1) is not bounded from below. The equilibrium points are (0,0) (a local maximum) and $(\pm 1, 0)$ and $(0, \pm 1)$ (saddle points).

In this paper we look for stationary patterns of (I). This leads to a system of two coupled second-order ordinary differential equations:

$$\text{(S)}\left\{\begin{array}{rcl} A''(x) & = & A(x)\{A^2(x)+g(x)B^2(x)-1\}, \\ B''(x) & = & B(x)\{B^2(x)+g(x)A^2(x)-1\}. \end{array}\right.$$

In particular, we will investigate the existence of heteroclinic orbits, i.e., connections between different equilibrium points.

System (S) with a minus sign on left hand side arises in the description of vector solitons in optical fibres [HS, Ya]. There the emphasis lies on homoclinic orbits to zero.

When $g(x)$ is constant the equilibrium points of (S) are the constant stationary states of system (I) described above. Linearisation of (S) around the equilibrium points again reveals the bifurcation at $g=1$: for $g>1$ the single-mode states are real saddles (four real eigenvalues, two positive and two negative) and the bimodal states are saddle-centres (two real, two imaginary eigenvalues), whereas for $-1<g<1$ the roles are reversed. This corresponds to the change from minima to saddle points mentioned under i) above.

In the following two subsections we state our main results for the inhomogeneous and the homogeneous case, respectively.

1.1 The inhomogeneous case

In this section we focus on the case where there is a junction of a single-mode and a bimodal state, separated by a so-called *domain-wall* (for a physical background see [HM, He]).

From the previous considerations it is natural to write

$$g(x) \equiv 1+\gamma(x).$$

In the simplest approach the function $\gamma(x)$ changes from positive to negative over the domain $\mathbb{R}$. This corresponds to the physical picture where two different patches, one with a supercritical γ-value and one with a subcritical γ-value, are joined together. One then expects a transition layer to arise.

In this paper we prove the existence of a domain-wall between a single-mode state and a bimodal state. We make the following assumptions on $\gamma(x)$:

G_1 $\quad \gamma(x) \in L^2_{\text{loc}}(\mathbb{R})$.

G_2 $\quad \inf_{x\in\mathbb{R}} \gamma(x) \stackrel{\text{def}}{=} \gamma_{\min} > -2$.

G_3 $\quad \liminf_{x\to-\infty} \gamma(x) > 0$.

G_4 $\quad \lim_{x\to\infty} \gamma(x) \stackrel{\text{def}}{=} \gamma_\infty < 0$.

G_5 $\quad \gamma(x)-\gamma_\infty \in L^2(\mathbb{R}^+)$.

A typical coupling function is $\gamma(x) = -c_0 \tanh(c_1 x)$ with $c_0 \in (0,2)$ and $c_1 > 0$, but our approach does not require $\gamma(x)$ to be monotone or continuous.

The limit conditions on $\gamma(x)$ at $-\infty$ are fairly weak, because for $\gamma(x) > 0$ the minima of the Lyapunov functional L are firmly fixed to be the single-mode states $(\pm 1, 0)$ and $(0, \pm 1)$. The conditions on $\gamma(x)$ at $+\infty$ are considerably more restricting, because in this limit $\gamma(x)$ is negative and therefore the minimising states of L are crucially coupled to the value of $\lim_{x\to\infty} \gamma(x)$.

We are now able to state our first result.

Theorem 1. *Suppose that* $\gamma(x) \equiv g(x) - 1$ *satisfies* $\mathsf{G}_{1\text{-}5}$. *Then there exists a weak solution* $(A(x), B(x)) \in C^1(\mathbb{R})$ *to* (S) *such that*

$$\lim_{x\to-\infty} (A(x), B(x)) = (1, 0) \qquad \text{and} \qquad \lim_{x\to+\infty} (A(x), B(x)) = \left(\tfrac{1}{\sqrt{2+\gamma_\infty}}, \tfrac{1}{\sqrt{2+\gamma_\infty}}\right). \tag{2}$$

On the set where $\gamma(x)$ *is continuous* $A(x)$ *and* $B(x)$ *are twice continuously differentiable and they obey the differential equations of* (S) *in the classical sense.*

The definition of a weak solution will be given in Section 2. There we also look in more detail at the regularity of the solution. We want to remark here that the solution in Theorem 1 is found as a minimiser of a functional that closely resembles the Lyapunov functional L (see (6)).

Since the system (S) is invariant under the transformations $(A, B) \to (-A, B)$ and $(A, B) \to (B, A)$, an immediately consequence of Theorem 1 is the existence of seven more transitions from a single-mode state to a bimodal state.

In the remainder of this section $u = (A, B)$ denotes the solution obtained in Theorem 1. We can derive several qualitative properties of u.

Theorem 2. *Suppose that* $\gamma(x)$ *satisfies* $\mathsf{G}_{1\text{-}5}$. *Then*

$$0 \le B(x) \le A(x) \qquad \text{for all } x \in \mathbb{R}.$$

We now introduce an additional condition on γ:

G_6 $\quad \sup_{x\in\mathbb{R}} \gamma(x) \stackrel{\text{def}}{=} \gamma_{\max} < \infty,$

In the next theorem we establish upper and lower bounds on u.

Theorem 3. *Suppose that* $\gamma(x)$ *satisfies* $\mathsf{G}_{1\text{-}6}$.

a) *We have the lower bound (see Figure 1)*

$$\begin{aligned} &\text{if } \gamma_{\max} \le 2 : && A \ge 1 - \left[\sqrt{2+\gamma_{\max}} - 1\right] B, \\ &\text{if } \gamma_{\max} > 2 : && A \ge \frac{1}{\sqrt{1+\gamma_{\max}}} - \left[\frac{\sqrt{2+\gamma_{\max}}}{\sqrt{1+\gamma_{\max}}} - 1\right] B. \end{aligned}$$

In particular,

$$A(x) \geq \frac{1}{\sqrt{2+\gamma_{\max}}} \qquad \textit{for all } x \in \mathbb{R}.$$

b) *As upper bound we have (see Figure 1)*

$$A(x) \leq \max\left\{1, \frac{1}{\sqrt{2+\gamma_{\min}}}\right\} \qquad \textit{for all } x \in \mathbb{R}.$$

Moreover, if $\gamma_{\min} > -1$ *then*

$$A \leq \min\left\{1, \sqrt{\frac{1-B^2}{\gamma_{\min}+1}}\right\}.$$

We remark that in fact hypothesis G_6 is not needed to prove part b) of this theorem.

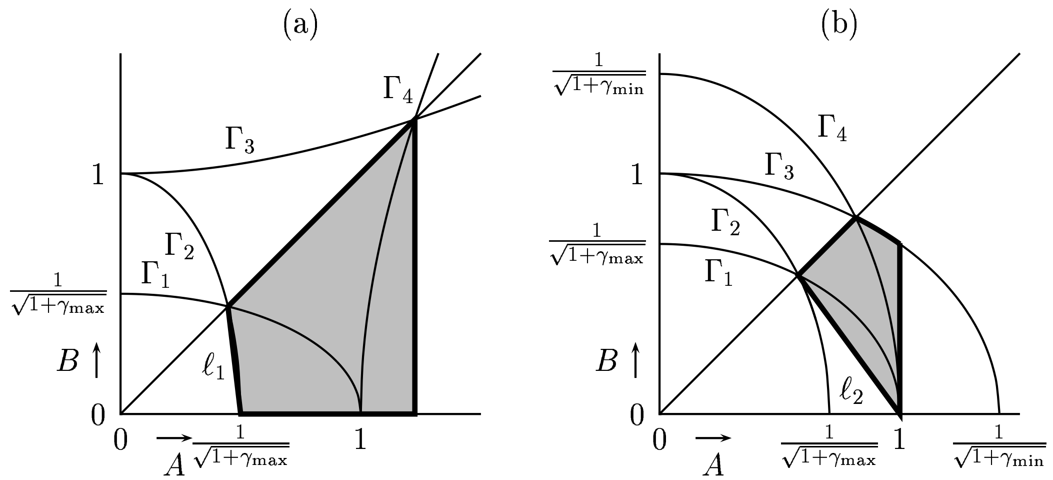

Figure 1: The bounds from Theorem 3: (a) for $\gamma_{\max} > 2$ and $\gamma_{\min} \leq -1$, (b) for $\gamma_{\max} \leq 2$ and $\gamma_{\min} > -1$. In these pictures we have denoted the ellipses/hyperbolas

$$\begin{aligned} \Gamma_1 &\equiv \left\{A^2 + (1+\gamma_{\max})B^2 = 1\right\}, \\ \Gamma_2 &\equiv \left\{(1+\gamma_{\max})A^2 + B^2 = 1\right\}, \\ \Gamma_3 &\equiv \left\{(1+\gamma_{\min})A^2 + B^2 = 1\right\}, \\ \Gamma_4 &\equiv \left\{A^2 + (1+\gamma_{\min})B^2 = 1\right\}, \end{aligned}$$

and the lines

$$\begin{aligned} \ell_1 &\equiv \left\{A = \tfrac{1}{\sqrt{1+\gamma_{\max}}} - \left[\tfrac{\sqrt{2+\gamma_{\max}}}{\sqrt{1+\gamma_{\max}}} - 1\right] B\right\}, \\ \ell_2 &\equiv \left\{A = 1 - \left[\sqrt{2+\gamma_{\max}} - 1\right] B\right\}. \end{aligned}$$

We now define the polar coordinate

$$\phi = \arctan\frac{B}{A},$$

and introduce the additional hypothesis that $\gamma(x)$ has a unique transition from positive to negative values, i.e., $\gamma(x)$ obeys

$$\mathsf{G}_7 \qquad \text{There exist } x_a, x_b \in \mathbb{R},\ x_a \le x_b \text{ such that } \gamma(x) \begin{cases} > 0 & \text{for } x < x_a, \\ = 0 & \text{for } x_a \le x \le x_b, \\ < 0 & \text{for } x > x_b. \end{cases}$$

In particular we observe that $\gamma(x)$ satisfies G_7 when it decreases monotonically. The last theorem of this section states that if $\gamma(x)$ satisfies G_7, then $\phi(x)$ increases monotonically (and thus the ratio of A and B decreases monotonically).

Theorem 4. *Suppose that $\gamma(x)$ satisfies $\mathsf{G}_{1\text{-}7}$. Then*

$$\phi'(x) \geq 0.$$

1.2 The homogeneous case

In this section we address the question of existence of heteroclinic orbits in the homogeneous case, i.e., when $\gamma \equiv g - 1$ is independent of x and $\gamma > -2$.

As already mentioned in Section 1.1, system (S) is invariant under the transformations $(A, B) \to (-A, B)$ and $(A, B) \to (B, A)$, corresponding to reflections with respect to the lines $\{A = 0\}$ and $\{A = B\}$, respectively. These transformations also imply invariance under reflection with respect to the line $\{B = 0\}$, as well as under rotation over 90° in the (A, B)-plane.

In the homogeneous case the system has additional symmetries. Firstly, system (S) is now autonomous and hence invariant under spatial translations. Secondly, it is invariant under space reflection ($x \to -x$). Thirdly, when $[A, B, \gamma]$ represents a solution to (S), then the transformation

$$T:\ [A, B, \gamma] \to \left[\frac{\sqrt{2+\gamma}}{2}(A+B), \frac{\sqrt{2+\gamma}}{2}(B-A), \frac{-2\gamma}{2+\gamma}\right] \tag{3}$$

also gives a solution to (S). This transformation corresponds to a rotation over 45° in the (A, B)-plane, followed by a scaling. It transforms the cases with $-2 < \gamma < 0$ into the cases with $\gamma > 0$, and vice versa.

The autonomous system is equivalent to a two-dimensional energy-conserving mechanical system with potential

$$V(A, B) = \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 - \tfrac{1}{4}A^4 - \tfrac{1}{4}B^4 - \tfrac{1}{2}(1+\gamma)A^2B^2,$$

and energy

$$H = \tfrac{1}{2}A'^2 + \tfrac{1}{2}B'^2 + V(A, B). \tag{4}$$

Two types of heteroclinic orbits are easily recognised. In the first place, for all $\gamma > -2$ there are orbits of the form $(A(x), 0)$ (and $(0, B(x))$) connecting the single-mode states $(1,0)$ and $(-1,0)$ (see Figure 2a). Here $A(x)$ is the kink of the *Fisher-Kolmogorov* equation

$$A'' = A^3 - A,$$

namely

$$A(x) = \pm \tanh\left(\frac{x}{\sqrt{2}}\right).$$

Secondly, by transformation (3) we obtain orbits of the form $(A(x), \pm A(x))$ connecting two bimodal states, $\left(\frac{1}{\sqrt{2+\gamma}}, \pm\frac{1}{\sqrt{2+\gamma}}\right)$ and $\left(-\frac{1}{\sqrt{2+\gamma}}, \mp\frac{1}{\sqrt{2+\gamma}}\right)$, where $A(x)$ obeys a scaled version of the Fisher-Kolmogorov equation (see Figure 2b).

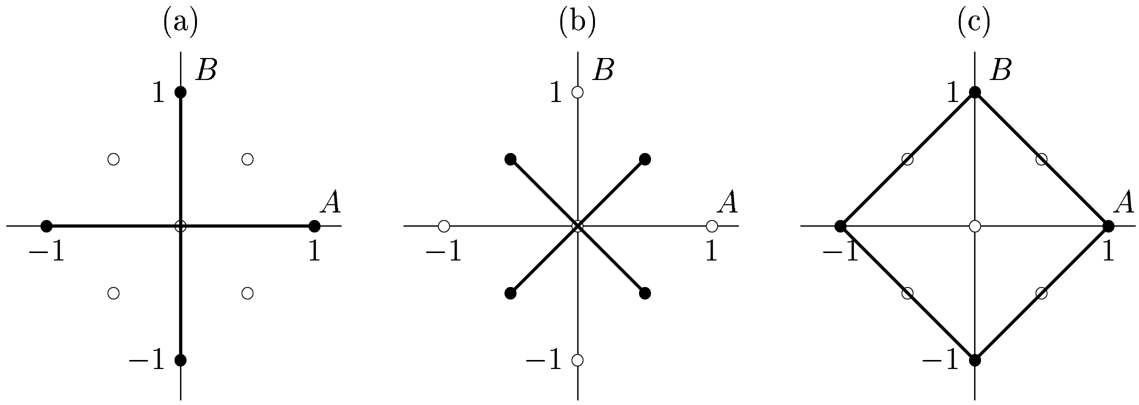

Figure 2: Several types of heteroclinic orbits for the homogeneous case (the pictures are for $\gamma = 2$): (a) connecting $(1,0)$ with $(-1,0)$ and $(1,0)$ with $(0,-1)$, (b) connecting $\left(\frac{1}{\sqrt{2+\gamma}}, \pm\frac{1}{\sqrt{2+\gamma}}\right)$ with $\left(-\frac{1}{\sqrt{2+\gamma}}, \mp\frac{1}{\sqrt{2+\gamma}}\right)$. and (c) connecting $(\pm 1, 0)$ with $(0, \pm 1)$. Notice that each line represents two orbits going in opposite directions.

We now examine the collection of heteroclinic orbits for some special values of γ.

- For $\gamma = 0$ ($g = 1$) the potential is radially symmetric and thus the system is completely integrable: besides the energy H the angular momentum $r^2\phi'$ (using polar coordinates in the (A,B)-plane) is also conserved. Heteroclinic orbits connect opposite points on the circle $\{A^2 + B^2 = 1\}$.
- For $\gamma = -1$ ($g = 0$) the system is decoupled and as a consequence completely integrable. There is a heteroclinic orbit of the form $(A(x), 1)$ connecting single-mode states $(1,1)$ with $(-1,1)$, where $A(x)$ obeys the Fisher-Kolmogorov equation. It should be clear from symmetry that there are eight heteroclinic solutions of this type. Moreover there is a continuous family of solutions that connect $(1,1)$ with $(-1,-1)$:

$$(A(x), B(x)) = \left(\tanh\left(\frac{x}{\sqrt{2}}\right), \tanh\left(\frac{x + x_0}{\sqrt{2}}\right)\right) \qquad \text{with } x_0 \in \mathbb{R}.$$

- From the invariance under transformation (3) we see that the system is also completely integrable for $\gamma = 2$. The heteroclinic orbits for $\gamma = 2$ are found from the decoupled case via the linear transformation (3): there is one orbit connecting $(\pm 1, 0)$ with $(0, \pm 1)$ (see Figure 2c) and there is a continuous family of solutions connecting $(1, 0)$ and $(-1, 0)$.

Thus, for $\gamma = 2$ we have an explicit solution connecting $(1, 0)$ and $(0, 1)$. The existence of such a connection for all $\gamma > 0$ is established in the following theorem, that will be proved in Section 5.

Theorem 5. *Suppose that γ is independent of x, and $\gamma > 0$. Then there exists a solution $(A(x), B(x)) \in C^\infty(\mathbb{R})$ to (S) such that*

$$\lim_{x\to-\infty}(A(x), B(x)) = (1, 0) \qquad \text{and} \qquad \lim_{x\to+\infty}(A(x), B(x)) = (0, 1).$$

This solution has the property that $A(x) \geq 0$, $B(x) \geq 0$ and it satisfies the symmetry relation

$$A(x) = B(x) \;\Leftrightarrow\; x = 0 \qquad \text{and} \qquad A(x) = B(-x) \quad \text{for all } x \in \mathbb{R}.$$

Moreover, we have the following bounds:

$$A(x)^2 + B(x)^2 \leq 1 \quad \text{and} \quad A(x) + B(x) \geq \min\Big\{1, \frac{2}{\sqrt{2+\gamma}}\Big\} \qquad \text{for all } x \in \mathbb{R}.$$

Finally, the angular coordinate $\phi = \arctan\frac{B}{A}$ increases monotonically, i.e.,

$$\phi'(x) \geq 0.$$

Using transformation (3) again, we see from this theorem that for $\gamma < 0$ there exists a heteroclinic orbit connecting $\big(\frac{1}{\sqrt{2+\gamma}}, -\frac{1}{\sqrt{2+\gamma}}\big)$ and $\big(\frac{1}{\sqrt{2+\gamma}}, \frac{1}{\sqrt{2+\gamma}}\big)$.

When $\gamma(x)$ is not a constant, the heteroclinic orbits of the form $(A(x), 0)$ and $(0, B(x))$ persist. It is not clear whether the other heteroclinic connections persist for non-constant γ. This will of course depend on function $\gamma(x)$. For example, if $\gamma(x) \in C^1(\mathbb{R})$ and $\gamma'(x) < 0$ for all x, then for all solutions the energy H decreases monotonically as a function of x and thus no orbit connecting $(1, 0)$ and $(0, 1)$ can exist.

Another interesting question concerns the existence of multibump solutions. This will be the subject of future investigation.

We will now begin the proof of Theorem 1 about the inhomogeneous case. In Section 2 we give an outline of the method that leads to the existence of a connection between a single-mode state and a bimodal state when $\gamma(x)$ obeys hypotheses $G_{1\text{-}5}$. In Section 3 we complete the proof of Theorem 1, whereas the qualitative properties of the solution are discussed in Section 4. In the final section we return to the homogeneous case and prove Theorem 5.

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2 Outline of the method

When we identify A and B with position coordinates, the system (S) can be written as the Hamilton equations corresponding to the Hamiltonian (compare the Lyapunov functional in (1))

$$H = \tfrac{1}{2}A'^2 + \tfrac{1}{2}B'^2 + \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 - \tfrac{1}{4}A^4 - \tfrac{1}{4}B^4 - \tfrac{1}{2}\{1+\gamma(x)\}A^2B^2,$$

where x is the 'time' coordinate. As in Section 1.2 this leads us to define the *potential*

$$\begin{aligned} V(A,B,\gamma(x)) &\stackrel{\text{def}}{=} \tfrac{1}{2}A^2 + \tfrac{1}{2}B^2 - \tfrac{1}{4}A^4 - \tfrac{1}{4}B^4 - \tfrac{1}{2}\{1+\gamma(x)\}A^2B^2 \\ &= -\tfrac{1}{4}\left\{1-(A^2+B^2)\right\}^2 - \tfrac{1}{2}\gamma(x)A^2B^2 + \tfrac{1}{4}. \end{aligned}$$

Remark 1. The potential can and will be looked at in three ways. Firstly as a continuous function of A,B and γ. Secondly, for a given function $\gamma(x)$, V is a function of three variables: the two position variables (A,B) and the 'time' variable x. Thirdly, when we substitute $(A,B) = (A(x),B(x))$ and fix the function $\gamma(x)$, we can look upon V as a function of x only. •

Let us look at the maximum of the potential $V(A,B,\gamma)$. When $\gamma \le -2$, then the potential is unbounded from above. We preclude this possibility by assuming that $\gamma(x)$ satisfies hypothesis G_2. For $\gamma > -2$ we have the following lemma, that can be verified by an elementary calculation.

Lemma 1. *The potential $V(A,B,\gamma)$ is bounded from above if $\gamma > -2$. We distinguish three cases:*

1. *For $-2 < \gamma < 0$ the maximum is attained at the bimodal states $|A| = |B| = \dfrac{1}{\sqrt{2+\gamma}}$.*
2. *For $\gamma = 0$ the maximum is attained everywhere on the circle $\left\{A^2+B^2=1\right\}$.*
3. *For $\gamma > 0$ the maximum is attained at the single-mode states $(\pm1,0)$ and $(0,\pm1)$.*

Besides, for the maximum value $c(\gamma)$ of $V(A,B,\gamma)$ we have

$$c(\gamma) = \begin{cases} \dfrac{1}{4} & \text{if } \gamma \ge 0, \\ \dfrac{1}{2(2+\gamma)} & \text{if } -2 < \gamma < 0. \end{cases}$$

From the definition of $c(\gamma)$ it follows that for any $\gamma(x)$ obeying hypothesis G_2

$$W(A,B,\gamma) \equiv -V(A,B,\gamma) + c(\gamma) \geq 0 \qquad \text{for all } A,B \in \mathbb{R},\ \gamma > -2. \tag{5}$$

Note that Remark 1 also applies to $W(A,B,\gamma)$.

Writing $u(x) = (A(x), B(x))$ we now define the functional

$$J[u] \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left\{ \tfrac{1}{2}|u'(x)|^2 + W\big(u(x), \gamma(x)\big) \right\} dx. \tag{6}$$

The difference between $J[u]$ and the Lyapunov functional in (1) is the term $c(\gamma(x))$, which guarantees that $J[u] \geq 0$. We will minimise this functional in the Hilbert space

$$E = \left\{ u \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^2) \;\Big|\; \int_{\mathbb{R}} |u'(x)|^2\, dx < \infty \right\},$$

with inner product

$$(u,v)_E = \int_{\mathbb{R}} u'(x) v'(x)\, dx + u(0)v(0)\,. \tag{7}$$

Notice that $u(0)$ is well-defined since $E \subset C(\mathbb{R})$.

Definition 1. *We call $u \in E$ a weak solution of* (S) *on* $\mathbb{R}$ *if for every interval* $[a,b]$, $-\infty < a < b < \infty$,

$$\int_a^b \left\{ u'(x) \cdot \varphi'(x) - \nabla_u W\big(u(x), \gamma(x)\big) \cdot \varphi(x) \right\} dx = 0 \qquad \textit{for all } \varphi \in W^{1,2}_0([a,b], \mathbb{R}^2),$$

where $\nabla_u W = \left(\frac{\partial W}{\partial A}, \frac{\partial W}{\partial B}\right)$.

It follows from the definition (5) of W that a classical solution of (S) is also a weak solution.

Remark 2. Concerning the regularity of weak solutions, we make the following remarks, that will be proved in the next section. Hypothesis G_1 can be weakened to $\gamma(x) \in L^1_{\text{loc}}$. This does not change the proof of the existence result nor the qualitative properties given in Theorems 2, 3 and 4. The solution is no longer continuously differentiable, but we find that $(A(x), B(x)) \in E$. Moreover, if $\gamma(x) \in H^n_{\text{loc}}$ for some n then $A(x), B(x) \in H^{n+2}_{\text{loc}}$; if $\gamma(x) \in C^n$ then $A(x), B(x) \in C^{n+2}$. The statement of Theorem 1 is based on the special case $n = 0$. •

In the remainder of this section we will assume that $\gamma(x)$ satisfies hypotheses $\mathsf{G}_{1\text{-}5}$. We will show that as a minimiser of $J[u]$ on E we find a solution of (S) that represents a domain-wall between a single-mode state and a bimodal state. Our method is similar to that of Rabinowitz, who proved the existence of heteroclinic orbits for autonomous Hamiltonian systems [Ra].

Lemma 2. *The infimum*

$$m \equiv \inf_{u\in E} J[u]$$

is finite.

Proof. It is clear that $J[u] \geq 0$ since $W(A,B,\gamma) \geq 0$ by (5). To show that $m < \infty$, we choose a function $\chi \in C^\infty(\mathbb{R},\mathbb{R}^2)$ with the properties

$$\chi(x) = (1,0) \quad \text{for } x<0 \qquad \text{and} \qquad \chi(x) = \left(\tfrac{1}{\sqrt{2+\gamma_\infty}}, \tfrac{1}{\sqrt{2+\gamma_\infty}}\right) \quad \text{for } x>1.$$

We shall show that $J[\chi] < \infty$. We write

$$\begin{aligned} J[\chi] &= \left(\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{\infty}\right) \left\{\tfrac{1}{2}|\chi'(x)|^2 + W\big(\chi(x),\gamma(x)\big)\right\} dx \\ &\equiv J_1[\chi] + J_2[\chi] + J_3[\chi]. \end{aligned}$$

Observe that

$$W(\chi(x),\gamma(x)) = -\tfrac{1}{4} + c(\gamma(x)) \qquad \text{for } x<0.$$

Let

$$a \stackrel{\text{def}}{=} \sup\big\{x_0 \in (-\infty,0) \;\big|\; \gamma(x) > 0 \text{ for } x \in (-\infty,x_0)\big\},$$

which is well-defined (and finite) since $\gamma(x)$ obeys G_3. By the definition of $c(\gamma)$ we have $W(\chi(x),\gamma(x)) = 0$ for $x \in (-\infty,a)$. Hence if $a=0$, then $J_1[\chi] = 0$, and if $a<0$ then

$$J_1[\chi] = \int_a^0 \left\{-\tfrac{1}{4} + c(\gamma(x))\right\} dx.$$

Since $c(\gamma)$ is non-increasing, it follows that $c(\gamma(x)) \leq c(\gamma_{\min})$, hence J_1 is bounded.

It is clear that the integral J_2 on $(0,1)$ is also bounded. To show that J_3 is finite, we note that

$$W(\chi(x),\gamma(x)) = \frac{\gamma(x) - 2\gamma_\infty - 2}{2(2+\gamma_\infty)^2} + c(\gamma(x)) \qquad \text{for } x>1.$$

Let

$$b \stackrel{\text{def}}{=} \inf\big\{x_0 \in (1,\infty) \;\big|\; \gamma(x) < 0 \text{ for } x \in (x_0,\infty)\big\},$$

which is well-defined (and finite) thanks to G_4. We now find that

$$W(\chi(x),\gamma(x)) = \frac{\big(\gamma_\infty - \gamma(x)\big)^2}{2(2+\gamma_\infty)^2(2+\gamma(x))} \qquad \text{for } x>b.$$

Because $2+\gamma(x) \geq 2+\gamma_{\min} > 0$ by hypothesis G_2, and $\gamma(x) - \gamma_\infty \in L^2(\mathbb{R}^+)$ by G_5, it follows that J_3 is bounded as well. This completes the proof. □

Let $\{v_n\}_{n=0}^{\infty} \subset E$ be a minimising sequence of J: $J[v_n] \downarrow m$. The following lemma shows that we can restrict ourselves to a sequence of functions with range in Q, where

$$Q \stackrel{\text{def}}{=} \big\{(A,B) \in \mathbb{R}^2 \;\big|\; 0 \leq B \leq A\big\}.$$

Lemma 3. *For every $v \in E$ with $J[v] < \infty$ there exists a $u \in E_Q$, where*

$$E_Q \stackrel{\text{def}}{=} \{u \in E \mid u(x) \in Q \text{ for all } x \in \mathbb{R}\},$$

such that $J[u] = J[v]$.

Proof. We proceed in two steps. First we show that we can find a function of which the range is contained in the first quadrant $\{A \geq 0,\ B \geq 0\}$. Then we show that we can restrict this function even further to one with its range contained in Q.

We write $v(x) = (A(x), B(x))$ and define $w(x) \equiv (|A(x)|, |B(x)|)$. Now w is contained in E, the range of w is contained in $\{(A, B) \in \mathbb{R}^2 \mid A, B \geq 0\}$, and $|w'(x)| = |v'(x)|$. Moreover, as before we remark that $W(A, B, \gamma)$ is invariant under the transformations $(A, B) \to (-A, B)$ and $(A, B) \to (A, -B)$. Thus $J[w] = J[v]$.

Next, let the range of v be contained in the first quadrant. We introduce new variables $C = (A + B)/\sqrt{2}$ and $D = (A - B)/\sqrt{2}$. For the function $v(x) = (A(x), B(x))$ expressed in the new variables we write $\hat{v}(x) = (C(x), D(x))$. The functional $\hat{J}[\hat{v}] \equiv J[v]$ can be written as

$$\hat{J}[\hat{v}] = \int_{\mathbb{R}} \left\{ \tfrac{1}{2}(C'^2 + D'^2) + \tfrac{1}{4}\left(1 - (C^2 + D^2)\right)^2 + \tfrac{1}{8}\gamma(x)(C^2 - D^2)^2 - \tfrac{1}{4} + c(\gamma) \right\} dx. \quad (8)$$

We now set $\hat{u}(x) \equiv (C(x), |D(x)|)$ and arguing as before, we see that $\hat{u} \in E$ and $\hat{J}[\hat{u}] = \hat{J}[\hat{v}]$. Transforming back to variables A and B demonstrates that the range of u is in Q. □

We may thus replace the minimising sequence $\{v_n\}_{n=0}^{\infty}$ in E by a sequence in E_Q.

Definition 2. *In the following section we will refer to the minimising sequence $\{u_n\}_{n=0}^{\infty}$, that has the properties*

$$u_n \in E_Q \qquad \text{for } n = 0, 1, 2, \ldots$$

and

$$J[u_n] \downarrow \inf_{u \in E_Q} J[u] = \inf_{u \in E} J[u] \qquad \text{as } n \to \infty.$$

In the next section we will show that u_n is bounded in E, so that we can extract a subsequence that converges weakly to a function $u \in E_Q$, and that this function is a solution to (S) that satisfies the limit conditions (2).

Intuitively it is clear that

$$W(u(x), \gamma(x)) \to 0 \qquad \text{as } x \to \pm\infty.$$

We define three sets of zeros of W (compare Lemma 1):

$$M_+ \stackrel{\text{def}}{=} \{(A, B) \in Q \mid W(A, B, \xi) = 0 \text{ for some } \xi > 0\} = \{(1, 0)\}, \quad (9a)$$

$$M_0 \stackrel{\text{def}}{=} \{(A, B) \in Q \mid W(A, B, 0) = 0\} = \{A^2 + B^2 = 1\} \cap Q \qquad \text{and} \quad (9b)$$

$$M_- \stackrel{\text{def}}{=} \{(A, B) \in Q \mid W(A, B, \xi) = 0 \text{ for some } \xi \in (-2, 0)\} = \{A = B > \tfrac{1}{\sqrt{2}}\}. \quad (9c)$$

Since $\gamma(x)$ decreases from positive to negative over the domain $\mathbb{R}$, we expect that

$$\lim_{x\to-\infty} u(x) = (1,0), \qquad \text{and} \qquad \lim_{x\to\infty} u(x) \in M_-.$$

We will make this rigorous in the next section.

3 Proof of Theorem 1

In this section we continue to assume that $\gamma(x)$ satisfies $\mathsf{G}_{1\text{-}5}$.

We define the region in Q (depending on $\gamma(x)$) where W can vanish for some $x \in \mathbb{R}$ (see (9)):

$$M \stackrel{\text{def}}{=} \Big\{(A,B) \in Q \;\Big|\; A^2 + B^2 = 1 \text{ or } A = B = \frac{1}{\sqrt{2+\xi}},\ \xi \in [\gamma_{\min}, 0]\Big\}.$$

Note that, since $\gamma_{\min} > -2$ by hypothesis G_2, M is a closed, bounded region in Q. We also define an ε-neighbourhood N_ε of M as

$$N_\varepsilon(M) \stackrel{\text{def}}{=} \big\{y \in Q \;\big|\; |y - M| < \varepsilon\big\}. \tag{10}$$

In the following lemma we show that $W(A,B,\gamma)$ is bounded away from zero on $Q \setminus N_\varepsilon(M)$.

Lemma 4. *The number α_ε defined by*

$$\alpha_\varepsilon \stackrel{\text{def}}{=} \inf\big\{W(A,B,\gamma(x)) \;\big|\; (A,B) \in Q \setminus N_\varepsilon(M), x \in \mathbb{R}\big\}$$

is strictly positive for all $\varepsilon > 0$.

Proof. Since $W(A,B,\xi) \to \infty$ uniformly with respect to $\xi \in [\gamma_{\min}, 0]$ as $A + B \to \infty$, we can choose a constant $c > 0$ such that $W(A,B,\xi) > \frac{1}{4}$ for $A + B > c$ and $\xi \in [\gamma_{\min}, 0]$. Moreover, $W(A,B,\xi) \geq W(A,B,0)$ for $\xi \in [0,\infty)$. Observing that the range of $\gamma(x)$ is contained in $[\gamma_{\min}, \infty)$, we infer that (for $0 < \varepsilon < 1$)

$$\inf\big\{W(A,B,\gamma(x)) \;\big|\; (A,B) \in Q \setminus N_\varepsilon(M), x \in \mathbb{R}\big\} \geq \inf\big\{W(A,B,\xi) \;\big|\; (A,B,\xi) \in K\big\},$$

where K is the compact set

$$K \stackrel{\text{def}}{=} \big\{(A,B,\xi) \in \mathbb{R}^3 \;\big|\; (A,B) \in Q \setminus N_\varepsilon(M), A + B \leq c \text{ and } \xi \in [\gamma_{\min}, 0]\big\}.$$

Because $W(A,B,\xi)$ is a continuous function of A, B and ξ, it attains its infimum on the compact set K. Besides, the region M covers all points where $W(A,B,\xi)$ can be zero for some $(A,B) \in Q$ and some $\xi \in [\gamma_{\min}, \infty)$. We now conclude that

$$\alpha_\varepsilon \geq \inf\big\{W(A,B,\xi) \;\big|\; (A,B,\xi) \in K\big\} = \min\big\{W(A,B,\gamma) \;\big|\; (A,B,\gamma) \in K\big\} > 0. \qquad \square$$

Remark 3. It follows from Lemma 4 and (6) that if $u \in E_Q$ and $J[u] < \infty$, then for any $\varepsilon > 0$ the function u spends only a finite amount of 'time' x outside the set $N_\varepsilon(M)$. It is seen from the proof and the fact that M only depends on $\gamma_{\min}$, that the bound on this amount of time depends on $J[u]$, ε and $\gamma_{\min}$ only. •

The following lemma shows that the minimising sequence $\{u_n\}$ is bounded in E (in the norm associated with the inner product (7)). The proof is the same as in [Ra].

Lemma 5. *Let $u \in E_Q$ and $\varepsilon > 0$. Suppose that $u(x) \notin N_\varepsilon(M)$ on an interval $[a,b]$, where $-\infty < a < b < \infty$. Then*

$$J[u] \geq \sqrt{2\alpha_\varepsilon} \int_a^b |u'(x)|\, dx\,.$$

Proof. Let $\tau \equiv b-a$, then

$$L \equiv \int_a^b |u'(x)|\, dx \;\leq \tau^{1/2} \left(\int_a^b |u'(x)|^2\, dx \right)^{1/2}.$$

It then follows that

$$J[u] \geq \frac{L^2}{2\tau} + \int_a^b W(u(x),x)\, dx \geq \frac{L^2}{2\tau} + \alpha_\varepsilon \tau \equiv f(\tau).$$

The minimum of f occurs for $\tau = \tau_0 \equiv \sqrt{\frac{L^2}{2\alpha_\varepsilon}}$ so that $f(\tau) \geq f(\tau_0) = \sqrt{2\alpha_\varepsilon}L$. □

Remark 4. We can interpret Lemma 5 in terms of the length of the curve

$$\Gamma(u) = \{u(x) = (A(x), B(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\},$$

described by the orbit of a function $u \in E_Q$ with $J[u] < \infty$. Lemma 5 implies that if $J[u] < C_1$, then the length of $\Gamma(u)$ outside $N_\varepsilon(M)$ is bounded by some constant that depends on C_1, ε and $\gamma_{\min}$ only. In particular, for any $\varepsilon > 0$ there exists a constant $C > 0$ such that for the minimising sequence $\{u_n\}$, the length of $\Gamma(u_n)$ outside $N_\varepsilon(M)$ is uniformly bounded by C. Moreover, M is a bounded region in Q, hence we obtain a uniform bound on $\|u_n\|_\infty$. •

The following is a consequence of the preceding considerations:

Lemma 6. *If $u \in E_Q$ and $J[u] < \infty$, then*

$$\lim_{x\to-\infty} u(x) = (1,0) \qquad \text{and} \qquad \lim_{x\to+\infty} u(x) = \left(\tfrac{1}{\sqrt{2+\gamma_\infty}}, \tfrac{1}{\sqrt{2+\gamma_\infty}}\right).$$

Proof. The idea of the proof is that if $u(x)$ does not tend to these limits as $x \to \pm\infty$, then we use Remarks 3 and 4 to show that $J[u]$ is not finite.

Let $\theta \equiv \left(\frac{1}{\sqrt{2+\gamma_\infty}}, \frac{1}{\sqrt{2+\gamma_\infty}}\right)$. Suppose that $\lim_{x\to\infty} u(x) \neq \theta$. Then there exist an $\varepsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty}$ tending to infinity, such that $u(x_n) \notin B_\varepsilon(\theta) \equiv \{y \in \mathbb{R}^2 \mid |y-\theta| < \varepsilon\}$. Moreover, because $\gamma_\infty \in (-2,0)$ by G_2 and G_4, we can take

$$\varepsilon < \frac{1}{\sqrt{2+\gamma_\infty}} - \frac{1}{\sqrt{2}}, \tag{11}$$

so that $B_\varepsilon(\theta) \cap M \subset \{(A,B) \in \mathbb{R}^2 \mid A = B > \frac{1}{\sqrt{2}}\}$.

Let $x_0 \in \mathbb{R}$ be such that

$$\left| \frac{1}{\sqrt{2+\gamma(x)}} - \frac{1}{\sqrt{2+\gamma_\infty}} \right| < \frac{\varepsilon}{4} \qquad \text{for } x \geq x_0$$

(such an x_0 exists by hypothesis G_4). We now define the region where W can attain its minimum value of 0 for $x \geq x_0$:

$$\tilde{M} \stackrel{\text{def}}{=} \left\{(A,B) \in Q \mid A = B \in \left[\tfrac{1}{\sqrt{2+\gamma_\infty}} - \tfrac{\varepsilon}{4}, \tfrac{1}{\sqrt{2+\gamma_\infty}} + \tfrac{\varepsilon}{4}\right]\right\}.$$

From the observation that $N_{\varepsilon/4}(\tilde{M}) \subset B_{\varepsilon/2}(\theta)$ we conclude (see (10) and Lemma 4) that

$$\tilde{\alpha} \stackrel{\text{def}}{=} \inf\left\{W(A,B,\gamma(x)) \mid (A,B) \in Q \setminus N_{\varepsilon/4}(\tilde{M}),\ x \in [x_0,\infty)\right\}$$

is positive.

We thus see from Remark 4 that the curve $u(x)$ for $x \geq x_0$ has only finite length outside $B_{\varepsilon/2}(\theta)$. Besides, from Remark 3 we see that $u(x)$ spends only a finite 'time' x outside $B_{\varepsilon/2}(\theta)$ for $x \geq x_0$, so that there is a sequence $\{y_n\}_{n=1}^\infty \subset [x_0,\infty)$ tending to infinity, such that $u(y_n) \in B_{\varepsilon/2}(\theta)$.

From the existence of the sequences $\{x_n\}$ and $\{y_n\}$ we conclude that the orbit $u(x)$ must go back and forth between $\partial B_{\varepsilon/2}(\theta)$ and $\partial B_\varepsilon(\theta)$ infinitely many times, so that the length of the curve outside $B_{\varepsilon/2}(\theta)$ is infinite. This is a contradiction, so the second limit has been proved.

A similar though simpler argument proves the first limit. In this case we write $\zeta \equiv (1,0)$ and suppose that $\lim_{x\to-\infty} u(x) \neq \zeta$. Then there exists an $\varepsilon > 0$ and a sequence $\{x_n\}_{n=1}^\infty$ tending to $-\infty$, such that $u(x_n) \notin B_\varepsilon(\zeta)$. By hypothesis G_3 there are constants $\delta^* > 0$ and $x^* < 0$ such that $\gamma(x) > \delta^*$ for $x < x^*$. It is not difficult to see that

$$\tilde{\alpha} \stackrel{\text{def}}{=} \inf\left\{W(A,B,\gamma(x)) \mid (A,B) \in Q \setminus B_{\varepsilon/2}(\zeta),\ x \in (-\infty,x^*)\right\}$$

is positive. A contradiction can now be obtained as in the previous case. □

We make two observations:

1. $\displaystyle\int_{\mathbb{R}} |u_n'|^2 \leq C_1$ for all n, since $J[u_n] \downarrow m$ and W is positive.
2. It follows from Remark 4 that $|u_n(0)| \leq C_2$ for all n.

Together these observations imply that $\|u_n\|_E \leq C$ for all n (C being independent of n). Since E is a Hilbert space, we can extract a subsequence, again denoted by $\{u_n\}$, that converges weakly in E to some $\tilde{u} \in E$ as $n \to \infty$. Moreover, $\{u_n\}$ converges (strongly) to $\tilde{u}$ in L^∞_{loc} and thus, since $\{u_n\} \subset E_Q$, $\tilde{u} \in E_Q$.

To prove Theorem 1 we now only have to show that $\tilde{u}$ minimises $J[u]$, which will be done in Lemma 7. From this it is readily seen that for every interval $[a,b]$, $-\infty < a < b < \infty$,

$$\int_a^b \left\{\tilde{u}'(x) \cdot \varphi'(x) - \nabla_u W(\tilde{u}(x),\gamma(x)) \cdot \varphi(x)\right\} dx = 0 \qquad \text{for all } \varphi \in W_0^{1,2}([a,b],\mathbb{R}^2),$$

i.e., $\tilde{u}(x)$ is a weak solution of (S). The behaviour of $\tilde{u}(x)$ as $x \to \pm\infty$ is then given by Lemma 6. The regularity of the solutions (see Remark 2) follows from the differential equations of system (S) by standard regularity theory and the fact that $\tilde{u} \in E \subset H^1_{\mathrm{loc}}$ [Br].

Lemma 7. *The limit function $\tilde{u}$ minimises $J[u]$ on E.*

Proof. We again follow the proof in [Ra]. Let $-\infty < a < b < \infty$. For $u \in E$ we define

$$\Psi(u, a, b) \stackrel{\text{def}}{=} \int_a^b \left\{ \tfrac{1}{2}|u'(x)|^2 + W(u(x), x) \right\} dx.$$

Since the region of integration is bounded, the first term on the right hand side is lower semicontinuous on E and the second term is weakly sequentially continuous on E.

We now have the following inequalities

$$m \equiv \inf_{u \in E} J[u] = \lim_{n\to\infty} J[u_n] \geq \lim_{n\to\infty} \Psi(u_n, a, b) \geq \Psi(\tilde{u}, a, b).$$

Since a and b are arbitrary this implies that $J[\tilde{u}] \leq m$, which proves the lemma. □

4 Qualitative properties

Theorem 2 follows from the fact that the solution u found in the previous section is in E_Q. In this section we prove Theorems 3 and 4. Recall that $u = (A, B)$ is a weak solution and thus satisfies

$$\int_a^b \left\{ A'\psi' - A\big(1 - A^2 - (1+\gamma)B^2\big)\,\psi \right\} dx = 0, \tag{12a}$$

$$\int_a^b \left\{ B'\psi' - B\big(1 - B^2 - (1+\gamma)A^2\big)\,\psi \right\} dx = 0, \tag{12b}$$

for every interval $[a, b]$, $-\infty < a < b < \infty$ and all $\psi \in H^1_0([a, b], \mathbb{R})$.

There are several ways to obtain qualitative information about the solution that was found as a minimiser of $J[u]$. One approach is to assume, by contradiction, that the solution has certain properties and subsequently construct a function $\tilde{u}$ in E such that $J[\tilde{u}] < J[u]$, contradicting the fact that u minimises J on E. This method will be employed in Lemma 13.

Another possibility is to draw a parallel with physics and to consider the potential $V(A, B, \gamma)$ and the corresponding force $F = -\nabla_u V$. When the solution $u(x)$ would enter certain regions in the (A, B)-plane, then one can show that the direction of this force F is such that u cannot get back to its prescribed endpoint. In order to prove this, we will use the maximum principle applied to weak solutions of equations that resemble (12). We remark that although we have used the latter approach to prove most of the following lemmas, they can also be proved equally using the former method.

We start with the lower bounds from Theorem 3.

Lemma 8. *If* $\gamma(x)$ *satisfies* $\mathsf{G}_{1\text{-}6}$ *then*

$$A \geq \frac{1}{\sqrt{1+\gamma_{\max}}} - \left[\frac{\sqrt{2+\gamma_{\max}}}{\sqrt{1+\gamma_{\max}}} - 1\right] B.$$

Proof. To simplify the notation we write

$$\ell(B) \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+\gamma_{\max}}} - \left[\frac{\sqrt{2+\gamma_{\max}}}{\sqrt{1+\gamma_{\max}}} - 1\right] B.$$

We argue by contradiction and thus assume that there is an $x_0 \in \mathbb{R}$ such that

$$A(x_0) - \ell(B(x_0)) < 0.$$

We examine the continuous function

$$f(x) = A(x) - \ell(B(x)).$$

Clearly $f(x_0) < 0$ and, since $\gamma_{\max} > 0$ by hypothesis G_3,

$$\begin{aligned} \lim_{x\to-\infty} f(x) &= 1 - \frac{1}{\sqrt{1+\gamma_{\max}}} > 0 \qquad \text{and} \\ \lim_{x\to\infty} f(x) &= -\frac{1}{\sqrt{1+\gamma_{\max}}} + \frac{\sqrt{2+\gamma_{\max}}}{\sqrt{1+\gamma_{\max}}}\frac{1}{\sqrt{2+\gamma_\infty}} > 0. \end{aligned}$$

We conclude that

$$\begin{aligned} x_1 &\stackrel{\text{def}}{=} \inf\bigl\{\tilde{x} < x_0 \bigm| f(x) < 0 \text{ for all } x \in (\tilde{x}, x_0)\bigr\} \qquad \text{and} \\ x_2 &\stackrel{\text{def}}{=} \sup\bigl\{\tilde{x} > x_0 \bigm| f(x) < 0 \text{ for all } x \in (\tilde{x}, x_0)\bigr\} \end{aligned}$$

are well-defined and finite and we have

$$f(x_1) = 0, \quad f(x_2) = 0 \quad \text{and} \quad f(x) < 0 \quad \text{for } x \in (x_1, x_2).$$

Besides, from (12) we see that for all $\psi \in H_0^1([x_1, x_2], \mathbb{R})$

$$\int_{x_1}^{x_2} f'(x)\,\psi'(x)\,dx = \int_{x_1}^{x_2} U\bigl(A(x), B(x), \gamma(x)\bigr)\,\psi(x)\,dx, \tag{13}$$

where

$$U(A, B, \xi) = A\bigl(1 - A^2 - (1+\xi)B^2\bigr) + \left[\frac{\sqrt{2+\gamma_{\max}}}{\sqrt{1+\gamma_{\max}}} - 1\right] B\bigl(1 - B^2 - (1+\xi)A^2\bigr).$$

It is seen from Figure 1a (the line segment $\ell_1 \cap Q$ lies inside the ellipses Γ_1 and Γ_2) that the choice of $\ell(B)$ guarantees that for $(A, B) \in \bigl\{(A, B) \in Q \bigm| A \leq \ell(B)\bigr\}$ we have

$$1 - A^2 - (1+\gamma_{\max})B^2 \geq 0 \qquad \text{and} \qquad 1 - B^2 - (1+\gamma_{\max})A^2 \geq 0.$$

Besides, U decreases as a function of ξ $(A, B \geq 0)$ and therefore

$$U(A,B,\xi) \geq U(A,B,\gamma_{\max}) \geq 0 \qquad \text{for } (A,B) \in Q,\ A \leq \ell(B), \quad \xi \leq \gamma_{\max}.$$

Since $A(x) - \ell(B(x)) \leq 0$ for $x \in [x_1, x_2]$ and $\gamma(x) \leq \gamma_{\max}$, we conclude that

$$U(A(x), B(x), \gamma(x)) \geq 0 \qquad \text{for } x \in [x_1, x_2].$$

It now follows from (13) and the maximum principle for weak solutions of elliptic equations that $f(x) \geq \min\{f(x_1), f(x_2)\} = 0$ for $x \in (x_1, x_2)$, which contradicts the fact that $f(x_0) < 0$. □

For the case where $\gamma_{\max} \leq 2$, we can improve on the Lemma 8.

Lemma 9. *If* $\gamma(x)$ *satisfies* $\mathsf{G}_{1\text{-}6}$ *and* $\gamma_{\max} \leq 2$ *then*

$$A \geq 1 - \left[\sqrt{2+\gamma_{\max}} - 1\right] B.$$

Proof. To simplify the notation we write

$$\ell(B) \stackrel{\text{def}}{=} 1 - \left[\sqrt{2+\gamma_{\max}} - 1\right] B.$$

We again argue by contradiction and thus assume that there exist constants $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ such that

$$A(x_0) - \ell(B(x_0)) = -2\varepsilon < 0.$$

We now examine the continuous function

$$f(x) = A(x) - \ell(B(x)) + \varepsilon.$$

Clearly $f(x_0) < 0$ and

$$\begin{aligned} \lim_{x\to-\infty} f(x) &= \varepsilon > 0 \qquad \text{and} \\ \lim_{x\to\infty} f(x) &= -1 + \frac{\sqrt{2+\gamma_{\max}}}{\sqrt{2+\gamma_\infty}} + \varepsilon > 0. \end{aligned}$$

As in the proof of the previous lemma, we conclude that

$$\begin{aligned} x_1 &\stackrel{\text{def}}{=} \inf\left\{\tilde{x} < x_0 \;\middle|\; f(x) < 0 \text{ for all } x \in (\tilde{x}, x_0)\right\} \qquad \text{and} \\ x_2 &\stackrel{\text{def}}{=} \sup\left\{\tilde{x} > x_0 \;\middle|\; f(x) < 0 \text{ for all } x \in (\tilde{x}, x_0)\right\} \end{aligned}$$

are well-defined and finite and we have

$$f(x_1) = 0, \quad f(x_2) = 0 \quad \text{and} \quad f(x) < 0 \quad \text{for } x \in (x_1, x_2).$$

Besides, from (12) we see that for all $\psi \in H_0^1([x_1, x_2], \mathbb{R})$

$$\int_{x_1}^{x_2} f'(x)\, \psi'(x)\, dx = \int_{x_1}^{x_2} U\big(A(x), B(x), \gamma(x)\big)\, \psi(x)\, dx,$$

where

$$U(A, B, \xi) = A\big(1 - A^2 - (1+\xi)B^2\big) + \left[\sqrt{2 + \gamma_{\max}} - 1\right] B\big(1 - B^2 - (1+\xi)A^2\big).$$

We now assert that

$$U(A, B, \xi) \geq U(A, B, \gamma_{\max}) \geq 0 \qquad \text{for } (A, B) \in Q,\ A \leq \ell(B), \quad \xi \leq \gamma_{\max}. \tag{14}$$

The first inequality is straightforward and the second inequality follows from Lemma 10, below. We obtain a contradiction in the same way as in Lemma 8. □

The next lemma is needed in order to justify (14). As before we write

$$\ell(B) = 1 - \left[\sqrt{2+\eta} - 1\right] B,$$

and we define

$$\Omega \stackrel{\text{def}}{=} \big\{(A, B) \in Q \mid A \leq \ell(B)\big\}.$$

Lemma 10. *The inequality*

$$A\big(1 - A^2 - (1+\eta)B^2\big) + \left[\sqrt{2+\eta} - 1\right] B\big(1 - B^2 - (1+\eta)A^2\big) \geq 0$$

holds for

$$(A, B) \in \Omega \quad \textit{and} \quad 0 \leq \eta \leq 2.$$

Proof. First of all we define

$$f(A, B) \equiv A(1 - A^2 - (1+\eta)B^2).$$

We need to prove that

$$f(A, B) + \big[\sqrt{2+\eta} - 1\big] f(B, A) \geq 0 \qquad \text{for all } (A, B) \in \Omega \text{ and } 0 \leq \eta \leq 2. \tag{15}$$

Notice that

$$f(A, B) \geq 0 \qquad \text{for all } (A, B) \in \Omega \text{ and } 0 \leq \eta \leq 2$$

(see Figure 1b: the line segment $\ell_2 \cap Q$ lies inside the ellipse Γ_1). Hence, if $f(B, A) \geq 0$ then the assertion is proved. Assume now that $f(B, A) < 0$. Then, for $0 \leq \eta \leq 2$,

$$\begin{aligned} f(A, B) + \big[\sqrt{2+\eta} - 1\big] f(B, A) &\geq f(A, B) + f(B, A) \\ &= (A + B)\left[1 - (A^2 + B^2 + \eta AB)\right]. \end{aligned}$$

Besides, we have that

$$\begin{aligned} 1-(A^2+B^2+\eta AB) &\geq 1-\left[\left(\ell(B)\right)^2+B^2+\eta B\ell(B)\right] \\ &= a^2(2-a)\,B\,(a^{-1}-B), \end{aligned}$$

where $a=\sqrt{2+\eta}$. Since $\sqrt{2}\leq a\leq 2$ and $0\leq B\leq a^{-1}$ in Ω, it follows that (15) holds. □

We now turn to the upper bounds from Theorem 3.

Lemma 11. *If $\gamma(x)$ satisfies $\mathsf{G}_{1\text{-}5}$, then*

$$A(x)\leq \max\Big\{1,\frac{1}{\sqrt{2+\gamma_{\min}}}\Big\} \qquad \text{for all } x\in\mathbb{R}.$$

Proof. For notational purposes we put $\beta \overset{\text{def}}{=} \max\big\{1,\frac{1}{\sqrt{2+\gamma_{\min}}}\big\}$. It is easily seen that for $(A,B)\in Q$ with $A\geq\beta$ we have, since $B\leq A$,

$$A(1-A^2-(\gamma(x)+1)B^2)\leq A(1-A^2-(\gamma_{\min}+1)B^2)\leq 0. \tag{16}$$

We argue again by contradiction and thus assume that there exist constants $\varepsilon>0$ and $x_0\in\mathbb{R}$ such that

$$A(x_0)>\beta+\varepsilon.$$

From the limit behaviour of $A(x)$

$$\lim_{x\to-\infty}A(x)=1<\beta+\varepsilon \qquad \text{and} \qquad \lim_{x\to\infty}A(x)=\frac{1}{\sqrt{2+\gamma_\infty}}<\beta+\varepsilon,$$

we conclude that

$$\begin{aligned} x_1 &\overset{\text{def}}{=} \inf\big\{\tilde{x}<x_0 \;\big|\; A(x)-\beta-\varepsilon>0 \text{ for all } x\in(\tilde{x},x_0)\big\} \qquad \text{and} \\ x_2 &\overset{\text{def}}{=} \sup\big\{\tilde{x}>x_0 \;\big|\; A(x)-\beta-\varepsilon>0 \text{ for all } x\in(\tilde{x},x_0)\big\} \end{aligned}$$

are well-defined and finite and we have

$$A(x_1)=\beta+\varepsilon,\quad A(x_2)=\beta+\varepsilon \quad \text{and} \quad A(x)>\beta+\varepsilon \quad \text{for } x\in(x_1,x_2).$$

It now follows from (12a) and (16) and the maximum principle for weak solutions of elliptic equations that $A(x)\leq\max\big\{A(x_1),A(x_2)\big\}=\beta+\varepsilon$ for $x\in(x_1,x_2)$, which contradicts the fact that $A(x_0)>\beta+\varepsilon$. □

Together with Lemma 11 the next lemma proves part b) of Theorem 3.

Lemma 12. *If $\gamma(x)$ satisfies $\mathsf{G}_{1\text{-}5}$ and $\gamma_{\min}>-1$ then*

$$A\leq\sqrt{\frac{1-B^2}{\gamma_{\min}+1}}.$$

Proof. To simplify the notation we write

$$e(B) \stackrel{\text{def}}{=} \sqrt{\frac{1-B^2}{\gamma_{\min}+1}}.$$

We argue by contradiction and thus assume that there exists an $x_0 \in \mathbb{R}$ such that

$$A(x_0) > e(B(x_0)).$$

The graph $\{(e(B), B) \in \mathbb{R}^2 \mid 0 \le B \le 1\}$ is part of the ellipse Γ_3 in Figure 1b. Observe that the region $\tilde{\Omega} \stackrel{\text{def}}{=} \{(A, B) \in Q \mid A \le e(B)\}$ is convex. Therefore it is possible to find a straight line $A = p - qB \stackrel{\text{def}}{=} \ell(B)$ for some positive constants p and q, that passes between the point $(A(x_0), B(x_0))$ and $\tilde{\Omega}$, i.e.,

$$A(x_0) > \ell(B(x_0)) \qquad \text{and} \qquad A < \ell(B) \quad \text{for all } (A, B) \in \tilde{\Omega}.$$

We again examine the continuous function

$$f(x) = A(x) - \ell(B(x)).$$

Clearly $f(x_0) > 0$ and

$$\lim_{x\to-\infty} f(x) < 0 \qquad \text{and} \qquad \lim_{x\to\infty} f(x) < 0.$$

As before, we conclude that

$$\begin{aligned} x_1 &\stackrel{\text{def}}{=} \inf\{\tilde{x} < x_0 \mid f(x) > 0 \text{ for all } x \in (\tilde{x}, x_0)\} \qquad \text{and} \\ x_2 &\stackrel{\text{def}}{=} \sup\{\tilde{x} > x_0 \mid f(x) > 0 \text{ for all } x \in (\tilde{x}, x_0)\} \end{aligned}$$

are well-defined and finite and we have

$$f(x_1) = 0, \quad f(x_2) = 0 \quad \text{and} \quad f(x) > 0 \quad \text{for } x \in (x_1, x_2).$$

Besides, from (12) we see that for all $\psi \in H_0^1([x_1, x_2], \mathbb{R})$

$$\int_{x_1}^{x_2} f'(x)\,\psi'(x)\,dx = \int_{x_1}^{x_2} U\big(A(x), B(x), \gamma(x)\big)\,\psi(x)\,dx, \tag{17}$$

where (recall that $\ell(B) = p - qB$ with $p, q > 0$)

$$U(A, B, \xi) = A\big(1 - A^2 - (1+\xi)B^2\big) + qB\big(1 - B^2 - (1+\xi)A^2\big).$$

Since U decreases in ξ, we have that

$$U(A, B, \xi) \le U(A, B, \gamma_{\min}) \qquad \text{for } (A, B) \in Q, \quad \xi \ge \gamma_{\min}.$$

Because the region $\{(A, B) \in Q \mid A \ge \ell(B)\}$ lies outside the ellipses (see Figure 1b)

$$\Gamma_3 \equiv \{(1+\gamma_{\min})A^2 + B^2 = 1\} \quad \text{and} \quad \Gamma_4 \equiv \{A^2 + (1+\gamma_{\min})B^2 = 1\},$$

it now follows that

$$U(A,B,\gamma_{\min}) \le 0 \qquad \text{for } (A,B) \in Q,\ A \ge \ell(B).$$

Since $A(x) - \ell(B(x)) \ge 0$ for $x \in [x_1, x_2]$ and $\gamma(x) \ge \gamma_{\min}$, we conclude that

$$U(A(x),B(x),\gamma(x)) \le 0 \qquad \text{for } x \in [x_1,x_2].$$

It now follows from (17) and the maximum principle for weak solutions of elliptic equations that $f(x) \le \max\bigl\{f(x_1), f(x_2)\bigr\} = 0$ for $x \in (x_1, x_2)$, which contradicts the fact that $f(x_0) > 0$. ☐

Finally, we prove Theorem 4.

Lemma 13. *If $\gamma(x)$ satisfies $\mathsf{G}_{1\text{-}7}$ then, writing $\phi = \arctan\frac{B}{A}$,*

$$\phi'(x) \ge 0.$$

Proof. We change to polar coordinates $r \equiv \sqrt{A^2+B^2}$ and $\phi \equiv \arctan\frac{B}{A}$ and write $u = (r,\phi)$. Note that $r(x) > 0$ for all $x \in \mathbb{R}$ by Lemma 8, which implies that $\phi(x)$ is well-defined. In these coordinates we have

$$Q = \bigl\{(r,\phi) \bigm| r \ge 0,\ 0 \le \phi \le \frac{\pi}{4}\bigr\},$$

and

$$J[u] = \int_{\mathbb{R}} \Bigl\{\tfrac{1}{2}(r'^2 + r^2\phi'^2) + \tfrac{1}{4}(1-r^2)^2 + \tfrac{1}{2}\gamma(x) r^4 \sin^2 2\phi - \tfrac{1}{4} + c(\gamma)\Bigr\}\, dx.$$

The idea of the proof is that if ϕ decreases somewhere, then we find a contradiction by showing that u does not minimise $J[u]$.

Thus, suppose by contradiction that there exist constants $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, such that $\phi'(x) < 0$ for $x \in (x_1,x_2)$. Plainly we can choose x_1, x_2 and $x_0 \in (x_1,x_2)$ such that

$$\frac{\pi}{4} > \phi(x_1) > \phi(x_0) > \phi(x_2) > 0.$$

Since

$$\lim_{x\to-\infty} \phi(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} \phi(x) = \frac{\pi}{4},$$

we conclude that

$$\begin{aligned} x_3 &\stackrel{\text{def}}{=} \inf\bigl\{\tilde{x} \le x_0 \bigm| \phi(x) > \phi(x_0) \text{ for all } x \in (\tilde{x}, x_0)\bigr\} \qquad \text{and} \\ x_4 &\stackrel{\text{def}}{=} \sup\bigl\{\tilde{x} \ge x_0 \bigm| \phi(x) < \phi(x_0) \text{ for all } x \in (x_0, \tilde{x})\bigr\} \end{aligned}$$

are well-defined and finite. Besides, $x_3 \le x_1$ and $x_4 \ge x_2$, and from the continuity of $\phi(x)$

we see that both $\phi(x_3) = \phi(x_0)$ and $\phi(x_4) = \phi(x_0)$.

We distinguish two cases: $\gamma(x_0) \geq 0$ and $\gamma(x_0) < 0$. If $\gamma(x_0) \geq 0$ then $\gamma(x) \geq 0$ for all $x \leq x_0$ by G_7. We define $u_*(x) = (r_*(x), \phi_*(x))$ with

$$r_*(x) = r(x) \qquad \text{and} \qquad \phi_*(x) = \begin{cases} \phi(x) & \text{if } x \notin (x_3, x_0), \\ \phi(x_0) & \text{if } x \in (x_3, x_0). \end{cases}$$

We easily see that u_* is in E. Since $\gamma(x) \geq 0$ and $0 < \phi_*(x) < \phi(x) \leq \frac{\pi}{4}$ for $x \in (x_3, x_0)$, we see that $\gamma(x) r_*^4(x) \sin^2 2\phi_*(x) \leq \gamma(x) r^4(x) \sin^2 2\phi(x)$. Besides, $|{\phi_*}'(x)| < |\phi'(x)|$ on (x_1, x_0), so that $J[u_*] < J[u]$. This contradicts the fact that u minimises J on E

If $\gamma(x_0) < 0$ then $\gamma(x) < 0$ for all $x \geq x_0$ by G_7. We now define $u^*(x) = (r^*(x), \phi^*(x))$ with

$$r^*(x) = r(x) \qquad \text{and} \qquad \phi^*(x) = \begin{cases} \phi(x) & \text{if } x \notin (x_0, x_4), \\ \phi(x_0) & \text{if } x \in (x_0, x_4). \end{cases}$$

We now have $\gamma(x) \leq 0$ and $0 < \phi(x) < \phi_*(x) \leq \frac{\pi}{4}$ for $x \in (x_0, x_4)$, and a contradiction is obtained in the same manner as before. □

5 Proof of Theorem 5

Throughout this section γ is positive and does not depend on x:

$$\gamma(x) \equiv \gamma > 0$$

The proof of the existence of a heteroclinic solution connecting $(1,0)$ and $(0,1)$ for the homogeneous case is analogous to the proof of Theorem 1. Therefore, we will only concentrate on the differences.

We define the set of functions

$$\tilde{E} = \big\{ u \in E \;\big|\; \lim_{x\to-\infty} u(x) = (1,0) \text{ and } \lim_{x\to\infty} u(x) = (0,1) \big\}.$$

We are going to minimise $J[u]$ on $\tilde{E}$ and therefore define

$$\tilde{m} \stackrel{\text{def}}{=} \inf_{u \in \tilde{E}} J[u]. \tag{18}$$

From a simple analogue to Lemma 2 it is found that $\tilde{m}$ is finite.

In the following lemmas we will establish the existence of a function $u \in \tilde{E}$ such that $J[u] = \tilde{m}$. It then follows from standard theory that this minimiser is a solution to (S) and that $u \in C^\infty(\mathbb{R})$.

Let $\{v_n\}_{n=0}^{\infty} \subset \tilde{E}$ be a minimising sequence of J: $J[v_n] \downarrow \tilde{m}$. We remark that the set $\tilde{E}$ is not weakly closed, hence weak limits are generally not in $\tilde{E}$. We shall get around this difficulty by constructing a special minimising sequence.

As in Lemma 3, replacing $(A(x), B(x))$ by $(|A(x)|, |B(x)|)$, we can choose $\{v_n\}$ to be a minimising sequence such that

$$A_n(x) \geq 0 \quad \text{and} \quad B_n(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}. \tag{19}$$

The following lemma shows that we can restrict ourselves to functions that intersect the line $\{A = B\}$ once and are symmetric with respect to this line.

Lemma 14. *For all $v \in \tilde{E}$ with $J[v] < \infty$ there exists a function $u = (\tilde{A}, \tilde{B}) \in \tilde{E}$ with the properties*

$$\tilde{A}(x) = \tilde{B}(x) \Leftrightarrow x = 0, \quad \tilde{A}(x) \geq \tilde{B}(x) \Leftrightarrow x \leq 0 \quad \text{and} \quad \tilde{A}(x) = \tilde{B}(-x), \tag{20}$$

such that $J[u] \leq J[v]$.

Proof. We first note that the second property is a consequence of the first and third property combined with the limit behaviour of $\tilde{A}$ and $\tilde{B}$.

We write $v = (A, B)$. It follows from the limit behaviour of A and B that

$$\begin{aligned} x_1 &\stackrel{\text{def}}{=} \sup\{x \in \mathbb{R} \mid A(x) > B(x)\} \qquad \text{and} \\ x_2 &\stackrel{\text{def}}{=} \inf\{x \in \mathbb{R} \mid A(x) < B(x)\} \end{aligned}$$

are finite and $x_1 \leq x_2$. Besides, $A(x_1) = B(x_1)$ and $A(x_2) = B(x_2)$ by the continuity of A and B.

We now introduce the mirrored functions

$$u_1(x) = \begin{cases} \big(A(x_1 + x), B(x_1 + x)\big) & \text{if } x \leq 0, \\ \big(B(x_1 - x), A(x_1 - x)\big) & \text{if } x > 0, \end{cases}$$

and

$$u_2(x) = \begin{cases} \big(B(x_2 - x), A(x_2 - x)\big) & \text{if } x \leq 0, \\ \big(A(x_2 + x), B(x_2 + x)\big) & \text{if } x > 0. \end{cases}$$

It is not too difficult to check that u_1 and u_2 are in $\tilde{E}$ and that they satisfy (20). From the definitions of u_1 and u_2 we derive that

$$2J[v] - J[u_1] - J[u_2] = 2\int_{x_1}^{x_2} \Big\{\tfrac{1}{2}|v'(x)|^2 + W\big(v(x), \gamma\big)\Big\}\, dx \geq 0. \tag{21}$$

Therefore, at least one of the inequalities

$$J[u_1] \leq J[\tilde{v}], \qquad J[u_2] \leq J[\tilde{v}]$$

holds. This proves the lemma. □

From the preceding lemma we infer that we can replace the minimising sequence $\{v_n\}$ by another minimising sequence $\{u_n\}_{n=0}^{\infty} \subset \tilde{E}$ with $J[u_n] \to \tilde{m}$, such that $u_n = (A_n, B_n)$ obeys (19) and the symmetry condition (20) for all n.

We are now ready to prove the existence of a minimiser of J on $\tilde{E}$.

Lemma 15. *There exists a* $u \in \tilde{E}$ *such that* $J[u] = \tilde{m}$.

Proof. The boundedness of $\{u_n\}$ in E follows from an analogue to Lemma 5 and Remark 4, hence we can take a subsequence that converges weakly to some $u \in E$. Since this subsequence converges to $u = (A, B)$ strongly in L^∞_{loc}, it is easily seen that $A(x) \geq 0$ and $B(x) \geq 0$ for all $x \in \mathbb{R}$. Besides, we derive from (20) that

$$A(0) = B(0), \qquad A(x) \geq B(x) \quad \text{for } x \leq 0 \qquad \text{and} \qquad A(x) = B(-x).$$

It follows from the proof of Lemma 7 that $J[u] \leq \tilde{m}$. It remains to show that $u \in \tilde{E}$. From an argument similar to Lemma 6 we conclude that the limits points $\lim_{x\to\pm\infty} u(x)$ are either $(1, 0)$ or $(0, 1)$. Finally, from the observation that $A(x) \geq B(x)$ for $x \leq 0$ (and $A(x) \leq B(x)$ for $x \geq 0$), we obtain that

$$\lim_{x\to-\infty} u(x) = (1, 0) \qquad \text{and} \qquad \lim_{x\to\infty} u(x) = (0, 1). \qquad \square$$

As mentioned before, it follows from standard theory that every minimiser u of J on $\tilde{E}$ (the previous lemma shows that there exists at least one) is a solution to (S) and that $u \in C^\infty(\mathbb{R})$. We will now prove some properties of such minimisers, starting with a symmetry property.

Lemma 16. *Suppose* $u(x) = (A(x), B(x))$ *is a minimiser of* J *on* $\tilde{E}$. *Then* $A(x) \geq 0$ *and* $B(x) \geq 0$ *for all* $x \in \mathbb{R}$. *Besides, there exists a unique* $x_* \in \mathbb{R}$ *such that* $A(x_*) = B(x_*)$. *Moreover,*

$$A(x_* + x) = B(x_* - x) \qquad \textit{for all } x \in \mathbb{R}. \tag{22}$$

Proof. To prove that $A(x)$ is positive, we introduce $v(x) \equiv (|A(x)|, B(x))$. Clearly $v \in \tilde{E}$ and $J[u] = J[v]$. This implies that v is also a minimiser of J on $\tilde{E}$. Since $\lim_{x\to-\infty} A(x) = 1$, there exists a constant $x_0 \in \mathbb{R}$ such that $A(x) > 0$ for $x < x_0$. Observe that $u(x)$ and $v(x)$ are solutions to (S) that coincide for $x < x_0$. From the uniqueness of solutions to initial value problems, we conclude that $v(x) = u(x)$ for all $x \in \mathbb{R}$. Therefore $A(x) \geq 0$ for all $x \in \mathbb{R}$. By exchanging the roles of A and B we obtain that $B(x)$ is also positive for all $x \in \mathbb{R}$.

Writing $u = (A, B)$, we define x_1, x_2, u_1 and u_2 as in the proof of Lemma 14. We now assert that $x_1 = x_2$. When we assume by contradiction that $x_1 < x_2$, then we see from (21) that

$$2J[u] - J[u_1] - J[u_2] > 0.$$

Thus, at least one of the inequalities

$$J[u_1] < J[u], \qquad J[u_2] < J[u]$$

holds. Since this contradicts the fact that u is a minimiser of J on $\tilde{E}$, we conclude that $x_1 = x_2 \stackrel{\text{def}}{=} x_*$ and

$$A(x) = B(x) \;\Leftrightarrow\; x = x_*.$$

Because u minimises J, we now see from (21) that

$$J[u] = J[u_1] = J[u_2].$$

Thus, u_1 is also a minimiser of J on $\tilde{E}$ and therefore a solution to (S). Observe that $u_1(x)$ and $u(x+x_*)$ are solutions to (S) that coincide for $x < 0$. From the uniqueness of solutions to initial value problems, we conclude that $u_1(x) = u(x+x_*)$ for all $x \in \mathbb{R}$. It now follows from the definition of u_1 that $u(x)$ satisfies (22). □

We remark that since the system (S) is autonomous, we can shift minimising solutions in such a way that $x_* = 0$. Using arguments similar to those in the previous lemma, we now show that minimisers of J have no self-intersections in the (A, B)-plane.

Lemma 17. *Let $u(x) = (A(x), B(x))$ be a minimiser of J on $\tilde{E}$. Suppose there exist constants x_1 and x_2 in $\mathbb{R}$ such that $u(x_1) = u(x_2)$. Then $x_1 = x_2$.*

Proof. Without loss of generality we assume that $x_1 \le x_2$. Defining

$$v(x) = \begin{cases} u(x) & \text{if } x \le x_1, \\ u(x + x_2 - x_1) & \text{if } x \ge x_1, \end{cases}$$

we have $v \in \tilde{E}$ and

$$J[v] = J[u] - \int_{x_1}^{x_2} \Big\{ \tfrac{1}{2}|u'(x)|^2 + W\big(u(x), \gamma\big) \Big\}\, dx \le J[u].$$

Thus $v(x)$ is also a minimiser of J on $\tilde{E}$. Observe that $u(x)$ and $v(x)$ are solutions to (S) that coincide for $x < x_1$. From the uniqueness of solutions to initial value problems, we conclude that $v(x) = u(x)$ for all $x \in \mathbb{R}$. Therefore either $x_1 = x_2$ or $u(x) = (0,1)$ for all $x \ge x_1$. However, the latter would imply (again by the uniqueness for the initial value problem) that $u(x) = (0,1)$ for all $x \in \mathbb{R}$, which contradicts the fact that $u \in \tilde{E}$. We thus conclude that $x_1 = x_2$. □

In the next two lemmas we prove lower and upper bounds on minimisers of J on $\tilde{E}$.

Lemma 18. *Any minimiser $u(x) = (A(x), B(x))$ of J on $\tilde{E}$ satisfies the lower bound*

$$A(x) + B(x) \ge \min\Big\{1, \frac{2}{\sqrt{2+\gamma}}\Big\} \qquad \textit{for all } x \in \mathbb{R}. \tag{23}$$

Proof. We argue by contradiction. If $u(x)$ does not satisfy (23), then we can construct a function $v \in \tilde{E}$ such that $J[v] < J[u]$.

To simplify the notation we introduce (for $\gamma > 0$)

$$b(\gamma) \stackrel{\text{def}}{=} \min\Big\{\frac{1}{\sqrt{2}}, \sqrt{\frac{2}{2+\gamma}}\Big\}.$$

As in Lemma 3 we change variables to $C = (A+B)/\sqrt{2}$ and $D = (A-B)/\sqrt{2}$ and we replace $u(x) = (A(x), B(x))$ by $\hat{u}(x) = (C(x), D(x))$. We remark here that $|D(x)| \le C(x)$ for all $x \in \mathbb{R}$, since by Lemma 16 the range of $u(x)$ is contained in the first quadrant of the (A, B)-plane. Rearranging the terms in (8) we obtain (for $\gamma > 0$)

$$\hat{J}[\hat{u}] = \int_{\mathbb{R}} \Big\{\tfrac{1}{2}(C'^2 + D'^2) + \tfrac{1}{8}\Big(h(D) + (2+\gamma)C^4 + \big[(4-2\gamma)D^2 - 4\big]\,C^2\Big)\Big\}\,dx,$$

with $h(D) = 2 + (2+\gamma)D^4 - 4D^2$.

We now define $\hat{v}(x) = (\tilde{C}(x), \tilde{D}(x))$, where

$$\tilde{C}(x) = \max\big\{C(x), b(\gamma)\big\} \quad \text{and} \quad \tilde{D}(x) = D(x).$$

As before, we have that $v \in \tilde{E}$. It remains to show that $\hat{J}[\hat{v}] \le \hat{J}[\hat{u}]$ and that the inequality is strict if $\hat{v}(x) \ne \hat{u}(x)$ for some $x \in \mathbb{R}$.

Since $|\tilde{C}'(x)| \le |C(x)|$ and D is unchanged, the remaining difficulty concerns the function

$$f(C) \stackrel{\text{def}}{=} (2+\gamma)C^4 + \big[(4-2\gamma)D^2 - 4\big]\,C^2.$$

We shall show that f is a strictly decreasing function of C on $0 < C < b(\gamma)$ for all $|D| \le b(\gamma)$ (recall that $|D(x)| \le C(x)$). This implies, since $C(x)$ is continuous, that replacing $C(x)$ by $\tilde{C}(x) = \max\big\{C(x), b(\gamma)\big\}$ strictly decreases the value of J if $\tilde{C}(x) \ne C(x)$ for some x, which proves the assertion.

We distinguish two cases: $\gamma \ge 2$ and $\gamma < 2$. If $\gamma \ge 2$, then $b(\gamma) = \sqrt{\frac{2}{2+\gamma}}$, and the minimum of $f(C)$ is attained at

$$C = \sqrt{\frac{2 + (\gamma-2)D^2}{2+\gamma}} \ge \sqrt{\frac{2}{2+\gamma}}.$$

It is seen from the graph of f that f decreases strictly for $0 < C < b(\gamma)$.

If $\gamma < 2$, then $b(\gamma) = \frac{1}{\sqrt{2}}$, and for $0 < C < \frac{1}{\sqrt{2}}$ and $|D| \le \frac{1}{\sqrt{2}}$ we have

$$f'(C) = 4(2+\gamma)C\left(C^2 + \frac{2-\gamma}{2+\gamma}D^2 - \frac{2}{2+\gamma}\right) < 0.$$

This completes the proof. □

Lemma 19. *Any minimiser $u(x) = (A(x), B(x))$ of J on $\tilde{E}$ satisfies the upper bound*

$$A^2(x) + B^2(x) \le 1 \qquad \textit{for all } x \in \mathbb{R}.$$

Proof. As in Lemma 13 we change to polar coordinates $u(x) = (r(x), \phi(x))$. In these coordinates we have (for $\gamma > 0$)

$$J[u] = \int_{\mathbb{R}} \left\{ \tfrac{1}{2}(r'^2 + r^2\phi'^2) + \tfrac{1}{4}(1 - r^2)^2 + \tfrac{1}{2}\gamma r^4 \sin^2 2\phi \right\} dx. \tag{24}$$

We argue by contradiction and thus suppose that $r(x) > 1$ for some $x \in \mathbb{R}$. We define $v(x) \equiv (\tilde{r}(x), \tilde{\phi}(x))$, where

$$\tilde{r}(x) = \min\{1, r(x)\} \qquad \text{and} \qquad \tilde{\phi}(x) = \phi(x).$$

As before it is clear that $v \in \tilde{E}$. From the continuity of u and (24) it is seen that $J[v] < J[u]$ if $r(x) > 1$ for some $x \in \mathbb{R}$, which contradicts the fact that u is a minimiser of J. □

Finally, we prove monotonicity of the angular coordinate $\phi(x)$.

Lemma 20. *For any minimiser $u(x) = (r(x), \phi(x))$ of J on $\tilde{E}$ we have*

$$\phi'(x) \geq 0 \qquad \textit{for all } x \in \mathbb{R}.$$

Proof. The proof is analogous to the proof of Lemma 13. Note that $r(x) > 0$ for all $x \in \mathbb{R}$ by Lemma 18, which implies that $\phi(x)$ is well-defined. We suppose by contradiction that there exist constants $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, such that $\phi'(x) < 0$ for $x \in (x_1, x_2)$. Plainly we can choose x_1 and x_2 such that

$$\frac{\pi}{2} > \phi(x_1) > \phi(x_2) > 0.$$

Since

$$\lim_{x \to -\infty} \phi(x) = 0 \qquad \text{and} \qquad \lim_{x \to \infty} \phi(x) = \frac{\pi}{2},$$

we infer that

$$\begin{aligned} x_3 &\stackrel{\text{def}}{=} \inf\{\tilde{x} \leq x_2 \mid \phi(x) > \phi(x_2) \text{ for all } x \in (\tilde{x}, x_2)\} \qquad \text{and} \\ x_4 &\stackrel{\text{def}}{=} \sup\{\tilde{x} \geq x_1 \mid \phi(x) < \phi(x_1) \text{ for all } x \in (x_1, \tilde{x})\} \end{aligned}$$

are well-defined and finite, and from the continuity of $\phi(x)$ we see that $\phi(x_3) = \phi(x_2)$ and $\phi(x_4) = \phi(x_1)$.

We recall that it follows from Lemma 16 that

$$\phi(x) = \frac{\pi}{4} \Leftrightarrow x = x_*.$$

We now distinguish two cases: $\phi(x_1) > \frac{\pi}{4}$ and $\phi(x_1) \leq \frac{\pi}{4}$. If $\phi(x_1) > \frac{\pi}{4}$ then $\phi(x) > \frac{\pi}{4}$ for all $x > x_1$, since otherwise $\phi(x)$ would be equal to $\frac{\pi}{4}$ more than once (namely for some $x \in (-\infty, x_1)$ and for some in $x \in (x_1, \infty)$). We now define $u_*(x) = (r_*(x), \phi_*(x))$ with

$$r_*(x) = r(x) \qquad \text{and} \qquad \phi_*(x) = \begin{cases} \phi(x) & \text{if } x \notin (x_1, x_4), \\ \phi(x_1) & \text{if } x \in (x_1, x_4). \end{cases}$$

It is easily seen that u_* is in $\tilde{E}$. Since $\frac{\pi}{2} > \phi_*(x) > \phi(x) > \frac{\pi}{4}$ for $x \in (x_1, x_4)$, we see that $\gamma\, r_*^4(x) \sin^2 2\phi_*(x) < \gamma\, r^4(x) \sin^2 2\phi(x)$. Besides, ${\phi_*}'(x) = 0$ on (x_1, x_4), so that $J[u_*] < J[u]$. This contradicts the fact that u minimises J on $\tilde{E}$

If $\phi(x_1) \leq \frac{\pi}{4}$ then clearly $\phi(x_2) < \frac{\pi}{4}$ and $\phi(x) < \frac{\pi}{4}$ for all $x < x_2$, since otherwise $\phi(x)$ would be equal to $\frac{\pi}{4}$ more than once. We now define $u^*(x) = (r^*(x), \phi^*(x))$ with

$$r^*(x) = r(x) \qquad \text{and} \qquad \phi^*(x) = \begin{cases} \phi(x) & \text{if } x \notin (x_3, x_2), \\ \phi(x_2) & \text{if } x \in (x_3, x_2). \end{cases}$$

A contradiction is obtained in the same manner as before. □

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G. J. B. van den Berg

Mathematical Institute
Leiden University
Niels Bohrweg 1
2333 CA Leiden
The Netherlands

gvdberg@wi.leidenuniv.nl
http://www.wi.leidenuniv.nl/~gvdberg/

and

R. C. A. M. van der Vorst

Center for Dynamical Systems
and Nonlinear Studies
Georgia Institute of Technology
Atlanta, GA 30332-0190
USA

rvander@math.gatech.edu
http://www.math.gatech.edu/~rvander/

or

Mathematical Institute
Leiden University
Niels Bohrweg 1
2333 CA Leiden
The Netherlands