Dynamics and equilibria of fourth order differential equations

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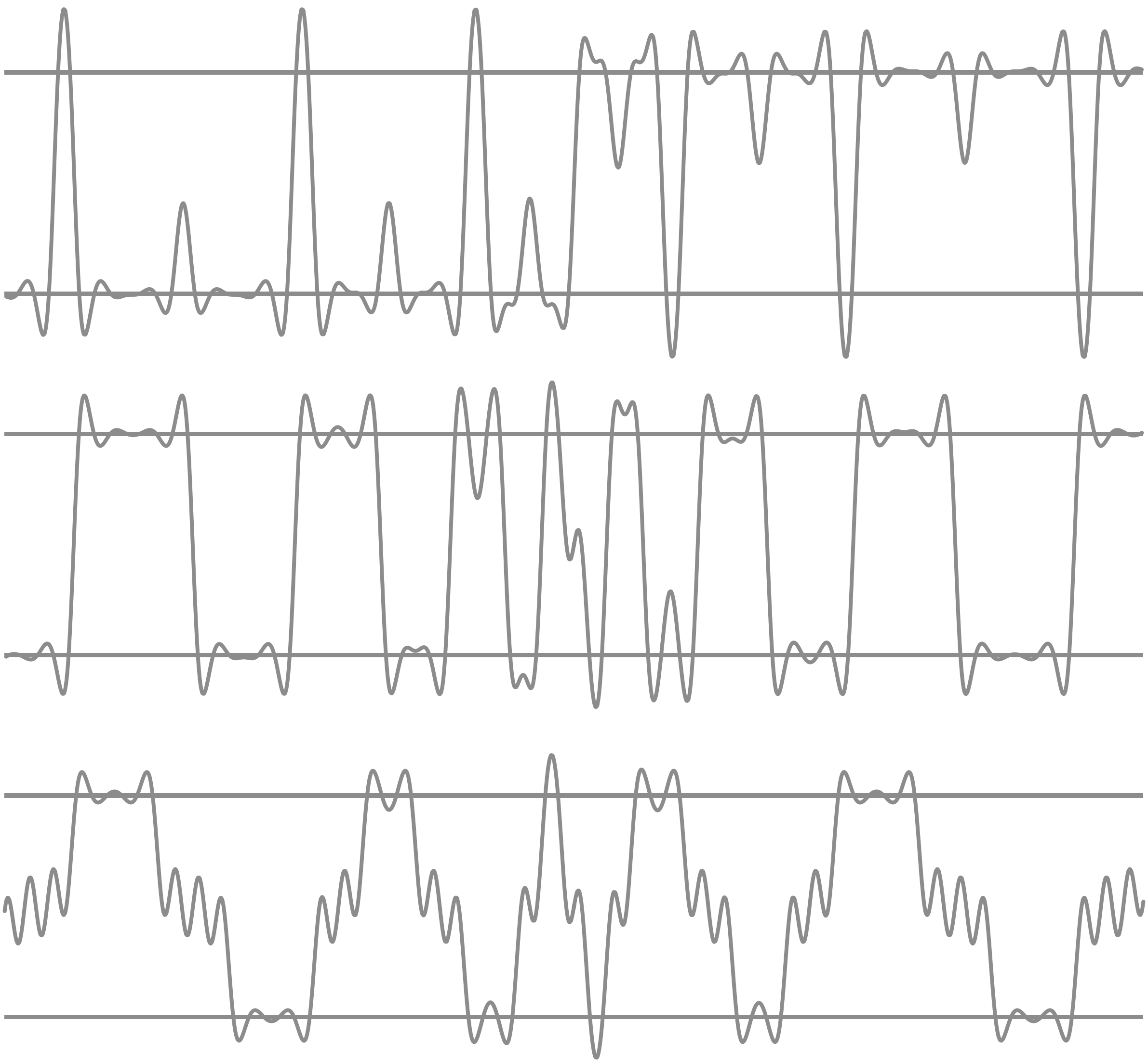

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Contents

Preface

The subject of this thesis is the mathematical study of a class of fourth order differential equations. In the introductory Chapter 1 a summary is presented of the results which are proved in the subsequent chapters. The mathematical results collected here originate from several papers, and they have been obtained in cooperation with a number of highly appreciated co-authors. In order of appearance: Chapter 2 is a minor modification of [21]. The content of Chapter 3 is joint work with Bill Kalies, Jarek Kwapisz and Rob van der Vorst, and is published in [88]. Chapter 4 is largely based on [25] and is joint work with Rob van der Vorst. The joy of Chapter 5 was shared with Joost Hulshof and Rob van der Vorst, and the contents are published in [22]. The results in Chapter 6 were obtained in collaboration with Bert Peletier and Bill Troy, and it appears in [23]. Chapter 7 differs only slightly from [24], and is again a fruit of the collaboration with Rob van der Vorst. Finally, Chapter 8 is based on [73] and also contains parts of [74], which are both joint work with Robert Ghrist and Rob van der Vorst.

Chapter 1

Introduction to fourth order equations

1.1 Prologue

The laws of nature are stated in the language of mathematics. Since physical laws describe changing quantities it is natural that they are expressed in the form of differential(-delay) equations. A well-known illustration is *Newton's law* in classical mechanics. Newton's law is a second order differential equation since it relates the acceleration of an object, the *second* derivative of its position, to the force exerted on it.

Apart from the fundamental laws of physics, many physical phenomena are modelled by differential equations. A simple example is the *heat equation*, which describes the changes of temperature as a function of time and place. This is another second order differential equation, although of a nature that is very different from Newton's law. The latter is an *ordinary* differential equation (the position depends on *one* variable: time), whereas the heat equation is a *partial* differential equation (the temperature varies in both time and space).

Notwithstanding their differences, these two examples of differential equations have a common ground: given the initial state (and possibly boundary conditions) the differential equation completely determines the evolution of the system. This is what characterises these differential equations as so-called *dynamical systems*. Physics is not the only science in which differential equations play a prominent role. There are many areas where differential equations are used as a model for the problem at hand. To name a few examples: the reaction and diffusion of chemicals, the dynamics of populations in biology, the development and treatment of diseases in medicine, or the flow of a fluid or gas, which has applications ranging from fundamental astronomy to meteorology to industrial engineering.

Besides the applicability there is another motivation to study differential equations: the mathematical challenge. The theory of differential equations has connections with many different branches of mathematics. A further development of the theory of differential equations both facilitates applications and provides new insights in mathematics. And the penultimate reason for the research presented in this thesis is to understand the underlying structure of a mathematical problem, which we shall now begin to describe.

This thesis revolves about the fourth order parabolic differential equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + u - u^3 \qquad \gamma > 0,\ \beta \in \mathbb{R}, \tag{1.1}$$

and its generalisations. Here $u = u(t, x)$ is a function of the time t and the space variable x. The name *Extended Fisher-Kolmogorov* is usually attached to equations of type (1.1) when $\beta > 0$ [53, 117], while for $\beta < 0$ the name *Swift-Hohenberg* equation is more appropriate [137]. We restrict our attention to one spatial dimension because some of the application are truly one dimensional, but also because more spatial dimensions make the analysis much harder and of a different nature.

Substantial attention is given to stationary (i.e. time-independent) solutions of this equation. While Equation (1.1) is a partial differential equation, such equilibria satisfy the ordinary differential equation

$$-\gamma u'''' + \beta u'' + f(u) = 0, \qquad \text{with } f(u) = u - u^3. \tag{1.2}$$

The nonlinearity $f(u) = u - u^3$ is the prototypical example which leads to a *bi-stable* system (the states $u \equiv \pm 1$ are stable for the homogenised equation $\frac{du}{dt} = u - u^3$), and in the introduction we will focus on this particular nonlinearity. This being said, it should be stressed that many of the results are obtained for broad classes of nonlinear functions $f(u)$, and often a wider class of fourth order equations is considered.

Casually we have already introduced one of the most important properties of Equations (1.1) and (1.2), namely that they are *nonlinear* equations. This, of course, is the main reason that the problem is mathematically interesting. The other reason is that the equations are *fourth* order. Before we turn our attention to the origin of these equations in applications, let us briefly discuss the mathematical viewpoint.

Equations (1.1) and (1.2) can be regarded as fourth order extension of the heat equation and Newton's law respectively. Concerning the partial differential equation (1.1), when we set $\gamma = 0$ and $\beta > 0$, then we obtain a nonlinear heat equation. The reason that we require γ to be positive is precisely that we want the equation to have a character which is similar to the second order equation, namely to be of parabolic (diffusive) type.

The second order equation has an important property which the fourth order equation in general does not possess: a comparison or maximum principle. Let us explain the main point of this principle: when the initial state for the second order equation ($\gamma = 0$) is positive, then the solution will stay positive for all time. This property is an essential feature of the model in case the variable u represents a temperature or a concentration of some chemical. The fourth order equation ($\gamma > 0$) does not obey this conservation of positivity. This, on the one hand, allows for a wide range of dynamics which second order equations do not and cannot possess, while on the other hand it means the loss of one of our main analytic tools in studying the dynamics.

For the ordinary differential equation (1.2) the fourth order character provides the possibility of chaotic behaviour. Second order (autonomous) equations cannot have chaotic behaviour, basically because such a system does not have enough degrees of freedom. Thus the variety of patterns that evolve in the fourth order equations is much larger than in the second order equation. And again, because these types of dynamics occur in a higher dimensional space, they are also more difficult to analyse.

The fourth order equations (1.1) and (1.2) are part of an immense collection of higher order differential equations. One of the reasons to study *fourth* order equations, and in particular those of the form (1.1) and (1.2), is that they lie on the edge of what presently can be analysed rigorously. They are comparable to a set of two coupled (second order) reaction-diffusion equations, which are extensively studied (see e.g. [75, 76, 58, 108]). Although in general no form of a comparison principle holds for the fourth order equations (in some cases such a principle holds for coupled second order equations), this is outweighed, or at least balanced, by the benefit of dealing with a single equation. On a different level, an advantage of the fourth order equations (1.1) and (1.2) over third order equations is that they have a variational structure (the equivalent of the Lagrangian and Hamiltonian formulation in classical mechanics)[1]. This structure acts as a handle for the mathematical analysis.

A final reason to investigate Equations (1.1) and (1.2) is that, with various nonlinearities $f(u)$, they serve as a model in an abundance of applications:

- the behaviour close to a so-called Lifshitz point in phase transition physics (e.g. nematic liquid crystals, ferroelectric crystals) [85].
- the rolls in a Rayleigh-Bénard convection cell (two parallel plates of different temperature with a liquid in between) [52].
- spontaneous pattern formation in second order materials (e.g. polymeric fibres) [99].
- the waves on a suspension bridge [97, 44].
- geological folding of rock layers [31].
- the buckling of a strut on a nonlinear elastic foundation [3].
- travelling water waves in a shallow channel [35].
- pulse propagation in optical fibres [1].
- the patterns near a degeneracy of co-dimension 2 in a system of two reaction-diffusion equations [59].
- the propagation of a front into an unstable state leading to pattern formation behind the front [53, 49].

In general, any system described by a second order Lagrangian leads to a fourth order equation (we will come back to this in Section 1.3). The interpretation of the quantity u depends on the application (e.g. it may indicate the temperature, vertical position, intensity or order parameter), but in all cases it signifies the deviation from an underlying average value (and hence u may be negative as well as positive).

Having stressed and illustrated the significance of the fourth order differential equations which are the subject of this thesis, it is by no means intended to give the impression as if this is the most important topic in the universe or, for that matter, in mathematics.

Let us briefly give an outline of the sort of questions that we deal with. As already mentioned, a differential equation is often accompanied by boundary and/or initial conditions. In this thesis various cases are studied. For example, we examine the large time behaviour of solutions of the initial value problem associated to (1.1) on a finite interval $[0, L]$, where we take Neumann boundary conditions at $x = 0$ and $x = L$. But we also study different boundary conditions, and we consider unbounded intervals as well. Regarding Equation (1.2) we consider both special solutions such as periodic and heteroclinic solutions, as well as the properties of the set of all bounded solutions (and even unbounded ones).

The behaviour of solutions of (1.1) and (1.2) depends critically on the values of the parameters γ and β. We note that these two parameters can be combined into a single

[1] Some differential equations of odd order (for example the Korteweg-de Vries equation) have a variational structure of a very different nature, where the Lagrangian action acts as the Hamiltonian in an infinite dimensional setting, see e.g. [109].

parameter via a scaling of the spatial coordinate. For example, without loss of generality one may set $\gamma = 1$. However, since we are also interested in the limiting behaviour as $\gamma \to 0$, we will retain both parameters for the moment. Three significant parameter regions can be identified, which correspond to the three different natures of the equilibrium points $u = \pm 1$ of Equation (1.2). These are: real saddle for $\frac{\beta}{\sqrt{\gamma}} \geq \sqrt{8}$, saddle-focus for $-\sqrt{8} < \frac{\beta}{\sqrt{\gamma}} < \sqrt{8}$, and center for $\frac{\beta}{\sqrt{\gamma}} \leq -\sqrt{8}$. We will come back to this in much more detail in Section 1.3.

The techniques used to study Equations (1.1) and (1.2) have to be chosen suitably for each parameter region. To summarise, there are roughly two ways to investigate the solutions of (1.2): the shooting method and the variational method. With the shooting method one tries to make a fairly detailed study of the flow (associated to (1.2)) in the (four dimensional) phase space. The variational approach is set in an (infinite dimensional) function space, where critical points of an appropriate functional are sought. Variational methods can often unveil general phenomena in a large class of systems, while more detailed information for specific examples may be obtained via a shooting procedure. In this thesis both methods are employed, and we also present a variational approach to the shooting method. We remark that the mathematical techniques are the source of an additional partition in the range of parameter values: most techniques are applicable either for positive or for negative values of β.

Finally, let us make some comments about the symmetries of Equation (1.1). The equation is invariant under the transformations $x \to -x$ as well as $u \to -u$. The first invariance signifies the equivalence of left and right. This symmetry is exploited in many (though not all) of the results presented in this thesis. On the other hand, the second symmetry is rather less essential, although it is sometimes very useful to be able to simplify the presentation. Hence, the majority of the results is obtained for general nonlinearities $f(u)$, not necessarily having the symmetry $f(-u) = -f(u)$, and sometimes even for a more general class of fourth order equations. Some of the more detailed results are only valid under the assumption of this additional symmetry.

Before we give an outline of the results about the fourth order equations (1.1) and (1.2), we first review the second order analogues.

1.2 The second order equation

Second order equations have long been a workhorse of applied mathematics, the most wide-spread example being the *Fisher-Kolmogorov* or *Allen-Cahn* equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3. \tag{1.3}$$

This partial differential equation is often referred to as a nonlinear reaction-diffusion equation (different nonlinearities have been considered as well). It has been used to model, among others, phase transitions, chemical reactions and populations genetics [94, 12, 68]. The study of (1.3) has been extensive and we can only give a brief summary of the relevant results. As a general reference we refer to [81, chapter 5] for a thorough analysis of (1.3) and to [79, 80] for an overview of results on scalar parabolic equations.

Note that (1.3) can be obtained as the second order analogue of (1.1) when β is posi-

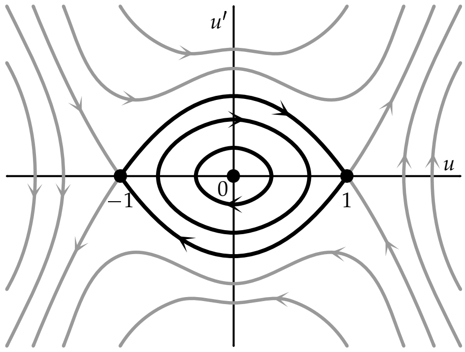

Figure 1.1: The phase-plane for Equation (1.4). The bounded orbits/solutions are indicated in black.

tive by setting $\gamma = 0$ and rescaling x. The time-independent solutions of (1.3) satisfy the ordinary differential equation

$$u'' = -u + u^3. \tag{1.4}$$

In the analysis of (1.4) it is often useful to view it as the equation of motion of a particle in the potential $F(u) = -\frac{1}{4}(u^2 - 1)^2$.

Of special interest are the *bounded* solutions of (1.4). By bounded solutions we mean functions which satisfy (1.4) for all $x \in \mathbb{R}$ and which are uniformly bounded. These bounded solution can easily be gathered from the phase-plane depicted in Figure 1.1:

- The homogeneous/constant states $u \equiv 0$, $u \equiv -1$ and $u \equiv 1$. The states $u \equiv \pm 1$ are saddle point for the flow of Equation (1.4), and they are *stable* equilibria of (1.3), whereas $u \equiv 0$ is an *unstable* equilibrium of (1.3) (and a center of (1.4)).
- Two *kinks* or heteroclinic solutions. These solutions describe monotone transition layers connecting the stable homogeneous states $u \equiv \pm 1$. They are antisymmetric and are accidentally given explicitly by $u = \pm \tanh\left(\frac{x}{\sqrt{2}}\right)$.
- A family of periodic solutions filling up the space between $u \equiv 0$ and the cycle of heteroclinic solutions. These periodic solutions oscillate around 0 and have an amplitude between 0 and 1. They are symmetric with respect to their extrema and antisymmetric with respect to their zeros.

For the classification of the periodic solutions it is important to introduce the *energy* or *Hamiltonian*:

$$\mathcal{E}[u] = \frac{1}{2}u'^2 - \frac{1}{4}(u^2 - 1)^2, \tag{1.5}$$

which is constant along solutions of (1.4). This just corresponds to the classical energy of a particle in a potential. The notation is somewhat ambiguous: depending on the context $\mathcal{E}[u]$ either signifies the energy of a solution or the energy level (manifold) in phase space.

The constant states $u \equiv \pm 1$ as well as the heteroclinic solutions have energy $\mathcal{E}[u] = 0$, while the other constant state $u \equiv 0$ has energy $\mathcal{E}[u] = -\frac{1}{4}$. For all intermediate values of the energy there is precisely one (modulo translation) periodic solution. In other words, in each of the energy levels $\mathcal{E}[u] = E \in (-\frac{1}{4}, 0)$ lies a unique periodic orbit. The period of the periodic solutions can be expressed in terms of an elliptic integral, and it can be shown that the period L strictly increases with increasing amplitude. The period ranges from $L = 2\pi$ in the limit of zero amplitude to infinity as the amplitude tends to 1. The periodic solutions can thus be parametrised by their amplitude, period or energy.

In connection with the Hamiltonian (1.5) there is a *Lagrangian action* associated with Equation (1.4):

$$J[u] = \int \Big(\tfrac{1}{2} u'(x)^2 + \tfrac{1}{4} \big(1 - u(x)^2\big)^2 \Big)\, dx. \tag{1.6}$$

Solutions of (1.4) correspond to critical points of the action functional (1.6) in an appropriate function space. For example, the kink can be characterised as the minimiser of (1.6), with $\mathbb{R}$ as the domain of integration, in the affine function space $\zeta + H^1(\mathbb{R})$, where e.g. $\zeta = \tanh x$.

The Lagrangian action has an additional, related property: $J[u](t)$ it is a Lyapunov function for the flow of (1.3), i.e., it is non-increasing in time (again in an appropriate function space). To be precise, Equation (1.3) is the L^2-gradient flow of (1.6). The existence of a Lyapunov functional has important consequences for the attractor of (1.3). The attractor is, roughly speaking, the ω-limit set of all possible initial values, and it describes the long-time behaviour of all solutions. For Equation (1.3) on a finite interval $[0, L]$ the attractor consists of the equilibrium solutions and all connecting orbits between these equilibria. Here one may take, for example, Neumann boundary conditions $\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, L) = 0$, or Dirichlet boundary conditions $u(t, 0) = u(t, L) = 0$. We remark that equilibrium solutions which obey Neumann or Dirichlet boundary conditions, can be extended to periodic solutions of (1.4) and are thus part of the family of periodic solutions described previously.

Equation (1.3) possesses another important property, which is intimately connected to a broad class to *second order* parabolic equations, namely the comparison principle: when u_1 and u_2 are solutions of (1.3) with $u_1(0, x) \geq u_2(0, x)$ for all x, then $u_1(t, x) \geq u_2(t, x)$ for all $t > 0$. Second order parabolic equations in *one* space dimension, such as (1.3) have a stronger property: the number of intersections of two solutions, the *zero number*, is a non-increasing function of the time t [101]. This principle has been used to give a complete characterisation of the attractor of (1.3) on finite intervals [4].

We now turn our attention from a finite interval to the problem on the entire real line. The existence of a pair of heteroclinic solutions has already been mentioned, and we now discuss a different type of special solutions: uniformly translating profiles, or *travelling waves* (heteroclinic solutions can be interpreted as standing waves). Consider the following generalisation of (1.3):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (1 - u^2)(u + a), \qquad a \in [0, 1). \tag{1.7}$$

Substituting the travelling wave Ansatz $u(t, x) = U(x - ct)$, where c is the (a priori unknown) wave speed, one obtains the ordinary differential equation (where we have switched to lower case again)

$$u'' = -cu' - (1 - u^2)(u + a). \tag{1.8}$$

As before it is sometimes useful to view this as the equation of motion of a particle in a potential of the form depicted in Figure 1.2 under the influence of friction. We now see that the energy $\mathcal{E}[u]$ is no longer a conserved quantity, but instead $\mathcal{E}'[u] = -cu'^2$.

There are two types of travelling waves: those connecting the two stable states $u = \pm 1$, and those connecting the unstable state to one of the stable states. For all $a \in (0, 1)$ we have $\mathcal{E}[1] > \mathcal{E}[-1] > \mathcal{E}[-a]$, hence we may assume that $c > 0$ (i.e., waves travelling from left to right) and restrict our attention to heteroclinic solutions of (1.8) from 1 to -1, and from

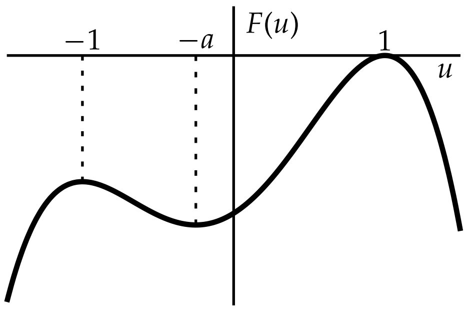

Figure 1.2: The potential associated to (1.8) for $a = 0.25$.

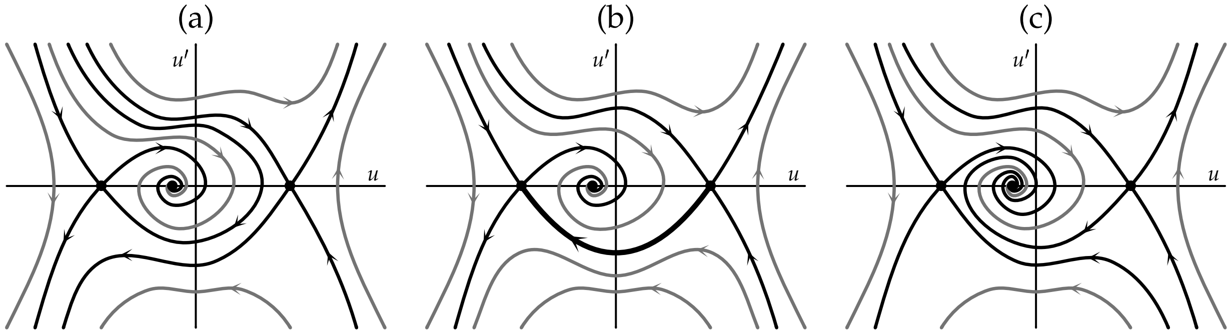

Figure 1.3: The phase-plane for Equation (1.8): (a) for $0 < c < a\sqrt{2}$; (b) for $c = a\sqrt{2}$; (c) for $c > a\sqrt{2}$.

± 1 to $-a$. The phase-planes for three distinctive values of c are shown in Figure 1.3. We notice the following:

- A travelling wave connecting -1 to $-a$ exists for all wave speeds $c > 0$.
- A travelling wave connecting 1 to -1 exists only for $c = a\sqrt{2}$.
- A travelling wave connecting 1 to $-a$ exists for $c > a\sqrt{2}$, but not for $c \leq a\sqrt{2}$.

The stability of these travelling waves has been determined in [68]. The travelling wave connecting the two stable states is the limiting profile of solutions of (1.7) for a large class of initial conditions. The waves connecting the stable to the unstable state (the stable state invades the unstable state) are far less stable, as the existence of a continuum of such travelling waves already indicates.

Let us finally recall that the two major tools in examining the second order equation are on the one hand an analysis of the phase-plane, and on the other hand applications of a comparison/maximum/lap-number principle. Both these techniques are not readily available for fourth order equations, since a four dimensional phase space is much harder to analyse (there are more degrees of freedom), and a comparison principle is just non-existent for fourth order equations (it is violated in very simple examples such as the linear equation).

1.3 The fourth order equation

A natural extension of Equation (1.3) is the Extended Fisher-Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0. \tag{1.9}$$

It has been proposed in [49, 53] as a pattern generating generalisation of the classical Fisher-Kolmogorov equation. In this section we give an overview of the results obtained in this thesis concerning Equation (1.9), or more generally on equations of the form

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + f(u) \qquad \gamma > 0,\ \beta \in \mathbb{R}, \tag{1.10}$$

where $f(u)$ is a nonlinear function. A second example which falls in this category is the Swift-Hohenberg (SH) equation

$$\frac{\partial u}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + \alpha u - u^3, \qquad \alpha \in \mathbb{R}. \tag{1.11}$$

Note that this is an equation of the form (1.10). One may perform a rescaling to obtain the equation $u_t = -u_{xxxx} + \beta u_{xx} + u - u^3$ where $\beta = -\frac{2}{\sqrt{\alpha-1}}$, i.e., β has the opposite sign as compared to the EFK equation (1.9). The SH equation was first introduced as a model equation in the study of Rayleigh-Bénard convection cells [137]. It serves as a model equation for the behaviour of a system near the onset of a finite wavelength instability [126, 50].

There are many more examples of physical systems where Equation (1.10) or similar equations play a role (see also Section 1.1). We do not attempt to give a complete account, but instead refer to [38, 39] and [123].

To understand the behaviour of solutions of the evolution equations (1.9), (1.10) and (1.11), it is essential to understand the stationary (time-independent) solutions. A major part of the results are therefore concerned with solutions of the equation

$$-\gamma u'''' + \beta u'' + f(u) = 0. \tag{1.12}$$

This equation describes the stationary solutions of (1.10), but is also interesting in its own right. As should be clear from the above examples, we are especially interested in the case where $f(u) = u - u^3$. This is the typical example of a bi-stable nonlinearity. In the description of the results we will for simplicity concentrate on this particular choice of $f(u)$:

$$-\gamma u'''' + \beta u'' + u - u^3 = 0. \tag{1.13}$$

It should be stressed that most of the results are valid for a broad class of nonlinearities.

There are two important functionals associated to (1.12). First, when we multiply the equation by u' and integrate, we obtain the *energy* or *Hamiltonian*

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -\gamma u'u''' + \frac{\gamma}{2}u''^2 + \frac{\beta}{2}u'^2 + F(u),$$

where

$$F(u) \stackrel{\text{def}}{=} \int_1^u f(s)\, ds \tag{1.14}$$

is the *potential*. The energy is a conserved quantity along orbits of (1.12). Second, the Lagrangian action associated with this Hamiltonian is

$$J[u] \stackrel{\text{def}}{=} \int \Big(\frac{\gamma}{2} u''(x)^2 + \frac{\beta}{2} u'(x)^2 - F(u(x)) \Big)\, dx. \tag{1.15}$$

The solutions of (1.12) correspond to critical points of the action $J[u]$ and vice versa. The domain of integration depends on the kind of solution under investigation. We will go into more details of this *variational* structure in Section 1.3.3.

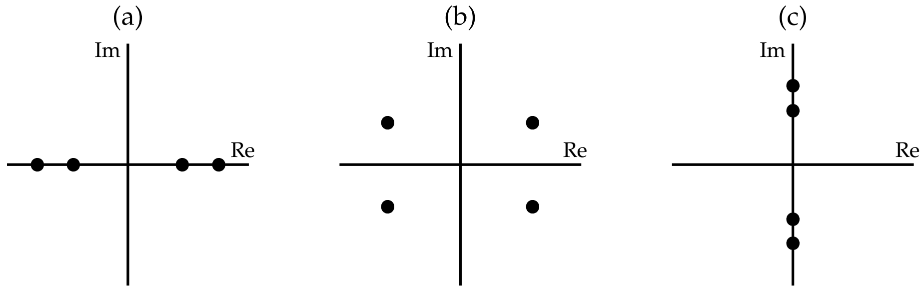

Figure 1.4: Eigenvalues (in the complex plane) of the linearised problem around ± 1: (a) $\frac{\beta}{\sqrt{\gamma}} \geq \sqrt{8}$, a real saddle; (b) $-\sqrt{8} < \frac{\beta}{\sqrt{\gamma}} < \sqrt{8}$, a saddle-focus; (c) $\frac{\beta}{\sqrt{\gamma}} \leq -\sqrt{8}$, a center.

Let us remark here that Equation (1.12) can be formulated as a Hamiltonian system. Define $v = u'$, $p_u = \beta u' - \gamma u'''$ and $p_v = \gamma u''$. The Hamiltonian in these new variables is

$$H(u, v, p_u, p_v) \stackrel{\text{def}}{=} \frac{1}{2\gamma} p_v^2 + v p_u - \frac{\beta}{2} v^2 + F(u) = \mathcal{E}[u], \tag{1.16a}$$

and (1.12) is equivalent to the Hamiltonian system

$$\begin{aligned} u' &= \tfrac{\partial H}{\partial p_u}, & p_u' &= -\tfrac{\partial H}{\partial u}, \\ v' &= \tfrac{\partial H}{\partial p_v}, & p_v' &= -\tfrac{\partial H}{\partial v}. \end{aligned} \tag{1.16b}$$

Although only limited direct use of this formulation is made in this thesis, the Hamiltonian always plays an important role in the background.

For the special nonlinearity $f(u) = u - u^3$ one has $F(u) = -\frac{1}{4}(u^2-1)^2$, and for $\gamma, \beta > 0$ the integrand in (1.15) is positive. From a variational point of view it makes sense to define $F(u) = -\int_1^u f(s)\,ds$ instead of (1.14), i.e. with opposite sign. This will lead to a change of notation in some chapters of this thesis. Although it might cause some confusion, this seems inevitable. The definition in (1.14) is the natural one when one draws upon the analogy with a classical mechanical system, whereas when the variational structure is prevalent, the definition with the opposite sign is more sensible. In this introduction we will stick to the definition as given in (1.14). There is another notational issue: the two parameters in (1.12) can be replaced by just one (through a scaling). In some chapters the equation will be rewritten as (with $q = -\frac{\beta}{\sqrt{\gamma}}$)

$$u'''' + q u'' - f(u) = 0, \qquad q \in \mathbb{R}.$$

Equation (1.13) has three equilibrium points: $u = 0$ and $u = \pm 1$. The eigenvalues of the linearised problem around ± 1 are depicted in Figure 1.4. While for $\frac{\beta}{\sqrt{\gamma}} \geq \sqrt{8}$ the eigenvalues are real, they are complex for $|\frac{\beta}{\sqrt{\gamma}}| < \sqrt{8}$, and purely imaginary for $\frac{\beta}{\sqrt{\gamma}} \leq -\sqrt{8}$, with the corresponding nature of the equilibrium points $u = \pm 1$ being *real saddle*, *saddle-focus* and *center* respectively. As we will see later on, the dynamics of (1.13) in these parameter regions differs quite drastically. Note that the third equilibrium point $u = 0$ is a *saddle-center* for all parameter values.

We will in particular investigate bounded solutions of (1.12). These include periodic solutions, homoclinic solutions (pulses) and heteroclinic solutions (kinks), but other types of bounded solutions (among others so-called chaotic profiles) can also be present (in contrast to the second order equation). These different types of solutions appear in different situations in applications, and the mathematical treatment often varies as well.

1.3.1 Uniqueness

In the study of differential equations the second question is always about uniqueness (multiplicity); the first question, that of existence, will be discussed in the next sections. The problem that we address here is what happens to the bounded solutions of the second order equation (1.3) when we add a fourth order term with a small coefficient γ (Equation (1.13)). Without loss of generality (by a scaling of x) we may put $\beta = 1$.

One way of proceeding is via singular perturbation theory [87, 67]. This leads to results for $\gamma < \varepsilon$, where ε is some unknown small positive constant. This is a very general and fruitful method and often information about the stability of solutions can be found as well [72, 2, 130].

Here we pursue a different method which covers a well-defined and rather large range of parameter values, and is specific to equations of the form (1.12). Let us first illustrate the power of this method by considering the results that are obtained for Equation (1.13). We find that for all $\gamma \leq \frac{1}{8}$ the bounded solutions of the fourth order equation (1.13) correspond exactly to those of the second order equation ($\gamma = 0$). Moreover, the projections of two bounded orbits onto the (u, u')-plane do not intersect. Hence we may refer to Figure 1.1 again.

Theorem 1.1 *The only bounded solutions of* (1.13) *with* $\beta = 1$ *and* $\gamma \in (0, \frac{1}{8}]$ *are the three equilibrium points, two monotone antisymmetric kinks and a one-parameter family of periodic solutions, parametrised by the energy* $E \in (-\frac{1}{4}, 0)$.

Additionally, one finds that the periodic solutions are antisymmetric with respect to their zeros, that they form a continuous family, and that the period strictly increases with increasing amplitude. Furthermore, the monotonically increasing heteroclinic orbit is the transverse intersection of the unstable manifold $W^u(-1)$ and the stable manifold $W^s(+1)$ in the energy level $\mathcal{E} = 0$.

These results show that the picture which can be obtained for the EFK equation via perturbation methods actually extends all the way to $\gamma = \frac{1}{8}$. This is already interesting in its own right, but part of this analysis will also be used later on to derive further results.

We now turn to a more general setting. Consider functions $f(u) \in C^1(\mathbb{R})$ and define, for $-\infty \leq a < b \leq \infty$,

$$\omega(a, b) \stackrel{\text{def}}{=} \max\{0, \max_{u \in [a,b]} -f'(u)\}.$$

Introduce sets of bounded functions

$$\mathcal{B}(a, b) \stackrel{\text{def}}{=} \{u \in C^4(\mathbb{R}) \mid u(x) \in [a, b] \text{ for all } x \in \mathbb{R}\}.$$

One can often find an a priori bound on the set of all bounded solutions, i.e., for some $-\infty \leq a < b \leq \infty$ all bounded solutions of (1.12) are in $\mathcal{B}(a, b)$. It is important to keep in mind that these a priori bounds are usually valid for a range of values of γ. As will be clear from the statement of the theorems below, a better bound leads to a lower value of ω, which in turn leads to a stronger result. These definitions may seem a bit technical at first sight, but we will see shortly that they lead to powerful results (see also Chapter 2).

An essential property of smooth second order autonomous ordinary differential equations is that there is exactly one solution through every point in the phase-plane. The following theorem states that the (u, u')-plane preserves this uniqueness property for *bounded* solutions of the fourth order equation as long as γ is not too large.

Theorem 1.2 *Let u_1 and u_2 be bounded solutions of* (1.12) *with $\beta = 1$, i.e., u_1 and u_2 are in $\mathcal{B}(a, b)$ for some $-\infty < a < b < \infty$. Suppose that $\gamma \in \big(0, \frac{1}{4\omega(a,b)}\big]$. Then the paths of u_1 and u_2 in the (u, u')-plane do not cross.*

Let us give some examples: for the double-well potential $F(u) = \frac{1}{4}(u^2 - 1)^2$ (note that this is *not* the EFK potential) and for the periodic potential $F(u) = \cos u$, we have that $\omega(-\infty, \infty) = 1$. So we do not need an a priori bound, and Theorem 1.2 applies for $\gamma \leq \frac{1}{4}$. For the EFK equation, $F(u) = -\frac{1}{4}(u^2 - 1)^2$, it turns out (we come back to this later) that for $0 < \gamma \leq \frac{1}{8}$ all bounded solutions satisfy $\|u\|_\infty \leq 1$, and $\omega(-1, 1) = 2$. Hence Theorem 1.2 applies for $\gamma \leq \frac{1}{8}$. We note that in all three cases Theorem 1.2 holds exactly up to the value of γ where the nature of some of the equilibrium points changes from real saddle to saddle-focus.

Now assume that for some $\gamma > 0$ we have an a priori bound on the set of bounded solutions, i.e., all bounded solutions of (1.12) are in $\mathcal{B}(a, b)$ for some $-\infty \leq a < b \leq \infty$, and let us assume that $\omega = \omega(a, b) < \infty$ and that $\gamma \leq \frac{1}{4\omega}$. Then if $\gamma \in [0, \frac{1}{4\omega}]$ bounded solutions of (1.12) do not cross by Theorem 1.2. An immediate consequence of Theorem 1.2 and the reversibility of (1.12), is that when $\gamma \in [0, \frac{1}{4\omega}]$ any bounded solution of (1.12) is symmetric with respect to its extrema. This implies that the only possible *bounded* solutions are

- equilibrium points,
- homoclinic solutions with one extremum,
- monotone heteroclinic solutions,
- periodic solutions with a unique maximum and minimum value.

It turns out that, as for the second order equation, the energy $\mathcal{E}[u]$ orders the bounded solutions in the phase-plane, i.e., the energy increases when walking in the $\pm u'$ direction away from the u-axis. The results for Equation (1.13) in Theorem 1.1 (and those just below the theorem) are direct consequences of the above considerations.

Let us now briefly discuss the idea behind these results. The method used to prove the above theorems is based on a splitting of Equation (1.12) into two second order equations. Define (setting $\beta = 1$)

$$\lambda = \frac{1}{\gamma}\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \omega\gamma}\right) \qquad \text{and} \qquad \mu = \frac{1}{\gamma}\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \omega\gamma}\right).$$

It is easily seen that λ and μ are positive real numbers if and only if $\gamma \leq \frac{1}{4\omega}$. Equation (1.12) can then be factorised as

$$\begin{cases} \sqrt{\gamma}\,(u'' - \lambda u) = v \\ \sqrt{\gamma}\,(v'' - \mu v) = f(u) + \omega u, \end{cases} \tag{1.17}$$

and the definition of ω ensures that $f(u) + \omega u$ is a non-decreasing function of u for $u \in [a, b]$. The fact that the right-hand sides of the two equations in (1.17) are increasing functions can now be used to apply arguments which bear some resemblance to the maximum principle.

The observation that this factorisation could be used for uniqueness proofs was first made in [3, 35]. There it was applied to prove uniqueness of the homoclinic solution (to 0) for (1.12) with $f(u) = -u + u^2$. In the above arguments, which are a summary of Chapter 2, it is shown how a *global* picture of all bounded solutions can be obtained.

We note that the uniqueness of the heteroclinic solution of (1.13) for $\gamma \leq \frac{1}{8}$ is also proved in [96] via the use of a Twist map (see also Section 1.3.4). Although the method used there in many ways resembles part of the analysis presented here, it highlights a totally different way of looking at the problem. Recently it has been shown in [15] that uniqueness of the monotone kink for $\gamma \leq \frac{1}{8}$ can also be proved using a reformulation in terms of a convolution equation. This approach allows extensions to higher order equations.

Theorems 1.1 and 1.2 provide statements about all energy levels at once. When the attention is restricted to a specific energy level, results on uniqueness and non-existence of bounded solutions can often be extended to larger values of γ. In [112] the above techniques are combined with a geometric analysis of the flow in the (u, u'')-plane to do just that.

Finally, let us come back to the a priori bounds on bounded solutions of (1.13). The fact that $\|u\|_\infty \leq 1$ for all bounded solutions if $\gamma \leq \frac{1}{8}$ follows from a repeated application of the maximum principle to (1.17). However, to be able to do this one needs another, less sharp, a priori bound. Such a bound can be obtained for all $\gamma > 0$ and $\beta \geq 0$: any bounded solution of (1.13) for $\gamma > 0$ satisfies $\|u\|_\infty < \sqrt{2}$. This bound and analogous ones for other nonlinearities are obtained as a generalisation of a result in [119].

1.3.2 Shooting methods

One of the basic tools in analysing the solutions of (1.12) is a topological shooting method which is designed to handle oscillatory solution graphs. It is based on a careful analysis of the location and height of the successive local maxima and minima of the graph. This method has been developed to study Equation (1.13) in a series of papers [117, 118, 119, 120, 121]. In this section we fix $\gamma = 1$ without loss of generality. Searching for *odd* solutions of (1.13) in some energy level $\mathcal{E}[u] = E$, one takes initial values

$$u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = \eta(\alpha), \tag{1.18}$$

where $\eta(\alpha) = \frac{\beta}{2}u' - (E + \frac{1}{4})\frac{1}{u'}$. One tries to find values of the slope α such that the solution of the initial value problem has certain properties. For example, one looks for solutions which tend to 1 as $x \to \infty$ to find heteroclinic solutions, or one proves the existence of one or more values of α such that for some $\xi > 0$ it holds that $u'(\xi) = 0$ and $u'''(\xi) = 0$, which leads to a periodic solution. The invariance of Equation (1.13) under the transformations $u(x) \to -u(x)$ and $u(x) \to -u(-x)$ is used to extend the solutions to all $x \in \mathbb{R}$. Note that such solutions may also be even about some point $\xi \in \mathbb{R}$, but we will nevertheless refer to them as odd, to distinguish them from the genuinely even solutions, which will be discussed next.

A similar shooting procedure can be set up for *even* solutions by taking as initial values

$$u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \pm\sqrt{2(E - F(\alpha))}, \quad u'''(0) = 0. \tag{1.19}$$

In this case one can study the existence of homoclinic and (again) periodic solutions. Note that in this latter approach only the reversibility of Equation (1.12) is used. When no

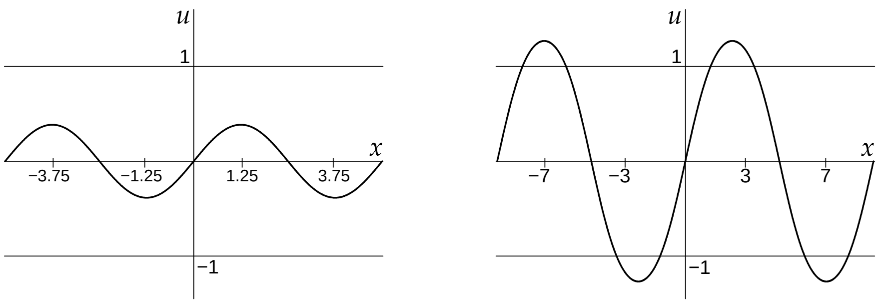

Figure 1.5: Two simple periodic solutions for $E = 0$, $\beta = -1$, $\gamma = 1$.

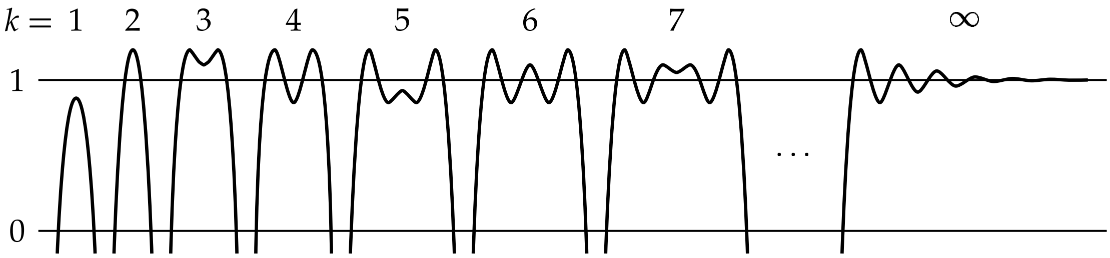

Figure 1.6: Building blocks with their corresponding numbers.

symmetry is present in the system then a two dimensional shooting method would have to be used.

We now describe some of the results. Taking initial conditions (1.18) with $\alpha > 0$ and monitoring the first maximum of the solutions, it was found in [120] that for any $E \in (-\frac{1}{4}, 0)$ and any $\beta \in \mathbb{R}$ ($\gamma = 1$ fixed) there exists a periodic solution of (1.13), odd with respect to its zeros and even with respect to its extrema. For $E = 0$ such a periodic solution is found for $\beta < \sqrt{8}$ [120, 106]. In fact, two such solutions are found in that case, one with $\|u\|_\infty < 1$ and one with $\|u\|_\infty > 1$ (see Figure 1.5). For $\beta \geq \sqrt{8}$ no periodic solution exists in the energy level $E = 0$, but instead there are monotone heteroclinic solutions connecting -1 to $+1$ [117]. For $\beta < \sqrt{8}$ monotone heteroclinics do not exist, since then the equilibrium points ± 1 are saddle-foci or centers. The existence of symmetric homoclinic solutions which have a unique critical point is discussed in [116]. Such solutions do not exist for Equation (1.13), but they do exist for other nonlinearities $f(u)$.

For $\beta < \sqrt{8}$ the fact that ± 1 are saddle-foci or centers can be used to obtain periodic, heteroclinic, homoclinic and even chaotic solutions which have oscillations around ± 1. We fix the energy level $E = 0$ (containing the equilibria $u = \pm 1$). Instead of giving definitions we will try to explain what kind of solutions are found. In Figure 1.6 the *building blocks* of solution shapes are shown, with a number attached to each building block. By symmetry, there are similar building blocks for negative u, and we thus distinguish *positive* and *negative* building blocks. There are also building blocks of type ∞; they consist of solutions which tend to ± 1 as $x \to \pm\infty$, see Figure 1.6. These building blocks are not exact solutions, but they show the approximate shape. The important features are the positions of the extrema with respect to the lines $u = \pm 1$ and $u = 0$.

By alternating positive and negative building blocks one may build solutions and one obtains a corresponding sequence of positive integers. We will say that a solution is of *type* $(\ldots, k_1, k_2, k_3, \ldots)$ if its shape corresponds to this sequence of building blocks. Since

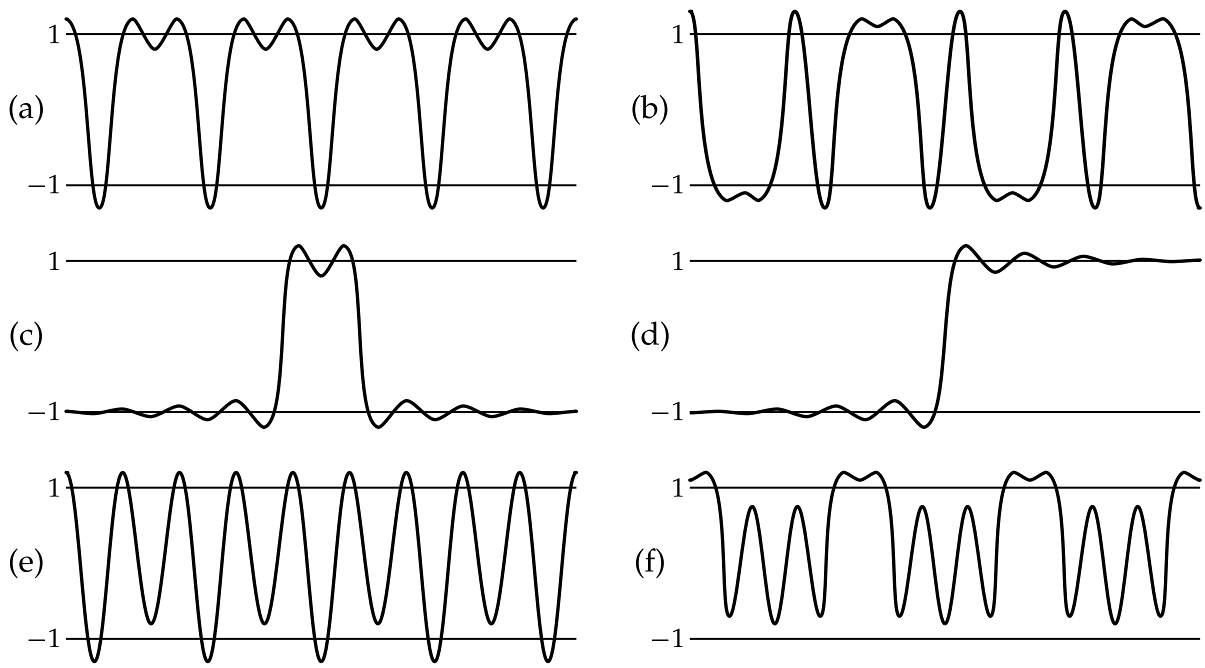

Figure 1.7: Schematic examples of solutions for $\beta \in [0, \sqrt{8})$ corresponding to sequences (a) $(\dots,2,4,2,4,2,\dots)$, (even) periodic; (b) $(\dots,3,2,2,3,2,2,3,2,2,3,\dots)$, (odd) periodic; (c) $(\infty,4,\infty)$, homoclinic; (d) (∞,∞), heteroclinic; (e) $(\dots,2,2,1,2,2,2,1,2,2,\dots)$, periodic; (f) $(\dots,1,1,1,3,1,1,1,1,1,3,1,1,1,\dots)$, periodic.

we deal with odd or even solutions we restrict our attention to symmetric types. The techniques developed in [118, 119] can be used to show that for $\beta \in [0, \sqrt{8})$ there are solutions of (1.12) with energy $\mathcal{E}[u] = 0$ whose shape corresponds to any of the following sequences (types):

- $(\dots,k_2,k_1,k_1,k_2,\dots,k_{n-1},k_n,k_{n-1},\dots,k_2,k_1,k_1,k_2\dots)$ or
  $(\dots,k_2,k_1,k_2,\dots,k_{n-1},k_n,k_{n-1},\dots,k_2,k_1,k_2,\dots)$ for some $n \in \mathbb{N}$ with $k_i \in \mathbb{N}$.
  These sequence are periodic and correspond to an odd or even periodic solution respectively (see Figures 1.7a,b).
- $(\dots,k_3,k_2,k_1,k_2,k_3,\dots)$ or $(\dots,k_2,k_1,k_1,k_2,\dots)$ with $k_i \in \mathbb{N}$.
  If there is no 'regularity' in the sequence then the solution is sometimes called *chaotic*.
- $(\infty,k_n,\dots,k_2,k_1,k_2,\dots,k_n,\infty)$ for some $n \in \mathbb{N}$ and $k_i \in \mathbb{N}$.
  This corresponds to an even homoclinic solution (see Figure 1.7c).
- $(\infty,k_n,\dots,k_2,k_1,k_1,k_2,\dots,k_n,\infty)$ for some $n \in \mathbb{N}$ and $k_i \in \mathbb{N}$.
  This corresponds to an odd heteroclinic solution. There is also a simple heteroclinic solutions of type (∞,∞), see Figure 1.7d.

Notice that both u and $-u$ correspond to the same sequence unless we specify a certain building block to be positive. A few examples are given in Figure 1.7. The restriction to even and odd solutions does not seem to be crucial. A limit procedure could be used to prove the existence of solutions corresponding to non-symmetric sequences[2]. These results can be generalised to nonlinearities with a shape similar to $f(u) = u - u^3$, i.e., having

[2]Periodicity of solutions corresponding to non-symmetric periodic types seems more difficult to prove (it becomes a truly two dimensional problem). In a variational setting one can make the periodicity an intrinsic part of the problem by choosing suitable function spaces, see Chapter 3.

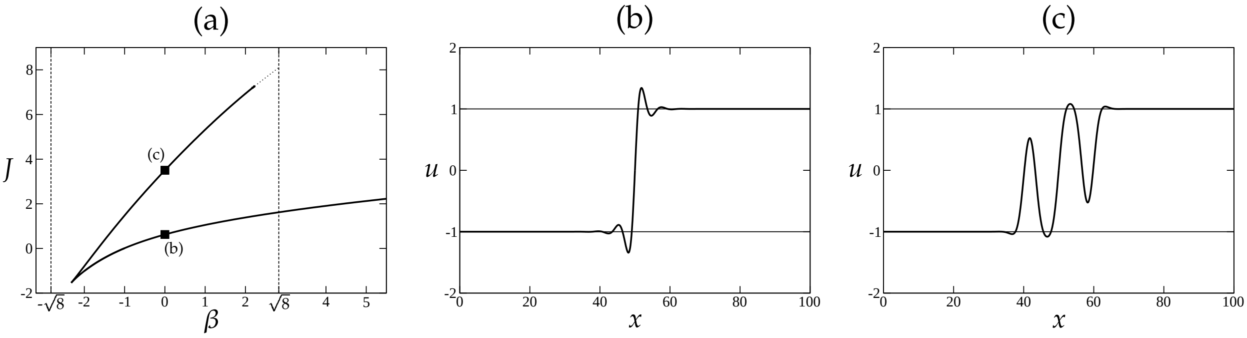

Figure 1.8: The simple kink is continued numerically in the parameter β. The value of the action $J[u]$ is shown as a function of β. The cusp is solely due to the choice of the action J as the quantity on the vertical axis; the L^{∞}-norm displays smooth behaviour near the turning point (cf. Figure 1.9b). Solutions (indicated by small squares) on the lower and upper branch are shown in (b) and (c) respectively.

three zeros $u_1 < u_2 < u_3$ such that u_1 and u_3 lie in the same energy level. In particular, $f(u)$ does not need to be anti-symmetric (one can use the shooting setup in (1.19)).

For $\beta \in (-\sqrt{8}, 0)$ the equilibria $u = \pm 1$ are still saddle-foci, so one might expect the above solutions to continue to exist. And to some extent this is indeed true. Although the shape of the solutions starts to change somewhat (sometimes even (almost) beyond recognition), one can (numerically) follow continuous branches of solutions into the parameter region $\beta < 0$ (using the program AUTO [57]). However, some types of solutions cease to exist while the equilibrium points are still saddle-foci. The critical value $\beta_* < 0$ after which the solution no longer exists, is different for each type of solution. The example of the simple kink (∞, ∞) is shown in Figure 1.8. At a critical value of about $\beta_* \approx -2.3$ this solution coalesces with another heteroclinic solution (with sequence $(\infty, 1, 2, 2, 1, \infty)$). This phenomenon has been extensively studied in [35] for Equation (1.12) with $f(u) = -u + u^2$.

In Chapter 6 we use the shooting method to study what happens to certain types of periodic solutions when β becomes negative. We find three different kinds of behaviour:

1. We obtain a family of periodic solutions bifurcating from the unique kink at $\beta = \sqrt{8}$ and extending to $-\infty$, i.e., these solutions exist for all $\beta < \sqrt{8}$ (see Figure 1.9a). The family consists of a countable infinity of distinct periodic solutions of type $(\ldots, 2, 2, 3, 2, 2, \ldots, 2, 2, 3, 2, 2, \ldots)$, where the number of 2's between two 3's is arbitrary (for an example see Figure 1.7b).
2. As already explained above, there are (infinitely) many *pairs* of families, which exist at least for $\beta \in [0, \sqrt{8})$. These solutions continue to exist for some, but not all, negative values of β. Numerical evidence shows that these solutions pairwise lie on *loops* in the $(\beta, \|u\|_{\infty})$-plane (see Figure 1.9b) of which the projection on the β-axis is of the form $[\beta_*, \sqrt{8})$.
3. Finally, we find a third kind of (even) periodic solutions. These again come as a family of countable many distinct periodic solutions which bifurcate from the kink at $\beta = \sqrt{8}$. However, this family does not extend to infinity nor do they lie on loops. Instead, the numerical results indicate that these periodic solutions bifurcate from the constant solution $u = 1$ as β tends to a critical value $\beta_n = -\sqrt{2}(n + \frac{1}{n})$, $n \in \mathbb{N}$ (see Figure 1.9c). For $n \geq 2$ these solutions come in pairs. The critical values β_n arise when

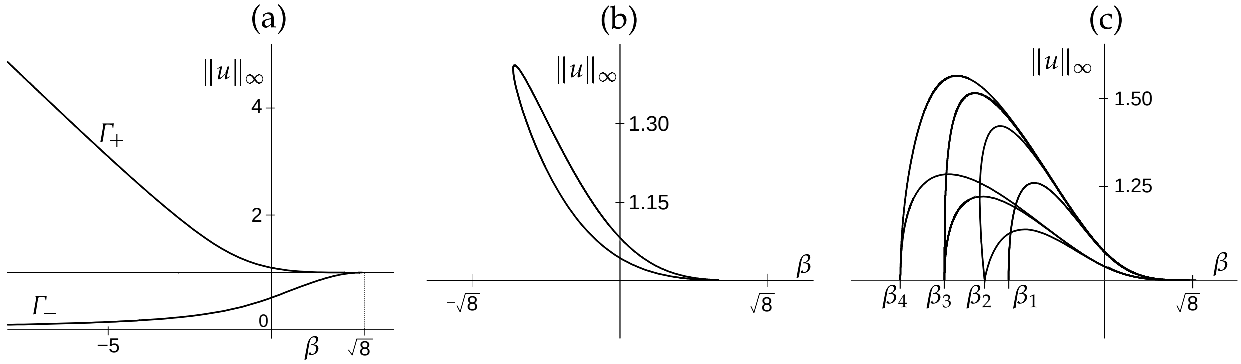

Figure 1.9: The three types of branches: (a) extending to $-\infty$; (b) loop-shaped; (c) bifurcating from $u \equiv 1$.

the moduli of the eigenvalues of the linearisation around $u = 1$ are a multiple of one another. The sequences corresponding to these solutions, at least for $\beta \geq 0$, are of the form $(\ldots,1,1,2,1,1,\ldots,1,1,2,1,1,\ldots)$ and $(\ldots,1,1,3,1,1,\ldots,1,1,3,1,1,\ldots)$. The number j of 1's between two 2's or 3's is odd (which means that $u(x) > -1$ for all $x \in \mathbb{R}$) and $j = 2n-1$ or $j = 2n-3$ respectively. See Figure 1.7f for an example.

We emphasise that these branches of solutions illustrate that the structure of the solutions set does not immediately change when β changes sign.

In addition to the branches of solutions extending over $(\beta_n, \sqrt{8})$, it is also possible to construct branches of even solutions which bifurcate from $u = 1$ at $\beta_{n,m} = -\sqrt{2}(\frac{n}{m} + \frac{m}{n})$, $n, m \in \mathbb{N}$. We refer to Chapter 6 for a more detailed discussion and examples. Besides, in Chapter 8 (see also Section 1.3.4.2) a variational setting of the shooting method is presented, involving the calculation of the Conley index of certain types of solutions. Most of the solutions described above are recovered, as well as many additional branches of solutions. This approach also sheds light on the matter of coalescence of solutions. A non-trivial Conley index implies the existence of at least one solution of the corresponding type for any parameter value. A trivial Conley index allows for the possibility of two coalescing solutions, so that a solution of the corresponding type may exists in only part of the parameter regime.

We want to make some comments about quantities which are conserved along continuous branches of solutions in the energy level $E = 0$. It turns out that for all $\beta < \sqrt{8}$ the number of extrema (counted with multiplicity) cannot change and that the number of intersections with $u = 1$ and $u = -1$ is also conserved. This already puts quite a few restrictions on the behaviour of branches of solutions, and combined with numerical experiments this gives a lot of intuition. The following prediction can be made. We recall that the use of the sequences to label solutions is only valid for $\beta \geq 0$. A solution whose sequence contains at least one $k_i \geq 4$ lies on a loop-shaped branch. A solution whose sequence consists of 2's and 3's lies on an unbounded branch. The only solution which can lie on a branch which bifurcates from $u \equiv \pm 1$ are those of the form $(\ldots,1,k_1,1,k_2,1,\ldots)$ with $k_i \in \{1,2,3\}$. However, some of these happen to lie on loop-shaped branches. This still leaves some undecided cases and we will not try to make bold predictions about those. Nevertheless a quite complete picture exists of all these branches. We should add that the relative position of the coalescence points β_* (the turning points of the loop-shaped branches) has been investigated and partially resolved in [35].

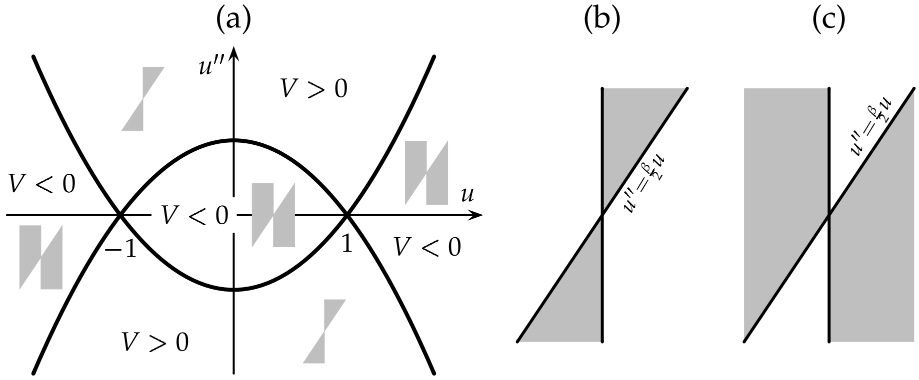

Figure 1.10: (a) The (u, u'')-plane with the set $\{V = 0\}$. If $V \neq 0$ the velocity (u', u''') lies in a cone depicted in (b) for $V > 0$ and in (c) for $V < 0$.

There is a second shooting method for equations of the form (1.12), which has been developed in [84, 3, 43]. In contrast to the method above, the central role is played by the (u, u'')-plane. This setting stems from the identity (for energy $E = 0$)

$$u'(u''' - \frac{\beta}{2}u') = \frac{1}{2}u''^2 - F(u) \tag{1.20}$$

Defining $V = \frac{1}{2}u''^2 - F(u)$ the (u, u'')-plane is divided into regions where V has different signs (see Figure 1.10a). The sign of V implies a form of monotonicity: if $V > 0$ then by Equation (1.20) the *velocity* (u', u''') lies in the cone as indicated in Figure 1.10b (for $\beta > 0$), whereas for $V < 0$ the velocity lies in the cone depicted in Figure 1.10c. Note that the opening angles of the cones depend on the parameter β.

If $V = 0$ then either $u' = 0$ or $u''' = \frac{\beta}{2}u'$. Therefore, if an orbit intersects the set $\{V = 0\}$ at $x = x_0$ then there are two possibilities. Either the velocity (u, u'') is non-zero and the direction $\frac{u'''}{u'}$ of the orbit is ∞ (vertical) or $\frac{\beta}{2}$; or $u' = u''' = 0$ implying that u is symmetric about x_0, and the direction of the incoming and outgoing orbit is $\beta + \frac{f(u)}{u''}$, which again depends on the value of β.

The transition from real saddle to saddle-focus is characterised by the fact that the slope of the cones becomes smaller than the slope of the graph of $\{V = 0\}$ at the equilibrium point. Besides, at the transition value the direction $\beta + \frac{f(u)}{u''}$ of an orbit with zero velocity (i.e. at a turning point) close to the equilibrium point, also becomes equal to the slope of the graph of $\{V = 0\}$. The value of β therefore has a serious influence on the possible dynamics.

We have only touched upon the basic features of this second approach to a shooting method and we do not want to go into the details. We remark that this method has been successfully applied to prove several results for the Equation (1.12) with $f(u) = -u + u^2$ [3, 43], see also [33] for an application in the setting of Lorentz-Lagrangian systems. The above description of the setting in the (u, u'')-plane is largely based on the work in [112], which was already mentioned in Section 1.3.1.

1.3.3 Variational techniques

It is well-known that many problems with a physical origin have a variational formulation and Equation (1.12) indeed falls in this category. Solutions of (1.12) correspond to critical points of the *action*

$$J[u] = \int L(u(x), u'(x), u''(x))\, dx,$$

where the integrand, the *Lagrangian*, is given by

$$L(u, u', u'') = \frac{\gamma}{2} u''^2 + \frac{\beta}{2} u'^2 - F(u). \tag{1.21}$$

The Lagrangian is closely related to the Hamiltonian formulation, see (1.16). There is an intimate connection between Hamiltonian and Lagrangian systems and we refer to [9] for more details.

The Lagrangian (1.21) is a *second order* Lagrangian since it depends on the derivatives up to second order. One can thus embed Equation (1.12) in the class of second order Lagrangian systems, which makes it possible to extend those results on (1.12) which are obtained via a variational approach to a larger class of fourth order equations. The *Euler-Lagrange* equation, which is satisfied by critical/stationary points of a second order Lagrangian system $L(u, u', u'')$, reads

$$\frac{\partial L}{\partial u} - \frac{d}{dx}\frac{\partial L}{\partial u'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial u''} = 0.$$

Let us return to the Lagrangian (1.21) with $F(u) = -\frac{1}{4}(u^2 - 1)^2$ to study stationary solutions of (1.13). To find periodic solutions with period ℓ one can consider the minimisation problem

$$\min_{u \in X} \int L(u, u', u'')\, dx \qquad \text{where } X = H^2(0, \tfrac{\ell}{2}) \cap H^1_0(0, \tfrac{\ell}{2}).$$

Minimisers automatically satisfy the boundary conditions $u(0) = u''(0) = u(\frac{\ell}{2}) = u''(\frac{\ell}{2}) = 0$, so that by the symmetry of (1.13) they extend to periodic solutions with period ℓ. This minimisation problem has been investigated in [124] for $\beta > 0$. In that case one fixes $\beta = 1$ and it was found that for $\gamma > 0$ there exist periodic solutions for any period $\ell > 2\pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}$ (which originates from the linearisation around $u = 0$; for $\ell \leq 2\pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}$ the uniform state $u = 0$ is the minimiser). These periodic solutions have precisely two zeros and two extrema in one period.

Although such a minimisation procedure is greatly facilitated by the non-negativity of the integrand $L(u, u', u'')$ for $\beta \geq 0$, it seems that, since we are dealing with finite intervals, at least part of this method could be applied for $\beta < 0$ as well, but this has not been pursued. In [124] a similar method is used to prove the existence of a heteroclinic solution for all $\beta > 0$, which in the notation of Section 1.3.2 corresponds to the type (∞, ∞) in the case of saddle-foci.

Chapter 3 deals with the variational analysis of periodic solutions with more than two extrema in one period. We restrict our search to the energy level $E = 0$. The minimisation method used here does not fix the period but instead minimises in a class of functions with a certain shape, thus finding *local* minimisers of the action.

Consider the punctured phase-plane $\mathcal{P} = \{(u, u') \in \mathbb{R}^2 \setminus (\pm 1, 0)\}$. The fundamental group of $\mathcal{P}$ is generated by two loops e_1 and e_2 winding around the points $(+1, 0)$ and $(-1, 0)$ respectively in clockwise direction, see Figure 1.11a. The path of a continuously differentiable periodic function u which has no extrema on the lines $u = \pm 1$, belongs to

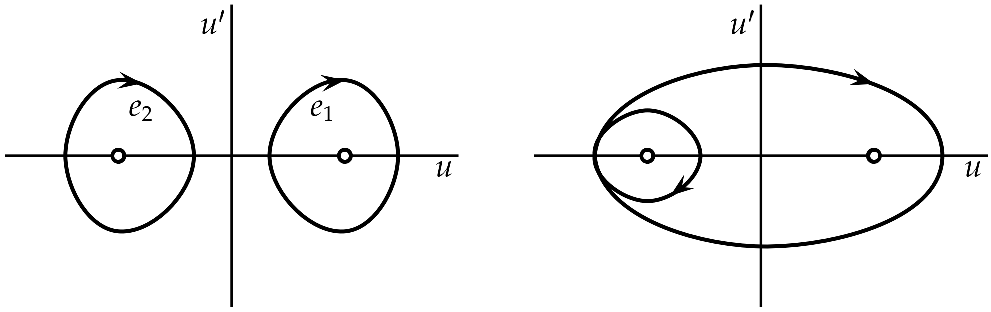

Figure 1.11: (a) The punctured plane $\mathcal{P}$ with the two generators of the fundamental group. (b) The path of a function in equivalence class $[e_2^2 e_1]$, homotopy type $[1,2]$.

some equivalence class, say $[e_2^{m_n} e_1^{m_{n-1}} \cdots e_1^{m_3} e_2^{m_2} e_1^{m_1}]$ for some $n \in \mathbb{N}$. The integers m_i are positive since u is a function (i.e., we are dealing with the half-group generated by e_1 and e_2). Without loss of generality we take n to be even. We define a function class X corresponding to a set of positive integers $(m_i)_{i=1}^n$ as all periodic functions in H^2 whose path in $\mathcal{P}$ lies in the equivalence class, say $[e_2^{m_n} \cdots e_2^{m_2} e_1^{m_1}]$, see Figure 1.11b for an example. Define the *homotopy type* as the finite sequence of even numbers $[2m_1, 2m_2, \ldots, 2m_n]$; this homotopy type counts the number of consecutive crossings of the lines $u = 1$ and $u = -1$ for the functions in the equivalence class. In each of these equivalence classes we obtain a minimising periodic solution. The result holds whenever the equilibrium points $u = \pm 1$ are saddle-foci and $\beta > 0$.

Theorem 1.3 *Let $\beta > 0$ and $\frac{\gamma}{\beta^2} > \frac{1}{8}$. For any homotopy type $[2m_1, 2m_2, \ldots, 2m_n]$, with n even and $m_i \in \mathbb{N}$ for all $1 \le i \le n$, there exists a periodic solution of* (1.13) *which locally minimises the action $J[u]$.*

Note that since the period is not fixed, minimisation is also carried out over all possible periods, which implies that all minimisers have energy $\mathcal{E}[u] = 0$.

When we try to compare these solutions to the notation introduced in Section 1.3.2 we encounter the following difficulty. Consider the solutions of periodic type $(\ldots, 2, 4, 2, 4, 2, \ldots)$ and $(\ldots, 1, 2, 2, 2, 1, 2, 2, 2, 1, \ldots)$, then the *homotopy* type of both these solutions is $[2, 4]$, see Figure 1.7a,e. Numerical evidence as well as the nature of the minimisation procedure suggest that the minimiser is the solution of type $(\ldots, 2, 4, 2, 4, 2, \ldots)$, while the other one is a solution of index 1 (i.e. one unstable direction). In general this has however not been proved. Nevertheless, the idea is that the periodic solution which are made from only *even* building blocks are the minimisers and thus are *stable* stationary solutions of Equation (1.9), whereas each *odd* building block used corresponds to one unstable direction so that the index of the solution equals the number of odd building blocks. Finally, we remark that the homotopy type is conserved along a continuous branch of solutions with $\mathcal{E}[u] = 0$, see also Section 1.3.2. The pair of solutions discussed here lie on a loop-shaped branch of solutions (see again Section 1.3.2).

The method works for a large class of second order non-negative Lagrangians which have two global minima which are saddle-foci. If L is of the form (1.21) with a symmetric potential then one obtains Theorem 1.3. For non-symmetric Lagrangians one finds minimisers for homotopy types $[2m_1, \ldots, 2m_n]$ with $m_i = 2$ or m_i sufficiently large.

We also prove the existence of minimising solutions corresponding to non-periodic homotopy types. Some properties of the homotopy type, such as symmetry, are reflected in the corresponding minimisers. Another example is that if a homotopy type is asymptotically periodic in both directions, then there exists a minimiser of that type which is a heteroclinic connection between two periodic minimisers. An important feature of minimisers is that their projections onto the (u, u')-plane have no more self-intersections then necessary, i.e., the only self-intersections are those forced by the homotopy type. A similar result in the situation of a constrained minimisation problem can be found in [114].

In many ways the situation resembles the that for geodesics: one tries to find the most cost-effective path to make $J[u]$ as low as possible, so that shortcuts are very favourable (see Chapter 3 for more details). This technique was developed in [89] to prove the existence of heteroclinic and homoclinic solutions in much the same setting as described above for periodic solutions. Heteroclinics and homoclinics are found of type $(\infty, 2m_1, \ldots, 2m_n, \infty)$, where $m_i \in \mathbb{N}$, and n is odd for homoclinics and even for heteroclinics.

A series of related variational problems has been investigated in the last few years, and we will briefly discuss the most relevant ones.

In the study of second order materials a minimisation problem of the form

$$\inf \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} L(u, u', u'')\, dx \tag{1.22}$$

is encountered, where the infimum is taken over all functions $u \in H^2_{\mathrm{loc}}$ for which this limit exists. For $\beta \geq 0$ the minimisers are the homogeneous states $u \equiv \pm 1$, but when β is sufficiently negative this is no longer true, and a periodic function takes over the role of minimiser [46, 99, 106]. This happens when β reaches the (a priori unknown) value β^* for which the first periodic solutions appears with action $J[u] = 0$. The numerical value of β^* is about -0.92 for $\gamma = 1$. We remark that the minimisation technique in homotopy type classes described in this section relies on the positivity of the Lagrangian L to apply cut-and-paste techniques, i.e., it only works for $\beta \geq 0$. Nevertheless it seems that one should be able to extend this to $\beta < 0$ as long as $\beta > \beta^*$, since for $\beta > \beta^*$ every function corresponding to a loop in the configuration plane $\mathcal{P}$ has positive action.

In the description of geological pattern formation the following constraint minimisation plays a role [113]:

$$\inf_X \int_{\mathbb{R}} L(u, u', u'')\, dx, \qquad \text{where } X = \{u \in H^2 \mid \textstyle\int_{\mathbb{R}} u'^2 dx = \lambda\}.$$

Here L is taken to be of the form $L = \frac{1}{2}u''^2 + \frac{1}{2}u^2 - \frac{1}{4}u^4 + \frac{a}{6}u^6$. For $a \geq \frac{1}{4}$ a minimising homoclinic orbit to 0 is found, which converges to a periodic function as $\lambda \to \infty$, where the periodic function solves a related minimisation problem.

A mountain pass argument has been applied to prove the existence of a homoclinic solution to 0 of (1.12) with $f(u) = -u + u^2$ for $\beta \in (-\sqrt{8}, \infty)$ [33]. The estimates needed for this method (to prove that the Palais-Smale condition is satisfied) are unfortunately only available for nonlinearities of the special form $f(u) = -u + u^s$, $s > 1$.

Various gluing methods have been used to show that, when the equilibria $u = \pm 1$ are saddle-foci, the existence of a heteroclinic loop implies the existence of infinitely many heteroclinic, homoclinic and periodic solutions. These methods are partly variational and partly functional analytic in the sense that one tries to prove the existence of a zero of $dJ[u]$. If the heteroclinic orbits of the heteroclinic loop are transverse intersections of $W^u(\mp 1)$ and $W^u(\pm 1)$ then, as was already shown in [54], dynamical system techniques imply the existence of a Smale horseshoe, hence there are infinitely many heteroclinic, homoclinic and periodic solutions. For Equation (1.13) the transversality condition can be checked for $\frac{\gamma}{\beta^2}$ very close to $\frac{1}{8}$. In [90, 36] it was shown that similar results can be obtained via a gluing method without checking the transversality conditions. However, one needs the fact that the heteroclinic orbits are isolated, which for example holds for analytic nonlinearities. Finally, in Section 1.3.5 we will encounter a truly variational variant of this gluing technique. This variational variant has in common with the minimisation method described above (see also [89]) that no information about the non-degeneracy or isolation of the heteroclinic orbits is needed.

There are several advantages and disadvantages of the variational methods when we compare them to a shooting method. The most important advantage is that, since use is made of the variational structure, conclusions can be drawn about the index of solutions and their stability. Another advantage is that a large class of systems can be dealt with simultaneously. On the other hand, solutions of different index often require a totally different method. Besides, the information about the shape of the solutions is usually less detailed and sometimes completely absent. In the next section we will show how we can combine the variational setting with the shooting information to get the best of both worlds.

1.3.4 Twist maps and braids: a combination of shooting and variational methods

In this section we investigate the existence of periodic orbits in a fixed energy level $\{\mathcal{E} = E\}$ via an approach which can either be viewed as a shooting method or as a variational method. Alternatively, the method can be regarded as an attempt to unify the variational and shooting approaches. This will lead to the introduction of a *Twist property*. The first application of a Twist map to Equation (1.13) was presented in [96], although the approach there is very different from ours. We now try to give the essential ingredients, and for precision, more generality and details we refer to Chapter 7.

1.3.4.1 Twist maps

Let us start by explaining the dynamical systems (shooting) point of view. Since the present subject is rather geometrical we will use the Hamiltonian formulation (1.16). Trajectories of the Hamiltonian system lie on three dimensional sets $M_E \stackrel{\text{def}}{=} \{H = E\}$. The sets M_E are smooth manifolds for all regular energy values E of H (i.e. $\nabla H|_{M_E} \neq 0$), and are non-compact for all $E \in \mathbb{R}$. A bounded solution of (1.12) has either finitely or infinitely many isolated local extrema (or zero for a monotone heteroclinic solution). This means that a bounded orbit always intersects the section $\{u' = 0\}$ in M_E. In case there are only finitely many intersections, u must be asymptotic to an equilibrium point as $x \to \pm\infty$. If E is a regular energy level then this possibility is excluded.

Recall that $v = u'$, $p_u = \beta u' - \gamma u'''$ and $p_v = \gamma u''$. The section $\{v = 0\} \cap M_E$ will be denoted by Σ_E. This *Poincaré section* $\Sigma_E = \{(u, v, p_u, p_v) \,|\, v = 0,\ p_v = \pm\sqrt{2\gamma(E - F(u))}\}$ splits into two graphs $\Sigma_E^\pm$ over the (p_u, u)-plane. Let us also define the set $I = \{u \in \mathbb{R} \,|\, E - F(u) \geq 0\}$.

Bounded trajectories can thus be identified with sequences of points (p_{u_i}, u_i) in the (p_u, u)-plane or rather in the subset $\mathbb{R} \times I$. The vector field is transverse to the interior of the sections Σ^+ and Σ^-. It therefore makes sense to consider the Poincaré return maps, i.e. maps from Σ^+ to Σ^- and from Σ^- to Σ^+ by following the flow starting at Σ^+ until it intersects Σ^-. For the points in Σ^+ for which the flow does intersect Σ^- we have defined a map $\mathcal{T}_+$ from Σ^+ to Σ^-. The same can be done for the map $\mathcal{T}_-$ from Σ^- to Σ^+. These are of course just the maps considered in the shooting method in Section 1.3.2. The maps $\mathcal{T}_\pm$ can be considered as maps from (part of) the (p_u, u)-plane to itself. Since periodic solutions consist of increasing laps alternated by decreasing laps we seek fixed points of iterates of the composition map $\mathcal{T} = \mathcal{T}_- \circ \mathcal{T}_+$.

The flow is a Hamiltonian, which implies that maps $\mathcal{T}_\pm$ are *area preserving* (we refer to [9] for more details about this and what follows). This in turn implies that

$$p_{u_2} du_2 - p_{u_1} du_1 = dS_*(p_{u_1}, u_1), \tag{1.23}$$

where $(p_{u_2}, u_2) = \mathcal{T}_+(p_{u_1}, u_1)$ and S_* is a C^1-function of (p_{u_1}, u_1). The map $\mathcal{T}_+$ is a *Twist map* if $u_2 = u_2(p_{u_1}, u_1)$ is strictly increasing in $-p_{u_1}$. It then follows from (1.23) that there exists a C^1-function $S_E(u_1, u_2) = S_*(p_{u_1}(u_1, u_2), u_1)$ such that $\partial_1 S_E = -p_{u_1}$ and $\partial_2 S_E = p_{u_2}$. This function is called the *generating function* of the Twist map. A similar construction can be carried out for $\mathcal{T}_-$. Notice that the above definition of a Twist map is nothing else than the following requirement. When one shoots from a fixed minimum the only free parameter is the third derivative $u'''(0)$. Now $\mathcal{T}_+$ is a Twist map if the height of the first maximum is a strictly increasing function of $u'''(0)$.

The advantage of having such a generating function will become clear when we make the connection with the variational interpretation, and that is what we will do next.

Periodic solutions in an energy level are often called *closed characteristics*. These are functions u which are stationary for $J[u]$ and are τ-periodic for some period τ. If we seek closed characteristics at a given energy level E we can invoke the following variational principle:

$$\text{Extremise}\,\{\, J_E[u] \;|\; u \in \bigcup_{\tau > 0} C^2(S^1, \tau) \,\}, \tag{1.24a}$$

where

$$J_E[u] = \int_0^\tau \big(L(u, u', u'') + E\big)\, dx. \tag{1.24b}$$

It may be clear that τ is also a parameter in this problem, which guarantees that any critical point has energy $H \stackrel{\text{def}}{=} \mathcal{E}[u] = E$.

The connection between the variational problem (1.24) and the shooting maps $\mathcal{T}_\pm$ is as follows.

Lemma 1.4 *Let $S_*(p_{u_1}, u_1) = J_E[u]$, where u is the trajectory starting at $(u_1, p_{u_1}) \in \Sigma^+$, and $\tau = \tau(p_{u_1}, u_1)$ is the first intersection time at Σ^-. Then S_* satisfies Equation* (1.23).

This Lemma does not depend on $\mathcal{T}_+$ being a Twist map or not. If $\mathcal{T}_+$ is a Twist map, then for J_E this implies that there exists a continuous family $u(t; u_1, u_2)$ of extrema (and

$\tau(u_1, u_2)$ varies continuously). Conversely, the continuity conditions on the family of extrema $u(t; u_1, u_2)$ imply the Twist property.

We now need to take care of the domains of u_1 and u_2. For that purpose we denote the connected components of $I = \{u \in \mathbb{R} \,|\, E - F(u) \geq 0\}$ by I_E, and we will refer to them as *interval components*. Define the diagonal $\Delta = \{(u_1, u_2) \in I_E \times I_E \,|\, u_1 = u_2\}$. The Lagrangian L is now said to satisfy the *Twist property* on an interval component I_E if (with E a regular energy value):

(T) $\inf\{J_E[u] \,|\, u \in X_\tau(u_1, u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimiser* $u(t; u_1, u_2)$ *for all* $(u_1, u_2) \in I_E \times I_E \setminus \Delta$, *and* u *and* τ *are* C^1*-smooth functions of* (u_1, u_2).

Here $X_\tau = X_\tau(u_1, u_2) = \{u \in C^2([0, \tau]) \,|\, u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = u'(\tau) = 0,\ u'|_{(0,\tau)} > 0 \text{ if } u_1 < u_2 \text{ and } u'|_{(0,\tau)} < 0 \text{ if } u_1 > u_2\}$. In practice the minimiser is often unique. A similar definition holds for singular energy values. Finally, if L satisfies the Twist property (T) we may take as a definition

$$S_E(u_1, u_2) = \inf_{\substack{u \in X_\tau \\ \tau \in \mathbb{R}^+}} \int_0^\tau \big(L(u, u', u'') + E\big)\, dx.$$

On the diagonal Δ we set $S_E|_\Delta = 0$, so that S_E is continuous on $I_E \times I_E$. Let us remark that these definitions can be applied to general second order Lagrangians provided that $\partial^2_{u''} L \geq \delta > 0$.

The Twist property allows one to reduce the variational problem (1.24) to a finite dimensional setting where only the extrema u_i are varied, because the monotone laps between two extrema are unique. The concatenations of these monotone laps are the analogues of broken geodesics (cf. [103]).

We would not have introduced all these concepts if it was not for the following.

Lemma 1.5 *Let the Lagrangian L be as in* (1.21)*, and let I_E be an interval component. Then for all $\beta \leq 0$ this L satisfies the Twist property* (T) *on I_E.*

The proof is based on the following reduction to a second order equation. Stationary solutions of (1.12) with energy $\mathcal{E}[u] = E$ satisfies the equation $-\gamma u' u''' + \frac{\gamma}{2} u''^2 + \frac{\beta}{2} u'^2 + F(u) - E = 0$. For an increasing lap from u_1 to u_2 the derivative u' can be represented as a function of u. Set $z(u) = u'^{3/2}(x(u))$, where $x(u)$ is the inverse of $u(x)$ on a monotone lap of u. We find that $z(u)$ satisfies the equation

$$\begin{cases} \frac{d^2 z}{du^2} = g(u, z) & \text{for } u \in (u_1, u_2), \\ z(u_1) = z(u_2) = 0 & \text{and} \quad z > 0 \text{ on } (u_1, u_2), \end{cases} \tag{1.25}$$

where $g(u, z) = \frac{2}{3\gamma} \frac{\frac{1}{2}\beta z^{4/3} + F(u) - E}{z^{5/3}}$. The same holds for decreasing laps ($z < 0$). The fact that $\frac{\partial g}{\partial z} \geq 0$ for $\beta \leq 0$ and $(u_1, u_2) \subset I = \{u \in \mathbb{R} \,|\, E - F(u) \geq 0\}$ allows us to apply the comparison principle to conclude the uniqueness of the lap z.

The generating function S_E has the following essential properties which follow from its definition:

Lemma 1.6 *Let E be a regular energy value.*

(a) $\partial_1 S_E(u_1, u_2) = -p_{u_1}$ *and* $\partial_2 S_E(u_1, u_2) = p_{u_2}$ *for all* $(u_1, u_2) \in I_E \times I_E \backslash \Delta$.

(b) $\partial_1 \partial_2 S_E(u_1, u_2) > 0$ *for all* $(u_1, u_2) \in \mathrm{int}(I_E \times I_E \backslash \Delta)$.

(c) $\partial_{n_\pm} S_E|_{\mathrm{int}(\Delta)} = +\infty$*, where* $n_\pm = (\mp 1, \pm 1)^T$.

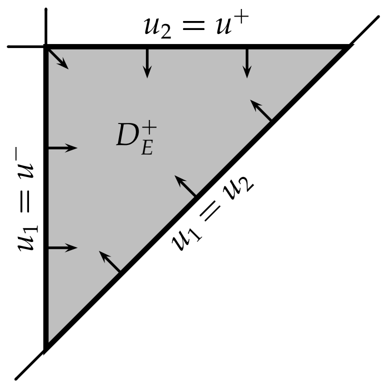

Figure 1.12: A picture of D_E^+ for the case of a compact interval component $I_E = [u^-, u^+]$. The arrows schematically denote the direction of the gradient ∇W_2. Clearly the maximum of W_2 is attained in the interior of D_E^+.

The question of finding closed characteristics can now be formulated in terms of S_E. Extremising the action J_E over a space of 'broken geodesics' corresponds to finding critical points of the sum $W_{2p}(u_1, \dots, u_{2p}) = \sum_{i=1}^{2p} S_E(u_i, u_{i+1})$, where $u_{2p+1} = u_1$. Critical points are characterised by the set of equations

$$\partial_2 S_E(u_{i-1}, u_i) + \partial_1 S_E(u_i, u_{i+1}) = 0, \qquad \text{for } i = 1, \dots, 2p. \tag{1.26}$$

Such equations are called *second order recurrence relations* (cf. [5]). If (1.26) is satisfied for all $i = 1, \dots, 2p$ then u-laps can be glued to a C^3-function for which all derivatives up to order three match (Lemma 1.6a), so that one finds a C^3-function u that is stationary for $J[u]$, hence a periodic solution of (1.12). Moreover, periodic sequences as critical points of W_{2p} have a Morse index, which is exactly the Morse index of the corresponding closed characteristic u as critical point of J_E.

Let us focus on simple closed characteristics. A useful aid in finding critical point of $W_2(u_1, u_2)$ (or W_{2p} in general) is to consider the gradient flow:

$$\begin{cases} \frac{du_1}{dt} &= \partial_1 W_2(u_1, u_2), \\ \frac{du_2}{dt} &= \partial_2 W_2(u_1, u_2). \end{cases}$$

Since $W_2(u_1, u_2) = W_2(u_2, u_1)$ we can restrict our analysis to $D_E^+ = \{(u_1, u_2) \in I_E \times I_E \,|\, u_2 > u_1\}$.

The following result is derived immediately from the properties of S_E in Lemma 1.6 (see also Figure 1.12).

Theorem 1.7 *Let L satisfy the Twist property on some* compact *interval component I_E for some regular energy level E. Then W_2 has at least one maximum on D_E^+, corresponding to a simple periodic solution, generically with index* 2.

For singular energy levels a similar theorem can be proved. The bottom line is that under the compactness assumption there exists a simple closed characteristic in the broader sense of the word, i.e., depending on possible singularities a closed characteristics is either a regular simple closed trajectory, a simple homoclinic orbit, or a simple heteroclinic loop.

Let us now draw some conclusions for Equation (1.13). Energy levels $E \in (-\frac{1}{4}, 0)$ are regular and all contain a compact interval component, so that the existence of a simple periodic solution (with $\|u\|_\infty < 1$) in each of these energy levels can be concluded. This

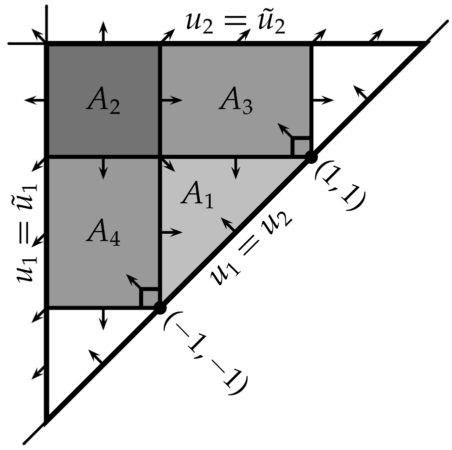

Figure 1.13: The triangle $D_E^+ = [\tilde{u}_1, \tilde{u}_2] \times [\tilde{u}_1, \tilde{u}_2] \cap \{u_2 > u_1\}$. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has at least one maximum in A_1 and one minimum in A_2. Additionally, when the equilibrium points are saddle-foci then W_2 has saddle points in A_3 and A_4.

holds for all values of γ and β since for $u \in [-1, 1]$ the restriction on β in Lemma 1.5 can be disregarded. There is also a closed characteristic in the broad sense in the singular energy level $E = 0$, the compact interval component being $[-1, 1]$. For $\beta > 0$, $\frac{\gamma}{\beta^2} \le \frac{1}{8}$ this is the heteroclinic loop, while for all other parameter values it is a simple periodic orbit.

We investigate the energy level $E = 0$ more carefully. Using the asymptotic behaviour of $F(u) = -\frac{1}{4}(u^2 - 1)^2$ for $u \to \infty$, one can show that for large $-\tilde{u}_1$ and $\tilde{u}_2$ the gradient $\nabla W_2(\tilde{u}_1, \tilde{u}_2)$ points in the north-west direction; the system is then said to be *dissipative* on $[\tilde{u}_1, \tilde{u}_2]$. Besides, if an equilibrium point u_* is a saddle-focus, then we can find a point $(\hat{u}_1, \hat{u}_2)$ close to (u_*, u_*) such that $\hat{u}_1 < u_* < \hat{u}_2$ and $\nabla W_2(\hat{u}_1, \hat{u}_2)$ points in the north-west direction.

The following theorem follows from these observations and the (monotonicity) properties of S_E in Lemma 1.6. The proof is illustrated in Figure 1.13.

Theorem 1.8 *Let L be as in* (1.21) *with $\beta \le 0$ and $F(u) = -\frac{1}{4}(u^2 - 1)^2$. Then in the energy level $E = 0$ there are at least two geometrically distinct simple closed characteristics: one with $\|u\|_\infty < 1$ and (generically) index 2, and one with $\|u\|_\infty > 1$ and (generically) index 0. If $u = \pm 1$ are saddle-foci then there exist two more geometrically distinct simple closed characteristics (generically index 1).*

We mention once again that all details of this analysis as well as generalisations are described in Chapter 7. Remark that the solutions in Theorem 1.8 were already found previously via the shooting method (Section 1.3.2). The importance of the Twist map approach lies more in the variational information and the generality. Moreover, it can be used to find more complicated closed characteristics. This is the subject of the next section.

1.3.4.2 Braids

The Twist property allows one to encode a characteristic by its extrema $\{u_i\}$. Assume without loss of generality that u_1 is a local minimum. We can construct a piecewise linear graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ by straight line segments (see

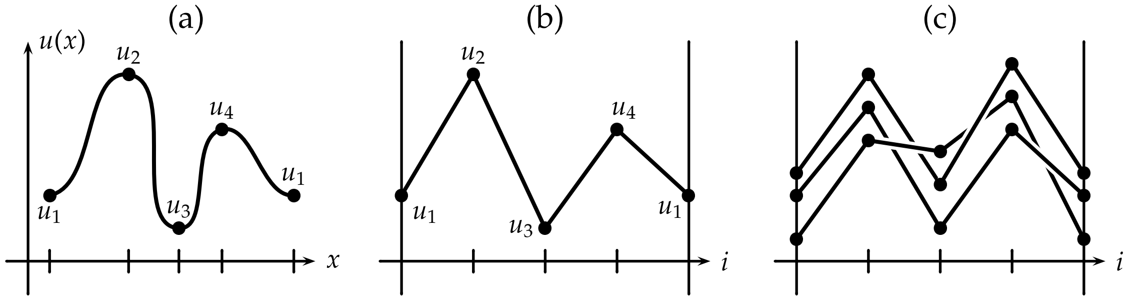

Figure 1.14: (a) A periodic function and (b) its piecewise linear graph; (c) a braid consisting of 3 strands.

Figure 1.14a,b). If u is a closed characteristics then its critical points are encoded in a finite sequence $\{u_i\}_{i=1}^{2p}$, where $2p$ is the discrete period. The piecewise linear graph, called a *strand*, is really cyclic: one restricts to $1 \le i \le 2p+1$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n strands. We place on these diagrams a *braid structure* by assigning a crossing type (positive) to every transverse intersection of the graphs: larger slope crosses over smaller slope (see Figure 1.14c). We thus represent periodic sequences of extrema in the space of closed, positive, piecewise linear braid diagrams. Since for bounded characteristics local minima and maxima occur alternately, we require that $(-1)^i(u_i - u_{i\pm1}) > 0$: the (natural) *up-down* restriction. This space of piecewise linear up-down braids is denoted by $\mathcal{E}_{2p}^n$, where $2p$ is the period and n is the number of strands. The completion $\overline{\mathcal{E}}_{2p}^n$ includes singular braid diagrams (having non-transverse crossings). We refer to Chapter 8 for more details and to [74] for a complete development of the theory.

The gradient flow of $W_{2p}(u_1, \ldots, u_{2p})$ on $2p$-periodic sequences immediately translates to a flow on $\overline{\mathcal{E}}_{2p}^n$. Lemma 1.6b implies that along this flow the number of crossings of a braid does not increase. This property is the discrete analogue of the lap number theorem for second order parabolic equations.

The strategy is to construct isolating neighbourhoods for the gradient flow of W_{2p} on $\overline{\mathcal{E}}_{2p}^n$ and to compute its Conley homology. Non-trivial Conley homology implies the existence of closed characteristics.

Consider the special situation of $(n+1)$-strand braid diagrams where n designated strands, the *skeleton*, corresponds to a collection of closed characteristics. Since these closed characteristics are stationary for the gradient flow of W_{2p}, it induces a flow on a ($2p$-dimensional) invariant subset of $\overline{\mathcal{E}}_{2p}^{n+1}$, the *relative braid diagrams*: only one of the strands exhibits dynamics under the gradient flow of W_{2p}.

The space $\overline{\mathcal{E}}_{2p}^{n+1}$ is partitioned into braid classes by co-dimension 1 'walls' of singular braids. This also induces a partitioning of the relative braid diagrams. These equivalence classes of braid types are candidates for isolating neighbourhoods.

In order to have a smooth flow on a compact space we consider the two *boundary conditions* introduced in Section 1.3.4.1: the compact case (large amplitudes are repelling) and the dissipative case (large amplitudes are attracting). Under either of these boundary conditions consider a braid class for which the $(n+1)^{\text{st}}$ strand is non-isotopic to the skeleton (i.e., none of the strands of the skeleton is contained in the boundary). The fact that the number of intersections only decreases along the flow implies that the closure of such

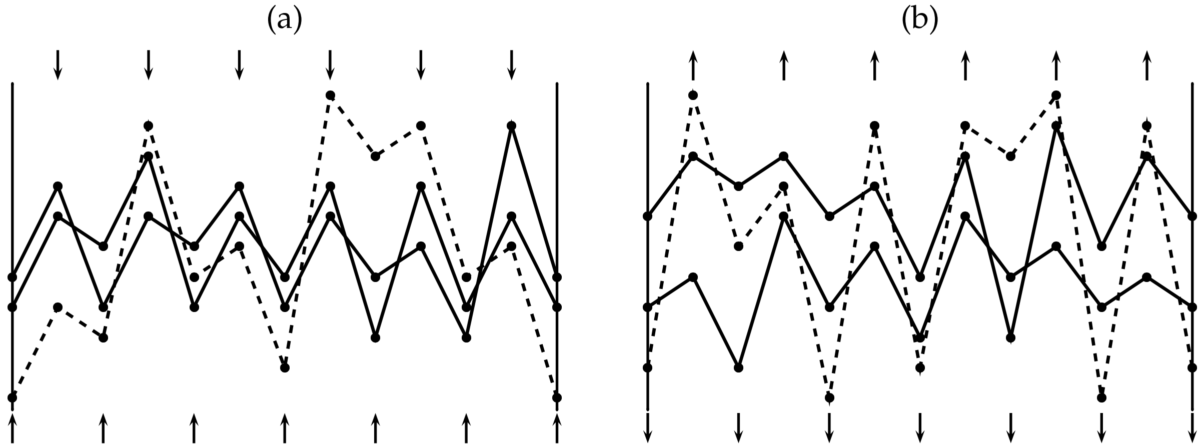

Figure 1.15: Two examples of relative braid classes (dashed) whose Conley homology with respect to the fixed strands (solid) is nontrivial. (a) Compact boundary conditions: $X^r_{p,q}$ with $p = 6$, $r = 3$, $q = 2$; (b) dissipative boundary conditions: $Y^r_{p,q}$ with $p = 6$, $r = 1$, $q = 4$.

a braid class is a proper isolating neighbourhood for the induced flow. Consequently the Conley homology is well-defined, see Chapter 8 for precise statements.

We carry out the above construction for two special braid classes depicted in Figure 1.15. In the compact case we consider a skeleton of two linked strands with period $2p$ and nonzero linking number r (i.e. crossing number $2r$), where $0 < r \le p$. The third strand (dashed) has linking number $q < r$ with the skeleton. We denote this braid class by $X^q_{p,r}$. In the dissipative case we consider a skeleton of two strands of period $2p$ with non-maximal linking number $0 \le r < p$. The third strand (dashed) has linking number $q > r$ with the skeleton. We denote this braid class by $Y^q_{p,r}$.

Proposition 1.9 *Consider the braid classes $X^q_{p,r}$ (with $0 < q < r \le p$) and $Y^q_{p,r}$ (with $0 \le r < q < p$) indicated in Figure 1.15. The Conley homology of the gradient flow of W_{2p} on these braid classes is well-defined and given by*

$$CH_k(X^q_{p,r}) = \begin{cases} \mathbb{Z} & k = 2q-1 \text{ or } 2q, \\ 0 & \text{else.} \end{cases} \qquad CH_k(Y^q_{p,r}) = \begin{cases} \mathbb{Z} & k = 2q \text{ or } 2q+1, \\ 0 & \text{else.} \end{cases}$$

One easily constructs an infinite family of closed characteristics with distinct braid types forced by the pair of non-maximally linked (including unlinked) orbits for dissipative boundary conditions or linked orbits for compact boundary conditions, by taking higher covers of the base orbits (i.e., taking multiples of p and r) and applying Theorem 8.1 iteratively.

Theorem 1.10 *Consider Equation* (1.12) *for a regular energy level E under the Twist hypothesis* (T). *The following are sufficient conditions for the existence of infinitely many distinct (in particular having distinct braid types) closed characteristics:*

(a) *a compact interval component I_E and the existence of a pair of closed orbits whose braid representations are linked.*

(b) *an interval component $I_E = \mathbb{R}$ with dissipative asymptotic behaviour and the existence of a pair of closed orbits whose braid representations are unlinked or non-maximally linked.*

Note that in both cases the existence of a single non-simple closed characteristic u is a sufficient condition. Indeed, two even shifts of the braid representation of u yield a 2-strand braid that is necessarily linked but not maximally linked.

1.3.5 Dynamics of the partial differential equation

In this section we discuss some of the consequences of the results on stationary solutions for the dynamics of the partial differential equation (1.9). A simple example is that minimisers of the action $J[u]$ are expected to be stable solutions of the partial differential equation, since (1.9) is, at least formally, the gradient flow of the action. Here one has to choose the function classes appropriately, so that this formal argument can be made precise. We also remark here that the index of a periodic solution is in general not equal to the index of the same function as solution of a boundary value problem. Besides, there are subtle differences between being a minimiser and being stable.

Let us focus on solutions on a finite interval $[0, \ell]$ with Neumann boundary conditions

$$u_x(t,0) = u_{xxx}(t,0) = u_x(t,\ell) = u_{xxx}(t,\ell) = 0. \tag{1.27}$$

Now (1.9) is indeed the L^2-gradient flow of J, and $J[u](t) = J[u(t,x)]$ is a Lyapunov functional: $\frac{dJ[u](t)}{dt} \leq 0$ for all solutions of (1.9) with boundary conditions (1.27). The natural function space for this case is $H^2_N = \{u \in H^2(0,\ell) \,|\, u_x(0) = u_x(\ell) = 0\}$.

By symmetry the stationary solutions can be extended to periodic solutions, and for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$, $\beta > 0$ these have all been classified in Section 1.3.1. The situation for the stationary solutions in this case is completely analogous to the second order equation. In Chapter 4 it is shown that this analogy extends to the characterisation of the attractor.

Theorem 1.11 *Let $\beta > 0$ and $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$. Then for all $\ell > 0$ there is a semi-conjugacy from the flow on the attractor of the fourth order equation* (1.9) *with Neumann boundary conditions* (1.27) *to the flow on the attractor of the second order equation.*

In particular this implies that for any interval length ℓ there are exactly two stable solutions, namely the homogeneous states $u \equiv \pm 1$.

For $\frac{\gamma}{\beta^2} > \frac{1}{8}$ the situation is completely different. The origin of this change is again the fact that the nature of the equilibrium points $u = \pm 1$ changes from real saddle to saddle-focus at $\frac{\gamma}{\beta^2} = \frac{1}{8}$. The minimisation method of Section 1.3.3 already gives one stable periodic solution (or at least a minimiser) of every homotopy type. Together with a symmetry property this shows that there are stable stationary solutions for many different interval lengths ℓ. This only proves the existence of stable solutions for a discrete set of values of ℓ. This is caused by the fact that all solutions have energy $\mathcal{E}[u] = 0$. In Chapter 4 a variational variant of the gluing method is used to construct stable solutions for all the intermediate values of ℓ. This construction is carried out in such a way that the shape of the stable equilibria is also revealed. The main result is a lower bound on the number of stable equilibria of (1.9) as a function of the interval length ℓ.

Theorem 1.12 *Let $\beta > 0$ and $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$. Then for any $n \in \mathbb{N}$ there exists a constant ℓ_n such that for all $\ell \geq \ell_n$ Equation* (1.9) *with Neumann boundary conditions* (1.27) *has at least n disjunct stable sets of stationary solutions. Moreover, the number of disjunct stable sets grows exponentially in ℓ.*

Each stable set consists of stationary solutions with a specific geometrical shape. One can think of these sets as consisting of exactly one stable stationary solution, but this cannot be proved generally.

As in Section 1.3.3 this method can be applied to a large class of double-well potentials $F(u)$ and even to more general Lagrangians $L(u, u', u'')$. The crucial condition in Theorem 1.12 is that $u = \pm 1$ are saddle-foci. The boundary conditions are of minor importance; as long as they are variational the result of Theorem 1.12 stays the same.

Let us consider the consequences for the formation of patterns for Equation (1.9). The usual definition of a *pattern* is a stable non-homogeneous stationary state, although different definitions are also in use. For (1.9) with Neumann boundary conditions the above theorems imply that there are no patterns for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$, whereas for $\frac{\gamma}{\beta^2} > \frac{1}{8}$ more and more patterns appear as $\ell \to \infty$. For other nonlinearities this transition may not be so sharp, but for small values of γ the behaviour is always similar to the second order equation, while as soon as the equilibria $u = \pm 1$ become saddle-foci pattern formation occurs. We note that the situation for $\beta < 0$ is not well understood and certainly merits further exploration. Finally, we remark that the slow motion results for the flow near the attractor [92] do not distinguish between the case where there are only two stable states and the case where there are may stable states, i.e., slow motion may be no motion.

We now turn our attention to the behaviour of (1.9) on the entire real line, i.e. $x \in \mathbb{R}$. For $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$, $\beta > 0$ the heteroclinic solutions are asymptotically stable. This follows from the fact that they are minimisers combined with the transversality result in Section 1.3.1. The results in [89] shows that there are many heteroclinic and homoclinic solutions for $\frac{\gamma}{\beta^2} \geq \frac{1}{8}$, $\beta > 0$, which are weakly stable in the sense that they are local minimisers of the action (this does however not ensure (asymptotic) stability).

There is another class of special solutions of (1.9) with $x \in \mathbb{R}$, namely travelling wave solutions. These are the subject of the next section.

1.3.6 Travelling wave solutions

We consider a special case of Equation (1.10) which is a slight generalisation of (1.9):

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \beta \frac{\partial^2 u}{\partial x^2} + (1 - u^2)(u + a), \qquad a \in [0, 1). \tag{1.28}$$

Travelling waves $u(t, x) = U(x + ct)$, where c is the wave speed, obey the ordinary differential equation (where we have switched to lower case again)

$$-\gamma u'''' + \beta u'' - cu' + (1 - u^2)(u + a) = 0. \tag{1.29}$$

We restrict our analysis to $c > 0$ (waves travelling to the left). The energy $\mathcal{E}[u](x)$ is a non-decreasing function: $\mathcal{E}'[u] = cu'^2 \geq 0$. Since for the equilibrium points we have $\mathcal{E}[1] > \mathcal{E}[-1] > \mathcal{E}[-a]$ (see Figure 1.2), we look for heteroclinic solutions of (1.29) from -1 to 1, and from $-a$ to ± 1.

For $\beta > 0$ fixed, say $\beta = 1$, and γ very small the problem of finding travelling waves and determining their stability may be approached via (singular) perturbation methods. This is done in [2, 72] for the travelling wave connecting -1 to 1. It is found that a unique travelling wave exists in the neighbourhood of the wave for the second order equation (i.e. $u = \tanh\left(\frac{x + a\sqrt{2}t}{\sqrt{2}}\right)$), and its wave speed is $c = a\sqrt{2} - \frac{1}{5}\sqrt{2}a(2a^2 - 3)\gamma + O(\gamma^2)$. Stability

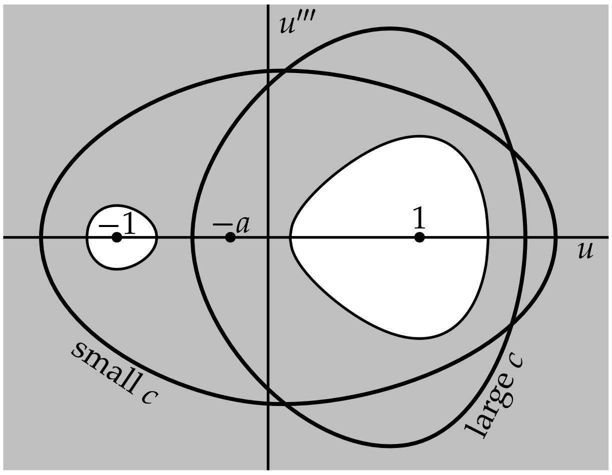

Figure 1.16: The projection (in grey) of the energy level $\{\mathcal{E} = E_0\}$ onto the (u, u''')-plane. The two closed curves depict the projection of the intersections $W^s(1) \cap \{\mathcal{E} = E_0\}$ for small c and large c.

of this wave could be proved following the method of [72]. In [130] travelling waves connecting $-a$ to ± 1 are proved to exist for small γ, and 'energy' estimates are used to prove stability (in a restricted class of initial profiles) for a range of wave speeds.

A perturbation result in a different direction is obtained in [34]. There it is proved that the heteroclinic solutions for $a = 0$ (and $c = 0$) obtained via the shooting method (see Section 1.3.2) are topologically transverse intersection of $W^u(\mp 1)$ and $W^s(\pm 1)$. A perturbation argument is then used to show that for all $\gamma > 0$, $\beta > 0$, each such a kink solution can be perturbed to a travelling wave solution of (1.28) for a sufficiently small, with $c = c(a)$ small as well.

In Chapter 5 we investigate the existence of travelling wave solutions for a more global parameter range. Let us first consider travelling waves connecting -1 to 1. We want to find an intersection of the two dimensional manifolds $W^u(-1)$ and $W^s(1)$, so we follow the orbits in $W^s_{\text{loc}}(1)$ back in time. Choose an energy level $E_0 \in \big(\mathcal{E}[-1], \mathcal{E}[1]\big)$. If an orbit in $W^s(1)$ is not in $W^u(-1)$ then it will intersect this energy level $\{\mathcal{E} = E_0\}$. For both very small and very large $c > 0$ it is found that the intersection $W^s(1) \cap \{\mathcal{E} = E_0\}$ is a closed curve. However, it turns out that one curve cannot be deformed continuously into the other curve inside the energy level $\{\mathcal{E} = E_0\}$. The projection of the energy level $\{\mathcal{E} = E_0\}$ onto the (u, u''')-plane contains two holes, and the projection of the closed curves for small c and large c wind around these holes in topologically distinct ways, see Figure 1.16.

Since the closed curves cannot be deformed into each other, the loop has to break at some moment when we vary c from 0 to ∞. At this breaking point, say at $c = c_0$, there is an orbit $u(x)$ in $W^s(1)$ which does not intersect $\{\mathcal{E} = E_0\}$. The existence of the Lyapunov functional $\mathcal{E}[u](x)$ now implies that this orbit has to converge to -1 as $x \to -\infty$. These arguments, which are made rigorous in Chapter 5 for a range of parameter values, show that there exists a travelling wave connecting $u = -1$ to $u = 1$:

Theorem 1.13 *Let* $0 < a < 1$ *and* $\beta > 0$, $\frac{\gamma}{\beta^2} < \sigma(a)$ *where* $\sigma(a) = \min\{\frac{-F(u)}{2f(u)^2} \mid u \in (-1, -a)\}$. *Then for some* $c = c_0 > 0$ *there exists a travelling wave solution of* (1.28) *connecting* $u = -1$ *and* $u = 1$.

To give an idea of the value of $\sigma(a)$ we mention the estimate $\sigma(a) > \frac{1}{8(1-a)}$ for $f(u) = (1 -$

$u^2)(u+a)$. The reason for the upper bound on the parameters is that the analysis of the closed curve for small $c > 0$ can only be carried out for those values of $\frac{\gamma}{\beta^2}$. The proof involves a global analysis of the flow of (1.29) and in particular the flow *at infinity* has to be investigated. This leads to a complete analysis of the equation $u'''' + u^3 = 0$, which is of independent interest (in particular, the dynamics at infinity is completely governed by two periodic orbits, one attracting, the other repelling).

Travelling waves connecting -1 to 1 exist only for special values of the wave speed c, and finding the wave speed is part of the problem. In contrast, intersections of the three dimensional manifold $W^u(a)$ and the two dimensional manifolds $W^s(\pm 1)$, and hence travelling waves connecting $-a$ to ± 1, are expected to and indeed do exist for large ranges of the wave speed c.

Theorem 1.14 *Let* $0 \le a < 1$ *and let* $\gamma > 0$, $\beta \in \mathbb{R}$. *Then*

(a) *for all* $c > 0$ *there exists a travelling wave solution of* (1.28) *connecting* $u = -a$ *and* $u = -1$.

(b) *there exists a* $c^* > 0$ *such that for all* $c > c^*$ *there exists a travelling wave solution of* (1.28) *connecting* $u = -a$ *and* $u = 1$.

This theorem as well as Theorem 1.13 can be generalised to large classes of nonlinearities $f(u)$.

A different form of a travelling structure has been studied in [47]. These fronts are not travelling waves, but rather satisfy

$$u(t, x - c\tau) = u(t + \tau, x), \qquad \text{for some } \tau, c > 0. \tag{1.30}$$

Equation (1.30) implies that after a time τ the profile has moved a distance $c\tau$ to the right, but the profile does not translate uniformly. The profiles under consideration in [47] are solutions $u(t,x)$ of Equation (1.11) for small $\alpha > 0$ which tend to 0 as $x \to \infty$, and for $x \to -\infty$ they look like some small stationary periodic solution $v(x)$, i.e. $u(t,x) \to v(x)$ as $x \to -\infty$. In other words, these solutions are heteroclinic (in time) from a homogeneous state to a periodic state. Although the existence of such solutions has only been proved for very small α, numerical results suggest that they are omnipresent. For example, in [53, 136] it is suggested that such solutions play an important role in the dynamics of Equation (1.9) for $\frac{\gamma}{\beta^2} > \frac{1}{12}$ (although the limiting periodic solution $v(x)$ is unstable for $\frac{\gamma}{\beta^2} \le \frac{1}{8}$). A global analysis of this type of solutions has so far been lacking. Finally, we also mention related work in [132] on pulse-shaped travelling structures of the form (1.30).

1.4 Reflections

Having already touched upon an interesting unresolved issue at the end of the previous section, a few other ideas and open problems are described below.

The dynamics for $\beta < 0$

The results in this thesis present a rather complete picture of the dynamics and equilibria of Equation (1.9) in the parameter region $\beta \ge 0$. Besides, for $\beta < 0$ the time-independent problem is extensively studied. The dynamics of the partial differential equation in the parameter region $\beta < 0$ is much less explored. While many stationary periodic solutions have been found, not much is known about the behaviour of time-dependent solutions. The first step should be to perform a systematic numerical investigation.

Let us summarise the few results that are known in this direction. For small $\beta < 0$ the behaviour of the stationary states is the same as for $\beta \in [0, \sqrt{8})$, but when β becomes more negative this starts to change. For example, for $\beta < -\sqrt{8}$ the homogeneous states $u = \pm 1$ are no longer asymptotically stable for the Neumann boundary value problem on large intervals (they are however always the only stable solution if the interval length is sufficiently small). The large simple periodic solution of Equation (1.13) with $\beta \leq 0$, which is found in Theorem 1.8 and which lies in the energy level $E = 0$, is a stable solution. A similar stable solution can be found for small non-zero energies. Of course, such a periodic solution is only a solution of the Neumann problem when the interval length is a multiple of half its period. For values of β less than about -0.92 we know that there is a simple periodic minimiser of the minimisation problem (1.22). This solution is clearly stable as well, and it is in general a different solution than the ones described above, since the energy of this periodic solutions grows towards infinity as $\beta \to -\infty$. Let us remark that although there are numerous indications that these solutions are stable, this has in fact never been proved rigorously. Moreover, it is unknown whether there are other more complicated stable solutions.

Homoclinic orbits for $\beta < 0$

For $\beta \geq 0$ many homoclinic solutions to ± 1 of Equation (1.13) have been found, both via shooting and via variational methods. For $\beta < 0$ such homoclinic solutions appear to be much harder to grasp. One expects them to exist as long as $u = \pm 1$ are saddle-foci, and this is also observed numerically. Shooting methods fail in showing the convergence of the tail of the solution to the equilibrium point. The unboundedness from below of the action $J[u]$ is the biggest obstacle for variational approaches. The estimates needed for a classical mountain pass argument have only been obtained when the nonlinearity is of the special form $f(u) = -u + u^s$, $s > 1$ [33].

Recently it has been shown in [134], using the monotonicity of $J[u]$ with respect to β, that for almost all values of $\frac{\beta}{\sqrt{\gamma}} \in (-\sqrt{8}, 0)$ there is a solution of (1.13) which is homoclinic to 1. This solution has the property that $u(x) > -1$ for all $x \in \mathbb{R}$. A similar result is obtained for $f(u) = 1 - e^u$, which corresponds to the suspension bridge problem.

Let us mention that numerical calculations [19, 35] suggests the existence of many branches of homoclinic solutions of (1.13), but only two of those extend all the way to $\beta = -\sqrt{8}$. Finally, for (1.12) with $f(u) = -u + u^2 + bu^3$ branches of homoclinic solutions have also been studied numerically, and it has been found that these branches show a phenomenon known as homoclinic snaking, see [141]. There it also becomes apparent that the heteroclinic solutions for $b = \frac{2}{9}$ (the equation is then equivalent to (1.13)) play a central organising role.

Limiting behaviour for $\beta \to -\infty$

The limit $\beta \to -\infty$ can also be considered as the limit $\gamma \to 0$ with $\beta < 0$ fixed. Two cases have to be distinguished: families of solution whose amplitude $\|u\|_\infty$ grows to infinity,

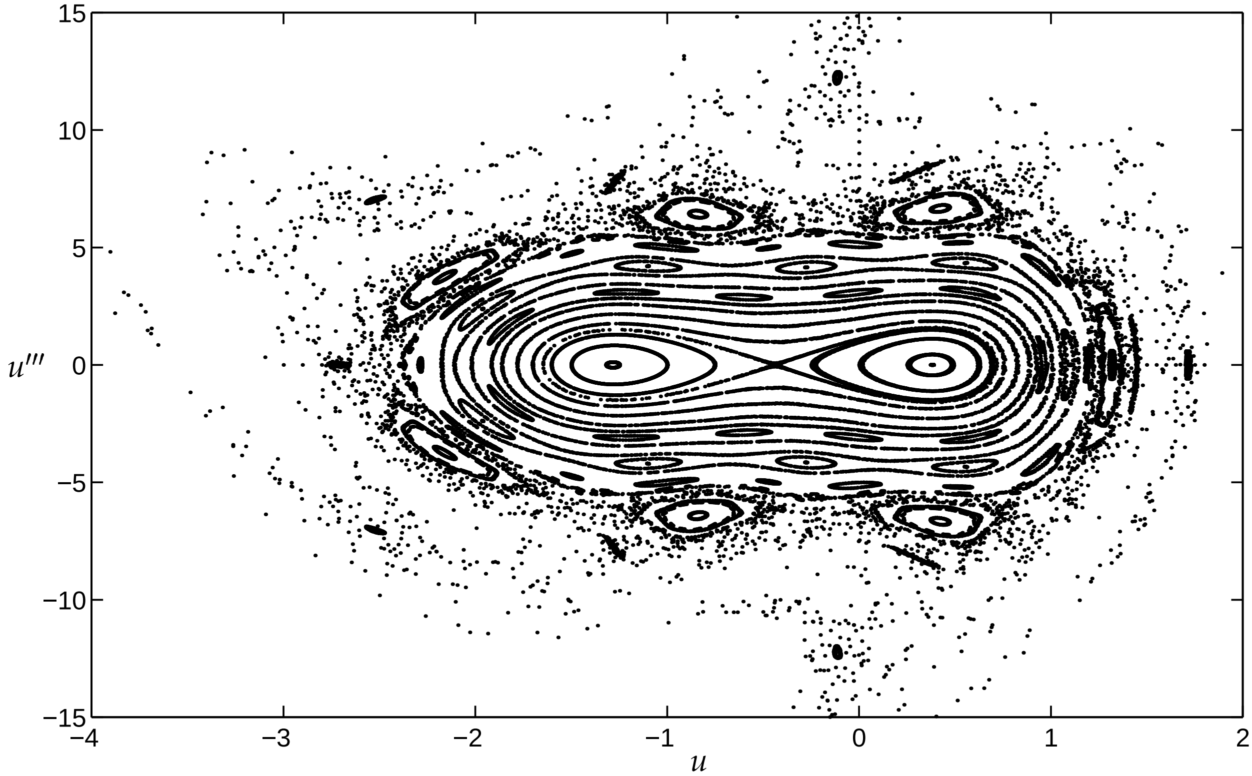

Figure 1.17: Orbits of the Poincaré map $\mathcal{T}$ in the (u, u''')-plane for Equation (1.13) with $\gamma = 1$, $\beta = -10$ and $E = 10$.

and families of solutions which are uniformly bounded.

In [106] it is shown that the simple periodic solution of (1.13) with $\|u\|_\infty > 1$ and energy $\mathcal{E}[u] = 0$ has asymptotic behaviour $\|u\|_\infty \sim |\beta|$, while the period behaves as $|\beta|^{-\frac{1}{2}}$ when $\beta \to -\infty$. After the rescaling $v(x) = \frac{1}{|\beta|} u(|\beta|^{-\frac{1}{2}} x)$ we obtain

$$-v'''' - v'' + \kappa v - v^3 = 0, \qquad \text{where } \kappa = \tfrac{1}{\beta^2}. \tag{1.31}$$

In this equation κ can be varied through zero continuously, and for $\kappa < 0$ the solutions still correspond to stationary solutions of the Swift-Hohenberg equation (1.11). In this way the periodic solutions whose amplitude grows to infinity as $\beta \to -\infty$ can be linked to the periodic solutions which bifurcate from 0 for $-\frac{1}{4} \leq \kappa < 0$, see also [122].

Families of periodic solutions which stay bounded exhibit a very different type of behaviour. Let us fix $\beta = -1$ and let $\gamma = \varepsilon^2$. Numerical and formal computations indicate that for small ε all bounded solutions of (1.12) are approximately of the form

$$u(x) \approx u_0(x) + C\varepsilon^2 \cos\big(\frac{x - \xi}{\varepsilon}\big), \qquad \text{for some } C, \xi \in \mathbb{R}, \tag{1.32}$$

where $u_0(x)$ is any solution of the second order equation $-u_0'' + f(u_0) = 0$. This has however only been proved for the family of small simple periodic solutions with energy $\mathcal{E}[u] = 0$, in which case $u_0 \equiv 0$ so that it asymptotically becomes a linear problem.

In order to obtain insight into the transition from small solutions (described by (1.32)) and large solutions (governed by (1.31)), a Poincaré map can be studied. In a fixed energy level E consider the Poincaré section $\Sigma = \{(u, u', u'', u''') \mid u' = 0,\, u'' = \sqrt{\frac{2}{\gamma}(E - F(u))}\}$, corresponding to the local minima of solutions. Let $F(u) = -\frac{1}{4}(u^2 - 1)^2$, then for $E > 0$ this section Σ is a global section. It is natural to project Σ onto the (u, u''')-plane. The Poincaré return map is denoted by $\mathcal{T}$ (see also Section 1.3.4.1). In Figure 1.17 orbits of this map $\mathcal{T}$

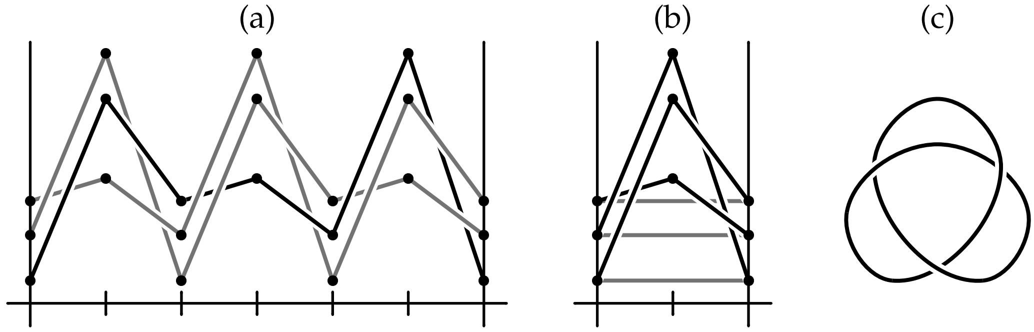

Figure 1.18: (a) A strand and its even translates. (b) All information is contained in an interval of length 2. (c) The braid can be closed to form a knot, in this case the trefoil.

(for Equation (1.13)) in the (u, u''')-plane are depicted (with $\gamma = 1$, $\beta = -10$ and $E = 10$). In the central region (small solutions) one observes many invariant curves and (1.32) seems a good description of the dynamics. Further away from the origin chaotic regions and (periodic) invariant regions alternate. Solutions with very large amplitude oscillate to infinity rapidly. For smaller ε the orderly central region becomes larger and larger; a KAM-like scenario takes place.

To conclude, it is known that no homoclinic orbit to $u = 1$ for (1.12) with $f(u) = -u + u^2$ exists in the limit $\beta \to -\infty$, as opposed to the situation for $\beta \to \infty$ where a homoclinic to $u = 0$ does exist. This involves a careful analysis of exponentially small terms, i.e., terms beyond all orders in the perturbation expansion; we refer to [63] and the references therein.

Braids and Twist maps

There are many interesting questions in connection with the braids and the Twist map in Section 1.3.4. First of all, the braid classes can be refined by incorporating all even translates of the strands (see Figure 1.18a). This gives (additional) information on the relative position of the extrema. Notice that in this case all information is already contained in an interval of length 2 (see Figure 1.18b). The set of boundary points on the left is connected to the same set on the right boundary, but when we follow the strands from left to right the points are permuted. The representation in Figure 1.18b can be used to close the braid by identifying the points on the left boundary with those on the right boundary. In this way one obtains a knot (or an unknot), see Figure 1.18c. An important issue is whether the orbit of a corresponding solution, when it is regarded as lying in the three dimensional energy surface, has the same knot type as its braid. These questions are part of the investigation in [74].

Another problem is what to do in those cases where we are unable to prove that the Poincaré map is a Twist map. For example, for $\beta > 0$ numerics strongly suggest that for (1.12) the Twist property still holds. The only restriction seems to be that for singular energy levels no equilibrium point with real saddle character lies in the interior of the interval component. However, suggestive as the numerics may be, we are not able to prove this in general. A possible solution could be to use the degree, or preferably the Conley index, to apply a continuation argument. For simple closed characteristics this can indeed be carried out [91].

Finally, the flow on the space of braids is of course just a flow on the end points of monotone laps. This can be regarded as a discretisation of an evolution of curves, and more particular of a curve shortening flow. Namely, for $\beta \geq 0$ the action $J[u]$ can be interpreted as a length of the curve in the (u, u')-plane. Parametrising curves in the (u, v)-plane (where $v = u'$) by $(u(s), v(s))$, one defines the length $\ell = \oint L\big(u, v, \frac{v}{u_s} v_s\big) \frac{u_s}{v}\, ds$ (so that $\ell = J[u] > 0$ for $\beta \geq 0$). Stationary points of the curve shortening flow now are critical points of J and thus solutions of (1.12). This is all formal arguing, but in this formulation the problem has strong similarities with the application of curve shortening to find closed geodesics on two dimensional manifolds [7].

General extensions

It is easy to come up with generalisations of Equation (1.10) or (1.12). First, one could investigate sixth order equations of the type $u_t = Au_{xxxxxx} - Bu_{xxxx} + Cu_{xx} + f(u)$. We refer to [72] for a perturbation result, and to [124] for some basic variational results. Second, numerical calculations for Equation (1.11) in two (or more) spatial dimensions show a wide variety of interesting patterns such as roles, hexagonal lattices and labyrinth patterns. We refer to [50, 56, 28] for some nice numerical pictures. A mathematical theory seems to be utterly lacking.

Let us nevertheless mention a few results in more space dimensions. The stability in higher space dimensions of essentially one dimensional structures such as domain walls, roles or travelling waves, could be approached as in [93], possibly in combination with Evans function techniques (see e.g. [72]). The Γ-limit for Equation (1.9) in higher space dimensions is examined in [83]. Global uniqueness results on the equation $(-\Delta)^n u + u^p = f$ in arbitrary space dimensions are studied in [26]. We refer to [45] for a study of the (exceptional) circumstances where a maximum or anti-maximum principle can be applied to equations of the form $(-\Delta)^n u = \lambda u + f$.

Finally, instead of higher order equations it would be worth investigating to what extent the methods in this thesis can be applied to systems of second order equations. In particular, a first order Lagrangian $L(\vec{u}, \vec{u}')$ with $\vec{u} = (u_1, u_2)$, leads to three dimensional energy manifolds, and perhaps in certain cases the problem of finding closed characteristics on these manifolds can be formulated in terms of a Twist map.

Two dimensional topology

It is remarkable how often the topology of the *two* dimensional plane, and the behaviour of *second* order equations is exploited in the various attempts in this thesis to examine *fourth* order equations, and *four* dimensional phase spaces. To name just a few examples: the projection onto the (u, u')-plane in Section 1.3.1, and onto the (u, u''')-plane in Section 1.3.6 (and the (u, u'')-plane is also used in some instances). Then there is the topology of the punctured plane in Section 1.3.3, and the two dimensional Poincaré section in Section 1.3.4.1. One can argue whether braids are two or three dimensional objects. And to top it of, the essentially second order behaviour of the gradient flow in the space of braids, and Equation (1.25) which is used to prove the Twist property, are striking examples of crucial occurrences of second order equations in the analysis of fourth order problems.

Chapter 2

The phase-plane picture

2.1 Introduction

In this chapter we study the bounded solutions of fourth order differential equations of the form

$$-\gamma u'''' + u'' + f(u) = 0, \qquad \gamma > 0, \tag{2.1}$$

where $f(u) = \frac{dF(u)}{du}$, and $F(u)$ is called the *potential*. By a bounded solution we mean a function $u(x) \in C^4(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which satisfies (2.1) for all $x \in \mathbb{R}$. For small positive γ, Equation (2.1) is a singular perturbation of the (mechanical) equation

$$u'' + f(u) = 0. \tag{2.2}$$

We investigate the correspondence between bounded solutions of (2.1) and (2.2).

Note that (2.1) is both translation invariant and reversible (invariant under the transformation $x \mapsto -x$). Besides, there is a constant of integration. When we multiply (2.1) by u' and integrate, we obtain the *energy* or Hamiltonian

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -\gamma u''' u' + \frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 + F(u) = E, \tag{2.3}$$

where E is constant along solutions.

In recent years fourth order equations of the form (2.1) have attracted a wide interest, and two special cases have been thoroughly studied. First, when the potential is

$$F(u) = -\frac{1}{4}(u^2 - 1)^2, \tag{2.4}$$

then Equation (2.1) is the stationary version of the Extended Fisher-Kolmogorov (EFK) equation, which has been studied by shooting methods [117, 120, 118, 119] and through variational approaches [124, 90, 89, 88]. Generalisations of the EFK potential have been studied in [116], including potentials with maxima of unequal height. Second, in the study of a strut on a nonlinear elastic foundation and in the study of shallow water waves, Equation (2.1) arises with the potential

$$F(u) = -\frac{1}{2}u^2 + \frac{1}{3}u^3. \tag{2.5}$$

The homoclinic orbits of this equation have been studied both analytically [3, 43, 36, 32] and numerically [35, 41]. In these studies a striking feature is that the behaviour of solutions changes dramatically when the parameter γ reaches the lowest value for which one of the equilibrium point becomes a saddle-focus. Below this critical value the solutions that have been found are as tame as for the second order equation. When one of the equilibrium points becomes a saddle-focus, an outburst of new solutions appears.

The situation for $\gamma < 0$ seems to be much less understood. We refer to [38] for an overview of equations of the form

$$u'''' - Au'' + Bu = f(u, u', u'', u''') \qquad A, B \in \mathbb{R}.$$

As remarked, the character of the equilibrium point plays a dominating role. If an equilibrium point is a center for the second order equation, then it is a saddle-center for all $\gamma > 0$. On the other hand, if an equilibrium point is a saddle for the second order equation, then it is a real saddle for small (positive) values of γ. The character of such a point changes to saddle-focus as γ increases beyond some critical value.

Since (2.1) is a singular perturbation of the equation for $\gamma = 0$, it is natural to ask when it inherits solutions from the second order equation. For *small* γ this question can be answered by using singular perturbation theory [2, 72]. Here we follow an approach that leads to uniqueness results for a wider range of γ-values. The method is based on repeated application of the maximum principle. In [35] this idea has been used to prove the uniqueness of the homoclinic orbit for the potential in (2.5).

We shall first state two general theorems and subsequently draw detailed conclusions for the case of the EFK equation. In fact, the general theorems presented here, are a natural extension of the result for the EFK equation, of which a short summary has been published in [20].

We consider functions $f(u) \in C^1(\mathbb{R})$ and define, for $-\infty \le a < b \le \infty$,

$$\omega(a, b) \stackrel{\text{def}}{=} \max\{0, \max_{u \in [a,b]} -f'(u)\}.$$

We are only interested in cases where $\omega(a, b) < \infty$. We will often drop the dependence of ω on a and b, when it is clear which constants a and b are meant. Also, we introduce sets of bounded functions

$$\mathcal{B}(a, b) \stackrel{\text{def}}{=} \{u \in C^4(\mathbb{R}) \mid u(x) \in [a, b] \text{ for all } x \in \mathbb{R}\}.$$

In the following we often have an a priori bound on the set of all bounded solutions, i.e., for some $-\infty \le a < b \le \infty$ all bounded solutions of (2.1) are in $\mathcal{B}(a, b)$. It is important to keep in mind that these a priori bounds are usually valid for a range of values of γ. As will be clear from the statement of the theorems below, a better bound leads to a lower value of ω, which in turn leads to a stronger result.

The bounded solutions of the second order equation ($\gamma = 0$) are found directly from the phase-plane. Our first theorem states that the (u, u')-plane preserves the uniqueness property for the fourth order equation as long as γ is not too large.

Theorem 2.1 *Let u_1 and u_2 be bounded solutions of* (2.1)*, i.e., u_1 and u_2 are in $\mathcal{B}(a, b)$ for some $-\infty < a < b < \infty$. Suppose that $\gamma \in \big(0, \frac{1}{4\omega(a,b)}\big]$. Then the paths of u_1 and u_2 in the (u, u')-plane do not cross.*

Remark 2.2 It turns out that we need to give a meaning to the case $\gamma = \infty$. A scaling in x, which is discussed later on, shows that the natural extension of (2.1) for $\gamma = \infty$ is $-u'''' + f(u) = 0$. •

The following theorem shows that the energy $\mathcal{E}[u]$ (see (2.3)) is a parameter that orders the bounded solutions in the phase-plane.

Theorem 2.3 *Let u_1, $u_2 \in \mathcal{B}(a,b)$ be bounded solutions of* (2.1) *for some $\gamma \in \big(0, \frac{1}{4\omega(a,b)}\big]$. Suppose that (after translation) $u_1(0) = u_2(0)$ and either $u_1'(0) > u_2'(0) \geq 0$ or $u_1'(0) < u_2'(0) \leq 0$. Then $\mathcal{E}[u_1] > \mathcal{E}[u_2]$.*

We now give some examples. For the double-well potential $F(u) = \frac{1}{4}(u^2 - 1)^2$ (note that this is not the EFK potential in (2.4)) we have that $\omega(-\infty, \infty) = 1$ and thus any two bounded solutions do not cross in the (u, u')-plane for $\gamma \in (0, \frac{1}{4}]$. Besides, in this parameter range the energy ordering of Theorem 2.3 holds for all bounded solutions of (2.1).

In the case of the periodic potential $F(u) = \cos u$, we again have $\omega(-\infty, \infty) = 1$. In this case Theorem 2.1 combined with the periodicity of the potential, shows that for $\gamma \in (0, \frac{1}{4}]$ every bounded solution has its range in an interval of length at most 2π. We note that in both cases $\gamma = \frac{1}{4}$ is exactly the value where the character of some of the equilibrium points changes from real saddle to saddle-focus.

In the previous two examples we did not need an a priori bound. However, for the EFK potential (2.4), the existence of a uniform bound on the bounded solutions is needed to obtain a finite ω. The results for the EFK equation are discussed in detail in Section 2.1.1. For the potential (2.5) only a lower bound is needed.

Let us now assume that for some $\gamma > 0$ we have an a priori bound on the set of bounded solutions, i.e., all bounded solutions of (2.1) are in $\mathcal{B}(a,b)$ for some $-\infty \leq a < b \leq \infty$, and let us assume that $\omega = \omega(a,b) < \infty$. Then if $\gamma \in [0, \frac{1}{4\omega}]$, bounded solutions of (2.1) do not cross (by Theorem 2.1), and Theorem 2.3 gives an ordering of the bounded solutions in the (u, u')-plane in terms of the energy. An immediate consequence of Theorem 2.1 and the reversibility of (2.1), is that when $\gamma \in [0, \frac{1}{4\omega}]$, any bounded solution of (2.1) is symmetric with respect to its extrema (therefore the analysis in Theorem 2.3 is restricted to the upper half-plane). This implies that the only possible bounded solutions are

- equilibrium points,
- homoclinic solutions with one extremum,
- monotone heteroclinic solutions,
- periodic solutions with a unique maximum and minimum value.

Another implication is that there are at most two bounded solutions in the stable and unstable manifolds of the equilibrium points.

We will use the following formulation. If $\tilde{u}(x)$ is a solution of (2.1), then by the transformation

$$u(x) = \tilde{u}(\sqrt[4]{\gamma}\, x) \qquad \text{and} \qquad q = -\frac{1}{\sqrt{\gamma}},$$

it is transformed to a solution of

$$u'''' + qu'' - f(u) = 0, \qquad q < 0. \tag{2.6}$$

We examine the case where $q \leq -2\sqrt{\omega}$, corresponding to $\gamma \in (0, \frac{1}{4\omega}]$. It should be clear that solutions of (2.6) with $q \leq -2\sqrt{\omega}$ correspond to solutions of (2.1) with $0 < \gamma \leq \frac{1}{4\omega}$, and vice versa. The energy in the new setting is

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -u'''u' + \frac{1}{2}(u'')^2 - \frac{q}{2}(u')^2 + F(u). \tag{2.7}$$

For $q \le -2\sqrt{\omega}$ we define λ and μ such that

$$\lambda\mu = \omega \qquad \text{and} \qquad \lambda + \mu = -q,$$

or explicitly,

$$\lambda = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \omega} \qquad \text{and} \qquad \mu = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \omega}\,.$$

It is easily seen that λ and μ are positive real number if and only if $q \le -2\sqrt{\omega}$. In that case we have

$$0 \le \lambda \le \sqrt{\omega} \le \mu.$$

Equation (2.6) can be factorised as

$$\begin{cases} u'' - \lambda u = w \\ w'' - \mu w = f(u) + \omega u, \end{cases} \tag{2.8}$$

and the definition of ω ensures that $f(u) + \omega u$ is a non-decreasing function of u for $u \in [a, b]$.

The central tool in this chapter is a comparison lemma which shows that if the initial data of two solutions obey certain inequalities, then at most one of the solutions can be bounded (cf. [35, Th. 2.1]).

Lemma 2.4 (Comparison Lemma) *Let u and v be solutions of* (2.6) *such that, for some $-\infty < a < b < \infty$,*

$$a \le u(x) \le b, \quad a \le v(x) \le b \qquad \text{for all } x \in [0, \infty).$$

Suppose that $q \le -2\sqrt{\omega(a,b)}$ and

$$\begin{aligned} u(0) &\ge v(0), & u'(0) &\ge v'(0), \\ u''(0) - \lambda u(0) &\ge v''(0) - \lambda v(0), & u'''(0) - \lambda u'(0) &\ge v'''(0) - \lambda v'(0). \end{aligned}$$

Then $u(x) - v(x) \equiv C$ on $[0, \infty)$ for some constant $C \in \mathbb{R}$, and $C = 0$ if $\omega(a, b) \ne 0$.

Note that when the bounds a and b are sharper, then ω and λ are smaller, hence the conditions in the statement of the lemma are weaker. The proof of this lemma relies on the factorisation (2.8) of Equation (2.6).

We remark that both the splitting (2.8) of the differential operator and the Comparison Lemma can be extended to sixth and higher order equations. However, the increasing dimension of the phase space and the lack of additional conserved quantities (like the energy) make it a difficult task to extend the uniqueness results to such higher order equations.

This chapter mainly deals with uniqueness of solutions, but the information we obtain about the shape of solutions of (2.1) for γ not too large also allows us to conclude that any periodic solution belongs to a continuous family of solutions.

Theorem 2.5 *Let u_0 be a periodic solution of* (2.1) *and let $a \equiv \min u_0(x)$ and $b \equiv \max u_0(x)$. Suppose that $\gamma \in \left(0, \frac{1}{4\omega(a,b)}\right]$. Then u_0 belongs to a continuous one-parameter family of periodic solutions, parametrised by the energy E. To be precise, let $E_0 = \mathcal{E}[u_0]$, then for $\varepsilon > 0$ sufficiently small there are periodic solutions u_E of* (2.1) *for all $E \in (E_0 - \varepsilon, E_0 + \varepsilon)$ such that $\mathcal{E}[u_E] = E$ and $u_{E_0} = u_0$, and such that u_E depends continuously on the parameter E.*

2.1.1 An example: the EFK equation

The stationary version of the Extended Fisher-Kolmogorov (EFK) equation is given by

$$-\gamma u'''' + u'' + u - u^3 = 0, \qquad \gamma > 0. \tag{2.9}$$

The EFK equation is a generalisation (see [49, 53]) of the classical Fisher-Kolmogorov (FK) equation ($\gamma = 0$). Clearly (2.9) is a special case of (2.1) with the potential $F(u) = -\frac{1}{4}(u^2 - 1)^2$. We note that in some literature about the EFK equation the function $+\frac{1}{4}(u^2-1)^2$ is called the potential. In the form of (2.6) the EFK equation becomes

$$u'''' + qu'' - u + u^3 = 0. \tag{2.10}$$

Linearisation around $u = -1$ and $u = +1$ shows that the character of these equilibrium points depends crucially on the value of γ. For $0 < \gamma \leq \frac{1}{8}$ they are real saddles (real eigenvalues), whereas for $\gamma > \frac{1}{8}$ they are saddle-foci (complex eigenvalues). The behaviour of solutions of (2.9) is dramatically different in these two parameter regions.

For $\gamma \in (0, \frac{1}{8}]$ the solutions are calm. It was proved in [117] that there exists a monotonically increasing heteroclinic solution (or *kink*) connecting -1 with $+1$ (by symmetry there is also a monotonically decreasing kink connecting $+1$ with -1). This solution is antisymmetric with respect to its (unique) zero. Moreover, it is unique in the class of monotone antisymmetric functions. In [120] it was shown that in every energy level $E \in (-\frac{1}{4}, 0)$ there exists a periodic solution, which is symmetric with respect to its extrema and antisymmetric with respect to its zeros. Remark that these solutions correspond exactly to the solutions of the FK equation ($\gamma = 0$).

In contrast, for $\gamma > \frac{1}{8}$ families of complicated heteroclinic solutions [89, 90, 118] and chaotic solutions [119] have been found. The outburst of solutions for $\gamma > \frac{1}{8}$ is due to the saddle-focus character of the equilibrium points ± 1.

We will prove that as long as the equilibrium points are real-saddles, i.e. $\gamma \leq \frac{1}{8}$ or $q \leq -\sqrt{8}$, bounded solutions are uniformly bounded above by $+1$ and below by -1. To prove this, we first recall a bound proved in [119, 116], stating that any bounded solution of (2.9) for $\gamma > 0$ ($q \leq 0$) obeys

$$|u(x)| < \sqrt{2} \qquad \text{for all } x \in \mathbb{R}. \tag{2.11}$$

This bound is deduced from the shape of the potential and the energy identity. It already shows that Theorems 2.1 and 2.3 hold for $\omega = 5$, i.e., for any pair of bounded solutions of (2.9) with $\gamma \in (0, \frac{1}{20}]$. The method used to obtain this a priori estimate on all bounded solutions is applicable to a class of potentials which strictly decrease to $-\infty$ as $|u| \to \infty$ (see Section 2.4).

The a priori bound can be sharpened in the case of the EFK equation.

Theorem 2.6 *For any $\gamma \leq \frac{1}{8}$, let u be a bounded solution of* (2.9) *on $\mathbb{R}$. Then $|u(x)| \leq 1$ for all $x \in \mathbb{R}$.*

Having established the a priori bound (2.11), the sharper bound is obtained by applying the maximum principle twice to the factorisation of (2.10). Remark that a sharper bound than the one in Theorem 2.6 is not possible since $u = \pm 1$ are equilibrium points of (2.9).

This theorem implies that we can sharpen the results of Theorems 2.1 and 2.3 to $\gamma \in (0, \frac{1}{8}]$, i.e., to all values γ for which the equilibrium points ± 1 are real saddles. It follows that for $\gamma \in (0, \frac{1}{8}]$ bounded solutions do not cross in the (u, u')-plane, and they are ordered by their energies.

Remark 2.7 For the potential in Equation (2.5) an upper bound is not needed since $f'(u) = -1 + 2u > 0$ for $u > \frac{1}{2}$. An a priori lower bound of $a = 0$ for $\gamma \leq \frac{1}{4}$ can be found in the same way as in the proof of Theorem 2.6. Therefore, for the potential (2.5) Theorems 2.1 and 2.3 hold for $\gamma \in (0, \frac{1}{4}]$ or $q \leq -2$ (see also [35]). •

We want to emphasise that the methods used in this chapter to obtain a priori bounds on bounded solutions are by no means exhaustive. They are sufficient for the EFK equation but for other potentials different methods may be more suitable. For example, the techniques from this chapter can be combined with geometric reasoning in the (u, u'')-plane to obtain a priori bounds on the bounded solutions in fixed energy levels, as is done in [112] for potentials that are polynomials of degree four. This allows an extension of the results on uniqueness to values of γ for which some of the equilibrium points are real saddles whereas other equilibrium points are saddle-foci.

The existence of bounded solutions corresponding to the solutions of the FK equation has been proved in [117, 120, 124]. From Theorems 2.1 and 2.3 it can be deduced that there is a complete correspondence between the bounded stationary solutions of the EFK equation and those of the FK equation ($\gamma = 0$).

Theorem 2.8 *The only bounded solutions of* (2.9) *for* $\gamma \in (0, \frac{1}{8}]$ *are the three equilibrium points, the two monotone antisymmetric kinks and a one-parameter family of periodic solutions, parametrised by the energy* $E \in (-\frac{1}{4}, 0)$.

The multitude of solutions which exist for $\gamma > \frac{1}{8}$, shows that this bound is sharp. Among other things, Theorem 2.8 proves the conjecture in [117] that the kink for $\gamma \in (0, \frac{1}{8}]$ is unique. We mention that the uniqueness of the kink for $\gamma \in (0, \frac{1}{8}]$ is also proved in [96] with the elegant use of a Twist map.

In the proof of Theorem 2.8 we do not use the symmetry of the potential F in an essential manner (it merely reduces the length of the proofs). By exploiting the symmetry F we obtain additional results. First, for $\gamma \leq \frac{1}{8}$ any bounded solution of (2.9) is antisymmetric with respect to its zeros. Second, the periodic solutions can also be parametrised by the period

$$L \in \left(2\pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}, \infty\right).$$

Third, we prove that the heteroclinic orbit is a transverse intersection of the stable and unstable manifold.

Theorem 2.9 *For* $\gamma \in (0, \frac{1}{8}]$ *the unique monotonically increasing heteroclinic solution of* (2.9) *is the transverse intersection of the unstable manifold of* -1 *and the stable manifold of* $+1$ *in the zero energy set.*

Since a transverse intersection cannot be perturbed away, we conclude from Theorem 2.9 that for $\gamma \in (\frac{1}{8}, \frac{1}{8} + \varepsilon)$ and $\varepsilon > 0$ sufficiently small, there still exists a transverse heteroclinic orbit for (2.9). Since the equilibrium points $u = \pm 1$ are saddle-foci for $\gamma > \frac{1}{8}$, this makes it possible to apply techniques from [54] (see also [140, Ch. 3]) to obtain 'multi-bump' solutions of (2.9) for $\gamma \in (\frac{1}{8}, \frac{1}{8} + \varepsilon)$.

The transversality result in Theorem 2.9 enables us to prove that the monotonically increasing kink $\tilde{u}(x)$ with $\tilde{u}(0) = 0$ and its translates, are asymptotically stable for the time-dependent EFK equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3. \tag{2.12}$$

Theorem 2.10 *Let $u(x,t)$ be a solution of* (2.12). *For any $\gamma \in (0, \frac{1}{8}]$ there exists an $\varepsilon > 0$ such that if $\|u(x,0) - \tilde{u}(x + x_0)\|_{H^1} < \varepsilon$ for some $x_0 \in \mathbb{R}$, then there exists a $\delta \in \mathbb{R}$, depending on $u(x,0)$ (and small when ε is small), such that*

$$\lim_{t\to\infty} \|u(x,t) - \tilde{u}(x + x_0 + \delta)\|_{H^1} = 0.$$

We remark that the kink is also asymptotically stable in the space of bounded uniformly continuous functions.

The outline of the chapter is the following. In Section 2.2 we prove the Comparison Lemma, and it then follows that bounded solutions do not cross each other in the (u, u')-plane, as formulated in Theorem 2.1. Section 2.3 is devoted to the proof of Theorem 2.3. In Section 2.4 we prove the a priori bound from Theorem 2.6, and in Section 2.5 the proof of Theorem 2.8 is completed. Besides, we prove the antisymmetry of bounded solutions, and we show that the periodic solutions can be parametrised by their period. In Section 2.6 we prove that the unstable manifold of -1 intersects the stable manifold of $+1$ transversely as stated in Theorem 2.9. Theorem 2.10 on the asymptotic stability of the kink for the EFK equation is proved in Section 2.7. Finally, in Section 2.8 we deal with the continuation and existence of solutions of (2.1) and in particular we prove Theorem 2.5.

2.2 Uniqueness property

In this section we prove the Comparison Lemma and Theorem 2.1, which states that for $q \le -2\sqrt{\omega}$ bounded solutions of (2.6) are unique in the (u, u')-plane.

Remark 2.11 For the results in this section, the condition that $f(u)$ is continuously differentiable can be weakened. When $f(u)$ is in $C^0(\mathbb{R})$, then $\omega(a, b)$ is defined as the lowest non-negative number such that $f(u) + \omega(a, b)u$ is non-decreasing as a function of u on $[a, b]$. •

We start with the proof of the Comparison Lemma, which is at the heart of most of the results in this chapter. The proof proceeds along the same lines as in [35, Th. 2.1].

Proof of Lemma 2.4 (Comparison Lemma). Let $u(x)$ and $v(x)$ satisfy the assumptions in Lemma 2.4. If $u(x) - v(x) \equiv C$, then by the assumptions we have that $C \ge 0$ and $\lambda C \le 0$, thus $C = 0$ if $\lambda \ne 0$, i.e. if $\omega \ne 0$.

Suppose now that $u(x) - v(x) \not\equiv C$. Let k be the smallest integer for which $u^{(k)}(0) \ne v^{(k)}(0)$. Then, by uniqueness of solutions, $k \in \{0, 1, 2, 3\}$ and $u^{(k)}(0) > v^{(k)}(0)$ by the hypotheses. Hence there exists a $\sigma > 0$ such that

$$u(x) > v(x) \qquad \text{on } (0, \sigma). \tag{2.13}$$

Now let

$$\phi(x) \equiv u''(x) - \lambda u(x), \qquad \text{and} \qquad \psi(x) \equiv v''(x) - \lambda v(x).$$

Then, by the hypotheses,

$$\phi(0) - \psi(0) \geq 0, \qquad \text{and} \qquad \phi'(0) - \psi'(0) \geq 0. \tag{2.14}$$

Besides, writing $h(u) = f(u) + \omega(a,b)u$,

$$(\phi - \psi)''(x) - \mu(\phi - \psi)(x) = h(u(x)) - h(v(x)) \qquad \text{on } (0,\sigma). \tag{2.15}$$

Since $a \leq u(x) < v(x) \leq b$ on $[0,\infty)$, and since $h(u)$ is a non-decreasing function on $[a,b]$ by the definition of $\omega(a,b)$, we have that

$$h(u(x)) - h(v(x)) \geq 0 \qquad \text{on } (0,\sigma). \tag{2.16}$$

It is not difficult to deduce from (2.14), (2.15) and (2.16) that

$$\phi(x) - \psi(x) \geq 0 \qquad \text{on } (0,\sigma),$$

which is equivalent to

$$(u - v)''(x) - \lambda(u - v)(x) \geq 0 \qquad \text{on } (0,\sigma). \tag{2.17}$$

By the hypotheses of the lemma we have that

$$(u - v)(0) \geq 0, \qquad \text{and} \qquad (u - v)'(0) \geq 0, \tag{2.18}$$

thus we see from (2.13) and (2.17) that $(u - v)''(x) \geq 0$ on $(0,\sigma)$ and this implies that $(u - v)(x)$ is non-decreasing on $(0,\sigma)$. Hence

$$u(x) > v(x) \qquad \text{on } (0,\sigma],$$

and by continuity $u(x) > v(x)$ on $(0,\sigma+\varepsilon)$ for $\varepsilon > 0$ small enough. Hence we infer that

$$\sup\{\sigma \mid u(x) > v(x) \text{ for all } x \in (0,\sigma)\} = \infty,$$

and

$$\begin{aligned}(\phi - \psi)''(x) - \mu(\phi - \psi)(x) \geq 0 \qquad &\text{on } (0,\infty),\\ (u - v)''(x) - \lambda(u - v)(x) \geq 0 \qquad &\text{on } (0,\infty).\end{aligned}$$

It follows that $(u - v)''(x) \geq 0$ on $(0,\infty)$, and (2.18) then implies that $(u - v)'(x)$ is non-negative and non-decreasing on $(0,\infty)$. Finally, the assumption that $u(x) - v(x) \not\equiv C$ implies that

$$(u - v)(x) \to \infty \qquad \text{as } x \to \infty.$$

Clearly, if $u(x)$ and $v(x)$ are bounded this is not possible. This concludes the proof of the Comparison Lemma. □

Theorem 2.1 is a consequence of the Comparison Lemma.

Proof of Theorem 2.1. Let u_1 and u_2 be bounded solutions of (2.6) for $q \leq -2\sqrt{\omega}$ (corresponding to bounded solutions of (2.1) for $0 < \gamma \leq \frac{1}{4\omega}$). Suppose by contradiction that the paths of u_1 and u_2 cross in the (u,u')-plane. Then, after translation, we have that $u_1(0) = u_2(0)$ and $u_1'(0) = u_2'(0)$. Without loss of generality we may assume that $u_1''(0) \geq u_2''(0)$. Now if $u_1'''(0) \geq u_2'''(0)$, then by the Comparison Lemma we conclude that $u_1(x) - u_2(x) = C$ for some $C \in \mathbb{R}$. Since $u_1(0) = u_2(0)$ this implies that $u_1(x) \equiv u_2(x)$.

On the other hand, if $u_1'''(0) < u_2'''(0)$, then we define $\tilde{u}_1(x) = u_1(-x)$ and $\tilde{u}_2(x) = u_2(-x)$. Clearly $\tilde{u}_1$ and $\tilde{u}_2$ are also bounded solutions of (2.10). We now apply the Comparison Lemma to $\tilde{u}_1$ and $\tilde{u}_2$, and find as before that $\tilde{u}_1(x) \equiv \tilde{u}_2(x)$, which concludes the proof of Theorem 2.1. □

We now touch upon a lemma which gives a lot of information about the shape of bounded solutions. It states that every bounded solution is symmetric with respect to its extrema.

Lemma 2.12 *Let $u \in \mathcal{B}(a,b)$ be a bounded solution of* (2.1) *for some $\gamma \in \left(0, \frac{1}{4\omega(a,b)}\right]$. Suppose that $u'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then $u(x_0 + x) = u(x_0 - x)$ for all $x \in \mathbb{R}$.*

Proof. After translation we may take $x_0 = 0$. Now we define $v(x) = u(-x)$. By reversibility $v(x)$ is also a bounded solution of (2.1). Clearly $u(0) = v(0)$ and $u'(0) = v'(0)$. From Theorem 2.1 we conclude that $u(x) \equiv v(x)$. □

Corollary 2.13 *Let $u \in \mathcal{B}(a,b)$ be a bounded solution of* (2.1) *for some $\gamma \in \left(0, \frac{1}{4\omega(a,b)}\right]$. Then $u(x)$ can only be an equilibrium point, a homoclinic solution with one extremum, a monotone heteroclinic solution or a periodic solution with a unique maximum and minimum.*

Proof. It should be clear that when a solution is bounded for $x > 0$, then it either has an infinite number of extrema or it tends to a limit monotonically as $x \to \infty$. We will show in Lemma 2.14 that such a limit can only be an equilibrium point. The corollary then follows directly from Lemma 2.12. □

2.3 Energy ordering

To fill in the remaining details of the phase-plane picture we use Theorem 2.3, which establishes an ordering in terms of the energy $\mathcal{E}$ of the paths in the (u, u')-plane. In this section we will use the notation of Equation (2.6). Before we start with the proof of Theorem 2.3, we obtain some preliminary results.

The following lemma shows that when a solution tends to a limit monotonically, then this limit has to be an equilibrium point. We denote the set of zeros of $f(u)$ by $\mathcal{A}$:

$$\mathcal{A} \stackrel{\text{def}}{=} \{u \in \mathbb{R} \,|\, f(u) = 0\}.$$

Lemma 2.14 *Let $u(x)$ be a solution of* (2.6) *for $q < 0$ which is bounded on $[x_0, \infty)$ for some $x_0 \in \mathbb{R}$. Suppose that $u'(x) \geq 0$ for all $x > x_0$, or $u'(x) \leq 0$ for all $x > x_0$. Then*

$$\lim_{x\to\infty} u(x) \in \mathcal{A} \qquad \text{and} \qquad \lim_{x\to\infty} u^{(k)}(x) = 0 \quad \text{for } k = 1, 2, 3.$$

Proof. We may assume that $u'(x) \geq 0$ for $x \geq x_0$ (the other case is completely analogous). It is then clear that

$$\lim_{x\to\infty} u(x) \stackrel{\text{def}}{=} L_0$$

exists and $u(x)$ increases towards L_0 as $x \to \infty$. Since $u(x)$ is bounded for $x > x_0$, L_0 is finite.

We now consider the function $y = u''' + qu'$. This function $y(x)$ satisfies

$$y'' = u' f'(u).$$

We first show that $u''(x)$ tends to zero as $x \to \infty$. If $f'(L_0) \neq 0$ (the other case will be dealt with later), then $f'(u)$ has a sign for x large enough, by which we mean that either $f'(u) \geq 0$ for large x, or $f'(u) \leq 0$ for large x. Since $u'(x) \geq 0$, it follows that $y''(x)$ has a sign for x large enough, hence so does $y(x)$. The fact that $y(x) = u'''(x) + qu'(x)$ has a sign for x large enough implies that

$$\lim_{x\to\infty} u''(x) + qu(x) \stackrel{\text{def}}{=} L_1$$

exists and $u''(x) \to L_1 - qL_0$ as $x \to \infty$. Moreover, since $u(x)$ is bounded, we must have

$$\lim_{x\to\infty} u''(x) = 0.$$

If $f'(L_0) = 0$, then we consider $\tilde{y} = u''' + \frac{q}{2}u'$. We now have

$$\tilde{y}'' + \frac{q}{2}\tilde{y} = u'\left(f'(u) + \frac{q^2}{4}\right).$$

Since $f'(L_0) + \frac{q^2}{4}$ is positive for x large enough, we conclude from the maximum principle that $\tilde{y}(x) = u'''(x) - qu'(x)$ has a sign for x large enough. As before we see that

$$\lim_{x\to\infty} u''(x) = 0.$$

The fact that $u(x) \to L_0$ and $u''(x) \to 0$, implies that $u'(x) \to 0$ as $x \to \infty$. Because $u^{(iv)} = -qu'' + f(u)$, we see that

$$\lim_{x\to\infty} u^{(iv)}(x) \stackrel{\text{def}}{=} L_2 = f(L_0),$$

and, since $u(x)$ is bounded, $L_2 = 0$ and thus $L_0 \in \mathcal{A}$. Finally, the fact that $u''(x) \to 0$ and $u^{(iv)}(x) \to 0$, implies that $u'''(x) \to 0$ as $x \to \infty$. □

Remark 2.15 For $q = 0$ the situation is slightly more subtle, but when $f'(u)$ has a sign as u tends to L_0 monotonically, then the proof still holds. Since we consider bounded solutions of (2.6) for $q \le -2\sqrt{\omega}$, this difficulty only arises when $\omega = 0$, which (by the definition of ω) implies that $f'(u) \ge 0$ for all values of u involved, hence the lemma holds for this case. •

We prove that $u''' - \lambda u'(x)$ is negative if and only if $u'(x)$ is positive.

Lemma 2.16 *Let $u \in \mathcal{B}(a,b)$ be a bounded solution of* (2.6) *for some $q \le -2\sqrt{\omega(a,b)}$. Then (with* $\operatorname{sign}(0) \stackrel{\text{def}}{=} 0$*)*

$$\operatorname{sign}\big(u'''(x) - \lambda u'(x)\big) = -\operatorname{sign}\big(u'(x)\big) \qquad \textit{for all } x \in \mathbb{R}. \tag{2.19}$$

Proof. Let $x_0 \in \mathbb{R}$ be arbitrary. We may assume that $u'(x_0) \ge 0$ (for $u'(x_0) < 0$ the proof is analogous). We see from Lemma 2.12 that (2.19) holds if $u'(x_0) = 0$. We thus assume that $u'(x_0) > 0$. Since $u(x)$ is bounded there exist $-\infty \le x_a < x_0 < x_b \le \infty$, such that $u'(x_a) = u'(x_b) = 0$ and $u'(x) > 0$ on (x_a, x_b). Here we write $u'(\infty) \equiv \lim_{x\to\infty} u'(x)$. By Lemmas 2.12 and 2.14 we have that $u'''(x_a) = u'''(x_b) = 0$. Let $y \equiv u''' - \lambda u'$. Then $y(x)$ satisfies the system

$$\begin{cases} y'' - \mu y = u'(f'(u) + \omega), \\ y(x_a) = u'''(x_a) - \lambda u'(x_a) = 0, \\ y(x_b) = u'''(x_b) - \lambda u'(x_b) = 0. \end{cases}$$

Since $u'(x) > 0$ on (x_a, x_b), we have by the definition of ω that $u'(f'(u) + \omega) \ge 0$. By the strong maximum principle we obtain that $y(x) < 0$ for all $x \in (x_a, x_b)$, and especially $y(x_0) < 0$. This completes the proof. □

Remark 2.17 If a bounded solutions of (2.1) for $q \le -2\sqrt{\omega}$ attains a maximum at some point $x_0 \in \mathbb{R}$, then

$$u''(x_0) < 0 \qquad \text{and} \qquad u^{(iv)}(x_0) - \lambda u''(x_0) > 0.$$

This follows from the boundary point lemma (see [127, p. 67]) applied to u' and $u''' - \lambda u'$, combined with the proof of Lemma 2.16 above. Besides, it is seen from the differential equation that

$$f(u(x_0)) = u^{(iv)}(x_0) + qu''(x_0) > -\mu u''(x_0) > 0,$$

i.e., maxima only occur at positive values of $f(u)$. •

We immediately obtain the following consequence.

Corollary 2.18 *Let $u \in \mathcal{B}(a,b)$ be a bounded solution of (2.6) for some $q \leq -2\sqrt{\omega(a,b)}$. Then*

$$H(x) \stackrel{\text{def}}{=} -\mathcal{E}[u] + F(u(x)) + \tfrac{1}{2}(u''(x))^2 \leq 0 \qquad \textit{for all } x \in \mathbb{R}.$$

Proof. By the energy identity we have

$$H = u'\left\{u''' + \frac{q}{2}u'\right\} = u'(u''' - \lambda u') - C(u')^2,$$

where $C = \sqrt{\left(\frac{q}{2}\right)^2 - \omega} \geq 0$. It is easily seen from Lemma 2.16 that the assertion holds. □

We will now prepare for the proof of Theorem 2.3. Let u_1 and u_2 satisfy the assumptions in Theorem 2.3. We point out that u_1 and u_2 are not translates of one another, because this would contradict the result on symmetry with respect to extrema, obtained in Lemma 2.12. We only consider the case where $u_1'(0) > u_2'(0) \geq 0$. The other case follows by symmetry. By contradiction we assume that $\mathcal{E}[u_1] \leq \mathcal{E}[u_2]$. It will be proved in Lemma 2.21 that we can then find points x_1 and x_2 such that $u_1(x_1) = u_2(x_2)$ and $u_1''(x_1) = u_2''(x_2)$. This enables us to apply the following lemma.

Lemma 2.19 *Let u_1, $u_2 \in \mathcal{B}(a,b)$ be bounded solutions of (2.6) for some $q \leq -2\sqrt{\omega(a,b)}$. Suppose that $\mathcal{E}[u_1] \leq \mathcal{E}[u_2]$ and*

$$u_1(0) = u_2(0), \qquad u_1'(0) > u_2'(0) \geq 0 \qquad \textit{and} \qquad u_1''(0) = u_2''(0).$$

Then $u_1 \equiv u_2$.

Proof. We will show that

$$u_1'''(0) - \lambda u_1'(0) \geq u_2'''(0) - \lambda u_2'(0) \tag{2.20}$$

and then an application of the Comparison Lemma completes the proof. From the energy identity we obtain at $x = 0$

$$u_i''' - \lambda u_i' = \frac{-\mathcal{E}[u_i] + F(u_i) + \frac{1}{2}(u_i'')^2}{u_i'} + Cu_i' \qquad \text{for } i = 1, 2,$$

where $C = \sqrt{\left(\frac{q}{2}\right)^2 - \omega} \geq 0$. By the assumptions and from Corollary 2.18, it follows that

$$-\mathcal{E}[u_2] + F(u_2(0)) + \tfrac{1}{2}(u_2''(0))^2 \leq -\mathcal{E}[u_1] + F(u_1(0)) + \tfrac{1}{2}(u_1''(0))^2 \leq 0.$$

Inequality (2.20) is now easily verified. □

We make the following change of variables on intervals $[x_a, x_b]$ where the function $u(x)$ is strictly monotone on the interior (see [117]). Denoting the inverse of $u(x)$ by $x(u)$, we set

$$t = u \qquad \text{and} \qquad z(t) = \left[u'(x(t))\right]^2.$$

We now get for $t \in [t_a, t_b] = [u(x_a), u(x_b)]$

$$z'(t) = 2u''(x(t)).$$

If $x_a = -\infty$, then we write $z'(t_a) = \lim_{t \to t_a} z'(t)$ (the limit exists by Lemma 2.14).

Before we proceed with the general case, we first consider the special case where two different solutions tend to the same equilibrium point as $x \to -\infty$. The next lemma in fact shows that there are at most two bounded solution in the unstable manifold of each equilibrium point.

Lemma 2.20 *Let u_1, $u_2 \in \mathcal{B}(a,b)$ be two different non-constant bounded solutions of* (2.6) *for some $q \le -2\sqrt{\omega(a,b)}$. Suppose there exists an $\tilde{u} \in \mathcal{A}$ such that*

$$\lim_{x \to -\infty} u_1(x) = \lim_{x \to -\infty} u_2(x) = \tilde{u}.$$

Then $u_1(x)$ decreases to $\tilde{u}$ and $u_2(x)$ increases to $\tilde{u}$ as $x \to -\infty$, or vice versa.

Proof. By Corollary 2.13, u_1 and u_2 can only tend to $\tilde{u}$ monotonically. Suppose u_1 and u_2 both decrease towards $\tilde{u}$ as $x \to -\infty$, i.e., $u_1'(x) > 0$ and $u_2'(x) > 0$ for $x \in (-\infty, x_0)$. We will show that $u_1 \equiv u_2$. The case where they both increase towards $\tilde{u}$ is analogous.

For $t \in (\tilde{u}, \tilde{u} + \varepsilon_0)$, where $\varepsilon_0 > 0$ is sufficiently small, let z_1 and z_2 correspond to u_1 and u_2 respectively by the change of variables described above. Note that $z_1(t) \ne z_2(t)$ for $t \in (\tilde{u}, \tilde{u} + \varepsilon_0)$, since otherwise $u_1 \equiv u_2$ by Theorem 2.1. Without loss of generality we may assume that $z_1(t) > z_2(t)$ on $(\tilde{u}, \tilde{u} + \varepsilon_0)$. Since $z_i(t)$ is differentiable on $(\tilde{u}, \tilde{u} + \varepsilon_0)$ and $z_1(\tilde{u}) = z_2(\tilde{u}) = 0$ (by Lemma 2.14), there exist a point $t_0 \in (\tilde{u}, \tilde{u} + \varepsilon_0)$, such that $z_1'(t_0) \ge z_2'(t_0)$.

We now first deal with the case that $z_2'(t_0) \ge 0$ (the case that $z_2'(t_0) < 0$ will be dealt with later). There are points x_1 and x_2 in $\mathbb{R}$ such that $u_1(x_1) = u_2(x_2) = t_0$ and $u_1''(x_1) \ge u_2''(x_2) \ge 0$. By translating u_1 and u_2 by x_1 and x_2 respectively, we obtain that

$$u_1(0) = u_2(0), \qquad u_1'(0) > u_2'(0) > 0 \qquad \text{and} \qquad u_1''(0) \ge u_2''(0) \ge 0. \tag{2.21}$$

Since u_1 and u_2 tend to $\tilde{u}$ monotonically as $x \to -\infty$, we infer from Lemma 2.14 that $(u_i, u_i', u_i'', u_i''')(x) \to (\tilde{u}, 0, 0, 0)$ as $x \to -\infty$ for $i = 1, 2$. Therefore $\mathcal{E}[u_1] = \mathcal{E}[u_2]$. It is easy to check that (2.21) is now sufficient for the proof of Lemma 2.19 to go on unchanged. Hence $u_1 \equiv u_2$. This ends the proof for the case that $z_2'(t_0) \ge 0$.

We now consider the case that $z_2'(t_0) < 0$. Since $z_2(t) > 0$ on $(\tilde{u}, t_0]$ and $z_2(\tilde{u}) = 0$ there exists a $t_1 \in (\tilde{u}, t_0)$ such that $z_2'(t_1) = 0$. If $z_1'(t_1) \ge 0$, then we have $z_1'(t_1) \ge z_2'(t_1) = 0$, which is equivalent to the case that we have already covered (taking t_1 instead of t_0). If $z_1'(t_1) < 0$ then we have

$$z_1'(t_1) < z_2'(t_1) \qquad \text{and} \qquad z_1'(t_0) \ge z_2'(t_0),$$

and by continuity there exists a $t_2 \in (t_1, t_0]$ such that $z_1'(t_2) = z_2'(t_2)$. Thus there are points x_1 and x_2 in $\mathbb{R}$ such that

$$u_1(x_1) = u_2(x_2) = t_2, \quad u_1'(x_1) > u_2'(x_2) > 0 \quad \text{and} \quad u_1''(x_1) = u_2''(x_2).$$

Since $\mathcal{E}[u_1] = \mathcal{E}[u_2]$ as above, we may apply Lemma 2.19 and conclude that $u_1 \equiv u_2$. □

Of course a similar result holds for solutions that tend to an equilibrium point as $x \to +\infty$: there are at most two bounded solutions in the stable manifold of each equilibrium point.

The next lemma shows that if some u_1 and u_2 violated the conclusion of Theorem 2.3, i.e. if $\mathcal{E}[u_1] \le \mathcal{E}[u_2]$, then we could find a point where $u_1 = u_2$ and $u_1'' = u_2''$.

Lemma 2.21 *Let $u_1,\ u_2 \in \mathcal{B}(a,b)$ be bounded solutions of* (2.6) *for some $q \le -2\sqrt{\omega(a,b)}$. Suppose that $u_1(0) = u_2(0)$ and $u_1'(0) > u_2'(0) \ge 0$, and $\mathcal{E}[u_1] \le \mathcal{E}[u_2]$. Then there exist x_1 and x_2 in $\mathbb{R}$ such that $u_1(x_1) = u_2(x_2)$ and $u_1''(x_1) = u_2''(x_2)$.*

Proof. Let $[\tilde{x}_a, \tilde{x}_b]$ be the largest interval containing $x = 0$ on which u_1' is positive, and let $[x_a, x_b]$ be the largest interval containing $x = 0$ on which u_2' is positive. We now change variables again. Let z_1 correspond to u_1 on $[\tilde{t}_a, \tilde{t}_b] = [u_1(\tilde{x}_a), u_1(\tilde{x}_b)]$, and let z_2 correspond to u_2 on $[t_a, t_b] = [u_2(x_a), u_2(x_b)]$. Clearly $z_1(t) > z_2(t)$ for all $t \in (t_a, t_b)$, since bounded solutions do not cross in the (u, u')-plane by Theorem 2.1. If x_a is finite, then it follows from Theorem 2.1 that $\tilde{t}_a < t_a$, while if $x_a = -\infty$ then this follows from Lemma 2.20. Similarly, $\tilde{t}_b > t_b$.

We have that $z_2(t_a) = 0$ and $z_2'(t_a) = 2u_2''(x_a) \ge 0$ (if $x_a = -\infty$ then this follows from Lemma 2.14, while if x_a is finite then it follows from the fact that $u_2(x_a)$ is a minimum). We will now prove that $z_1'(t_a) < z_2'(t_a)$ by showing that $(z_2')^2(t_a) - (z_1')^2(t_a) > 0$. Let $y_a \in (\tilde{x}_a, \tilde{x}_b)$ be the point such that $u_1(y_a) = u_2(x_a) = t_a$. By the energy identity we have that

$$\begin{aligned}
\frac{(z_2')^2(t_a) - (z_1')^2(t_a)}{8} &= \tfrac{1}{2}(u_2''(x_a))^2 - \tfrac{1}{2}(u_1''(y_a))^2 \\
&= \mathcal{E}[u_2] - F(t_a) - \left\{\mathcal{E}[u_1] - F(t_a) + u_1'(y_a)\left(u_1'''(y_a) + \frac{q}{2}u_1'(y_a)\right)\right\} \\
&= \mathcal{E}[u_2] - \mathcal{E}[u_1] - u_1'(y_a)\left(u_1'''(y_a) + \frac{q}{2}u_1'(y_a)\right).
\end{aligned}$$

From Lemma 2.16 and the observation that $u_1'(y_a) = \sqrt{z_1(t_a)} > \sqrt{z_2(t_a)} = 0$, we conclude that at y_a

$$u_1'\left(u_1''' + \frac{q}{2}u_1'\right) = u_1'(u_1''' - \lambda u_1') - \sqrt{\left(\tfrac{q}{2}\right)^2 - \omega}\, {u_1'}^2 < 0.$$

Having assumed that $\mathcal{E}[u_1] \le \mathcal{E}[u_2]$, we now conclude that $z_1'(t_a) < z_2'(t_a)$.

In the same way we can show that $z_1'(t_b) > z_2'(t_b)$. By continuity there exists a $t_c \in (t_a, t_b)$ such that $z_1'(t_c) = z_2'(t_c)$, which proves the lemma. □

We now complete the proof of Theorem 2.3.

Proof of Theorem 2.3. Let u_1 and u_2 satisfy the assumptions in the theorem. The previous lemma shows that if $\mathcal{E}[u_1] \le \mathcal{E}[u_2]$, then there exist points x_1 and x_2 such that

$$u_1(x_1) = u_2(x_2), \qquad u_1'(x_1) > u_2'(x_2) \ge 0 \qquad \text{and} \qquad u_1''(x_1) = u_2''(x_2).$$

By translation invariance we may take $x_1 = x_2 = 0$. Lemma 2.19 now shows that $u_1 \equiv u_2$, which contradicts the assumption. Therefore $\mathcal{E}[u_1] > \mathcal{E}[u_2]$, which proves the theorem. □

2.4 A priori bounds

In this section we derive a priori estimates for bounded solutions of the EFK equation (2.9) or (2.10). Where possible, we will indicate how the methods can be generalised to arbitrary $f(u)$, particularly in Remarks 2.28 and 2.32. We will prove Theorem 2.6 which states that every bounded solution for $q \le -\sqrt{8}$ (or $\gamma \in (0, \frac{1}{8}]$) satisfies $|u(x)| \le 1$ for all $x \in \mathbb{R}$. We first derive a weaker bound for all $q \le 0$, which follows from the shape of the potential and the energy identity (2.7). Subsequently, we sharpen this bound for all $q \leq -\sqrt{8}$ with the help of the maximum principle.

We now prove a slight variation of an important lemma from [119], which shows that when a solution of (2.10) becomes larger than $\sqrt{2}$, then it will oscillate towards infinity, and hence is unbounded. The proof can easily be extended to more general potentials F, as is done in [116]. The value $\sqrt{2}$ is directly related to the fact that

$$\min\{x_0 > 0 \mid F(x) \geq F(x_0) \text{ for all } x \in [-x_0, x_0]\} = \sqrt{2}.$$

Lemma 2.22 *For any $q \leq 0$, let $u(x)$ be a solution of* (2.10). *Suppose that there exists a point $x_0 \in \mathbb{R}$ such that*

$$u(x_0) \geq \sqrt{2}, \qquad u'(x_0) = 0, \qquad u''(x_0) \leq 0, \qquad \text{and} \qquad u'''(x_0) \leq 0. \tag{2.22}$$

Then either u decreases to $-\infty$ monotonically for $x > x_0$, or there exists a first critical point of u on (x_0, ∞), say y_0, and we have

$$u(y_0) < -u(x_0) \leq -\sqrt{2}, \qquad u'(y_0) = 0, \qquad u''(y_0) > 0, \qquad \text{and} \qquad u'''(y_0) > 0.$$

Besides, $F(u(y_0)) < F(u(x_0))$, and the following estimate holds:

$$F(u(y_0)) - F(u(x_0)) < -\frac{5\sqrt{2}}{3} \frac{[\mathcal{E}[u] - F(u(x_0))]}{u(y_0) - u^3(y_0)}. \tag{2.23}$$

Proof. We write $f(u) \equiv u - u^3$. Since $f(u(x_0)) < 0$ and $u''(x_0) \leq 0$, we see that

$$u^{(iv)}(x_0) = -qu''(x_0) + f(u(x_0)) < 0, \tag{2.24}$$

so that $u(x)$ is decreasing for x in a right neighbourhood of x_0. Thus, either $u(x)$ tends to $-\infty$ monotonically for $x > x_0$, or there exists a $y_0 \in (x_0, \infty]$ such that $u'(y_0) = 0$ (where $u'(\infty) \equiv \lim_{x\to\infty} u'(x)$), and $u'(x) < 0$ on (x_0, y_0). From now on we assume that $u(x)$ does not decrease monotonically to $-\infty$ for $x > x_0$, and we define

$$y_0 \stackrel{\text{def}}{=} \sup\{x > x_0 \mid u' < 0 \text{ on } (x_0, x)\}.$$

It follows from the assumptions and (2.24) that $u''' < 0$ in a right neighbourhood of x_0, hence we conclude that

$$x_1 \stackrel{\text{def}}{=} \sup\{x > x_0 \mid u''' < 0 \text{ on } (x_0, x)\}$$

is well-defined. Since $u'(y_0) = 0$ we conclude that x_1 is finite and $x_1 < y_0$. Since $u''' < 0$ on (x_0, x_1), we have that $u''(x_1) < u''(x_0) \leq 0$. Using the energy identity and the fact that $u'''(x_1) = 0$ and $u'(x_0) = 0$, we obtain

$$\begin{aligned} F(u(x_0)) &= \mathcal{E}[u] - \frac{1}{2}(u''(x_0))^2 \\ &> \mathcal{E}[u] - \frac{1}{2}(u''(x_1))^2 \\ &> \mathcal{E}[u] - \frac{1}{2}(u''(x_1))^2 + \frac{q}{2}(u'(x_1))^2 = F(u(x_1)). \end{aligned}$$

It follows from the definition of x_1 and the initial data at x_0, that $u''' < 0$, $u'' < 0$ and $u' < 0$ on (x_0, x_1), and thus $u(x_1) < u(x_0)$. It is seen from the shape of the potential that

$$F(s) \geq F(u(x_0)) \qquad \text{for all } s \in [-u(x_0), u(x_0)].$$

Consequently, the inequalities $F(u(x_0)) > F(u(x_1))$ and $u(x_1) < u(x_0)$ imply that $u(x_1) < -u(x_0) \leq -\sqrt{2}$. Since there are no equilibrium points in the region $u < -\sqrt{2}$ we conclude from Lemma 2.14 that $u(x_0)$ does not decrease monotonically to some finite limit, and therefore y_0 is finite.

We now define

$$x_2 \stackrel{\text{def}}{=} \sup\{x > x_1 \mid u'' < 0 \text{ on } (x_1, x)\},$$

which is well-defined since $u''(x_1) < 0$, and x_2 is finite because $x_2 < y_0 < \infty$. From the definition of x_2 we see that

$$u(x_2) < u(x_1) < -u(x_0), \quad u''(x_2) = 0, \quad u'''(x_2) \geq 0, \quad \text{and} \quad u^{(iv)}(x_2) = f(u(x_2)) > 0.$$

Since $f(u(x)) > 0$ on $[x_2, y_0]$ we have that $u^{(iv)}(x) = -qu''(x) + f(u(x)) > 0$ as long as $u''(x) > 0$ and $x \in (x_2, y_0]$, and it is not difficult to see that $u'' > 0$ and $u''' > 0$ on $(x_2, y_0]$. To summarise, we have that

$$u(y_0) < -u(x_0), \quad u'(y_0) = 0, \quad u''(y_0) > 0, \quad u'''(y_0) > 0 \quad \text{and} \quad F(u(y_0)) < F(u(x_0)).$$

We still have to prove the estimate (2.23). By the energy identity (2.7) we have that $F(u(x_0)) \leq \mathcal{E}[u]$. For $F(u(x_0)) = \mathcal{E}[u]$ the estimate has already been proved. Therefore we may assume that $F(u(x_0)) < \mathcal{E}[u]$, so that $u''(x_0) = -\sqrt{2[\mathcal{E}[u] - F(u(x_0))]} \stackrel{\text{def}}{=} -\beta < 0$.

From the definition of x_1 and x_2 we see that $u''(x_1) < -\beta$, $u'''(x_1) = 0$, and

$$u^{(iv)} = -qu'' + f(u) < f(u(y_0)) \qquad \text{on } (x_1, x_2).$$

By integrating we obtain

$$u''(x) < -\beta + \tfrac{1}{2} f(u(y_0))(x - x_1)^2 \qquad \text{for } x \in (x_1, x_2]. \tag{2.25}$$

By definition, x_2 is the first zero of $u''(x)$, thus $x_2 - x_1 > \sqrt{\frac{2\beta}{f(u(y_0))}} \stackrel{\text{def}}{=} \xi$. By integrating (2.25) twice and by using the fact that $u'(x_1) < 0$, we obtain

$$\begin{aligned} u(x_1 + \xi) - u(x_1) &< -\beta \frac{\xi^2}{2} + f(u(y_0)) \frac{\xi^4}{24} \\ &< -\frac{5}{6} \frac{\beta^2}{f(u(y_0))}. \end{aligned}$$

Because $u' < 0$ on $[x_1 + \xi, x_2]$, we see that

$$u(x_2) - u(x_1) < u(x_1 + \xi) - u(x_1) < -\frac{5}{6} \frac{\beta^2}{f(u(y_0))} \stackrel{\text{def}}{=} -\alpha.$$

Since $F'(u) = f(u) > 0$ for $u < -\sqrt{2}$ and $u(x_2) < u(x_1) - \alpha < u(x_1) < -\sqrt{2}$, we have that

$$F(u(y_0)) < F(u(x_2)) < F(u(x_1) - \alpha).$$

Moreover, $F''(u) = f'(u) > 0$ for $u < -\sqrt{2}$, and we finally obtain that

$$F(u(y_0)) < F(u(x_1) - \alpha) < F(u(x_1)) - \frac{dF(u(x_1))}{du} \alpha < F(u(x_1)) - f(-\sqrt{2})\alpha.$$

Since $f(-\sqrt{2}) = \sqrt{2}$, it is seen from the definitions of α and β, and the fact that $F(u(x_1)) > F(u(x_0))$, that (2.23) holds. □

Remark 2.23 It was proved in [119, Lemma 2.3] that if $u(x)$ is a solution of (2.10) on its maximal interval of existence (x_a, x_b), then for any $x_0 \in (x_a, x_b)$, there either exists an infinite number of extrema of $u(x)$ for $x > x_0$, or $u(x)$ eventually tends to a finite limit monotonically as $x \to \infty$. This result excludes the possibility in Lemma 2.22 that u tends to $-\infty$ monotonically for $x > x_0$. However, this fact is not essential to our reasoning, since we want to prove a uniform bound on the set of *bounded* solutions, hence we do not need to consider solutions that tend to infinity. •

Remark 2.24 Notice that the estimate (2.23) is by no means sharp. We will use the estimate to show that once a solution becomes larger than $\sqrt{2}$ it will start oscillating, and the amplitude of the oscillations tends to infinity. For the EFK potential we have given the explicit estimate (2.23), but in general it suffices that $F(u)$ strictly decreases to $-\infty$ as $|u| \to \infty$. In this chapter we do not need any information on the speed at which the solution tends to infinity, and therefore we are satisfied with this rather weak estimate. It can in fact be shown that if a solution of (2.10) obeys (2.22) at some $x_0 \in \mathbb{R}$, then the solution blows up in finite time (i.e., the maximal interval of existence for $x > x_0$ is finite), see Chapter 5. •

Remark 2.25 The following symmetric counterpart of Lemma 2.22 holds. For any $q \leq 0$, let $u(x)$ be a solution of (2.10). Suppose that there exists a point $x_0 \in \mathbb{R}$ such that

$$u(x_0) \leq -\sqrt{2}, \qquad u'(x_0) = 0, \qquad u''(x_0) \geq 0, \qquad \text{and} \qquad u'''(x_0) \geq 0.$$

Then either u increases to $+\infty$ monotonically for $x > x_0$, or there exists a first critical point of u on (x_0, ∞), say y_0, and we have

$$u(y_0) > -u(x_0) \geq \sqrt{2}, \qquad u'(y_0) = 0, \qquad u''(y_0) < 0, \qquad \text{and} \qquad u'''(y_0) < 0.$$

Besides, $F(u(y_0)) < F(u(x_0))$, and an estimate similar to (2.23) holds. •

The next lemma implies that if a solution $u(x)$ obeys (2.22) then it becomes wildly oscillatory for $x > x_0$. The function $u(x)$ then has an infinite number of oscillations on the right-hand side of x_0 and the amplitude of these oscillations grows unlimited. The function sweeps from one side of the potential to the other.

Lemma 2.26 *For any $q \leq 0$, let $u(x)$ be a solution of* (2.10). *Suppose that there exists a $\xi_0 \in \mathbb{R}$ such that*

$$u(\xi_0) \geq \sqrt{2}, \qquad u'(\xi_0) = 0, \qquad u''(\xi_0) \leq 0, \qquad \textit{and} \qquad u'''(\xi_0) \leq 0. \tag{2.26}$$

Then $u(x)$ has for $x \geq \xi_0$ an infinite, increasing sequence of local maxima $\{\xi_k\}_{k=0}^{\infty}$ and minima $\{\eta_k\}_{k=1}^{\infty}$, where $\xi_k < \eta_{k+1} < \xi_{k+1}$ for every $k \geq 0$. The extrema are ordered: $u(\xi_{k+1}) > -u(\eta_{k+1}) > u(\xi_k) \geq \sqrt{2}$, and $u(\xi_k) \to \infty$ as $k \to \infty$.

Proof. Remark 2.23 excludes the possibility that u tends to $-\infty$ or $+\infty$ monotonically, thus by combining Lemma 2.22 and Remark 2.25 we obtain the infinite sequences of local maxima and minima. The orderings $u(\xi_{k+1}) > -u(\eta_{k+1}) > u(\xi_k) \geq \sqrt{2}$ and

$$F(u(\xi_{k+1})) < F(u(\eta_{k+1})) < F(u(\xi_k)) \qquad \text{for all } k \geq 0, \tag{2.27}$$

are immediate. Clearly $\{u(\xi_k)\}_{k=0}^{\infty}$ is an increasing sequence, whereas $\{F(u(\xi_k))\}$ is a decreasing sequence. We assert that $F(u(\xi_k)) \to -\infty$ and thus $u(\xi_k) \to \infty$ as $k \to \infty$. Suppose by contradiction that $\{F(u(\xi_k))\}$ is bounded, then $\{F(u(\eta_k))\}$ is bounded as well (by Equation (2.27)). Hence $u(x)$ is bounded for $x > \xi_0$. However, then the right-hand side in (2.23) is bounded away from zero, which ensures that $F(u(\xi_k))$ tends to $-\infty$ as $k \to \infty$, contradicting the assumption that $\{F(u(\xi_k))\}$ is bounded. □

Note that if $u(x)$ attains a maximum at $x = 0$ above the line $u = \sqrt{2}$ then (2.26) holds with $\xi_0 = 0$ for either $u(x)$ or $u(-x)$. The next lemma states our first a priori bound.

Lemma 2.27 *For any $q \le 0$, let $u(x)$ be a bounded solution of* (2.10)*. Then $|u(x)| < \sqrt{2}$ for all $x \in \mathbb{R}$.*

Proof. We argue by contradiction and thus suppose that $u(x) \ge \sqrt{2}$ for some $x \in \mathbb{R}$. Since $u(x)$ is bounded, we infer from Lemma 2.14 that $u(x)$ attains a local maximum larger then $\sqrt{2}$, say at $x_0 \in \mathbb{R}$. By translation invariance we may assume that $x_0 = 0$. Clearly $u(0) \ge \sqrt{2}$, $u'(0) = 0$ and $u''(0) \le 0$. Without loss of generality we may assume that $u'''(0) \le 0$ (otherwise we switch to $\tilde{u}(x) \equiv u(-x)$, which also is a bounded solution of (2.10)). We are now in the setting of Lemma 2.26. Thus $u(x)$ is unbounded if $u(x_0) \ge \sqrt{2}$ for some $x_0 \in \mathbb{R}$. The case where $u(x_0) \le -\sqrt{2}$ for some $x_0 \in \mathbb{R}$ is excluded in a similar manner. □

Remark 2.28 This method of obtaining an a priori estimate on all bounded solutions is applicable to a class of non-symmetric potentials which strictly decrease to $-\infty$ as $|u| \to \infty$. In that case we can find $-\infty < a \le b < \infty$ such that

$$\begin{gathered} F(a) = F(b), \\ F(u) \ge F(a) = F(b) \quad \text{for all } u \in (a,b), \\ F'(u) > 0 \text{ for all } u \le a \quad \text{and} \quad F'(u) < 0 \text{ for all } u \ge b. \end{gathered}$$

Then every bounded solutions $u(x)$ of (2.1) for $\gamma > 0$ satisfies $a \le u(x) \le b$. We note that this method gives an explicit a priori bound, which is stronger then the method in [96, Th. 4]. For the potential in (2.5) a lower bound can be found in an analogous manner. In general, if for some $b \in \mathbb{R}$

$$F(u) \ge F(b) \text{ for all } u < b \qquad \text{and} \qquad F'(u) < 0 \text{ for all } u \ge b,$$

then b is an upper bound for all bounded solutions. •

We are now going to use the maximum principle to get sharper a priori bounds for the EFK equation. The following lemma shows that if a bounded solution has two local minima below the line $u = 1$, then the solution stays below this line between these minima. To shorten notation, we will write $u(\infty)$ instead of $\lim_{x\to\infty} u(x)$.

Lemma 2.29 *For any $q \le -\sqrt{8}$, let $u(x)$ be a solution of* (2.10)*, and let $-\infty \le x_a < x_b \le \infty$. Suppose that $u(x_a)$, $u(x_b) \le 1$ and $u''(x_a)$, $u''(x_b) \ge 0$. If $u(x) \ge -2$ for $x \in (x_a, x_b)$, then either $u \equiv 1$ or $u(x) < 1$ on (x_a, x_b).*

Proof. The proof is based on repeated application of the maximum principle. Let $v(x) \equiv u(x) - 1$. The function $v(x)$ obeys, for $x \in (x_a, x_b)$,

$$v^{(iv)} + qv'' + 2v = u - u^3 + 2(u-1) = -(u+2)(u-1)^2 \le 0,$$

where the inequality is ensured by the hypothesis that $u(x) \ge -2$. Now we define $w(x) \equiv v''(x) - \lambda v(x)$. From the definition of λ and μ we see that

$$w'' - \mu w = v^{(iv)} - (\lambda + \mu)v'' + \lambda\mu v = v^{(iv)} + qv'' + 2v.$$

By the hypotheses on u in x_a and x_b we find that $w(x)$ obeys the system

$$\begin{cases} w'' - \mu w = -(u+2)(u-1)^2 \le 0 \qquad \text{on } (x_a, x_b), \\ w(x_a) = v''(x_a) - \lambda v(x_a) \ge 0, \\ w(x_b) = v''(x_b) - \lambda v(x_b) \ge 0. \end{cases}$$

By the maximum principle we have that $w(x) \geq 0$ on (x_a, x_b). Finally, $v(x)$ obeys the system

$$\begin{cases} v'' - \lambda v \equiv w \geq 0 & \text{on } (x_a, x_b), \\ v(x_a) = u(x_a) - 1 \leq 0, \\ v(x_b) = u(x_b) - 1 \leq 0. \end{cases}$$

By the strong maximum principle we obtain that either $v \equiv 0$, or $v(x) < 0$ on (x_a, x_b). This proves Lemma 2.29. □

Remark 2.30 The symmetric counterpart of the previous lemma shows that if a solution $u(x)$ of (2.10) has two local maxima above -1 and $u(x) \leq 2$ between the maxima, then we have $u(x) > -1$ between the maxima. •

Note that for bounded solutions the condition that $-2 \leq u(x) \leq 2$ is automatically satisfied (Lemma 2.27). For heteroclinic solutions the previous lemma and remark (with $x_a = -\infty$ and $x_b = +\infty$) imply that every heteroclinic solution is uniformly bounded from above by 1 and from below by -1.

For the case of a general bounded solution, let us look at the consecutive extrema for $x > 0$ (and similarly for $x < 0$) of a bounded solution $u(x)$. Suppose that u is a bounded solution which does not tend to a limit. In that case we will prove that arbitrarily large negative x_a and arbitrarily large positive x_b can be found, such that $u(x_a)$ and $u(x_b)$ are local minima below the line $u = 1$, and thus the conditions in Lemma 2.29 are satisfied. We will need the following lemma, which has two related consequences. First, it shows that if $u(x)$ has a maximum above the line $u = 1$, then the first minimum on at least one of the sides of this maximum lies below the line $u = 1$. Second, we infer that a solution does not have two consecutive minima above the line $u = 1$.

Lemma 2.31 *For any $q \leq 0$ let $u(x)$ be a solution of* (2.10). *Suppose that there exists a point $x_0 \in \mathbb{R}$, such that*

$$u(x_0) > 1, \qquad u'(x_0) = 0, \qquad u''(x_0) \leq 0, \qquad \textit{and} \qquad u'''(x_0) \leq 0.$$

Then there exists a $y_0 \in (x_0, \infty)$ such that $u(y_0) = 1$ and $u'(x) < 0$ on $(x_0, y_0]$.

Proof. The proof is along the same lines as the proof of Lemma 2.22. Since $f(u(x_0)) < 0$ and $u''(x_0) \leq 0$, we see that $u^{(iv)}(x_0) = -qu''(x_0) + f(u(x_0)) < 0$ and thus $u''' < 0$ in a right neighbourhood of x_0. We now conclude that

$$x_1 \stackrel{\text{def}}{=} \sup\{x > x_0 \mid u''' < 0 \text{ on } (x, x_0)\}$$

is well-defined. By Remark 2.23 we conclude that x_1 is finite. Since $u''' < 0$ on (x_0, x_1), we have that $u''(x_1) < u''(x_0) \leq 0$. By using the energy identity and the facts that $u'''(x_1) = 0$ and $u'(x_0) = 0$, we obtain

$$F(u(x_0)) = \mathcal{E}[u] - \frac{1}{2}(u''(x_0))^2 > \mathcal{E}[u] - \frac{1}{2}(u''(x_1))^2 > F(u(x_1)).$$

It follows from the definition of x_1 and the initial data at x_0, that $u''' < 0$, $u'' < 0$ and $u' < 0$ on (x_0, x_1), and so $u(x_1) < u(x_0)$. Since $F'(u) < 0$ for $u > 1$, we see that $F(s) > F(u(x_0))$ for all $s \in [1, u(x_0))$, so that $u(x_1) < 1$. Hence

$$y_0 \stackrel{\text{def}}{=} \inf\{x > x_0 \,|\, u > 1 \text{ on } (x, x_0)\}$$

exists and $y_0 < x_1 < \infty$. This proves the lemma. □

We can now apply Lemma 2.29 to prove Theorem 2.6.

Proof of Theorem 2.6. We will only prove that $u(x) < 1$ for all $x \in \mathbb{R}$ (the proof of the assertion that $u(x) > -1$ is analogous). We argue by contradiction. Suppose there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) \geq 1$. We will show that there exists a constant $x_a \in [-\infty, x_0)$ such that

$$u(x_a) \leq 1, \qquad u'(x_a) = 0 \qquad \text{and} \qquad u''(x_a) \geq 0. \tag{2.28}$$

Similarly we obtain a point $x_b \in (x_0, \infty]$ such that

$$u(x_b) \leq 1, \qquad u'(x_b) = 0 \qquad \text{and} \qquad u''(x_b) \geq 0.$$

From Lemmas 2.27 and 2.29 we then conclude that $u(x) < 1$ on (x_a, x_b), which contradicts the fact that $u(x_0) \geq 1$. We will only prove the existence of x_a. The proof of the existence of x_b is similar.

By Remark 2.23 we see that either $u(x)$ has an infinite number of local minima on the left-hand side of x_0, or $u(x)$ tends to a limit monotonically as $x \to -\infty$. In the latter case Lemma 2.14 guarantees that u satisfies (2.28) with $x_a = -\infty$. In the former case we prove that at least one of the minima on the left-hand side of x_0 lies below the line $u = 1$. By contradiction, suppose there exist two consecutive local minima y_0 and y_1 above the line $u = 1$ (with $y_0 < y_1 < x_0$). Then there clearly exists a local maximum $x_1 \in (y_0, y_1)$. By translation invariance one may assume that $x_1 = 0$. We have that $u(0) > 1$, $u'(0) = 0$ and $u''(0) \leq 0$. Now first assume that $u'''(0) \leq 0$. Then we are in the setting of Lemma 2.31 and we conclude that $u(y_1) < 1$, thus a contradiction has been reached. On the other hand, if $u'''(0) \geq 0$, we switch to $\tilde{u}(x) \equiv u(-x)$ and, by the same argument, we conclude that $u(y_0) < 1$. This completes the proof of Theorem 2.6. $\square$

Remark 2.32 The method employed in this section to obtain a better a priori bound from a weaker one, has a nice geometrical interpretation, which makes it easy to apply the method to (2.1) with general $f(u)$. Let us assume that we have an a priori bound, i.e., for some $\gamma > 0$ all bounded solutions of (2.1) are in $\mathcal{B}(a, b)$. Suppose now that we can find constants $A > a$ and $0 < \Omega \leq \frac{1}{4\gamma}$ (i.e., $\gamma \in (0, \frac{1}{4\Omega}]$), such that

$$-\Omega(u - A) \leq f(u) \qquad \text{for all } u \in [a, b],$$

which means that the line $-\Omega(u - A)$ stays below $f(u)$ on the interval under consideration. Then A is a new (improved) lower bound on the set of bounded solutions.

Similarly, when we can find constants $B < b$ and $0 < \Omega \leq \frac{1}{4\gamma}$, such that

$$-\Omega(u - B) \geq f(u) \qquad \text{for all } u \in [a, b],$$

then B is a new (improved) upper bound on the set of bounded solutions. Remark that a new upper bound might allow us to find an improved lower bound, and vice versa. •

2.5 Conclusions for the EFK equation

We first make the observation that every bounded solution (except $u \equiv \pm 1$) has a zero.

Lemma 2.33 *For any $q \leq -\sqrt{8}$, let $u(x) \not\equiv 1$ be a bounded solution of* (2.10). *Then $u(x)$ has at least one zero.*

Proof. Suppose $u(x)$ does not have a zero. We may assume that $u(x) > 0$ for all $x \in \mathbb{R}$. Since $|u(x)| \leq 1$ for all $x \in \mathbb{R}$ by Theorem 2.6, we conclude that either $u(x)$ has a local minimum in the range $(0,1)$, or $u(x)$ is homoclinic to 0. The latter would imply that $\mathcal{E}[u] = -\frac{1}{4}$, and that $u(x)$ must attain a local maximum in the range $(0,1)$. It is easily seen from the energy identity that these two observations lead to a contradiction. We complete the proof by showing that $u(x)$ cannot have a local minimum in the range $(0,1)$.

Suppose that after translation we have

$$u(0) \in (0,1), \qquad u'(0) = 0 \qquad \text{and} \qquad u''(0) \geq 0.$$

We may suppose that in addition $u'''(0) \geq 0$ (otherwise we switch to $\tilde{u}(x) = u(-x)$). Analogous to the proof of Lemma 2.31 we set

$$x_1 \stackrel{\text{def}}{=} \sup\{x > x_0 \,|\, u''' > 0 \text{ on } (x_0, x)\},$$

and

$$y_0 \stackrel{\text{def}}{=} \sup\{x > x_0 \,|\, u < 1 \text{ on } (x_0, x)\}.$$

We find that $y_0 < x_1 < \infty$ from which we conclude that $u(y_0) = 1$ and $u'(y_0) > 0$, which contradicts Theorem 2.6. $\square$

We now prove Theorem 2.8.

Proof of Theorem 2.8. Lemma 2.12 shows that the only possible bounded solutions are equilibrium points, monotone heteroclinic solutions, homoclinic solutions with a unique extremum and periodic solutions with a unique maximum and minimum. Lemma 2.33 shows that any non-constant bounded solution has a zero, which means that except for the equilibrium points and the decreasing kink, every bounded solution has a zero at which it has a positive slope. Excluding the equilibrium points and the decreasing kink from these considerations, we conclude from Theorem 2.1 that no two solutions can have the same positive slope at their zeros, and from Theorem 2.3 that the solution with the larger slope has the higher energy. From these considerations we draw the following conclusions, to finish the proof of Theorem 2.8.

- Starting at low energies, it follows from the energy identity that solutions which lie in the levels $E < -\frac{1}{4}$ have no extrema in the range $[-\sqrt{2}, \sqrt{2}]$, and thus are unbounded.
- Similarly, for $E = -\frac{1}{4}$ the equilibrium solution $u \equiv 0$ is the only bounded solution, since any other would have a zero and this would contradict Theorem 2.3.
- There are no equilibrium points (and thus no connecting orbits) in the energy levels $E \in (-\frac{1}{4}, 0)$. Hence, it follows immediately from Lemma 2.33 and Theorem 2.3 that in each of these energy levels the periodic solution which has been proved to exist in [120], is the only bounded solution.
- For the energy level $E = 0$ we derive that beside the equilibrium points $u \equiv \pm 1$, the only bounded solutions are a unique monotonically increasing and a unique monotonically decreasing heteroclinic solution, of which the existence has been proved in [117]. In particular there exist no homoclinic connections to ± 1. These results for the energy level $E = 0$ were also obtained in [96] using a Twist property.
- Finally, there are no equilibrium points and thus no connecting orbits in the energy levels $E > 0$. Periodic solutions in these energy levels cannot have maxima smaller

than 1 by Theorem 2.3 (comparing them to the increasing kink). Therefore, Theorem 2.6 excludes the existence of periodic solutions for energies $E > 0$.

This completes the proof of Theorem 2.8. □

We recall how crucially these arguments depend on the real-saddle character of the equilibrium points. Both Theorem 2.6 and the Comparison Lemma do not hold when $\gamma > \frac{1}{8}$. The variety of solutions which exist for $\gamma > \frac{1}{8}$, shows that this bound is sharp.

Up to now, we did not use in an essential manner the invariance of (2.10) under the transformation $u \mapsto -u$. This invariance can be used to obtain further information on the shape of bounded solutions of (2.9). The next lemma states that every bounded solution is antisymmetric with respect to its zeros.

Lemma 2.34 *For any $\gamma \in (0, \frac{1}{8}]$, let $u(x)$ be a bounded solution of* (2.9). *Suppose that $u(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then $u(x_0 + x) = -u(x_0 - x)$ for all $x \in \mathbb{R}$.*

Proof. The proof is analogous to the proof of Lemma 2.12. Without loss of generality we may assume that $x_0 = 0$. Define $v(x) = -u(-x)$. By the symmetry of (2.9), $v(x)$ is also a bounded solution of (2.9). Clearly $u(0) = v(0)$ and $u'(0) = v'(0)$. From Theorem 2.1 we conclude that $u(x) \equiv v(x)$. □

We already saw that the periodic solutions of (2.9) can be parametrised by the energy. The next lemma shows that they can also be parametrised by their period.

Lemma 2.35 *Let $\gamma \in (0, \frac{1}{8}]$. Then the periodic solutions of* (2.9) *can be parametrised by the period $L \in (L_0, \infty)$, where*

$$L_0 \stackrel{\text{def}}{=} 2\pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}.$$

Proof. By Lemma 2.34 any periodic solution, of period L, is antisymmetric with respect to its zeros, and thus has exactly two zeros on the interval $[0, L)$. Via a variational method it has been proved in [124] that for every period $L \in (L_0, \infty)$ there exists at least one periodic solution $u(x)$ of (2.9) with exactly two zeros on the interval $[0, L)$. Besides, there are no periodic solutions with period smaller than or equal to L_0 [124, Lem. 2.4]. Therefore, we only need to show that there is at most one periodic solution with period L having exactly two zeros on the interval $[0, L)$.

We argue by contradiction. Suppose there are two such periodic solutions $u_1 \not\equiv u_2$ of (2.9) with period L. By Lemmas 2.12 and 2.34 we have that (after translation), for $i = 1, 2$

$$u_i'(0) = 0, \qquad u_i(\pm\tfrac{L}{4}) = 0, \qquad \text{and} \qquad u_i(x) > 0 \quad \text{for } x \in (-\tfrac{L}{4}, \tfrac{L}{4}).$$

Clearly, both solutions are increasing on $(-\frac{L}{4}, 0)$.

We see from Theorem 2.1 that $u_1'(-\frac{L}{4}) \neq u_2'(-\frac{L}{4})$, and without loss of generality we may assume that $u_1'(-\frac{L}{4}) > u_2'(-\frac{L}{4})$. Let

$$x_0 \stackrel{\text{def}}{=} \sup\{x > -\tfrac{L}{4} \mid u_2 < u_1 \text{ on } (-\tfrac{L}{4}, x)\}.$$

We assert that $x_0 = \frac{L}{4}$. Suppose that $x_0 \leq \frac{L}{4}$. Then $x_0 < 0$ since the solutions are symmetric with respect to $x = 0$. However, u_1 and u_2 are increasing on $(-\frac{L}{4}, x_0)$, and $u_1(-\frac{L}{4}) = u_2(-\frac{L}{4})$ and $u_1(x_0) = u_2(x_0)$. This implies that there exist x_1 and x_2 in $(-\frac{L}{4}, x_0]$ such that $u_1(x_1) = u_2(x_2)$ and $u_1'(x_1) = u_2'(x_2)$, contradicting Theorem 2.1.

Hence, we have established that

$$u_1(x) > u_2(x) > 0 \qquad \text{for } x \in (-\tfrac{L}{4}, \tfrac{L}{4}). \tag{2.29}$$

When we multiply the differential equation of u_1 by u_2, and integrate over $(-\frac{L}{4}, \frac{L}{4})$, then we obtain

$$\begin{aligned} 0 &= \int_{-\frac{L}{4}}^{\frac{L}{4}} \left\{ u_2(-\gamma u_1^{(iv)} + u_1'' + u_1 - u_1^3) \right\} dx \\ &= \int_{-\frac{L}{4}}^{\frac{L}{4}} \left\{ u_1(-\gamma u_2^{(iv)} + u_2'' + u_2) - u_2 u_1^3 \right\} dx. \end{aligned}$$

Here we have used partial integration and the fact that $u_i''(\pm\frac{L}{4}) = 0$ (by Lemma 2.34). Since u_2 is a solution of (2.9), this implies that

$$0 = \int_{-\frac{L}{4}}^{\frac{L}{4}} \left\{ u_1 u_2 (u_2^2 - u_1^2) \right\} dx,$$

which contradicts (2.29). □

2.6 Transversality

The unique monotonically increasing heteroclinic solution $v(x)$ of (2.10) for $q \le -\sqrt{8}$ is antisymmetric by Lemma 2.34. Removing the translational invariance by taking the unique zero of $v(x)$ at the origin, we have

$$v(0) = 0, \qquad v'(0) > 0 \qquad \text{and} \qquad v''(0) = 0.$$

In this section we will apply a technique similar to the one in [35] to prove that $v(x)$ is a transverse intersection of the unstable manifold $W^u(-1)$ and the stable manifold $W^s(+1)$ in the zero energy set (here we write $W^{u,s}(\pm 1)$ instead of $W^{u,s}(\pm 1, 0, 0, 0)$). Both $W^u(-1)$ and $W^s(+1)$ are two-dimensional manifolds since the equilibrium points $u = \pm 1$ are real saddles for $q \le -\sqrt{8}$ (for $q \in (-\sqrt{8}, \sqrt{8})$ they are saddle-foci and the manifolds $W^{u,s}(\pm 1)$ remain two-dimensional). If the intersection of $W^u(-1)$ and $W^s(+1)$ were not transverse, then it follows from the symmetry of the potential that there would be only two possibilities. We will exclude these possibilities with the help of the Comparison Lemma and some delicate and rather technical estimates. When the potential is not symmetric we still expect the intersection to be transverse, but a proof along the same lines seems more involved.

The following lemma provides a bound on the orbits $u(x)$ in the stable manifold of $+1$ that lie close to the kink $v(x)$. This bound will be useful later on, since it enables the application of the Comparison Lemma to these solutions.

Lemma 2.36 *For any $q \le -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solutions of* (2.10) *with its zero at the origin. Suppose that $u(x)$ is a solution of* (2.10) *such that $u \in W^s(+1)$, and (for some $\delta > 0$)*

$$|u^{(k)}(x) - v^{(k)}(x)| < \delta \qquad \text{for } k = 0, 1, 2, 3, \text{ and } x \in [0, \infty). \tag{2.30}$$

Then for $\delta > 0$ sufficiently small we have $|u(x)| < 1$ for all $x > 0$.

Proof. Recall that $v(x)$ increases monotonically from -1 to $+1$. The fact that $u(x) > -1$ on $[0,\infty)$ is immediate from (2.30). It is easily seen that the monotone kink $v(x)$ obeys the system

$$\begin{cases} v^{(iv)} + qv'' = v - v^3 < 0 & \text{on } (-\infty, 0), \\ v''(0) = 0, \\ v''(-\infty) = 0. \end{cases}$$

Since $q \le 0$, it follows from the strong maximum principle that $v''(x) > 0$ on $(-\infty, 0)$, and in particular $v''(-1) > 0$. Let $u(x)$ obey (2.30), then this implies that

$$u''(-1) > 0, \qquad u(-1) < 1 \qquad \text{and} \qquad u(x) > -2 \quad \text{on } [-1,\infty),$$

for δ sufficiently small. Besides, $u(\infty) = 1$ and $u''(\infty) = 0$. It now follows from Lemma 2.29 that $u(x) < 1$ on $[-1,\infty)$. □

We now start the proof of Theorem 2.9. We emphasise that we assume that the potential F is symmetric, which greatly reduces the number of possibilities that we have to check in order to conclude that the intersection of $W^u(-1)$ and $W^s(+1)$ is transverse.

For any $q \le -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solution of (2.10). Since $v(x)$ is antisymmetric by Lemma 2.34, we have that

$$v(0) = 0, \qquad v'(0) > 0 \qquad \text{and} \qquad v''(0) = 0,$$

and by Lemma 2.16 we have $v'''(0) - \lambda v'(0) < 0$. Besides, v lies in the zero energy manifold, i.e.,

$$v'v''' - \frac{1}{2}(v'')^2 + \frac{q}{2}(v')^2 - F(v) = 0,$$

where $F(v) = -\frac{1}{4}(v^2-1)^2$. Therefore

$$v'''(0) + qv'(0) = \frac{q}{2}v'(0) - \frac{1}{4v'(0)} = (-\lambda - C)v'(0) - \frac{1}{4v'(0)}, \tag{2.31}$$

where $C = \sqrt{\left(\frac{q}{2}\right)^2 - 2} \ge 0$. The tangent space to the zero energy manifold at the point $P = (0, v'(0), 0, v'''(0))$ is

$$\big(0, u'''(0) + qu'(0), 0, u'(0)\big)^{\perp} \subset \mathbb{R}^4.$$

The tangent spaces to the two-dimensional manifolds $W^u(-1)$ and $W^s(+1)$ at this point both contain the vector

$$X = \big(v'(0), 0, v'''(0), 0\big), \tag{2.32}$$

because of the differential equation.

Let us suppose, seeking a contradiction, that these stable and unstable manifolds do not intersect transversely in the zero energy set. Then their tangent spaces, which are two-dimensional, coincide. We denote this two-dimensional tangent space by T_P. Because of the symmetry of F and reversibility, $(\alpha,\beta,\gamma,\delta)$ lies in $W^u(-1)$ if and only if $(-\alpha,\beta,-\gamma,\delta)$ lies in $W^s(+1)$. It then follows that

$$(\alpha,\beta,\gamma,\delta) \in T_P \quad \text{if and only if} \quad (-\alpha,\beta,-\gamma,\delta) \in T_P. \tag{2.33}$$

This symmetry relation implies that there are only two possibilities for T_P. Namely, let T_P be spanned by X, given by (2.32), and $Y = (\alpha,\beta,\gamma,\delta)$. We may assume that $\alpha = 0$ (replacing Y by $Y - \frac{\alpha}{v'(0)}X$). If $\beta \ne 0$, then we see from (2.33) that $\gamma = 0$ (otherwise $(v'(0), 0, v'''(0), 0)$, $(0,\beta,\gamma,\delta)$ and $(0,\beta,-\gamma,\delta)$ would be three linearly independent vectors in T_P). Besides, δ is directly related to β since $T_P \in \big(0, u'''(0)+qu'(0), 0, u'(0)\big)^{\perp}$. On the other hand, if $\beta = 0$, then also $\delta = 0$. Thus, we are left with two possibilities:

$$\begin{array}{rrcl} \text{Case A:} & T_P & = & \{(\xi,0,\eta,0) \mid (\xi,\eta)\in\mathbb{R}^2\}, \qquad \text{or} \\ \text{Case B:} & T_P & = & \{\big(\xi v'(0), -\eta v'(0), \xi v'''(0), \eta(v'''(0)+qv'(0))\big) \mid (\xi,\eta)\in\mathbb{R}^2\}. \end{array}$$

Note that the symmetry of the potential has reduced the number of possibilities enormously. To complete the proof of Theorem 2.9 we have to exclude the possibilities described in Case A and Case B.

In Case A let $\xi = 1$ and $\eta = 1+\lambda$, and consider the point on $W^s(+1)$ given by

$$(u,u',u'',u''')(0) = \big(\varepsilon + O(\varepsilon^2), v'(0)+O(\varepsilon^2), (1+\lambda)\varepsilon + O(\varepsilon^2), v'''(0)+O(\varepsilon^2)\big).$$

It should be clear that for ε small enough the conditions of Lemma 2.36 are satisfied, so that $|u(x)| < 1$ on $[0,\infty)$. We will deal with this case in Lemma 2.38, where we show that under the present conditions, $u(x) \notin W^s(+1)$, which excludes Case A.

Now suppose that Case B holds, and let $\xi = 0$ and $\eta = -1$. Then there is a point $(u,u',u'',u''')(0)$ on $W^s(+1)$ of the form

$$\big(O(\varepsilon^2), v'(0)+\varepsilon v'(0)+O(\varepsilon^2), O(\varepsilon^2), v'''(0) - \varepsilon(v'''(0)+qv'(0)) + O(\varepsilon^2)\big).$$

Now

$$(u'-v')(0) = \varepsilon v'(0) + O(\varepsilon^2), \tag{2.34}$$

and, using (2.31),

$$\begin{aligned} (u'''-v''')(0) &= -\varepsilon(v'''(0)+qv'(0)) + O(\varepsilon^2) \\ &= \varepsilon(\lambda + C)v'(0) + \frac{\varepsilon}{4v'(0)} + O(\varepsilon^2) \\ &= \lambda(u'-v')(0) + \varepsilon C v'(0) + \frac{\varepsilon}{4v'(0)} + O(\varepsilon^2), \end{aligned}$$

where $C = \sqrt{\left(\frac{q}{2}\right)^2 - 2} \geq 0$. We infer that

$$(u'''-\lambda u')(0) - (v'''-\lambda v')(0) = \varepsilon\left(Cv'(0) + \frac{1}{4v'(0)}\right) + O(\varepsilon^2). \tag{2.35}$$

Besides, it should be clear that for ε small enough the conditions of Lemma 2.36 are satisfied, so that $|u(x)| < 1$ on $[0,\infty)$. We will deal with this case in Lemma 2.37, where we show that under the present conditions, $u(x) \notin W^s(+1)$, which excludes Case B.

We now prove two technical lemmas (adopted from [35] to the case of an antisymmetric heteroclinic orbit) to exclude the two possibilities which could occur if the intersection of $W^u(-1)$ and $W^s(+1)$ were not transverse. We show that in both Case A and Case B the initial data of u and v are such that for some small positive x, we arrive in the situation of the Comparison Lemma. We then conclude that u cannot be in the stable manifold $W^s(+1)$.

The next lemma deals with Case B (it is the counterpart of Th. 2.3 in [35]). In order for the points of $W^s(+1)$ in Case B to satisfy the assumptions of the lemma, we choose (looking at (2.34) and (2.35))

$$\alpha = \frac{1}{2}\min\Big\{v'(0), Cv'(0) + \frac{1}{4v'(0)}\Big\} \quad \text{and} \quad k = 2\max\Big\{v'(0), Cv'(0)+\frac{1}{4v'(0)}\Big\}.$$

Lemma 2.37 *For any $q \le -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solutions of* (2.10) *with its zero at the origin. Let k, α, $\beta > 0$ be constants. Suppose that $u(x)$ is a solution of* (2.10) *with $|u(x)| < 1$ on $[0, \infty)$, satisfying (for some $\varepsilon > 0$)*

$$k\varepsilon \ge (u''' - \lambda u')(0) - (v''' - \lambda v')(0) \ge \alpha\varepsilon \qquad \text{and} \qquad k\varepsilon \ge u'(0) - v'(0) \ge \alpha\varepsilon,$$

and

$$|u(0)| + |u''(0)| \le \beta\varepsilon^2.$$

Then, for ε sufficiently small, $u(0) \notin W^s(+1)$.

Proof. The solution v exists on $[0, \infty)$ and the initial data of u are ε-close to those of v. Therefore there exists an $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ the function u and its derivatives of all orders exist and are uniformly bounded on $[0, 1]$, for all u satisfying the assumptions independent of $\varepsilon \in (0, \varepsilon_0)$. Consequently, by Taylor's theorem we infer that for some $M > 0$ and for all $x \in [0, 1]$

$$\begin{aligned}
u(x) - v(x) &\ge u(0) - v(0) + \{u'(0) - v'(0)\}x \\
&\quad + \tfrac{1}{2}\{u''(0) - v''(0)\}x^2 - Mx^3 \\
&\ge -\beta\varepsilon^2 + \alpha\varepsilon x - \tfrac{1}{2}\beta\varepsilon^2 x^2 - Mx^3. \\
u'(x) - v'(x) &\ge u'(0) - v'(0) + \{u''(0) - v''(0)\}x - Mx^2 \\
&\ge \alpha\varepsilon - \beta\varepsilon^2 x - Mx^2. \\
(u'' - \lambda u)(x) - (v'' - \lambda v)(x) &\ge (u'' - \lambda u)(0) - (v'' - \lambda v)(0) \\
&\quad + \{(u''' - \lambda u')(0) - (v''' - \lambda v')(0)\}x \\
&\quad + \tfrac{1}{2}\{(u'''' - \lambda u'')(0) - (v'''' - \lambda v'')(0)\}x^2 - Mx^3 \\
&\ge -(1 + \lambda)\beta\varepsilon^2 + \alpha\varepsilon x - \tfrac{1}{2}K\beta\varepsilon^2 x^2 - Mx^3. \\
(u''' - \lambda u')(x) - (v''' - \lambda v')(x) &\ge (u''' - \lambda u')(0) - (v''' - \lambda v')(0) \\
&\quad + \{(u'''' - \lambda u'')(0) - (v'''' - \lambda v'')(0)\}x - Mx^2 \\
&\ge \alpha\varepsilon - K\beta\varepsilon^2 x - Mx^2.
\end{aligned}$$

Here we have used the fact that, for some constant $K > 0$,

$$|u''''(0) - \lambda u''(0)| = |(-q - \lambda)u''(0) + u(0) - u^3(0)| \le K\beta\varepsilon^2.$$

Let $\tilde{K} = \max\{1, K\}$, and let us define

$$\Gamma(\varepsilon) = \sqrt{\varepsilon}\left\{\sqrt{\frac{\alpha}{M}} - \frac{\tilde{K}\beta\varepsilon^{3/2}}{2M}\right\}.$$

Then we have, for all $x \in [0, \Gamma(\varepsilon)]$,

$$u'(x) - v'(x) \ge 0 \qquad \text{and} \qquad (u''' - \lambda u')(x) - (v''' - \lambda v')(x) \ge 0.$$

We now introduce $\tau(\varepsilon) = \varepsilon^{2/3}$. It then follows that $\tau(\varepsilon) \in [0, \Gamma(\varepsilon)] \cap [0, 1]$ for $\varepsilon > 0$ sufficiently small. We obtain that

$$(u - v)(\tau(\varepsilon)) \ge -\beta\varepsilon^2 + \alpha\varepsilon^{5/3} - \tfrac{1}{2}\beta\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for $\varepsilon > 0$ sufficiently small, and

$$(u'' - \lambda u)(\tau(\varepsilon)) - (v'' - \lambda v)(\tau(\varepsilon)) \ge -(1 + \lambda)\beta\varepsilon^2 + \alpha\varepsilon^{5/3} - \tfrac{1}{2}K\beta\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for $\varepsilon > 0$ sufficiently small.

We can now apply the Comparison Lemma to $u(x+\tau(\varepsilon))$ and $v(x+\tau(\varepsilon))$ to conclude that $u(x)$ does not tend to 1 as $x \to \infty$, which proves the lemma. □

The following lemma excludes Case A (it is the counterpart of Th. 2.4 in [35]). In order for the points of $W^s(+1)$ in Case A to satisfy the assumptions of the lemma, we choose $\alpha = \frac{1}{2}$ and $k = 2$.

Lemma 2.38 *For any $q \le -\sqrt{8}$, let $v(x)$ be the unique monotonically increasing heteroclinic solutions of* (2.10) *with its zero at the origin. Let k, α, $\beta > 0$ be constants. Suppose that $u(x)$ is a solution of* (2.10) *with $|u(x)| < 1$ on $[0,\infty)$, satisfying (for some $\varepsilon > 0$)*

$$k\varepsilon \ge u''(0) - \lambda u(0) \ge \alpha\varepsilon \qquad \text{and} \qquad k\varepsilon \ge u(0) \ge \alpha\varepsilon,$$

and

$$|u'(0) - v'(0)| + |u'''(0) - v'''(0)| \le \beta\varepsilon^2.$$

Then, for ε sufficiently small, $u(0) \notin W^s(+1)$.

Proof. We proceed as in the proof of Lemma 2.37. We find, by Taylor's theorem, that for some $M > 0$, $K > 0$ and $x \in [0,1]$,

$$\begin{aligned}
u(x) - v(x) &\ge \alpha\varepsilon - \beta\varepsilon^2 x - Mx^2 \\
u'(x) - v'(x) &\ge -\beta\varepsilon^2 + \alpha(1+\lambda)\varepsilon x - \tfrac{1}{2}\beta\varepsilon^2 x^2 - Mx^3 \\
(u'' - \lambda u)(x) - (v'' - \lambda v)(x) &\ge \alpha\varepsilon - (1+\lambda)\beta\varepsilon^2 x - Mx^2
\end{aligned}$$

and

$$\begin{aligned}
(u''' - \lambda u')(x) - (v''' - \lambda v')(x) &\ge (u''' - \lambda u')(0) - (v''' - \lambda v')(0) \\
&\quad + \left\{(u^{(iv)} - \lambda u'')(0) - (v^{(iv)} - \lambda v'')(0)\right\} x \\
&\quad + \tfrac{1}{2}\left\{(u^{(v)} - \lambda u''')(0) - (v^{(v)} - \lambda v''')(0)\right\} x^2 - Mx^3 \\
&\ge -2\beta\varepsilon^2 + (2+\mu)\alpha\varepsilon x - \tfrac{1}{2}K\varepsilon^2 x^2 - Mx^3.
\end{aligned}$$

Here we have used the following facts. First, $v^{(iv)}(0) = 0$ by (2.10) and

$$\begin{aligned}
(u^{(iv)} - \lambda u'')(0) &= \mu(u'' - \lambda u)(0) + 3u(0) - u^3(0) \\
&\ge \mu\alpha\varepsilon + 3\alpha\varepsilon - k^3\varepsilon^3 \\
&\ge (2+\mu)\alpha\varepsilon
\end{aligned}$$

for ε sufficiently small. Second, by differentiating (2.10) we obtain

$$u^{(v)} + qu''' + u'(3u^2 - 1) = 0,$$

from which we deduce that

$$\begin{aligned}
(u^{(v)} - \lambda u''')(0) - (v^{(v)} - \lambda v''')(0) &= \mu(u''' - \lambda u')(0) - \mu(v''' - \lambda v')(0) \\
&\quad + 3(u'(0) - v'(0)) - 3u'(0)u^2(0) \\
&\ge -\mu(1+\lambda)\beta\varepsilon^2 - 3\beta\varepsilon^2 - 6v'(0)k^2\varepsilon^2 \equiv -K\varepsilon^2,
\end{aligned}$$

since $|u'(0)| \le |v'(0)| + \beta\varepsilon^2 \le 2v'(0)$, for ε sufficiently small.

We define

$$\Gamma(\varepsilon) = \sqrt{\varepsilon}\left\{\sqrt{\frac{\alpha}{M}} - \frac{(1+\lambda)\beta\varepsilon^{3/2}}{2M}\right\}.$$

Then we have, for all $x \in [0, \Gamma(\varepsilon)]$,

$$u(x) - v(x) \ge 0 \qquad \text{and} \qquad (u'' - \lambda u)(x) - (v'' - \lambda v)(x) \ge 0.$$

If $\tau(\varepsilon) = \varepsilon^{2/3}$, then $\tau(\varepsilon) \in [0, \Gamma(\varepsilon)] \cap [0,1]$ for $\varepsilon > 0$ sufficiently small and

$$(u' - v')(\tau(\varepsilon)) \geq -\beta\varepsilon^2 + \alpha(1+\lambda)\varepsilon^{5/3} - \tfrac{1}{2}\beta\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for ε sufficiently small, and

$$(u''' - \lambda u')(\tau(\varepsilon)) - (v''' - \lambda v')(\tau(\varepsilon)) \geq -2\beta\varepsilon^2 + (2+\mu)\alpha\varepsilon^{5/3} - \tfrac{1}{2}K\varepsilon^{10/3} - M\varepsilon^2 > 0,$$

for ε sufficiently small.

We can now apply the Comparison Lemma to $u(x + \tau(\varepsilon))$ and $v(x + \tau(\varepsilon))$ to conclude that $u(x)$ does not tend to 1 as $x \to \infty$, which proves the lemma. □

Remark 2.39 The special symmetry of $u - u^3$ has enabled us to prove that the heteroclinic solution is transverse. For general $f(u)$ transversality of heteroclinic solutions is much harder to check. However, for homoclinic solutions this difficulty does not arise, since every homoclinic solution (for $\gamma \in (0, \frac{1}{4\omega}]$) is symmetric with respect to its extremum. We will give an outline of the proof that every homoclinic solution is a transverse intersection.

Without loss of generality we may assume that $v(x)$ is a positive homoclinic solution of (2.1) to 0 with a unique maximum at $x = 0$. As usual, we suppose that $\gamma \in (0, \frac{1}{4\omega(0,v(0))}]$. The method in [35] for homoclinic solutions can be extended to general $f(u)$, as was done above for heteroclinic solutions. To be able to apply the Comparison Lemma to a solution in $W^s(0)$ close to $v(x)$, we need a very mild assumption on $f(u)$, but only in a special case (when $\gamma = \frac{1}{4\omega(0,v(0))}$, then we need that $f'(u) \neq -\omega(0, v(0))$ in some left neighbourhood of $u = 0$). The only fairly specific condition in the rest of the proof is that $\frac{f(v(0))}{-v''(0)} > \lambda$, which follows directly from Remark 2.17. •

2.7 Stability of the kink

In this section we look at the stability of the kink for the EFK equation (2.12) and prove Theorem 2.10. The EFK equation is a semi-linear parabolic equation and for such equations the local existence of the flow (e.g., in the space of bounded uniformly continuous functions) has been well-established (e.g., see [111, 81]). To fix ideas, for $\gamma \leq \frac{1}{8}$ let $v(x)$ be the unique monotonically increasing heteroclinic solution of (2.9) such that $v(0) = 0$ (removing the translational invariance). The existence of this solution can be proved by a shooting method [117], but it can also be found as the minimiser of the functional

$$J[u] \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left\{ \frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 + \frac{1}{4}(u^2-1)^2 \right\} dx.$$

The minimum is taken over all functions $u(x)$ with $u - \chi \in H^2(\mathbb{R})$, where $\chi \in C^\infty(\mathbb{R})$ is an antisymmetric function such that such that $\chi(x) = -1$ for $x \leq -1$, and $\chi(x) = 1$ for $x \geq 1$ (see [89, 124]).

The minimising property of the kink $v(x)$ and its transversality in the zero energy set allow us to conclude that for $\gamma \leq \frac{1}{8}$ the kink is asymptotically stable in $H^1(\mathbb{R})$. Another possible choice is to work in the space of bounded uniformly continuous functions. The analysis below applies to both function spaces.

To study the stability of the kink, we write $u(x,t) = v(x) + \phi(x,t)$. The differential equation for the perturbation $\phi(x,t)$ is then

$$\frac{\partial \phi}{\partial t} = -\gamma\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^2 \phi}{\partial x^2} + (1 - 3v^2)\phi - 3v\phi^2 - \phi^3.$$

Note that the nonlinear term $-3v(x)\phi^2 - \phi^3$ is C^1 from H^1 to H^1.

We have to investigate the spectrum of the linearised operator

$$\mathcal{L}\phi \stackrel{\text{def}}{=} -\gamma\phi'''' + \phi'' - 2\phi + g(x)\phi,$$

where

$$g(x) \equiv 3 - 3v^2(x) \to 0 \qquad \text{as } x \to \pm\infty.$$

We consider $\mathcal{L}$ as an unbounded operator from $D(\mathcal{L}) = H^5(\mathbb{R}) \subset H^1(\mathbb{R})$ to $H^1(\mathbb{R})$. It is well-known that the essential spectrum of $\mathcal{L}$ is

$$\sigma_e(\mathcal{L}) = (-\infty, -2],$$

and that the remaining part of the spectrum $\sigma(\mathcal{L}) \setminus \sigma_e(\mathcal{L})$ consists entirely of isolated real eigenvalues of finite multiplicity (see e.g. [81, Ch. 5]).

The minimising property of the kink,

$$J[v] = \inf\{J[u] \,|\, u - \chi \in H^2\},$$

implies that

$$(\mathcal{L}\phi, \phi)_{L^2} \leq 0 \qquad \text{for all } \phi \in H^4. \tag{2.36}$$

Any eigenfunction of $\mathcal{L}$ in H^1 is in H^5, thus by substituting eigenfunctions in (2.36) we see that all eigenvalues of $\mathcal{L}$ are in $(-\infty, 0]$.

The EFK equation is autonomous, thus $v'(x)$ is an eigenfunction with eigenvalue 0. In fact, the zero eigenvalue is simple, which follows from the transversality of $W^s(+1)$ and $W^u(-1)$. To see this, we note that the flow of the tangent plane $TW^s(x)$ of the stable manifold of $+1$ at points $(v, v', v'', v''')(x)$ on the heteroclinic orbit, is given by the linearised equation around the kink. Since $W^s(+1)$ is two-dimensional this implies that there are exactly two linearly independent solutions of $\mathcal{L}\phi = 0$ which tend to 0 as $t \to \infty$, corresponding to two independent directions in the tangent planes $TW^s(x)$. A similar statement holds for the tangent plane $TW^u(x)$ of the unstable manifold of -1. Because an eigenfunction with eigenvalue 0 obeys this linearised equation $\mathcal{L}\phi = 0$ and tends to 0 as $x \to \pm\infty$, it corresponds to a common direction in the tangent planes $TW^s(x)$ and $TW^u(x)$. Therefore, a second independent eigenfunction with eigenvalue 0 would imply that the stable and unstable manifolds do not intersect transversely in the zero energy set, which contradicts Theorem 2.9.

We note that this reasoning also applies to the space of bounded uniformly continuous functions (e.g., see [81, Section 5.4]), the reason being that there is an exponential dichotomy when $x \to \pm\infty$ (see [81, 78]).

Having established that the zero eigenvalue of the linearisation around $v(x)$ is simple, we may apply the theory from [81, Section 5.1] or [16, Th. 4.3]. It follows that the stationary solution $v(x)$ of Equation (2.12) has a local stable manifold of co-dimension 1 in the flow of (2.12). The center manifold is one-dimensional and by exploiting the spatial translation invariance of (2.12) we see that this center manifold consists of the translates of the kink: $\{v(x + x_0) \,|\, x_0 \in \mathbb{R}\}$. Besides, we conclude from the translation invariance that the stable manifolds of the translates of the kink $v(x)$ fill a tubular neighbourhood of $\{v(x + x_0) \,|\, x_0 \in \mathbb{R}\}$ in function space. This implies asymptotic stability (see [16] for more details) and thus proves Theorem 2.10.

2.8 Continuation and existence

This section is devoted to the continuation of bounded solutions of (2.6) for values of q that are sufficiently negative. Theorem 2.1 shows that for each point P in the (u, u')-plane there is at most one bounded solution of which the path goes through P. In this section we show that if a point P lies on the path of a periodic solution, then there exists a periodic solution through any point in the neighbourhood of P, i.e., part of the phase-plane is filled up by bounded solutions. The fact that solutions can be continued, also implies the existence of certain bounded solutions.

The main result of this section is Theorem 2.5. In the proof of this theorem we use the notation of Equation (2.6). Let $u_0(x)$ be a periodic solution of (2.6) for $q = q_0$. We define $a \equiv \min u_0(x)$ and $b \equiv \max u_0(x)$. Suppose that $q_0 \le -2\sqrt{\omega(a,b)}$. Then this periodic solution is part of a continuous one-parameter family of periodic solutions. We will use the Implicit Function Theorem to prove this assertion. In Theorem 2.5 the energy is taken as parameter. Here we first take the maximum value of solutions as parameter and then we show that the energy can be used as parameter equally well.

Without loss of generality we may assume that u_0 attains a maximum at $x = 0$. Then $u_0'(0) = 0$ and $u_0'''(0) = 0$ by Lemma 2.12, and from Remark 2.17 we see that

$$u_0''(0) < 0 \quad \text{and} \quad u_0^{(iv)}(0) - \lambda u_0''(0) > 0.$$

Let $\xi_0 > 0$ be the first point where u_0 attains a minimum.

We now look at a family of solutions $u(x; \alpha, \beta)$ of (2.6) with initial data

$$(u, u', u'', u''')(0; \alpha, \beta) = (\alpha, 0, \beta, 0),$$

where (α, β) is in a small neighbourhood of $(\alpha_0, \beta_0) \stackrel{\text{def}}{=} (u_0(0), u_0''(0))$. Note that $u(x; \alpha_0, \beta_0)$ is the periodic solution $u_0(x)$.

To show that u_0 is part of a continuous family it suffices to prove that $(\xi_0; \alpha_0, \beta_0)$ lies on a smooth curve $(\xi_t; \alpha_t, \beta_t)$, with $t \in (-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ small, such that

$$u'(\xi_t; \alpha_t, \beta_t) = 0 \quad \text{and} \quad u'''(\xi_t; \alpha_t, \beta_t) = 0.$$

The functions $u(x; \alpha_t, \beta_t)$ extend to periodic solutions by reflection in $x = 0$ and $x = \xi_t$. Let α play the role of the parameter, then to be able to apply the Implicit Function Theorem, we have to show that the determinant

$$D \stackrel{\text{def}}{=} \det \begin{pmatrix} \frac{\partial u'}{\partial x} & \frac{\partial u'''}{\partial x} \\ \frac{\partial u'}{\partial \beta} & \frac{\partial u'''}{\partial \beta} \end{pmatrix} (\xi_0; \alpha_0, \beta_0)$$

is non-zero.

It follows from Remark 2.17 that

$$u''(\xi_0; \alpha_0, \beta_0) > 0 \qquad \text{and} \qquad u^{(iv)}(\xi_0; \alpha_0, \beta_0) - \lambda u''(\xi_0; \alpha_0, \beta_0) < 0. \tag{2.37}$$

We define $v(x) = \frac{\partial u}{\partial \beta}(x; \alpha_0, \beta_0)$ and observe that

$$v(0) = 0, \quad v'(0) = 0, \quad v''(0) - \lambda v(0) = 1 \quad \text{and} \quad v'''(0) - \lambda v'(0) = 0. \tag{2.38}$$

Besides, v satisfies the equation $v^{(iv)} + qv'' = f'(u)v$, which we write as

$$(v'' - \lambda v)'' - \mu(v'' - \lambda v) = (f'(u) + \omega)v. \tag{2.39}$$

Arguing along the lines of the Comparison Lemma we see from (2.38) that $v > 0$ on $(0, \sigma)$ for $\sigma > 0$ small enough. We now observe that $(f'(u) + \omega)v > 0$ on $(0, \sigma)$ by the definition of $\omega = \omega(a, b)$. We deduce from (2.38) and (2.39) that $v'' - \lambda v > 0$ on $(0, \sigma)$, and as in the proof of the Comparison Lemma, we conclude that $\sigma = \infty$. Hence, for all $x > 0$ we have

$$v > 0, \quad v' > 0, \quad v'' - \lambda v > 0, \quad \text{and} \quad v''' - \lambda v' > 0. \tag{2.40}$$

We now see from (2.37) and (2.40) that

$$\det \begin{pmatrix} \frac{\partial u'}{\partial x} & \frac{\partial (u''' - \lambda u')}{\partial x} \\ \frac{\partial u'}{\partial \beta} & \frac{\partial (u''' - \lambda u')}{\partial \beta} \end{pmatrix} (\xi_0; \alpha_0, \beta_0) = \det \begin{pmatrix} > 0 & < 0 \\ > 0 & > 0 \end{pmatrix} > 0,$$

which immediately implies that $D \neq 0$.

Above we have used the amplitude of the periodic solution as a parameter. We can also use the energy E as a parameter, taking x and α as variables. In that case we look at a family of solutions $u(x; \alpha, E)$ of (2.6) with initial data

$$(u, u', u'', u''')(0; \alpha, E) = (\alpha, 0, -\sqrt{2E - 2F(\alpha)}, 0),$$

where (α, E) is in a small neighbourhood of $(\alpha_0, \mathcal{E}[u_0])$. We define $v(x) = \frac{\partial u}{\partial \alpha}(x; \alpha_0, \mathcal{E}[u_0])$, and we notice that $v(0) = 1$, $v'(0) = 0$, $v'''(0) - \lambda v'(0) = 0$ and

$$\begin{aligned} v''(0) - \lambda v(0) &= \left. \frac{d\left(-\sqrt{2\mathcal{E}[u_0] - 2F(\alpha)} - \lambda \alpha\right)}{d\alpha} \right|_{\alpha = \alpha_0} \\ &= \frac{-F'(\alpha_0)}{-\sqrt{2\mathcal{E}[u_0] - 2F(\alpha_0)}} - \lambda \\ &\geq \frac{-f(u_0(0))}{u_0''(0)} - \mu = \frac{-u_0^{(iv)}(0) + \lambda u_0''(0)}{u_0''} > 0, \end{aligned}$$

by (2.6) and Remark 2.17. The previous analysis now applies once more and we conclude that Theorem 2.5 holds.

Another possibility for continuation of solutions is to fix the energy level E, take q as a parameter and use x and α as variables. Finally, instead of taking q as a parameter we can also deform the potential $F(u)$. This offers the possibility to obtain periodic solutions via continuation starting from a linear equation and then deforming the potential.

A different possible starting point for the continuation of bounded solutions is the second order equation ($\gamma = 0$), because for small positive γ the bounded solutions of (2.1) can be obtained from the second order equation by means of singular perturbation theory (e.g., see [2, 72]).

The continuation of periodic solutions can come to an end in a limited number of ways:

- the value of q becomes too large compared the critical value $\omega(\min u, \max u)$, i.e. $q > -2\sqrt{\omega(\min u, \max u)}$. This may either happen when we increase q, or when we deform the potential, or when the range of $u(x)$ expands.
- the amplitude of the periodic solutions tends to infinity.
- the amplitude of the periodic solutions tends to zero, i.e., the periodic orbits converge to an equilibrium point.
- the periodic solutions converge to a chain of connecting orbits (homoclinic and/or heteroclinic) as the period tends to infinity.

Considering homoclinic solutions we note that it follows from Remark 2.39 that under very weak assumptions on the potential, homoclinic solutions are transverse intersections and thus can be continued (for example starting at $\gamma = 0$ using singular perturbation theory). Another possibility is to obtain the homoclinic solutions as a limit of periodic solutions. Conversely, the existence of a transverse homoclinic orbit implies the existence of a family of periodic solutions close to this homoclinic orbit [55, 139].

Finally, with regard to heteroclinic solutions there is an important result from [89], which states that if there are two equilibrium points u_0 and u_1 ($u_0 < u_1$) such that

$$\begin{gathered} F(u_0) = F(u_1), \\ F(u) < F(u_0) = F(u_1) \quad \text{for all } u \in \mathbb{R} \setminus \{u_0, u_1\} \\ F''(u_0) < 0 \quad \text{and} \quad F''(u_1) < 0, \\ F(u) < C_1 - C_2 u^2 \quad \text{for all } u \in \mathbb{R} \text{ and some } C_1, C_2 > 0, \end{gathered}$$

then for all $\gamma > 0$ there exists a heteroclinic solution of (2.1) connecting these equilibrium points. On the other hand, the heteroclinic connections can also be obtained as a limit of periodic solutions, and when the potential is symmetric then, for γ not too large, the heteroclinic solution is a transverse intersection (as discussed in Section 2.6) and thus can be continued.

Chapter 3

Homotopy classes for stable patterns

3.1 Introduction

This chapter is a continuation of [89] where a constrained minimisation method has been developed to study heteroclinic and homoclinic local minimisers of the action functional[1]

$$J_I[u] = \int_I L(u, u', u'')\, dt = \int_I \Big[\frac{\gamma}{2} u''^2 + \frac{\beta}{2} u'^2 + F(u)\Big]\, dt, \tag{3.1}$$

with $\gamma, \beta > 0$. Minimisers are solutions of the equation

$$\gamma u'''' - \beta u'' + F'(u) = 0. \tag{3.2}$$

This equation with a double-well potential F has been proposed in connection with certain models of phase transitions. For brevity we will omit a detailed background of this problem and refer only to those sources required in the proofs of the results. A more extensive history and reference list are provided in [89] and Chapter 1, to which we refer the interested reader.

The above equation is Hamiltonian with

$$H = -\gamma u''' u' + \frac{\gamma}{2} u''^2 + \frac{\beta}{2} u'^2 - F(u). \tag{3.3}$$

The configuration space of the system is the (u, u')-plane, and solutions of (3.2) can be represented as curves in this plane. Initially these curves do not appear to be restricted in any way. However, the central idea presented here is that, when $(\pm 1, 0)$ are saddle-foci, the minimisers of J respect the topology of this plane punctured at these two points, which allows for a rich set of minimisers to exist. Using the topology of the doubly-punctured plane and its covering spaces, we describe the structure of all possible types of minimisers, including those which are periodic and chaotic. Since the action of the minimisers of these latter types is infinite, a different notion of minimiser is required that is reminiscent of the minimising (Class A) geodesics of Morse [107]. Such minimisers have been intensively studied in the context of geodesic flows on compact manifolds or the Aubry-Mather theory (see e.g. [29] for an introduction). A crucial difference is that we are dealing with a non-mechanical system on a non-compact space. Nevertheless, we are able to emulate many of Morse's original arguments about how the minimisers can intersect with themselves and each other. For a precise statement of the main results we refer to Theorem 3.10 and Theorem 3.34. For related work on mechanical Hamiltonian systems we refer to [29, 128] and the references therein.

Another important aspect of the techniques employed here and in [89] is the mildness of the hypotheses. In particular, our approach requires no transversality or non-

[1] Note that in this chapter the potential $F(u)$ is defined with the opposite sign compared to Chapter 1.

degeneracy conditions, such as those found in other variational methods and dynamical systems theory, see [89]. Specifically, we will assume the following hypothesis on F:

(H): $F \in C^2(\mathbb{R})$, $F(\pm 1) = F'(\pm 1) = 0$, $F''(\pm 1) > 0$, *and* $F(u) > 0$ *for* $u \neq \pm 1$.
Moreover there are constants c_1 *and* c_2 *such that* $F(u) \geq -c_1 + c_2 u^2$.

We will also assume for simplicity of the formulation that F is even, but many analogous results will hold for non-symmetric potentials, cf. [89]. Finally, we assume that the parameters γ and β are such that $u = \pm 1$ are saddle-foci, i.e. $\frac{\gamma}{\beta^2} > \frac{1}{4F''(\pm 1)}$. An example of a nonlinearity satisfying these conditions is $F(u) = \frac{1}{4}(u^2 - 1)^2$, in which case (3.2) is the stationary version of the so-called Extended Fisher-Kolmogorov (EFK) equation.

In [89] heteroclinic and homoclinic minimisers of J are classified by a finite sequence of even integers which represent the number of times a minimiser crosses $u = \pm 1$. In order to classify more general minimisers, we must consider infinite and bi-infinite sequences, as described below.

A function $u : \mathbb{R} \to \mathbb{R}$ can be represented as a curve in the $(u, u')-$plane, and the associated curve will be denoted by $\Gamma(u)$. Removing the equilibrium points $(\pm 1, 0)$ from the $(u, u')-$plane (the configuration space) creates a space with nontrivial topology, denoted by $\mathcal{P} = \mathbb{R}^2 \setminus \{(\pm 1, 0)\}$. In $\mathcal{P}$ we can represent functions u which have the property that $u' \neq 0$ when $u = \pm 1$, and various equivalence classes of curves can be distinguished. For example, in [89] classes of curves that terminate at the equilibrium points $(\pm 1, 0)$ are considered. Another important class consists of closed curves in $\mathcal{P}$, which represent periodic functions. We now give a systematic description of all classes to be considered.

Definition 3.1 *A* type *is a sequence* $\boldsymbol{g} = (g_i)_{i \in I}$ *with* $g_i \in 2\mathbb{N} \cup \{\infty\}$, *where* ∞ *acts as a terminator. To be precise,* $\boldsymbol{g}$ *satisfies one of the following conditions:*

(a) $I = \mathbb{Z}$, *and* $\boldsymbol{g} \in 2\mathbb{N}^{\mathbb{Z}}$ *is referred to as a* bi-infinite type.

(b) $I = \{0\} \cup \mathbb{N}$, *and* $\boldsymbol{g} = (\infty, g_1, g_2, \dots)$ *with* $g_i \in 2\mathbb{N}$ *for all* $i \geq 1$, *or*
$I = -\mathbb{N} \cup \{0\}$, *and* $\boldsymbol{g} = (\dots, g_{-2}, g_{-1}, \infty)$ *with* $g_i \in 2\mathbb{N}$ *for all* $i \leq -1$.
In these cases $\boldsymbol{g}$ *is referred to as a* semi-terminated type.

(c) $I = \{0, \dots, N+1\}$ *with* $N \geq 0$, *and* $\boldsymbol{g} = (\infty, g_1, \dots, g_N, \infty)$ *with* $g_i \in 2\mathbb{N}$.
In this case $\boldsymbol{g}$ *is referred to as a* terminated type.

These types will define function classes using the vector $\boldsymbol{g}$ to count the crossings of u at the levels $u = \pm 1$. Since there are two equilibrium points, we introduce the notion of *parity* denoted by $\boldsymbol{p}$, which will be equal to either 0 or 1.

Definition 3.2 *A function* $u \in H^2_{\mathrm{loc}}(\mathbb{R})$ *is in the* class $M(\boldsymbol{g}, \boldsymbol{p})$ *if there are nonempty sets* $\{A_i\}_{i \in I}$ *such that*

1. $u^{-1}(\pm 1) = \bigcup_{i \in I} A_i$,
2. $\# A_i = g_i$ *for* $i \in I$,
3. $\max A_i < \min A_{i+1}$,
4. $u(A_i) = (-1)^{i + \boldsymbol{p} + 1}$, *and*
5. $\bigcup_{i \in I} A_i$ *consists of transverse crossings of* ± 1, *i.e.,* $u'(x) \neq 0$ *for* $x \in A_i$.

Note that by Definition 3.1, a function u in any class $M(\boldsymbol{g}, \boldsymbol{p})$ has infinitely many crossings of ± 1. Definition 3.2 is similar to the definition of the class $M(\boldsymbol{g})$ in [89] except that here it is assumed that all crossings of ± 1 are transverse. Only finitely many crossings are assumed to be transverse in [89] so that the classes $M(g)$ are open subsets of $\chi + H^2(\mathbb{R})$. Since we will not directly minimise over $M(g,p)$, we now require transversality of all crossings of ± 1 to guarantee that $\Gamma(u) \in \mathcal{P}$. However, note that the minimisers found in [89] are indeed contained in classes $M(g,p)$ as defined above, where the types g are terminated.

The classes $M(g,p)$ are nonempty for all pairs (g,p). Conversely, any function $u \in H^2_{\mathrm{loc}}(\mathbb{R})$ is contained in the closure of some class $M(g,p)$ with respect to the complete metric on $H^2_{\mathrm{loc}}(\mathbb{R})$ given by $\rho(u,v) = \sum_i 2^{-i}\min\{1, \|u-v\|_{H^2(-i,i)}\}$, cf. [133]. That is, if we define

$$\overline{M(g,p)} \equiv \{u \in H^2_{\mathrm{loc}}(\mathbb{R}) \,|\, \text{there are } u_n \in M(g,p) \text{ with } u_n \to u \text{ in } H^2_{\mathrm{loc}}(\mathbb{R})\},$$

then $H^2_{\mathrm{loc}}(\mathbb{R}) = \cup_{(g,p)}\overline{M(g,p)}$. Note that the functions in $\partial M(g,p) \equiv \overline{M(g,p)} \setminus \mathrm{int}(M(g,p))$ have tangencies at ± 1 and thus are limit points of more than one class. In the case of an infinite type, shifts of g can give rise to the same function class. Therefore certain infinite types need to be identified. Let σ be the shift map defined by $\sigma(g)_i = g_{i+1}$, and the map $\tau : \{0,1\} \to \{0,1\}$ be defined by $\tau(p) = (p+1) \bmod 2 = |p-1|$. Two infinite types (g,p) and (g',p') are equivalent if $g' = \sigma^n(g)$ and $p' = \tau^n(p)$ for some $n \in \mathbb{Z}$, and this implies $M(g,p) = M(g',p')$. A bi-infinite type g is *periodic* if there exists an integer n such that $\sigma^n(g) = g$.

When the domain of integration is $\mathbb{R}$, the action $J[u]$ given in (3.1) is well-defined only for terminated types g and $u \in M(g,p) \cap \{\chi_p + H^2(\mathbb{R})\}$, where χ_p is a smooth function from $(-1)^{p+1}$ to $(-1)^p$. For semi-terminated types or infinite types the action J is infinite for every $u \in M(g,p)$. In Section 3.2 we will define an alternative notion of minimiser in order to overcome this difficulty. The primary goal of this chapter is to prove the following theorem, but we also prove additional results about the structure and relationships between various types of minimisers.

Theorem 3.3 *If F satisfies hypothesis* (H) *and is even, then for any type g and parity p there exists a minimiser of J in $M(g,p)$ in the sense of Definition 3.4. Moreover, if g is periodic, then there exists a periodic minimiser in $M(g,p)$.*

In Sections 3.4 and 3.5 we show that other properties of the symbol sequences, such as symmetry, are reflected in the corresponding minimisers. The classification of minimisers by symbol sequences has other properties in common with symbolic dynamics; for example, if a type is asymptotically periodic in both directions, then there exists a minimiser of that type which is a heteroclinic connection between two periodic minimisers.

The minimisers discussed here all lie in the 3-dimensional *energy manifold* $M_0 = \{(u,u',u'',u''') \,|\, H((u,u',u'',u''') = 0\}$. Exploiting certain properties of minimisers that are established in this chapter, we can deduce various linking and knotting characteristics when they are represented as smooth curves in M_0 (see also Chapter 8). The minimisers found in this chapter are also used in Chapter 4 to construct stable patterns for the evolutionary EFK equation on a bounded interval, and the dynamics of the evolutionary EFK is discussed in [92].

Some notation used in this chapter was previously introduced in [89]. While we have attempted to present a self-contained analysis, we have avoided reproducing details (particularly in Section 3.3) which are not central to the ideas presented here, and which are thoroughly explained in [89].

3.2 Definition of minimiser

For every compact interval $I \subset \mathbb{R}$ the restricted action J_I is well-defined for all types. When we restrict u to an interval I, we can define its *type and parity relative to* I, which we denote by $(\boldsymbol{g}(u|_I), \boldsymbol{p}(u|_I))$. Namely, let $u \in M(\boldsymbol{g}, \boldsymbol{p})$. It is clear that $(u, u')|_{\partial I} \notin (\pm 1, 0)$ for any bounded interval I. Then $\boldsymbol{g}(u|_I)$ is defined to be the finite-dimensional vector which counts the consecutive instances of $u|_I = \pm 1$, and $\boldsymbol{p}(u|_I)$ is defined such that the first crossing of $u = \pm 1$ in I is a crossing of $(-1)^{p+1}$. Note that the components of $\boldsymbol{g}(u|_I)$ are not necessarily all even, since the first and the last entries may be odd. We are now ready to state the definition of a (global) minimiser in $M(\boldsymbol{g}, \boldsymbol{p})$.

Definition 3.4 *A function $u \in M(\boldsymbol{g}, \boldsymbol{p})$ is called a* minimiser for J over $M(\boldsymbol{g}, \boldsymbol{p})$ *if and only if for every compact interval I the number $J_I[u|_I]$ minimises $J_{I'}[v|_{I'}]$ over all functions $v \in M(\boldsymbol{g}, \boldsymbol{p})$ and all compact intervals I' such that $(v, v')|_{\partial I'} = (u, u')|_{\partial I}$ and $(\boldsymbol{g}(v|_{I'}), \boldsymbol{p}(v|_{I'})) = (\boldsymbol{g}(u|_I), \boldsymbol{p}(u|_I))$.*

The pair $(\boldsymbol{g}(u|_I), \boldsymbol{p}(u|_I))$ defines a homotopy class of curves in $\mathcal{P}$ with fixed end points $(u, u')|_{\partial I}$. The above definition says that a function u, represented as a curve $\Gamma(u)$ in $\mathcal{P}$, is a minimiser if and only if for any two points P_1 and P_2 on $\Gamma(u)$, the segment $\Gamma(P_1, P_2) \subset \Gamma(u)$ connecting P_1 and P_2 is the most J-efficient among all connections $\tilde{\Gamma}(P_1, P_2)$ between P_1 and P_2 that are induced by a function v and are of the same homotopy type as $\Gamma(P_1, P_2)$, regardless of the length of the interval needed to parametrise the curve $\tilde{\Gamma}(P_1, P_2)$. As we mentioned in the introduction, this is analogous to the length minimising geodesics of Morse and Hedlund and the minimisers in the Aubry-Mather theory. The set of all (global) minimisers in $M(\boldsymbol{g}, \boldsymbol{p})$ will be denoted by $CM(\boldsymbol{g}, \boldsymbol{p})$.

Lemma 3.5 *Let $u \in M(\boldsymbol{g}, \boldsymbol{p})$ be a minimiser, then $u \in C^4(\mathbb{R})$ and u satisfies Equation* (3.2). *Moreover, u satisfies the relation $H(u, u', u'', u''') = 0$, i.e., the associated orbit lies on the energy level $H = 0$.*

Proof. From the definition of $M(\boldsymbol{g}, \boldsymbol{p})$, on any bounded interval $I \subset \mathbb{R}$ there exists $\epsilon_0(I) > 0$ sufficiently small such that $u + \phi \in M(\boldsymbol{g}, \boldsymbol{p})$ for all $\phi \in H_0^2(I)$ with $\|\phi\|_{H^2} < \epsilon \le \epsilon_0$. Therefore $J_I[u + \phi] \ge J_I[u]$ for all such functions ϕ, which implies that $dJ_I[u] = 0$ for any bounded interval $I \subset \mathbb{R}$, and thus u satisfies (3.2).

To prove the second statement we argue as follows. Since $u \in M(\boldsymbol{g}, \boldsymbol{p})$, there exists a bounded interval I such that $u'|_{\partial I} = 0$. Introducing the rescaled variable $s = t/T$ with $T = |I|$ and $v(s) = u(t)$, we have

$$J_I[u] = J[T, v] \equiv \int_0^1 \left[\frac{1}{T^3} \frac{\gamma}{2} v''^2 + \frac{1}{T} \frac{\beta}{2} v'^2 + T F(v) \right] ds,$$

which decouples u and T. Since $u'|_{\partial I} = 0$ we see from Definition 3.4 that $J[T \pm \epsilon, v] \ge J_T[u] = J[T, v]$. The smoothness of J in the variable $T > 0$ implies that $\frac{\partial}{\partial \tau} J[\tau, v]\Big|_{\tau = T} = 0$.

Differentiation yields, since H is constant along solutions (say $H(u,u',u'',u''')=E$),

$$\begin{aligned}\frac{\partial}{\partial\tau}J[\tau,v] &= \int_0^1\left[-\tau^{-4}\frac{3}{2}\gamma v''^2-\tau^{-2}\frac{\beta}{2}v'^2+F(v)\right]ds\\ &= \tau^{-1}\int_0^\tau\left[-\frac{3}{2}\gamma u''^2-\frac{\beta}{2}u'^2+F(u)\right]dt\\ &= -\tau^{-1}\int_0^\tau H(u,u',u'',u''')\,dt=-E.\end{aligned}$$

Thus $E=0$, and $H(u,u',u'',u''')=0$ for $t\in I$. This immediately implies that $H=0$ for all $t\in\mathbb{R}$. $\square$

The minimisers for J found in [89] also satisfy Definition 3.4, and we restate one of the main results of [89].

Proposition 3.6 *Suppose F is even and satisfies* (H)*, and $\beta,\gamma>0$ are chosen such that $u=\pm1$ are saddle-focus equilibria. Then for any terminated type $\boldsymbol{g}$ with parity either* 0 *or* 1 *there exists a minimiser $u\in M(\boldsymbol{g},\boldsymbol{p})$ of J.*

From Definition 3.2, the crossings of $u\in M(\boldsymbol{g},\boldsymbol{p})$ with ±1 are transverse and hence isolated. We adapt from [89], the notion of a normalised function with a few minor changes to reflect the fact that we now require every crossing of ±1 to be transverse.

Definition 3.7 *A function $u\in M(\boldsymbol{g},\boldsymbol{p})$ is* normalised *if, between each pair $u(a)$ and $u(b)$ of consecutive crossings of ±1, the restriction $u|_{(a,b)}$ is either monotone or $u|_{(a,b)}$ has exactly one local extremum.*

Clearly, the case of $u|_{(a,b)}$ being monotone can occur only between two crossings at different levels ±1, in which case we say that u has a *transition* on $[a,b]$.

Lemma 3.8 *If $u\in CM(\boldsymbol{g},\boldsymbol{p})$, then u is normalised.*

Proof. Since $u\in M(\boldsymbol{g},\boldsymbol{p})$, all crossings of $u=\pm1$ are transverse, i.e. $u'\neq0$. Thus for any critical point $t_0\in\mathbb{R}$, $u(t_0)\neq\pm1$, and the Hamiltonian relation from Lemma 3.5 and (3.3) implies that $\frac{\gamma}{2}u''(t_0)^2=F(u(t_0))>0$. Therefore u is a Morse function, and between any two consecutive crossings of ±1 there are only finitely many critical points. Now on any interval between consecutive crossings where u is not normalised, the clipping lemmas of Section 3 in [89] can be applied to obtain a more J-efficient function, which contradicts the fact that u is a minimiser. $\square$

3.3 Minimisers of arbitrary type

In this section we will introduce a notion of convergence of types which will be used in Section 3.5.1 to establish the existence of minimisers in every class $M(\boldsymbol{g},\boldsymbol{p})$ by building on the results proved in [89].

Definition 3.9 *Consider a sequence of types $(\boldsymbol{g}^n,\boldsymbol{p}^n)=\big((g_i^n)_{i\in I_n},\boldsymbol{p}^n\big)$ and a type $(\boldsymbol{g},\boldsymbol{p})=\big((g_i)_{i\in I},\boldsymbol{p}\big)$. The sequence $(\boldsymbol{g}^n,\boldsymbol{p}^n)$* limits *to the type $(\boldsymbol{g},\boldsymbol{p})$ if and only if there exist numbers $N_n\in2\mathbb{Z}$ such that $g^n_{i+N_n+p^n-p}\to g_i$ for all $i\in I$ as $n\to\infty$. We will abuse notation and write $(\boldsymbol{g}^n,\boldsymbol{p}^n)\to(\boldsymbol{g},\boldsymbol{p})$.*

We should point out that a sequence of types can limit to more than one type. For example, the sequence $(\boldsymbol{g}^n, 0) = \big((\infty, 2, 2, n, 4, 4, 4, 4, n, 2, 2, 2, \ldots), 0\big)$ limits to the types $\big((\infty, 2, 2, \infty), 0\big)$, $\big((\infty, 4, 4, 4, 4, \infty), 1\big)$ and $\big((\infty, 2, 2, 2, \ldots), 0\big)$.

Theorem 3.10 *Let $(\boldsymbol{g}^n, \boldsymbol{p}^n) \to (\boldsymbol{g}, \boldsymbol{p})$ and $u_n \in CM(\boldsymbol{g}^n, \boldsymbol{p}^n)$ with $\|u_n\|_{1,\infty} \leq C$ for all n. Then there exists a subsequence u_{n_k} such that $u_{n_k} \to \hat{u} \in M(\boldsymbol{g}, \boldsymbol{p})$ in $C^4_{\mathrm{loc}}(\mathbb{R})$, and $\hat{u}$ is a minimiser in the sense of Definition 3.4, i.e. $\hat{u} \in CM(\boldsymbol{g}, \boldsymbol{p})$.*

Proof. This proof requires arguments developed in [89] to which the reader is referred for certain details. The idea is to take the limit of u_n restricted to bounded intervals. We define the numbers N_n as in Definition 3.9, and we denote the convex hull of A_i by I_i. Due to translation invariance we can pin the functions u_n so that $u_n(0) = (-1)^{\boldsymbol{p}+1}$, which is the beginning of the transition between $I^n_{N_n+\boldsymbol{p}^n-\boldsymbol{p}}$ and $I^n_{1+N_n+\boldsymbol{p}^n-\boldsymbol{p}}$. Due to the assumed a priori bound and interpolation estimates which can be found in the appendix to [96], there is enough regularity to yield a limit function $\hat{u}$ as a C^4_{loc}-limit of u_n, after passing to a subsequence. Moreover, $\hat{u}$ satisfies the differential equation (3.2) on $\mathbb{R}$. The question that remains is whether $\hat{u} \in M(\boldsymbol{g}, \boldsymbol{p})$.

To simplify notation we will now assume that $N_n = 0$ and $\boldsymbol{p}^n = \boldsymbol{p} = 0$. Fixing a small $\delta > 0$, we define $I^n_i(\delta) \supset I^n_i$ as the smallest interval containing I^n_i such that $u|_{\partial I^n_i(\delta)} = (-1)^{i+1} - (-1)^{i+1}\delta$ (if $\boldsymbol{g}$ is a (semi-)terminated type then $I^n_i(\delta)$ may be a half-line). The interval of transition between $I^n_i(\delta)$ and $I^n_{i+1}(\delta)$ is denoted by $L^n_i(\delta)$. To see that $\hat{u} \in M(\boldsymbol{g}, \boldsymbol{p})$, one has to eliminate the two possibilities that a priori may lead to the loss or creation of crossings in the limit so that $\hat{u} \notin M(\boldsymbol{g}, \boldsymbol{p})$: the distance between two consecutive crossings in u_n could grow without bound or $\hat{u}$ could possess tangencies at $u = \pm 1$.

Due to the a priori estimate in $W^{1,\infty}$ we have the following bounds on J:

$$J[u_n|_{I^n_i(\delta)}] \leq C \qquad \text{and} \qquad J[u_n|_{L^n_i(\delta)}] \leq C', \tag{3.4}$$

where C and C' are independent of n and i. Indeed, note that for n large enough the homotopy type of u_n on the intervals $I^n_i(\delta)$ is constant by the definition of convergence of types. Since the functions u_n are minimisers, $J[u_n|_{I^n_i(\delta)}]$ is less than the action of any test function of this homotopy type satisfying the a priori bounds on u and u' on $\partial I^n_i(\delta)$ (see [89, Section 6] for a similar test function argument). The estimate $|L^n_i(\delta)| \leq C(\delta)$ is immediately clear from Lemma 5.1 of [89]. We now need to show that the distance between two crossings of $(-1)^{i+1}$ within the interval $I^n_i(\delta)$ cannot tend to infinity.

First we will deal with the case when g^n_i is finite for all n. Suppose that the distance between consecutive crossings of $(-1)^{i+1}$ in $I^n_i(\delta)$ tends to infinity as $n \to \infty$. Due to (3.4) and Lemma 3.8, minimisers have exactly one extremum between crossings of $(-1)^{i+1}$ for any $\epsilon > 0$, and hence there exist subintervals $K_n \subset I^n_i(\delta)$ with $|K_n| \to \infty$, such that $0 < |u_n - (-1)^{q_n}| < \epsilon$ on K_n where $q_n \in \{0, 1\}$, and $|u'|_{\partial K_n}| < \epsilon$. Taking a subsequence we may assume that q_n is constant.

We begin by considering the case where $q_n = i + 1$. Now ϵ can be chosen small enough, so that the local theory in [89] is applicable in K_n. If $|K_n|$ becomes too large then u_n can be replaced by a function with lower action and with many crossings of $(-1)^{i+1}$. Subsequently, redundant crossings can be clipped out, thereby lowering the action. This implies that u_n is not a minimiser in the sense of Definition 3.4, a contradiction.

The case where $q_n = i$ must be dealt with in a different manner. First, there are unique points $t_n \in K_n$ such that $u_n'(t_n) = 0$, and for these points $u_n(t_n) \to (-1)^i$ as $|K_n| \to \infty$. Let $u_n(s_n)$ be the first crossing of $(-1)^{i+1}$ to the left of K_n. Taking the limit (along subsequences) of $u_n(t - s_n)$ we obtain a limit function $\widetilde{u}$ which is a solution of (3.2). If $|t_n - s_n|$ is bounded then $\widetilde{u}$ has a tangency to $u = (-1)^i$ at some $t_* \in \mathbb{R}$. All u_n lie in $\{H = 0\}$ (see (3.3)) and so does $\widetilde{u}$, hence $\widetilde{u}''(t_*) = 0$. Moreover $\widetilde{u}'''(t_*) = 0$, because $\widetilde{u}(t_*)$ is an extremum. By uniqueness of the initial value problem this implies that $\widetilde{u} \equiv (-1)^i$, contradicting the fact that $\widetilde{u}(0) = (-1)^{i+1}$. If $|t_n - s_n| \to \infty$, then $\widetilde{u}$ is a monotone function on $[0, \infty)$, tending to $(-1)^i$ as $x \to \infty$, and its derivatives tend to zero (see Lemma 2.14 or [96, Lemma 1 Part (ii)] for details). This contradicts the saddle-focus nature of the equilibrium point.

In the case that $g_i^n = \infty$ we remark that (3.4) also holds when I_i^n is a half-line. It follows from the estimates in Lemma 5.1 in [89] that $u_i^n \to (-1)^{i+1}$ as $x \to \infty$ or $x \to -\infty$ (whichever is applicable). From the local theory in Section 4 of [89] and the fact that u_n is a minimiser, it follows that the derivatives of u_n tend to zero. The analysis above concerning the intervals K_n and the clipping of redundant oscillations now goes on unchanged.

We have shown that the distance between two crossings of ± 1 is bounded from above. Next we have to show that the limit function has only transverse crossings of ± 1. This ensures that no crossings are lost in the limit. If $\hat{u}$ were tangent to $(-1)^{i+1}$ in I_i, then we could construct a function in $v \in M(\boldsymbol{g}, \boldsymbol{p})$ in the same way as in [89, Theorem 5.5] by replacing tangent pieces by more J-efficient local minimisers and by clipping. The function v still has a lower action than $\hat{u}$ on a slightly larger interval (the limit function $\hat{u}$ also obeys (3.4), so that the above clipping arguments still apply). Since $u_n \to \hat{u}$ in C^4_{loc} it follows that $J_I[u_n] \to J_I[u]$ on bounded intervals I. This then implies that for n large enough the function u_n is not a minimiser in the sense of Definition 3.4, which is a contradiction.

The limit function $\hat{u}$ could also be tangent to $(-1)^i$ for some $t_0 \in I_i$. As before, such tangencies satisfy $\hat{u}(t_0) - (-1)^i = \hat{u}'(t_0) = \hat{u}''(t_0) = \hat{u}'''(t_0) = 0$, which leads to a contradiction the uniqueness of the initial value problem.

Finally, crossings of ± 1 cannot accumulate since this would imply that at the accumulation point all derivatives would be zero, leading to the same contradiction as above. In particular, if $g_i^n \to \infty$ for some i, then $|I_i^n| \to \infty$ and the crossings in A_j^n for $j > i$ move off to infinity and do not show in $\hat{u}$, which is compatible with the convergence of types.

We have now proved that $\hat{u} \in M(\boldsymbol{g}, \boldsymbol{p})$, and since $\hat{u}$ is the C^4_{loc}-limit of minimisers, $\hat{u}$ is also a minimiser in the sense of Definition 3.4. $\square$

Remark 3.11 It follows from regularity estimates on bounded solutions of (3.2), see e.g. [96, Theorem 3], that in the theorem above we in fact only need an L^∞-bound on the sequence u_n. •

Remark 3.12 It follows from the proof of Theorem 3.10 that there exists a constant $\delta_0 > 0$ such that for all uniformly bounded minimisers $u(t)$ it holds that $|u(t) - (-1)^{i+p}| > \delta$ for all $t \in I_i$ and all $i \in I$. This means that the uniform separation property discussed in [89] is uniformly satisfied by all minimisers. •

3.4 Periodic minimisers

A bi-infinite type $\boldsymbol{g}$ is *periodic* if there exists an integer n such that $\sigma^n(\boldsymbol{g}) = \boldsymbol{g}$. The (natural) definition of the period of $\boldsymbol{g}$ is $\min\{n \in 2\mathbb{N} \,|\, \sigma^n(\boldsymbol{g}) = \boldsymbol{g}\}$. We will write $\boldsymbol{g} = \langle \boldsymbol{r} \rangle$ where $\boldsymbol{r} = (g_1, ..., g_n)$ and n is even. Cyclic permutations of $\boldsymbol{r}$, with possibly a flip of $\boldsymbol{p}$, give rise to the same function class $M(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$. In reference to the type $\langle \boldsymbol{r} \rangle$ with parity $\boldsymbol{p}$ we will use the notation $(\boldsymbol{r}, \boldsymbol{p})$. Any such type pair $(\boldsymbol{r}, \boldsymbol{p})$ can formally be associated with a homotopy class in $\pi_1(\mathcal{P}, \mathbf{0})$ in the following way. Let e_0 and e_1 be the clockwise oriented circles of radius one centred at $(1,0)$ and $(-1,0)$ respectively, so that $[e_0]$ and $[e_1]$ are generators for $\pi_1(\mathcal{P}, \mathbf{0})$. Defining $\theta(\boldsymbol{r}, \boldsymbol{p}) = e_{\tau^n(\boldsymbol{p})}^{r_n/2} \cdots e_{\boldsymbol{p}}^{r_1/2}$, the map $\theta : \bigcup_{k \geq 1} 2\mathbb{N}^{2k} \times \{0,1\} \to \pi_1(\mathcal{P}, \mathbf{0})$ is an injection, and we define $\pi_1^+(\mathcal{P}, \mathbf{0})$ to be the image of θ in $\pi_1(\mathcal{P}, \mathbf{0})$. Powers of a type pair $(\boldsymbol{r}, \boldsymbol{p})^k$ for $k \geq 1$ are defined by concatenation of $\boldsymbol{r}$ with itself k times, which is equivalent to $(\boldsymbol{r}, \boldsymbol{p})^k = \theta^{-1}((\theta(\boldsymbol{r}, \boldsymbol{p}))^k)$.

Definition 3.13 *Two pairs $(\boldsymbol{r}, \boldsymbol{p})$ and $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}})$ are* equivalent *if there are integers $p, q \in \mathbb{N}$ such that $(\boldsymbol{r}, \boldsymbol{p})^p = (\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}})^q$ up to cyclic permutations. This relation, $(\boldsymbol{r}, \boldsymbol{p}) \sim (\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}})$, is an equivalence relation.*

Example: if $(\boldsymbol{r}, \boldsymbol{p}) = \big((2,4,2,4), 0\big)$ and $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) = \big((4,2,4,2,4,2), 1\big)$, then $\theta(\boldsymbol{r}, \boldsymbol{p})^3 = \theta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}})^2$. The equivalence class of $(\boldsymbol{r}, \boldsymbol{p})$ is denoted by $[\boldsymbol{r}, \boldsymbol{p}]$. A type $(\boldsymbol{r}, \boldsymbol{p})$ is a minimal representative for $[\boldsymbol{r}, \boldsymbol{p}]$ if for each $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) \in [\boldsymbol{r}, \boldsymbol{p}]$ there is $k \geq 1$ such that $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) = (\boldsymbol{r}, \boldsymbol{p})^k$ up to cyclic permutations. A minimal representative is unique up to cyclic permutations. It is clear that in the representation of a periodic type $\boldsymbol{g} = \langle \boldsymbol{r} \rangle$, the type $\boldsymbol{r}$ is minimal if the length of $\boldsymbol{r}$ is the minimal period. Due to the above equivalences we now have that

$$M(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) = M(\langle \hat{\boldsymbol{r}} \rangle, \hat{\boldsymbol{p}}), \quad \text{for all } (\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) \in [\boldsymbol{r}, \boldsymbol{p}].$$

It is not a priori clear that minimisers in $M(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ are periodic. However, we will see that among these minimisers, periodic minimisers can always be found.

For a given periodic type $\langle \boldsymbol{r} \rangle$ we consider the subset of periodic functions in $M(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$,

$$M_{\text{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) = \{u \in M(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) \,|\, u \text{ is periodic}\}.$$

For any $u \in M_{\text{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ and a period T of u, $\Gamma(u|_{[0,T]})$ is a closed loop in $\mathcal{P}$ whose homotopy type corresponds to a nontrivial element of $\pi_1^+(\mathcal{P}, \mathbf{0})$. In this correspondence there is no natural choice of a basepoint. For specificity, we will describe how to make the correspondence with the origin as the basepoint, and thereafter we omit it from the notation. Translate u so that $u(0) = 0$. Let $\gamma : [0,1] \to \mathcal{P}$ be the linepiece from $\mathbf{0}$ to $(0, u'(0))$, and let $\gamma^*(t) = \gamma(1-t)$. Let $\tilde{\Gamma}(u|_{[0,T]}) \stackrel{\text{def}}{=} \gamma^* \circ \Gamma(u|_{[0,T]}) \circ \gamma$, then $\big[\tilde{\Gamma}(u|_{[0,T]})\big] \in \pi_1^+(\mathcal{P}, \mathbf{0})$. Now define $\big[\Gamma(u|_{[0,T]})\big] \equiv \big[\tilde{\Gamma}(u|_{[0,T]})\big]$.

For any $u \in M_{\text{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ there thus exists a pair $\theta^{-1}\big[\Gamma(u|_{[0,T]})\big] = (\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) \in [\boldsymbol{r}, \boldsymbol{p}]$, with $\hat{\boldsymbol{r}} = \boldsymbol{r}^k$ for some $k \geq 1$. Therefore we define for any $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) \in [\boldsymbol{r}, \boldsymbol{p}]$

$$M_{\text{per}}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) = \{u \in M_{\text{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) \;|\; \big[\Gamma(u|_{[0,T]})\big] \sim \theta(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) \in \pi_1^+(\mathcal{P}) \text{ for a period } T \text{ of } u\}.$$

The type $\hat{\boldsymbol{r}} = \boldsymbol{g}(u|_{[0,T]})$ is the homotopy type of u relative to a period T. This type has an even number of entries. It follows that $M_{\text{per}}(\boldsymbol{r}, \boldsymbol{p}) \subset M_{\text{per}}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}})$ for all $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) = (\boldsymbol{r}, \boldsymbol{p})^k$, $k \geq 1$. Furthermore $M_{\text{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) = \cup_{(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}}) \in [\boldsymbol{r}, \boldsymbol{p}]} M_{\text{per}}(\hat{\boldsymbol{r}}, \hat{\boldsymbol{p}})$. In order to get a better understanding of periodic minimisers in $M(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ we consider the following minimisation problem:

$$\mathcal{J}_{\text{per}}(\boldsymbol{r},\boldsymbol{p}) = \inf_{u \in M_{\text{per}}(\boldsymbol{r},\boldsymbol{p})} J_T[u] = \inf_{\substack{M^T_{\text{per}}(\boldsymbol{r},\boldsymbol{p}) \\ T \in \mathbb{R}^+}} J_T[u], \tag{3.5}$$

where J_T is action given in (3.1) integrated over one period of length T, and $M^T_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ is the set of T-periodic functions $u \in M_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ for which $\boldsymbol{g}(u|_{[0,T]}) = \boldsymbol{r}$. Note that T is not necessarily the minimal period, unless $\boldsymbol{r}$ is a minimal representative for $[\boldsymbol{r}]$. It is clear that for $\gamma, \beta > 0$ the infima $\mathcal{J}_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ are well-defined and are nonnegative for any homotopy type $\boldsymbol{r}$. At this point it is not clear, however, that the infima $\mathcal{J}_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ are attained for all homotopy types $\boldsymbol{r}$. We will prove in Section 3.5 that existence of minimisers for (3.5) can be obtained using the existence of homoclinic and heteroclinic minimisers already established in [89].

Lemma 3.14 *If $\mathcal{J}_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ is attained for some $u \in M_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$, then $u \in C^4(\mathbb{R})$ and satisfies* (3.2). *Moreover, since u is minimal with respect to T, we have $H(u,u',u'',u''') = 0$, i.e., the associated periodic orbit lies on the energy surface $H = 0$.*

Proof. Since $\mathcal{J}_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ is attained by $u \in M_{\text{per}}(\boldsymbol{r},\boldsymbol{p})$ for some period T, we have that $J_T[u + \phi] - J_T[u] \geq 0$ for all $\phi \in H^2(S^1, T)$ with $\|\phi\|_{H^2} \leq \epsilon$, where $\epsilon > 0$ is sufficiently small. This implies that $dJ_T[u] = 0$, and thus u satisfies (3.2). The second part of this proof is analogous to the proof of Lemma 3.5. □

We now introduce the following notation:

$$\begin{aligned}
CM(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) &= \{u \in M(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) \,|\, u \text{ is a minimiser in the sense of Definition 3.4}\}, \\
CM_{\text{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) &= \{u \in CM(\langle \boldsymbol{r} \rangle, \boldsymbol{p}) \,|\, u \text{ is periodic}\}, \\
CM_{\text{per}}(\boldsymbol{r}, \boldsymbol{p}) &= \{u \in M_{\text{per}}(\boldsymbol{r},\boldsymbol{p}) \,|\, u \text{ is a minimiser for } \mathcal{J}_{\text{per}}(\boldsymbol{r},\boldsymbol{p})\}.
\end{aligned}$$

3.4.1 Existence of periodic minimisers of type $\boldsymbol{r} = (2,2)^k$

If we seek periodic minimisers of type $\boldsymbol{r} = (2,2)^k$, the uniform separation property for minimising sequences (see Section 5 in [89]) is satisfied in the class $M_{\text{per}}(\boldsymbol{r})$. Note that the parity is omitted because it does not distinguish different homotopy types here. The uniform separation property as defined in [89] prevents minimising sequences from crossing the boundary of the given homotopy class. For any other periodic type the uniform separation property is not a priori satisfied. For the sake of simplicity we begin with periodic minimisers of type $(2,2)$ and minimise J in the class $M_{\text{per}}((2,2))$.

Minimising sequences can be chosen to be normalised due to the following lemma, which we state without proof; the proof is analogous to Lemma 3.5 in [89].

Lemma 3.15 *Let $u \in M_{\text{per}}((2,2))$ and T be a period of u. Then for every $\epsilon > 0$ there exists a normalised function $w \in M_{\text{per}}((2,2))$ with period $T' \leq T$ such that $J_{T'}[w] \leq J_T[u] + \epsilon$.*

The goal of this subsection is to prove that when F satisfies (H) and $\beta, \gamma > 0$ are such that $u = \pm 1$ are saddle-foci, then $\mathcal{J}_{\text{per}}((2,2))$ is attained, by Theorem 3.17 below. The proof relies on the local structure of the saddle-focus equilibria $u = \pm 1$ and is a modification of arguments in [89]; hence we will provide only a brief argument. The reader is referred to [89] for further details.

In preparation for the proof of Theorem 3.17, we fix $\tau_0 > 0$ and $\delta > 0$ so that the conclusion of Theorem 4.2 in [89] holds, i.e., the characterisation of the oscillatory behaviour of solutions near the saddle-focus equilibria $u = \pm 1$ holds[2]. Let $u \in M^T_{\text{per}}((2,2))$ be normalised, and let t_0 be such that $u(t_0) = 0$. Then t_0 is part of a transition from ∓ 1 to ± 1. Assume without loss of generality that this transition is from -1 to 1. Define $t_- = \sup\{t < t_0 : |u(t) + 1| < \delta\}$ and $t_+ = \inf\{t > t_0 : |u(t) - 1| < \delta\}$. Let $S(u) = \{t : |u(t) \pm 1| < \delta\}$ and $B[u,T] = |S(u) \cap [t_+, t_- + T]|$, and note that $[t_0, t_0 + T] = \{S(u) \cap [t_+, t_- + T]\} \cup \{S(u)^c \cap [t_0, t_0 + T]\}$. With these definitions we can establish the following estimate (cf. Lemma 5.4 in [89]). For all $u \in M_{\text{per}}((2,2))$ with $J_T[u] \leq \mathcal{J}_{\text{per}}((2,2)) + \epsilon_0$, and ϵ_0 sufficiently small

$$\|u\|^2_{H^2} \leq C(1 + \mathcal{J}_{\text{per}}((2,2)) + B[u,T]). \tag{3.6}$$

First, $\|u'\|^2_{H^1} \leq C(\mathcal{J}_{\text{per}}((2,2)) + \epsilon_0)$, and second if $|u \pm 1| > \delta$ then $F(u) \geq \eta^2 u^2$ for some small $\eta > 0$, which implies that $\|u\|^2_{L^2} \leq 1/\eta^2 \int_{t_0}^{t_0+T} F(u)\,dt + (1+\delta)^2 B[u,T] \leq C(J_T[u] + B[u,T])$. Combining these two estimates proves (3.6).

For functions $u \in M^T_{\text{per}}((2,2))$ that satisfy $J_T[u] \leq \mathcal{J}_{\text{per}}((2,2)) + 1$, it follows from [89, Lemma 5.1] that there exist constants T_1 and T_2 (uniform in u) such that $T_2 \geq |S(u)^c \cap [t_0, t_0 + T]| \geq T_1 > 0$ and thus $T > T_1$. The next step is to give an a priori upper bound on T by considering the minimisation problem (cf. Section 5 in [89])

$$B_\epsilon = \inf\{B[u,T] \mid u \in M^T_{\text{per}}((2,2)) \text{ normalised}, \ T \in \mathbb{R}^+ \text{ and } J_T[u] \leq \mathcal{J}_{\text{per}}((2,2)) + \epsilon\}.$$

Lemma 3.16 *There exists a constant $K = K(\tau_0) > 0$ such that $B_\epsilon \leq K$ for all $0 < \epsilon < \epsilon_0$. Moreover, setting $T_0 \equiv K + T_2$, for any $0 < \epsilon < \epsilon_0$ there is a normalised $u \in M^T_{\text{per}}((2,2))$ with $J_T[u] \leq \mathcal{J}_{\text{per}}((2,2)) + 2\epsilon$ and $T_1 < T \leq T_0$.*

Proof. Let $(u_n, T_n) \in M^{T_n}_{\text{per}}((2,2)) \times \mathbb{R}^+$ be a minimising sequence for B_ϵ, with normalised functions u_n. As in the proof of Theorem 5.5 of [89], in the weak limit this yields a pair $(\hat{u}, \hat{T})$ such that $B[\hat{u}, \hat{T}] \leq B_\epsilon$. We now define $K((2,2), \tau_0) = 8((2\tau_0 + 2) + 2)$. This gives two possibilities for $B[\hat{u}, \hat{T}]$, either $B[\hat{u}, \hat{T}] > K$ or $B[\hat{u}, \hat{T}] \leq K$. If the former is true then we can construct (see Theorem 5.5 of [89]) a pair $(\hat{v}, \hat{T}') \in M^{\hat{T}'}_{\text{per}}((2,2)) \times \mathbb{R}^+$, with $\hat{v}$ normalised, such that

$$J_{\hat{T}'}[\hat{v}] < J_{\hat{T}}[\hat{u}] \leq \mathcal{J}_{\text{per}}((2,2)) + \epsilon \quad \text{and} \quad B[\hat{v}, \hat{T}'] < B[\hat{u}, \hat{T}] \leq B_\epsilon,$$

which is a contradiction excluding the first possibility. In the second case, where $B[\hat{u}, \hat{T}] \leq K$, we can construct a pair $(\hat{v}, \hat{T}')$ with $\hat{v}$ normalised such that

$$J_{\hat{T}'}[\hat{v}] < J_{\hat{T}}[\hat{u}] + \epsilon \leq J_{\text{per}}((2,2)) + 2\epsilon, \quad \text{and} \quad B[\hat{v}, \hat{T}'] < B[\hat{u}, \hat{T}] \leq K,$$

which implies that $T_1 < \hat{T}' < K + T_2 = T_0$, and concludes the proof. For details concerning these constructions, see Theorem 5.5 in [89]. □

Theorem 3.17 *Suppose that F satisfies* (H) *and $\beta, \gamma > 0$ are such that $u = \pm 1$ are saddle-foci, then $\mathcal{J}_{\text{per}}((2,2)^k)$ is attained for any $k \geq 1$. Moreover, the projection of any minimiser in $CM_{\text{per}}((2,2))$ onto the (u,u')-plane is a simple closed curve.*

Proof. By Lemma 3.16, we can choose a minimising sequence $(u_n, T_n) \in M^{T_n}_{\text{per}}((2,2)) \times \mathbb{R}^+$, with u_n normalised and with the additional properties that $\|u_n\|_{H^2} \leq C$ and $T_1 \leq T_n \leq$

[2]The characterisation is the following: let $X_T = \{v \in H^2[0,T] \,|\, (v(0), v'(0)) = \bar{x}, (v(T), v(T)) = \bar{y}\}$. For $\|\bar{x}\|, \|\bar{y}\| \leq \delta_0$ the unique global minimiser $\hat{v}$ of J on X_T changes sign in any subinterval of length τ_0 in $[0,T]$ for $T \geq 1$.

T_0. Since the uniform separation property is satisfied for the type $(2,2)$ this leads to a minimising pair $(\hat{u}, \hat{T})$ for (3.5) by following the proof of Theorem 2.2 in [89]. As for the existence of periodic minimisers of type $\boldsymbol{r} = (2,2)^k$ the uniform separation property is automatically satisfied and the above steps are identical.

Lemma 3.8 yields that minimisers are normalised functions and the projection of a normalised function in $M_{\text{per}}((2,2))$ is a simple closed curve in the (u, u')-plane. □

We would like to have the same theorem for arbitrary periodic types $\langle \boldsymbol{r} \rangle$. For homotopy types that satisfy the uniform separation property the analogue of Theorem 3.17 can be proved. However, in Section 3.5 we will prove a more general result using the information about the minimisers with terminated types (homoclinic and heteroclinic minimisers) which was obtained in [89].

Remark 3.18 The existence of a $(2,2)$-type minimiser is proved here in order to obtain a priori $W^{1,\infty}$-estimates for all minimisers (Section 3.5). However, if F satisfies the additional hypothesis that $F(u) \sim |u|^s$, $s > 2$ as $|u| \to \infty$, then such estimates are automatic (cf. Chapter 5 or [96]). In that case the existence of a minimiser of type $(2,2)$ follows from Theorem 3.26 below. To prove existence of minimisers of arbitrary type $\boldsymbol{r}$ we will use an analogue of Theorem 3.26 (see Lemma 3.33 and Theorem 3.34 below). •

3.4.2 Characterisation of minimisers of type $g = \langle (2,2) \rangle$

Periodic minimisers associated with $[e_0]$ or $[e_1]$ are the constant solutions $u = -1$ and $u = 1$ respectively. The simplest nontrivial periodic minimisers are those of type $\boldsymbol{r} = (2,2)^k$, i.e. $\boldsymbol{r} \in [(2,2)]$. These minimisers are crucial to the further analysis of the general case. The type $\boldsymbol{r} = (2,2)$ is a minimal type (associated with $[e_1 e_0]$), and we want to investigate the relation between minimisers in $M(\langle (2,2) \rangle)$ and periodic minimisers of type $(2,2)^k$.

Considering curves in the configuration space $\mathcal{P}$ is a convenient method for studying minimisers of type $(2,2)$. For example, minimisers in $CM(\langle (2,2) \rangle)$ and $CM_{\text{per}}((2,2))$ all satisfy the property that they do not intersect the line segment $L = (-1,1) \times \{0\}$ in $\mathcal{P}$. If other homotopy types $\boldsymbol{r}$ are considered, i.e. $\boldsymbol{r} \notin [(2,2)]$, then minimisers represented as curves in $\mathcal{P}$ necessarily have self-intersections and they must intersect the segment L, which makes their comparison more complicated. We will come back to this problem in Section 3.5. Note that for a C^1-function u the associated curve $\Gamma(u)$ is a closed loop if and only if u is a periodic function.

Lemma 3.19 *For any non-periodic minimiser $u \in CM(\langle (2,2) \rangle)$ and any bounded interval I the curve $\Gamma[u|_I]$ has only a finite number of self-intersections. For periodic minimisers $u \in CM_{\text{per}}(\langle (2,2) \rangle)$ this property holds when the length of I is smaller than the minimal period.*

Proof. Fix a time interval $I = [0, T]$. If u is periodic, T should be chosen smaller than the minimal period of u. By contradiction, suppose that $P = (u_0, u_0')$ is an accumulation point of self-intersections of $u|_I$. Then P is a self-intersection point, and there exists a monotone sequence of times $\tau_n \in I$ converging to t_0 such that $\Gamma(u(\tau_n))$ are self-intersection points and $\Gamma(u(t_0)) = P$. Also there exists a corresponding sequence $\sigma_n \in I$ with $\sigma_n \neq \tau_n$ such that $\Gamma(u(\tau_n)) = \Gamma(u(\sigma_n))$. Choosing a subsequence if necessary, $\sigma_n \to s_0$ monotonically.

Since u is a minimiser in $CM(\langle(2,2)\rangle)$, the intervals $[\sigma_n, \tau_n]$ must contain a transition, and hence $|\tau_n - \sigma_n| > T_0$ for some $T_0 > 0$. Therefore, $s_0 \neq t_0$, and we will assume that $s_0 < t_0$ (otherwise change labels). The homotopy type of $\Gamma(u|_{[s_0,t_0]})$ is $(2,2)^k$ for some $k \geq 1$ (since I is bounded).

Assume that σ_n and τ_n are increasing; the other cases are similar. Using the times $\sigma_n < s_0 < \tau_n < t_0$, for δ sufficiently small the curve $\Gamma_* = \Gamma[u|_{[\sigma_n-\delta,t_0+\delta]}]$ can be decomposed as $\Gamma_* = a \circ \gamma_2 \circ \gamma \circ \gamma_1 \circ b$, where $b = \Gamma(u|_{[\sigma_n-\delta,\sigma_n]})$, $\gamma_1 = \Gamma(u|_{[\sigma_n,s_0]})$, $\gamma = \Gamma(u|_{[s_0,\tau_n]})$, $\gamma_2 = \Gamma(u|_{[\tau_n,t_0]})$, and $a = \Gamma(u|_{[t_0,t_0+\delta]})$. For n sufficiently large, γ_1 and γ_2 have the same homotopy type, and $\gamma_1 \neq \gamma_2$, since otherwise u would be periodic with period smaller than $t_0 - \sigma_n < T$. We can now construct two more paths

$$\Gamma_1 = a \circ \gamma_1 \circ \gamma \circ \gamma_1 \circ b \quad \text{and} \quad \Gamma_2 = a \circ \gamma_2 \circ \gamma \circ \gamma_2 \circ b$$

which have the same homotopy type for n sufficiently large. Since $J[\Gamma_*]$ is minimal, $J[\Gamma_1] \geq J[\Gamma_*]$ and $J[\Gamma_2] \geq J[\Gamma_*]$, and thus $J[\gamma_1] \geq J[\gamma_2]$ and $J[\gamma_2] \geq J[\gamma_1]$ which implies that $J[\gamma_1] = J[\gamma_2]$. Therefore $J[\Gamma_*] = J[\Gamma_1] = J[\Gamma_2]$, and Γ_1, Γ_2 and Γ_* are all distinct minimisers with the same homotopy type and same boundary conditions. Since these curves all coincide along γ, the uniqueness of the initial value problem is contradicted. An argument very similar to the one above is also used in the proof of Lemma 3.24 and is demonstrated in Figure 3.1. □

Lemma 3.20 *For* $\boldsymbol{r} = (2,2)^k$ *with* $k > 1$ *one has that* $CM_{\text{per}}(\boldsymbol{r}) = CM_{\text{per}}((2,2))$*, and* $\mathcal{J}_{\text{per}}(\boldsymbol{r}) = k \cdot \mathcal{J}_{\text{per}}((2,2))$.

Proof. Let $u \in CM_{\text{per}}(\boldsymbol{r})$ with $\boldsymbol{r} = (2,2)^k$ for $k > 1$, and let T be the period (one may assume without loss of generality that T is the minimal period) such that the associated curve in $\mathcal{P}$, $\Gamma(u|_{[0,T]})$, has the homotopy type of $\theta((2,2)^k)$. First we will prove that $\Gamma(u|_{[0,T]})$ is a simple closed curve in $\mathcal{P}$, and hence $u \in M_{\text{per}}((2,2))$. Suppose not, then by Lemma 3.19 the curve $\Gamma(u|_{[0,T]})$ can be fully decomposed into k distinct simple closed curves Γ_i for $i = 1, \ldots, k$ (just call the inner loop Γ_1, cut it out, and call the new inner loop Γ_2, and so on). Denote by J_i the action associated with loop Γ_i, then $\sum_i J_i = J_T[u]$. Let $v_i \in M_{\text{per}}((2,2)^k)$ be the function obtained by pasting together k copies of $u|_{\Gamma_i}$. If v_i were a minimiser in $M_{\text{per}}((2,2)^k)$, then by Lemma 3.14 the functions u and v_i would be distinct solutions to the differential equation (3.2) which coincide over an interval. This would contradict the uniqueness of solutions of the initial value problem, and hence v_i is not a minimiser, i.e. $J_{\hat{T}}[v_i] = k \cdot J_i > \mathcal{J}_{\text{per}}((2,2)^k)$. Consequently $\mathcal{J}_{\text{per}}((2,2)^k) = \sum_i J_i > \mathcal{J}_{\text{per}}((2,2)^k)$, which is a contradiction. Thus $u \in M_{\text{per}}((2,2))$, and $\Gamma(u|_{[0,T]})$ is a simple loop traversed k times.

Now we will show that $u \in CM_{\text{per}}((2,2))$. Since $\Gamma(u)$ is the projection of a function into the (u, u')-plane, u traverses the loop once over the interval $[0, \frac{T}{k}]$, and $\mathcal{J}_{\text{per}}((2,2)^k) = k \cdot J_{T/k}[u]$. Suppose $J_{T/k}[u] > \mathcal{J}_{\text{per}}((2,2))$, then we can construct a function in $M_{\text{per}}((2,2)^k)$ with action less than $J[u] = \mathcal{J}_{\text{per}}((2,2)^k)$ by gluing together k copies of a minimiser in $M_{\text{per}}((2,2))$, which is a contradiction. □

Lemma 3.21 *For any* $k \geq 1$, $CM_{\text{per}}((2,2)^k) = CM_{\text{per}}((2,2)) = CM_{\text{per}}(\langle(2,2)\rangle)$.

Proof. We have already shown in Lemma 3.20 that $CM_{\text{per}}((2,2)^k) = CM_{\text{per}}((2,2))$. We first prove that $CM_{\text{per}}((2,2)) \subset CM_{\text{per}}(\langle(2,2)\rangle)$. Let $u \in CM_{\text{per}}((2,2))$ have period T. Suppose $u \notin CM_{\text{per}}(\langle(2,2)\rangle)$. Then there exist two points $\Gamma(u(t_1)) = P_1$ and $\Gamma(u(t_2)) = P_2$ on

$\Gamma(u)$ such that the curve γ between P_1 and P_2 obtained by following $\Gamma(u)$ is not minimal. Replacing γ by a curve with smaller action and the same homotopy type yields a function $v \in M_{\text{per}}(\langle(2,2)\rangle)$ for which $J_{[t_1,t_2]}[v] < J_{[t_1,t_2]}[u]$. Choose $k > 0$ such that $kT > t_2 - t_1$. Then u is a minimiser in $CM_{\text{per}}((2,2)^k) = CM_{\text{per}}((2,2))$, which contradicts the fact that $J_{[t_1,t_2]}[v] < J_{[t_1,t_2]}[u]$.

To finish the proof of the lemma we show that $CM_{\text{per}}(\langle(2,2)\rangle) \subset CM_{\text{per}}((2,2))$. Let $u \in CM_{\text{per}}(\langle(2,2)\rangle)$ have period T. Let $\Gamma(u|_{[0,T]})$ be the associated closed curve in $\mathcal{P}$ and let ω be its winding number with respect to the segment L. Suppose $J_T[u] > \mathcal{J}_{\text{per}}((2,2)^\omega) = \omega \cdot \mathcal{J}_{\text{per}}(2,2)$. This implies the existence of a function $v \in M_{\text{per}}((2,2)^\omega)$ and a period $\hat{T}$ such that $J_{\hat{T}}[v] < J_T[u]$. Choose a time $t_0 \in [0,T]$ such that $u(t_0) = 1$ and $u'(t_0) > 0$. Let $P_0 = (1, u'(t_0)) \in \mathcal{P}$. There exists a $\delta > 0$ sufficiently small such that $u(t_0 \pm \delta) > 0$, $u'(t_0 \pm \delta) > 0$, and u does not cross 1 in $[t_0 - \delta, t_0 + \delta]$ except at t_0. Let P_1 and P_2 denote the points $(u(t_0 \mp \delta), u'(t_0 \mp \delta))$. Let γ denote the piece of the curve $\Gamma(u)$ from P_1 to P_2 and γ^* the curve tracing $\Gamma(u)$ backward in time from P_2 to P_1. Now choose a point P_3 on $\Gamma(v)$ for which $v = 1$ and $v' > 0$. We can easily construct cubic polynomials p_1 and p_2 for which the curve $\Gamma(p_1)$ connects P_1 to P_3 and the curve $\Gamma(p_2)$ connects P_3 to P_2 in $\mathcal{P}$. These curves $\Gamma(p_i)$ are monotone functions, and hence the loop $\Gamma(p_1) \circ \Gamma(p_2) \circ \gamma^*$ has trivial homotopy type in $\mathcal{P}$. Therefore $\Gamma(u|_{[0,T]})^k \circ \gamma \sim \Gamma(p_2) \circ \Gamma(v|_{[0,\hat{T}]})^k \circ \Gamma(p_1)$ in $\mathcal{P}$ for any $k \geq 1$, and from Definition 3.4 it follows that $J[\Gamma(u|_{[0,T]})^k \circ \gamma] \leq J[\Gamma(p_2) \circ \Gamma(v|_{[0,\hat{T}]})^k \circ \Gamma(p_1)]$. Thus,

$$k \cdot J_T[u] + J[\gamma] \leq J[p_1] + J[p_2] + k \cdot J_{\hat{T}}[v],$$

which implies

$$0 < k(J_T[u] - J_{\hat{T}}[v]) \leq J[p_1] + J[p_2] - J[\gamma].$$

This estimate leads to a contradiction for k sufficiently large. □

Lemma 3.22 *For any two distinct minimisers u_1 and u_2 in $CM_{\text{per}}((2,2))$, the associated curves $\Gamma(u_i)$ do not intersect.*

Proof. Suppose $\Gamma(u_1)$ and $\Gamma(u_2)$ intersect at a point $P \in \mathcal{P}$. Translate u_1 and u_2 so that $\Gamma(u_1(0)) = \Gamma(u_2(0)) = P$. Define the function $u \in M_{\text{per}}((2,2)^2)$ as the periodic extension of

$$u(t) = \begin{cases} u_1(t) & \text{for } t \in [0, T_1], \\ u_2(t - T_1) & \text{for } t \in [T_1, T_1 + T_2], \end{cases}$$

where T_i is the minimal period of u_i. Then $J_{T_1+T_2}[u] = 2\mathcal{J}_{\text{per}}((2,2)) = \mathcal{J}_{\text{per}}((2,2)^2)$. By Lemma 3.20 we have $u \in CM_{\text{per}}((2,2))$, which in view of uniqueness of the initial value problem contradicts the fact that u_1 and u_2 are distinct minimisers with $\Gamma(u_1) \neq \Gamma(u_2)$. □

As a direct consequence of this lemma, the periodic orbits in $M_{\text{per}}((2,2))$ are ordered in the sense that $\Gamma(u_1)$ lies either strictly inside or strictly outside the region enclosed by $\Gamma(u_2)$. The ordering will be denoted by $>$.

Theorem 3.23 *There exists a largest and a smallest periodic orbit in $CM_{\text{per}}((2,2))$ in the sense of the above ordering, which we will denote by $u_{\max}$ and $u_{\min}$ respectively. Moreover, $1 < \|u_{\min}\|_{1,\infty} \leq \|u_{\max}\|_{1,\infty} \leq C_0$, and $u_{\min} < u < u_{\max}$ for every $u \in CM_{\text{per}}((2,2))$. In particular, the set $CM_{\text{per}}((2,2))$ is compact.*

Proof. Either the number of periodic minimisers is finite, in which case there is nothing left to prove, or the set of minimisers is infinite. Let $U = \bigcup\{\Gamma(u) \mid u \in CM_{\text{per}}((2,2))\} \subset \mathcal{P}$, and let $A = U \cap \{(u, u') \mid u' = 0, u > 0\}$. Every minimiser in $CM_{\text{per}}((2,2))$ intersects the positive u-axis transversely and exactly once. Moreover, distinct minimisers cross this axis at distinct points by Lemma 3.22. Thus we can use A as an index set and label the minimisers as u_α for $\alpha \in A$. Due to the a priori upper bound on minimisers (Lemma 5.1 in [89]), A is a bounded set. The set A is contained in the u-axis and hence has an ordering induced by the real numbers. This order corresponds to the order on minimisers, i.e. $\alpha < \beta$ in A if and only if $u_\alpha < u_\beta$ as minimisers.

Suppose α_* is an accumulation point of A. Then there exists a sequence α_n converging to α_*. From Theorem 3.10 (the a priori L^∞-bound on u_{α_n} is sufficient by Remark 3.11) we see that there exists a $\hat{u} \in CM(\langle(2,2)\rangle)$ which is a solution of Equation (3.2) such that $u_{\alpha_n} \to \hat{u}$ in $C^1_{\text{loc}}(\mathbb{R})$. Since u_{α_n} is periodic and the C^1_{loc}-limit of a sequence of periodic functions with uniformly bounded periods (compare with the proof of Theorem 3.10 to find a uniform bound on the periods) is periodic, hence $\hat{u} \in CM_{\text{per}}(\langle(2,2)\rangle)$. By Lemma 3.21, $\hat{u} \in CM_{\text{per}}((2,2))$. Furthermore, $\hat{u}$ corresponds to u_{α_*}, and hence A is compact. Consequently A contains maximal and minimal elements. Let $u_{\max}$ and $u_{\min}$ be the periodic minimisers through the maximal and minimal points of A respectively. This completes the proof. □

The above lemmas characterise periodic minimisers in $CM(\langle(2,2)\rangle)$. Now we turn our attention to non-periodic minimisers. We conclude this subsection with a theorem that gives a complete description of the set $CM(\langle(2,2)\rangle)$.

Let $u \in CM(\langle(2,2)\rangle)$ be non-periodic. Suppose that P is a self-intersection point of $\Gamma(u)$. Then there exist times $t_1 < t_2$ such that $\Gamma(u(t_1)) = \Gamma(u(t_2)) = P$, and $\Gamma(u|_{[t_1,t_2]})$ is a closed loop. By Lemma 3.19 there are only finitely many self-intersections on $[t_1, t_2]$. Without loss of generality we may therefore assume that γ is a simple closed loop, i.e., we need only consider the case where $P = \Gamma(u(t_1)) = \Gamma(u(t_2))$ and $\Gamma(u|_{[t_1,t_2]})$ is a simple closed loop. We now define $\Gamma_+ = \Gamma(u|_{(t_1,\infty)})$ and $\Gamma_- = \Gamma(u|_{(-\infty,t_2)})$, and we will refer to $\Gamma_\pm$ as the forward and backward orbits of u relative to P.

Lemma 3.24 *Let $u \in CM(\langle(2,2)\rangle)$ be a non-periodic minimiser with at least one self-intersection. Let P and $\Gamma_\pm$ be defined as above. Then the forward and backward orbits $\Gamma_\pm$ relative to P do not intersect themselves. Furthermore, P and $\Gamma_\pm$ are unique, and the curve $\Gamma(u)$ passes through any point in $\mathcal{P}$ at most twice.*

Proof. We will prove the result for Γ_+; the argument for Γ_- is analogous. Suppose that Γ_+ has self-intersections. Define

$$t_* = \min\{t > t_1 \mid \Gamma(u(t)) = \Gamma(u(\tau)) \text{ for some } \tau \in (t_1, t)\}.$$

The minimum t_* is attained by Lemma 3.19, and $t_* > t_2$ since $\gamma \equiv \Gamma(u|_{[t_1,t_2]})$ is a simple closed loop. Let $t_0 \in (t_1, t_*)$ be the point such that $\Gamma(u(t_0)) = \Gamma(u(t_*))$. This point is unique by the definition of t_*, and $\tilde{\gamma} \equiv \Gamma(u|_{[t_0,t_*]})$ is a simple closed loop. For small positive δ we define $Q = \Gamma(u(t_*))$, $B = \Gamma(u(t_1 - \delta))$, $E = \Gamma(u(t_* + \delta))$ and $\Gamma_* = \Gamma(u|_{[t_1-\delta,t_*+\delta]})$, see Figure 3.1. We can decompose this curve into five parts: $\Gamma_* = \sigma_3 \circ \tilde{\gamma} \circ \sigma_2 \circ \gamma \circ \sigma_1$, where σ_1 joins B to P, σ_2 joins P to Q, σ_3 joins Q to E, and γ and $\tilde{\gamma}$ are simple closed loops based at P and Q respectively, see Figure 3.1. The simple closed curves γ and $\tilde{\gamma}$ go around L exactly once and thus have the same homotopy type. Moreover, $\gamma \neq \tilde{\gamma}$ since u is non-periodic.

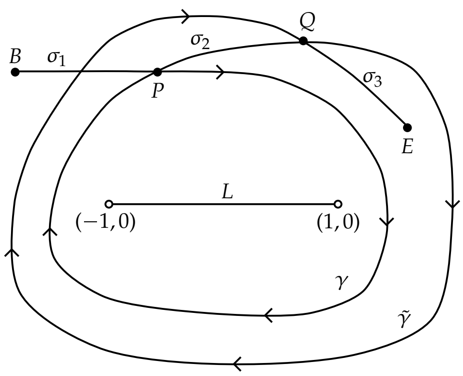

Figure 3.1: The forward orbit Γ_+ starting at P with a self-intersection at the point Q. Lemma 3.24 implies that this cannot happen for non-periodic $u \in CM(\langle(2,2)\rangle)$.

Besides Γ_*, we can construct two other (distinct) paths from B to E:

$$\Gamma_1 = \sigma_3 \circ \sigma_2 \circ \gamma \circ \gamma \circ \sigma_1 \quad \text{and} \quad \Gamma_2 = \sigma_3 \circ \tilde{\gamma} \circ \tilde{\gamma} \circ \sigma_2 \circ \sigma_1.$$

It is not difficult to see that Γ_1, Γ_2 and Γ_* all have the same homotopy type. Since $J[\Gamma_*]$ is minimal in the sense of Definition 3.4, we have, by the same reasoning as in Lemma 3.19, that $J[\Gamma_1] \geq J[\Gamma_*]$ and $J[\Gamma_2] \geq J[\Gamma_*]$. This implies that $J[\tilde{\gamma}] \geq J[\gamma]$ and $J[\gamma] \geq J[\tilde{\gamma}]$, hence $J[\gamma] = J[\tilde{\gamma}]$. Therefore $J[\Gamma_1] = J[\Gamma_2] = J[\Gamma_*]$, which gives that Γ_1, Γ_2 and Γ_* are all distinct minimisers of the same type as curves joining B to E. Since these curves all contain the paths σ_1, σ_2 and σ_3, and are solutions of (3.2), the uniqueness to the initial value problem is contradicted.

Finally, the curve $\Gamma(u)$ can pass through a point at most twice because it is a union of Γ_+ and Γ_-, each visiting a point at most once. Moreover, points in $\Gamma(u|_{(t_1,t_2)})$, common to both Γ_+ and Γ_-, are passed exactly once. It now follows that if there is another self-intersection besides P, say at $R = \Gamma(u(s_1)) = \Gamma(u(s_2))$, then $s_1 < t_1$ and $t_2 < s_2$ (since $\Gamma(u|_{[t_1,t_2]})$ is a simple closed loop). We conclude that the curve $\Gamma(u|_{(s_1,s_2)})$ contains $\Gamma(u|_{[t_1,t_2]})$ and therefore it is not a simple closed curve. Thus P is the unique self-intersection that cuts off a simple loop. □

Lemma 3.25 *Let $u \in CM(\langle(2,2)\rangle)$ be non-periodic, and suppose that $u \in L^\infty(\mathbb{R})$. Then u is a connecting orbit between two periodic minimisers $u_-, u_+ \in CM_{\text{per}}((2,2))$, i.e., there are sequences $t_n^-, t_n^+ \to \infty$ such that $u(t - t_n^-) \to u_-(t)$ and $u(t + t_n^+) \to u_+(t)$ in $C^4_{\text{loc}}(\mathbb{R})$.*

Proof. Lemma 3.24 implies that Γ_+ is a spiral which intersects the positive u-axis at a bounded, monotone sequence of points $(\alpha_n, 0)$ in $\mathcal{P}$ converging to a point $(\alpha_*, 0)$. Let t_n be the sequence of consecutive times such that $u(t_n) = \alpha_n$ and $u'(t_n) = 0$. Consider the sequence of minimisers in $CM(\langle(2,2)\rangle)$ defined by $u_n(t) = u(t + t_n)$. By Theorem 3.10 there exist a C^1_{loc}-limit $u_+ \in CM(\langle(2,2)\rangle)$. Suppose u_+ is non-periodic. Then the curve $\Gamma(u_+)$ crosses the u-axis infinitely many times. On the other hand, from the C^1_{loc} convergence it follows that $\Gamma(u_+)$ crosses this axis only at α_*. By Lemma 3.24 the curve $\Gamma(u_+)$ can intersect α_* at most twice, which is a contradiction. The C^4_{loc}-convergence follows from regularity (as in the proof of Theorem 3.10). The proof of the existence of u_- is similar. □

Theorem 3.26 *Let $u \in CM(\langle(2,2)\rangle)$. There are three possibilities: either u is unbounded, or u is periodic and $u \in CM_{\text{per}}((2,2))$, or u is a connecting orbit between periodic minimisers in $CM_{\text{per}}((2,2))$.*

Proof. Let $u \in CM(\langle(2,2)\rangle)$ be bounded, then u is either periodic or non-periodic. In the case that u is periodic it follows from Lemma 3.21 that $u \in CM_{\rm per}((2,2))$. Otherwise, if u is not periodic, it follows from Lemma 3.25 that u is a connecting orbit between two minimisers $u_-, u_+ \in CM_{\rm per}((2,2))$. □

In Section 3.5.2 we give analogues of the above theorems for arbitrary homotopy types $\boldsymbol{r}$. Notice that the option of $u \in CM(\langle(2,2)\rangle)$ being unbounded in the above theorem does not occur when $F(u) \sim |u|^s$, $s > 2$ as $|u| \to \infty$ (see Remark 3.18).

3.5 Properties of minimisers

In Section 3.4 we proved the existence of minimisers in $M_{\rm per}((2,2))$, which will provide a priori bounds on the minimisers of arbitrary type. These bounds and Theorem 3.10 establish the existence of such minimisers. In this section we will also prove that certain properties of a type $\boldsymbol{g}$ are often reflected in the associated minimisers. The most important examples are the periodic types $\boldsymbol{g} = \langle \boldsymbol{r} \rangle$. Although there are minimisers in every class $M(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$, it is not clear a priori that among these minimisers there are also periodic minimisers. In order to prove existence of periodic minimisers for every periodic type $\langle \boldsymbol{r} \rangle$ we use the theory of covering spaces.

3.5.1 Existence

The periodic minimisers of type $(2,2)$ are special for the following reason. For a normalised $u \in M_{\rm per}((2,2))$, define $D(u)$ to be the closed disk in $\mathbb{R}^2$ such that $\partial D(u) = \Gamma(u)$. Let $u_{\min}$ be the minimal element of $CM_{\rm per}((2,2))$, see Theorem 3.23.

Theorem 3.27 *For any periodic type $\langle \boldsymbol{r} \rangle \neq \langle(2,2)\rangle$ and any $u \in CM(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$, it holds that $\Gamma(u) \subset D(u_{\min})$. For any terminated type $\boldsymbol{g}$ and any $u \in CM(\boldsymbol{g}, \boldsymbol{p})$ it also holds then $\Gamma(u) \subset D(u_{\min})$.*

Proof. We start with the first assertion. If $\langle \boldsymbol{r} \rangle \neq \langle(2,2)\rangle$ then every $u \in CM(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ has the property that $\Gamma(u)$ intersects the u-axis between $u = \pm 1$. Suppose that $\Gamma(u)$ does not lie inside $D(u_{\min})$. Then $\Gamma(u)$ must intersect $\Gamma(u_{\min})$ at least twice, and let P_1 and P_2 be distinct intersection points with the property that the curve Γ_1 obtained by following $\Gamma(u)$ from P_1 to P_2 lies entirely outside of $D(u_{\min})$. Let $\Gamma_2 \subset \Gamma(u_{\min})$ be the curve from P_1 to P_2 following $u_{\min}$, such that Γ_1 and Γ_2 are homotopic (traversing the loop $\Gamma(u_{\min})$ as many times as necessary), and thus $J[\Gamma_1] = J[\Gamma_2]$ is minimal. Replacing Γ_1 by Γ_2 leads to a minimiser in $CM(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ which partially agrees with u. This contradicts the uniqueness of the initial value problem for (3.2).

The second assertion is proved analogously. As in the previous case the associated curve $\Gamma(u)$ either intersects $\Gamma(u_{\min})$ at least twice or lies completely inside $D(u_{\min})$, and the proof is identical. □

Corollary 3.28 *For all minimisers in the above theorem, $\|u\|_{1,\infty} \leq \|u_{\min}\|_{1,\infty} \leq C_0$.*

In order to prove existence of minimisers in every class we now use the above theorem in combination with an existence result from [89].

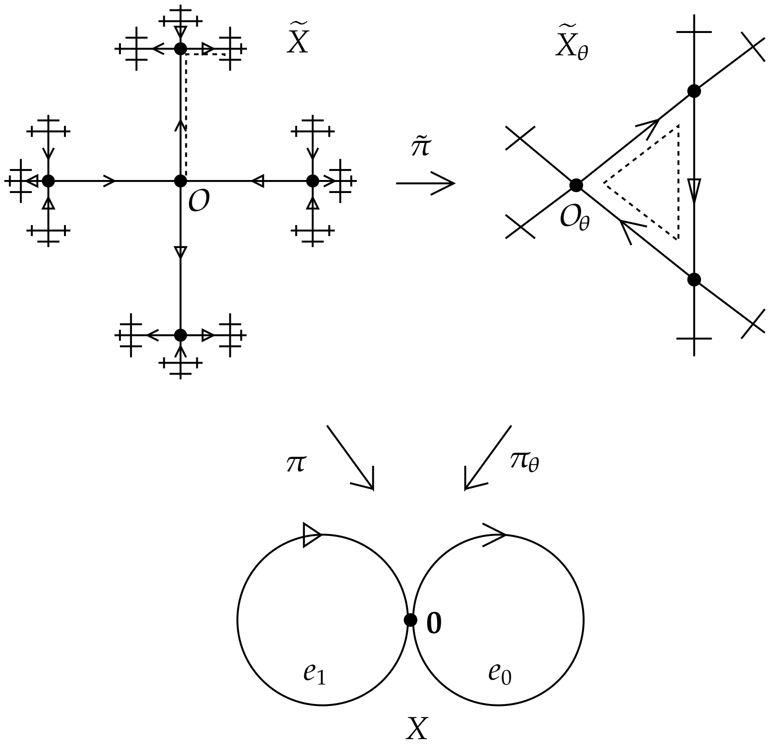

Figure 3.2: The universal cover $\widetilde{X}$ of X is a tree. Its origin is denoted by O. For $\theta = e_0 e_1 e_0$, the quotient space $\widetilde{X}_\theta = \widetilde{X}/\langle\theta\rangle$, depicted schematically on the right, is also a covering space over X, and $\widetilde{X}_\theta \sim S^1$.

Theorem 3.29 *For any given type* $\boldsymbol{g}$ *and parity* $\boldsymbol{p}$ *there exists a (bounded) minimiser* $u \in CM(\boldsymbol{g}, \boldsymbol{p})$*. Moreover* $\|u\|_{1,\infty} \le C_0$*, independent of* $(\boldsymbol{g}, \boldsymbol{p})$*.*

Proof. Given a type $\boldsymbol{g}$ we can construct a sequence $\boldsymbol{g}_n$ of terminated types such that $\boldsymbol{g}_n \to \boldsymbol{g}$ as $n \to \infty$. For any terminated type $\boldsymbol{g}_n$ there exists a minimiser $u_n \in CM(\boldsymbol{g}_n, \boldsymbol{p})$ by Proposition 3.6 (Theorem 1.3 of [89]). Clearly such a sequence u_n satisfies $\|u_n\|_{1,\infty} \le C$ by Corollary 3.28. Applying Theorem 3.10 completes the proof. □

3.5.2 Covering spaces and the fundamental group

The fundamental group of $\mathcal{P}$ is isomorphic to the free group on two generators e_0 and e_1 which represent loops (traversed clockwise) around $(1, 0)$ and $(-1, 0)$ respectively, with basepoint $(0, 0)$. Indeed, $\mathcal{P}$ is homotopic to a bouquet of two circles $X = S_1 \vee S_1$. The universal covering of X denoted by $\widetilde{X}$ can be represented by an infinite tree whose edges cover either e_0 or e_1 in X, see Figure 3.2. The universal covering of $\mathcal{P}$, denoted by $\pi : \widetilde{\mathcal{P}} \to \mathcal{P}$, can then be viewed by thickening the tree $\widetilde{X}$ so that $\widetilde{\mathcal{P}}$ is homeomorphic to an open disk in $\mathbb{R}^2$.

An important property of the universal covering is that the fundamental group $\pi_1(\mathcal{P})$ induces a left group action on $\widetilde{\mathcal{P}}$ in a natural way, via the lifting of paths in $\mathcal{P}$ to paths in $\widetilde{\mathcal{P}}$. This action will be denoted by $\theta \cdot p$ for $\theta \in \pi_1(\mathcal{P})$ and $p \in \widetilde{\mathcal{P}}$. We will not reproduce the construction of this action here, and the reader is referred to an introductory book on algebraic topology such as [69]. However, we will utilise the structure of the quotient spaces of $\widetilde{\mathcal{P}}$ obtained from this action, which are again coverings of $\mathcal{P}$. These quotient spaces will be the natural spaces in which to consider the lifts of curves $\Gamma(u)$ which lie in more complicated homotopy classes than those in the case of $u \in M_{\mathrm{per}}((2,2))$.

A periodic type $\boldsymbol{g} = \langle \boldsymbol{r} \rangle$ is generated by a finite type $\boldsymbol{r}$, which together with the parity $\boldsymbol{p}$ determines an element of $\pi_1(\mathcal{P})$ of the form $\theta(\boldsymbol{r}) = e_{|\boldsymbol{p}-1|}^{r_{2n}} \cdots e_{\boldsymbol{p}}^{r_1}$. Since we only consider curves in $\mathcal{P}$ which are of the form $\Gamma(u) = (u(t), u'(t))$, the numbers r_i are all positive. The infinite subgroup generated by any such element θ will be denoted by $\langle\theta\rangle \subset \pi_1(\mathcal{P})$. The quotient space $\widetilde{\mathcal{P}}_\theta = \widetilde{\mathcal{P}}/\langle\theta\rangle$ is obtained by identifying points p and q in $\widetilde{\mathcal{P}}$ for which $q = \theta^k \cdot p$ for some $k \in \mathbb{Z}$. The resulting space $\widetilde{\mathcal{P}}_\theta$ is homotopic to an annulus, and $\pi_\theta : \widetilde{\mathcal{P}}_\theta \to \mathcal{P}$ is a covering space. Figure 3.2 illustrates the situation for X, since it is easier to draw, and for $\mathcal{P}$ the reader should imagine that the edges in the picture are thin strips. The lift of the path $\theta = e_0 e_1 e_0$ to $\widetilde{X}$ based at O, is indicated by the dashed line. This piece of the tree becomes a circle in the quotient space $\widetilde{X}_\theta$. Note that infinitely many edges in $\widetilde{X}$ are identified with this circle. The dashed lines in both $\widetilde{X}$ and $\widetilde{X}_\theta$ are strong deformation retracts of $\widetilde{X}$ and $\widetilde{X}_\theta$ respectively, and hence $\widetilde{X}_\theta$ is homotopic to a circle. By thickening $\widetilde{X}_\theta$ one infers that $\widetilde{\mathcal{P}}_\theta$ is homotopic to an annulus. Thus $\pi_1(\widetilde{\mathcal{P}}_\theta)$ is a generated by a simple closed loop in $\widetilde{\mathcal{P}}_\theta$ which will be denoted by $\zeta(\boldsymbol{r})$. For convenience we suppress the dependence of θ and ζ on the parity $\boldsymbol{p}$.

Remark 3.30 If we interpret the shift operator σ on finite types $\boldsymbol{r}$ as a cyclic permutation, then $M_{\mathrm{per}}(\boldsymbol{r}, \boldsymbol{p}) = M_{\mathrm{per}}(\sigma^k(\boldsymbol{r}), \tau^k(\boldsymbol{p}))$ for all $k \in \mathbb{Z}$. Functions in $M_{\mathrm{per}}(\boldsymbol{r}, \boldsymbol{p})$ have a unique lift to simple closed curve in $\widetilde{\mathcal{P}}_\theta$, with $\theta = \theta(\boldsymbol{r})$. However, functions in the shifted class $M_{\mathrm{per}}(\sigma^k(\boldsymbol{r}), \tau^k(\boldsymbol{p}))$ are not simple closed curves in $\widetilde{\mathcal{P}}_\theta$. In order for such functions to be lifted to a unique simple closed curve we need to consider the covering space $\widetilde{\mathcal{P}}_{\theta_k}$, where $\theta_k = \theta(\sigma^k(\boldsymbol{r}), \tau^k(\boldsymbol{p}))$. •

3.5.3 Characterisation of minimisers of type $\langle \boldsymbol{r} \rangle$

In Section 3.4.2 we characterised minimisers in $CM(\langle(2,2)\rangle)$ by studying the properties of their projections into $\mathcal{P}$. What was special about the types $(2,2)^k$ was that the projected curves were a priori contained in $\mathcal{P} \setminus L$, which is topologically an annulus. The J-efficiency of minimising curves restricts the possibilities for their self and mutual intersections. In particular, we showed that all periodic minimisers in $CM(\langle(2,2)\rangle)$ project onto simple closed curves in $\mathcal{P} \setminus L$ and that no two such minimising curves intersect. These two properties, coupled with the simple topology of the annulus, already force the minimising periodic curves to have a structure of a family of nested simple loops.

Such a simple picture in the configuration plane $\mathcal{P}$ cannot be expected for minimisers in $CM((\langle \boldsymbol{r} \rangle, \boldsymbol{p}))$ with $\boldsymbol{r} \neq (2,2)$. The simple intersection properties (of Lemmas 3.22 and 3.24) no longer hold; in fact, periodic minimising curves must have self-intersections in $\mathcal{P}$, as do any curves in $\mathcal{P}$ representing the homotopy class of $(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$. However, by lifting minimising curves into the annulus $\widetilde{\mathcal{P}}_\theta$, we can remove exactly these necessary self-intersections, and this puts us in a position where we can emulate the discussion for the types $(2,2)^k$. More precisely, let $(\boldsymbol{r}, \boldsymbol{p})$ be a minimal type. For any $u \in M_{\mathrm{per}}((\boldsymbol{r}, \boldsymbol{p})^k)$ with period T such that $\theta^{-1}[\Gamma(u|_{[0,T]})] = (\boldsymbol{r}, \boldsymbol{p})^k$, there are infinitely many lifts of the closed loop $\Gamma(u|_{[0,T]})$ into $\widetilde{\mathcal{P}}_\theta(\boldsymbol{r})$ (see Remark 3.30), but there is exactly one lift, denoted $\Gamma_\theta(u|_{[0,T]})$, which is a closed loop homotopic to $\zeta^k(\boldsymbol{r})$ in $\widetilde{\mathcal{P}}_\theta(\boldsymbol{r})$. We can repeat all of the arguments in Section 3.4 by identifying intersections between the curves $\Gamma_\theta(u|_{[0,T]})$ in $\widetilde{\mathcal{P}}_\theta(\boldsymbol{r})$, instead of intersections between the curves $\Gamma(u|_{[0,T]})$ in $\mathcal{P} \setminus L$. Of course, when gluing together pieces of curves, the values of u and u' come from the projections into $\mathcal{P}$. In particular, the arguments of Lemma 3.21 show that $\Gamma_\theta(u|_{[0,T]})$ must be a simple loop traced k-times, which leads to the following.

Lemma 3.31 *For every periodic type $\langle \boldsymbol{r} \rangle$ and every $k \geq 1$ it holds that $CM_{\mathrm{per}}((\boldsymbol{r},\boldsymbol{p})^k) = CM_{\mathrm{per}}(\boldsymbol{r},\boldsymbol{p}) = CM_{\mathrm{per}}(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$.*

The proof of the next theorem is a slight modification of Theorem 3.23.

Theorem 3.32 *For any periodic type $\langle \boldsymbol{r} \rangle$ the set $CM_{\mathrm{per}}(\boldsymbol{r},\boldsymbol{p})$ is compact and totally ordered (in $\widetilde{\mathcal{P}}_\theta$).*

The following lemma is analogous to Lemma 3.25. Note however that by Theorem 3.27 we do not need to assume that the minimiser is uniformly bounded.

Lemma 3.33 *Let $u \in CM(\langle \boldsymbol{r} \rangle, \boldsymbol{p})$ for some periodic type $\langle \boldsymbol{r} \rangle \neq \langle (2,2) \rangle$. Then either u is periodic and $u \in CM_{\mathrm{per}}(\boldsymbol{r},\boldsymbol{p})$, or u is a connecting orbit between two periodic minimisers $u_-, u_+ \in CM_{\mathrm{per}}(\boldsymbol{r},\boldsymbol{p})$, i.e., there are sequences $t_n^-, t_n^+ \to \infty$ such that $u(t - t_n^-) \to u_-(t)$ and $u(t + t_n^+) \to u_+(t)$ in $C^4_{\mathrm{loc}}(\mathbb{R})$.*

Combining Theorem 3.29 and Lemma 3.33 we obtain the existence of periodic minimisers in every class with a periodic type (this result can also be obtained in a manner analogous to Theorem 3.17).

Theorem 3.34 *For any periodic type $\langle \boldsymbol{r} \rangle$ the set $CM_{\mathrm{per}}(\boldsymbol{r},\boldsymbol{p})$ is nonempty.*

The classification of functions by type has some properties in common with symbolic dynamics. For example, if a type $\boldsymbol{g}$ is asymptotic to two periodic types, i.e. $\sigma^n(\boldsymbol{g}) \to \boldsymbol{r}_+$ and $\sigma^{-n}(\boldsymbol{g}) \to \boldsymbol{r}_-$ as $n \to \infty$, with $\boldsymbol{r}_+ \neq \boldsymbol{r}_-$, then any minimiser $u \in CM(\boldsymbol{g},\boldsymbol{p})$ is a connecting orbit between two periodic minimisers $u_- \in CM_{\mathrm{per}}(\boldsymbol{r}_-,\boldsymbol{p}')$ and $u_+ \in CM_{\mathrm{per}}(\boldsymbol{r}_+,\boldsymbol{p}'')$, i.e., there exist sequences $t_n^-, t_n^+ \to \infty$ such that $u(t - t_n^-) \to u_-(t)$ and $u(t + t_n^+) \to u_+(t)$ in $C^4_{\mathrm{loc}}(\mathbb{R})$. This result follows from Cantor's diagonal argument using Theorems 3.10 and 3.33, and thus we have used the symbol sequences to conclude the existence of heteroclinic and homoclinic orbits connecting any two types of periodic orbits.

Symmetry properties of types $\boldsymbol{g}$ are also often reflected in the corresponding minimisers. For example, define the map Ψ_{i_0} on infinite types by $\Psi_{i_0}(\boldsymbol{g}) = (g_{2i_0-i})_{i\in\mathbb{Z}}$, and consider types that satisfy $\Psi_{i_0}(\boldsymbol{g}) = \boldsymbol{g}$ for some i_0. Moreover assume that $\boldsymbol{g}$ is periodic. In this case we can prove that the corresponding periodic minimisers are symmetric and satisfy Neumann boundary conditions.

Theorem 3.35 *Let $\boldsymbol{g} = \langle \boldsymbol{r} \rangle$ satisfy $\Psi_{i_0}(\langle \boldsymbol{r} \rangle) = \langle \boldsymbol{r} \rangle$ for some i_0. Then for any $u \in CM_{\mathrm{per}}(\boldsymbol{r},\boldsymbol{p})$ there exists a shift τ such that $u_\tau(x) = u(x - \tau)$ satisfies*

(a) *$u_\tau(x) = u_\tau(T - x)$ for all $x \in [0,T]$ where T is the period of u,*

(b) *$u_\tau'(0) = u_\tau'''(0) = 0$ and $u_\tau'(\frac{T}{2}) = u_\tau'''(\frac{T}{2}) = 0$, and*

(c) *u_τ is a local minimiser for the functional $J_{\frac{T}{2}}[u]$ on the Sobolev space $H^2_n(0,\frac{T}{2}) = \{u \in H^2(0,\frac{T}{2}) \,|\, u'(0) = u'(\frac{T}{2}) = 0\}$.*

Proof. Without loss of generality we may assume that $i_0 = 1$ and that $\boldsymbol{g} = \langle(g_1, \ldots, g_N)\rangle$ for some $N \in 2\mathbb{N}$. We can choose a point t_0 in the convex hull of A_1 such that $u'(t_0) = u'(t_0 + T) = 0$ and $\boldsymbol{g}(u|_{[t_0,t_0+T]}) = (g_1/2, g_2, \ldots, g_N, g_1/2)$. We now define $v(t) = u(t_0 + T - t)$. Then by the assumptions on the symmetry of $\boldsymbol{g}$ we have in fact that $(g_1, \ldots, g_N) = (g_1, \ldots, g_{\frac{N}{2}}, g_{\frac{N}{2}+1}, g_{\frac{N}{2}}, \ldots, g_2)$, hence $\boldsymbol{g}(v|_{[t_0,t_0+T]}) = \boldsymbol{g}(u|_{[t_0,t_0+T]})$. Since $J_{[t_0,t_0+T]}(v) = J_{[t_0,t_0+T]}(u)$ and $\Gamma(u(t_0)) = \Gamma(u(t_0 + T)) = \Gamma(v(t_0)) = \Gamma(v(t_0 + T))$, we conclude from the uniqueness of the initial value problem that $u(t) = v(t)$ for all $t \in [t_0, t_0 + T]$, which proves the first statement. The second statement follows immediately from (a). The third property follows from the definition of minimiser. $\square$

Chapter 4

Attracting sets and stable equilibria

4.1 Introduction

Higher order parabolic equations of the form[1]

$$u_t = -\gamma u_{xxxx} + \beta u_{xx} - F'(u), \qquad (t,x) \in \mathbb{R}^+ \times (0,L), \tag{4.1}$$

with $\gamma > 0$, $\beta > 0$, may display a multitude of stable equilibrium solutions depending on various parameters in the problem such as γ, β, the potential F, the interval length L and the boundary conditions at $x = 0$ and $x = L$. The goal of this chapter is to study the set of stable equilibria of Equation (4.1), and its qualitative properties.

In our notation u is a function of the variables t and x, and u_t and u_x denote the partial derivatives. The initial state $u(0,x)$ is denoted by u_0. The function $F \in C^2$ is a double-well potential and satisfies

$$F(\pm 1) = F'(\pm 1) = 0, \quad F''(\pm 1) > 0 \quad \text{and} \quad F(u) > 0 \text{ for } u \neq \pm 1. \tag{4.2}$$

The following growth condition is imposed on the potential: $F(u) > -C_0 + C_1 u^2$ for some $C_0, C_1 > 0$, i.e. F grows super-quadratically[2]. We will not require any generic properties for Equation (4.1) such as non-degenerate equilibria.

We have not yet specified boundary conditions at $x = 0$ and $x = L$. In certain physical models (Swift-Hohenberg equation, Extended Fisher-Kolmogorov equation) in which Equation (4.1) occurs, the boundary conditions

$$u_x(t,0) = u_{xxx}(t,0) = 0 \quad \text{and} \quad u_x(t,L) = u_{xxx}(t,L) = 0$$

are often used. These boundary conditions are referred to as the *Neumann* boundary conditions. In this case $u \equiv \pm 1$ are stable equilibria for all $\gamma, \beta, L > 0$. It should be noted at this point that the Neumann boundary conditions that we impose on Equation (4.1) are by no means a restriction for the results presented here, and different conditions can be used. We will come back to this point later on (especially in Section 4.6).

An essential property of Equation (4.1) is that it is the L^2-gradient flow equation for the action

$$J_L[u] = \int_0^L \tfrac{\gamma}{2}\big(|u_{xx}|^2 + \tfrac{\beta}{2}|u_x|^2 + F(u)\big)\, dx. \tag{4.3}$$

This variational structure allows an extension of most of the results to more general actions: $J_L[u] = \int_0^L j(u, u_x, u_{xx})\, dx$, where $j \geq 0$ satisfies the convexity condition $\partial^2_{u_{xx}} j \geq \delta > 0$. In order to best explain the overall features of our methods we restrict ourselves here to actions of the form given in (4.3).

[1]Note that in this chapter the potential $F(u)$ is defined with the opposite sign compared to Chapter 1.
[2]This growth condition is taken to simplify estimates, but it can be weakened in various directions.

To give an impression of the type of results proved in this chapter consider the special case $F(u) = \frac{1}{4}(u^2-1)^2$. When $\gamma = 0$ (the Neumann boundary conditions then reduce to $u_x(t,0) = u_x(t,L) = 0$) it follows that $u \equiv \pm 1$ are the only (asymptotically) stable equilibria. We show that this remains true for all $0 < \frac{\gamma}{\beta^2} \leq \frac{1}{8}$, whereas for $\frac{\gamma}{\beta^2} > \frac{1}{8}$ the number of stable equilibria changes dramatically and grows to infinity as L goes to infinity. In fact, it holds for all $F(u)$ satisfying (4.2) that when the equilibrium points $u = \pm 1$ are saddle-foci, then the number of stable equilibria grows exponentially as $L \to \infty$. This behaviour occurs for all kinds of boundary conditions.

Equations of type (4.1) occur in physical models for phase transitions. For instance, Equation (4.1) with $F(u) = \frac{1}{4}(u^2-1)^2$ and $\gamma, \beta > 0$, the *Extended Fisher-Kolmogorov (EFK) equation*, was proposed by Dee and Van Saarloos [53] as a model equation for phase transitions in the neighbourhood of a Lifshitz point [142]. For $\beta < 0$ the model corresponds to the Swift-Hohenberg equation [137].

There is a substantial literature now about the stationary solutions of (4.1) in the case that $\gamma, \beta > 0$ and the potential F satisfies the above hypotheses. In a series of papers Peletier and Troy [117, 118, 119, 120] have studied the stationary problem of (4.1) by means of a topological shooting method. In the case that $\frac{\gamma}{\beta^2} > \frac{1}{8}$ they find a great variety of different stationary solutions (heteroclinic orbits, periodic solutions, chaotic patterns). For $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$ it was proved in Chapter 2 that the stationary solutions of (4.1) are in 1-1 correspondence with the stationary solutions in the case $\gamma = 0$ (see also [96, 112] for related results).

In [88, 89, 90] stationary solution of (4.1) were found by means of minimisation of the associated action (4.3). In particular, the results in [89] will be drawn upon to construct stable solutions of the parabolic equation.

Recently light has been shed on the structure of the set of stationary solutions for $\beta < 0$. A shooting method has proved the existence of different types of branches of periodic solutions (see Chapter 6) previously identified by a numerical investigation [19]. Besides, from a variational point of view the study of such solutions has led to the analysis of a related Twist map in the framework of Morse-Conley theory on the space of braid diagrams (see Chapters 7 and 8). The latter method also gives insight into stability properties of the periodic solutions. In this chapter we restrict our analysis to the parameter region $\beta \geq 0$. In Section 4.9 we will give a brief indication of some numerical observations concerning the (im)possibility to extend the present analysis to the parameter regime $\beta < 0$.

In order to simplify matters we carry out the construction of stable equilibria in the case of the Neumann boundary conditions. The natural function space for this case is

$$H^2_N \stackrel{\text{def}}{=} \{u \in H^2(0,L) \,|\, u_x(0) = u_x(L) = 0\}.$$

Equation (4.1) has a compact attractor $\mathcal{A} = \mathcal{A}(L,\gamma,\beta,F)$ for all $0 < L < \infty$, $\gamma, \beta > 0$ and for all potentials F that satisfy the growth condition $\liminf_{|u|\to\infty} \frac{F'(u)}{u} > 0$; for $\beta < 0$ one needs that $\liminf_{|u|\to\infty} \frac{F'(u)}{u} > \frac{\beta^2}{4\gamma}$ (see e.g. [79, Section 4.3])[3]. If L is small enough then $\mathcal{A}$ contains exactly two stable equilibria ($u \equiv \pm 1$). The size of the attractor $\mathcal{A}$ depends on L in the sense that if L increases, the attractor also becomes larger and the number of equilibria in $\mathcal{A}$ increases. It is not a priori clear whether new *stable* equilibria are created.

[3]Note the difference with the earlier growth condition of F.

For $\gamma = 0$ the attractor is well understood. In fact, when $F(u) = \frac{1}{4}(u^2-1)^2$ then $u \equiv \pm 1$ are the only stable equilibria for all $L > 0$, and the attractor of the second order equation can be characterised completely (see also Section 4.2). For $0 < \frac{\gamma}{\beta^2} \leq \frac{1}{8}$ the following theorem gives a strong characterisation of the attractor, relating it to the second order equation. We first introduce some notation. The semi-flow associated with (4.1) and Neumann boundary conditions is denoted by $\phi(L, \gamma, \beta)$. The first bifurcation of the homogeneous solution $u \equiv 0$ occurs at $L = L_0(\gamma, \beta) \stackrel{\text{def}}{=} \pi\sqrt{\frac{2\gamma}{\sqrt{\beta^2+4\gamma}-\beta}}$ (and $L_0(0,\beta) = \beta\pi$).

Theorem 4.1 *Let $F(u) = \frac{1}{4}(u^2-1)^2$ and suppose that $\beta > 0$ and $0 < \frac{\gamma}{\beta^2} \leq \frac{1}{8}$, then for all $L > 0$ there is a semi-conjugacy between the flow on the attractor of* (4.1) *with Neumann boundary conditions and the corresponding flow for the second order equation ($\gamma = 0$). To be precise, there is a semi-conjugacy between $\phi(L, \gamma, \beta)|_{\mathcal{A}}$ and $\phi(L\frac{L_0(0,\beta)}{L_0(\gamma,\beta)}, 0, \beta)|_{\mathcal{A}}$.*

This theorem implies in particular that for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$ and all $L > 0$ the only stable solutions are the homogeneous states $u \equiv \pm 1$. Another consequence is the existence of connecting orbits between various stationary states (see Section 4.2 for more details). The above theorem holds for a more general class of potentials $F(u)$. For example, a sufficient condition is that F is even, satisfies (4.2) and $F'''(u) \geq 0$ for $u \geq 0$ (this condition can be somewhat relaxed). The parameter range for which the theorem holds is then $\frac{\gamma}{\beta^2} \leq \frac{1}{4F''(\pm 1)}$. An analogous theorem holds for Navier boundary conditions: $u(t,0) = u_{xx}(t,0) = 0$ and $u(t,L) = u_{xx}(t,L) = 0$.

If $\frac{\gamma}{\beta^2} > \frac{1}{4F''(\pm 1)}$, then the situation changes dramatically. The origin of this change is the fact that the nature of the equilibrium points $u = \pm 1$ changes from real saddle to saddle-focus at $\frac{\gamma}{\beta^2} = \frac{1}{4F''(\pm 1)}$. The aim of this chapter is to give a lower bound on the number of stable equilibria of Equation (4.1) as a function of the interval length L, and to describe the shape of these stable equilibria.

Since we do not require stationary solutions to be isolated, we need the more general notion of *stable set*:

Definition 4.2 *A set $\mathcal{S}$ of stationary solutions of Equation* (4.1) *is* stable *if for any $\epsilon > 0$ there exists an open neighbourhood $U \subset B_\epsilon(\mathcal{S})$ such that for all $u_0 \in U$ it holds that $u(t,x) \in B_\epsilon(\mathcal{S})$ for all $t > 0$.*

We want to identify various attracting sets, i.e. forwardly invariant sets, in which we can then find stable sets of equilibria.

Theorem 4.3 *Let the potential F satisfy the hypotheses* (4.2) *and grow super-quadratically. Suppose that $\beta > 0$ and $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. Then for any $n \in \mathbb{N}$ there exists a constant $L_n > 0$, such that for all $L \geq L_n$ Equation* (4.1)*, with Neumann boundary conditions, has at least n disjoint stable sets of stationary solutions.*

Each stable set in the above theorem consists of stationary solutions with a specific geometrical shape, which differs from set to set. Notice that this theorem holds under very mild conditions on the double-well potential F and that no non-degeneracy assumptions are made. We stress that the result of Theorem 4.3 is by no means restricted to just Neumann boundary data (see Section 4.6).

The idea of the method is to find regions in appropriate function spaces in which one can find a local minimum of the action. This also reveals the shape of the minimisers.

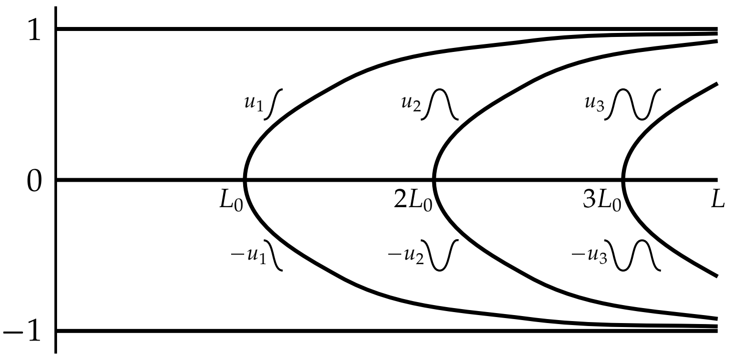

Figure 4.1: A sketch of the bifurcation diagram for $\gamma \in [0, \frac{1}{8}]$. The shape of the bifurcating solutions $\pm u_1$, $\pm u_2$ and $\pm u_3$ is indicated.

The neighbourhoods that we seek are roughly speaking product neighbourhoods of truncated homoclinic minimisers, as found in [89]. Such homoclinic minimisers can only be found in the case that $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. Our results are therefore only valid in this region.

For the EFK equation Theorem 4.3 holds for $\frac{\gamma}{\beta^2} > \frac{1}{8}$, while Theorem 4.1 covers the range $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$. This shows that there is a sharp transition from a relatively simple attractor to a rather complicated one. To better understand the transition from $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$ to $\frac{\gamma}{\beta^2} > \frac{1}{8}$ we investigate this bifurcation in Section 4.8. In this way one also obtains information about the unstable stationary solutions for $\frac{\gamma}{\beta^2} > \frac{1}{8}$. For other potentials $F(u)$ this transition may not be so sharp, but for all $\gamma, \beta > 0$ for which $u = \pm 1$ are saddle-foci, the number of stable sets tends to infinity as $L \to \infty$.

The number of stable stationary states will grow rapidly as the interval length L goes to infinity. In the proof of Theorem 4.3 various a priori estimates are used. If some of these estimates are carried out more carefully one can find a lower bound on the number of stable equilibria as a function of the interval length L. We prove that there is are constants $a_1 > 0$ and $a_2 > 0$ such that

$$\#\{\text{disjoint stable sets of equilibria}\} > a_1 e^{a_2 L}. \tag{4.4}$$

Hence the number of stable sets grows exponentially in L.

The dynamics near the attractor will thus depend in a very subtle manner on the parameters γ and β, which is not captured by, for example, the general slow motion results in [92].

The results in this chapter are not valid when $\beta < 0$. In Section 4.9 we undertake a numerical study of the parameter region $\beta < 0$ and find that new phenomena occur, which certainly merit further exploration. We refer to Chapters 6, 7 and 8 for recent results on periodic solutions for this parameter regime. It is in fact quite natural that our analysis cannot be performed globally, i.e. regardless of the sign of β, because it has been observed that most of the solutions found in this chapter cease to exist when β becomes sufficiently negative (while the equilibria are still saddle-foci). See Section 4.9 for a further discussion of this matter. An interesting open question is whether for $\beta < 0$ the number of stable stationary states grows to infinity as $L \to \infty$, or whether, on the contrary, this number is bounded for sufficiently large $|\beta|$.

The organisation of the chapter is as follows. In Section 4.2 we consider the case $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$

and prove Theorem 4.1. In Sections 4.3 we recall some results from [89] and introduce notation. A priori bounds and compactness results are proved in Section 4.4. These are used in Section 4.5 to perform the gluing method and prove Theorem 4.3. The fact that the number of stable solutions grows exponentially in L is proved in Section 4.7, but first we discuss the role of the boundary conditions in Section 4.6. Results on the bifurcation at $\frac{\gamma}{\beta^2} = \frac{1}{8}$ and numerical results for $\beta < 0$ are presented in Sections 4.8 and 4.9 respectively. Finally, the hyperbolicity of the stationary solutions for $\frac{\gamma}{\beta^2} < \frac{1}{8}$ is proved in Section 4.10.

4.2 The semi-conjugacy

We begin with the study of the attractor of (4.1) with $F(u) = \frac{1}{4}(u^2-1)^2$ and Neumann boundary conditions for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$, i.e.,

$$\begin{cases} u_t = -\gamma u_{xxxx} + \beta u_{xx} + u - u^3 & \text{for } x \in (0,L),\ t > 0 \\ u_x(t,0) = u_x(t,L) = u_{xxx}(t,0) = u_{xxx}(t,L) = 0 & \text{for all } t > 0. \end{cases} \tag{4.5}$$

Without loss of generality we put $\beta = 1$ throughout this section. We first consider the set of stationary solutions. Clearly all stationary solutions can be extended to the real line by reflection in the points $x = 0$ and $x = L$, and therefore they correspond to bounded solutions of

$$-\gamma u_{xxxx} + u_{xx} + u - u^3 = 0. \tag{4.6}$$

Solutions of (4.6) have a constant of integration, the *energy*:

$$\mathcal{E}[u] \stackrel{\text{def}}{=} \gamma u_{xxx}u_x - \tfrac{\gamma}{2}|u_{xx}|^2 - \tfrac{1}{2}|u_x|^2 + \tfrac{1}{4}(u^2-1)^2 = E, \tag{4.7}$$

where $E \in \mathbb{R}$ is constant along solutions of (4.6).

It was found in Chapter 2 that for $\gamma \in (0, \frac{1}{8}]$ the bounded solutions of (4.6) are in 1-1 correspondence with the bounded solutions of the second order equation ($\gamma = 0$). To be precise, for $\gamma \in (0, \frac{1}{8}]$ the only bounded solutions of (4.6) are the three homogeneous solutions $u \equiv 0$ and $u \equiv \pm 1$; two monotone heteroclinic solutions connecting $u = \pm 1$; and a family of periodic solutions which are symmetric with respect to their extrema and antisymmetric with respect to their zeros. These periodic solutions form a continuous family and can be parametrised either by their energy $E \in (0, \frac{1}{4})$, or by their period $\ell \in (0, 2\pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}})$. Existence of these solutions can be proved either via a shooting method where the energy is used as a parameter [117], via a minimisation method where the period is used as a parameter [124], or via continuation (see Section 2.8). The bifurcation diagram for the stationary solutions of (4.5) is given by Figure 4.1. For small L the only bounded solutions are the three homogeneous states. At $L = L_0 \stackrel{\text{def}}{=} \pi\sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}$ two non-uniform stationary solutions bifurcate. These solutions $\pm u_1(x; L)$ are monotone and have exactly one zero. The bifurcation is a generic supercritical pitchfork bifurcation (see e.g. [81, Section 6.2]). More generally, the same type of bifurcation occurs at $L = nL_0$ for all $n \geq 2$. The bifurcating stationary solutions are just multiples of the primary bifurcating branch.

For $\gamma = 0$ the attractor of problem (4.1) with Neumann boundary has been extensively studied (see [4, 37, 81]). For $0 < L \leq \pi$ the attractor consists of the three uniform states and their connecting orbits. For $\pi < L \leq 2\pi$ the attractor contains five equilibrium points, namely the three uniform states and two monotone non-uniform states $\pm u_1$. For $2\pi < L \leq$

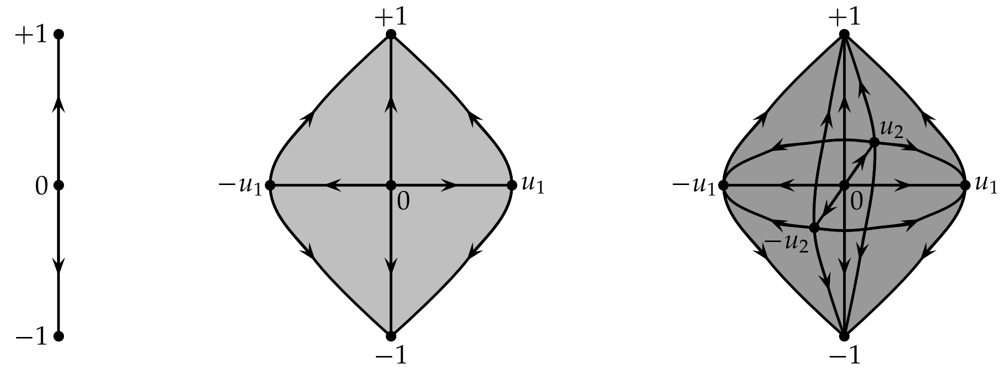

Figure 4.2: The attractor when $\gamma = 0$, for $0 < L \leq \pi$ on the left; for $\pi < L \leq 2\pi$ in the middle; and for $2\pi < L \leq 3\pi$ on the right.

3π the attractor is three-dimensional and consists of the seven equilibrium points $u \equiv 0$, $u \equiv \pm 1$, $\pm u_1$ and $\pm u_2$, and their connecting orbits. The situation is depicted in Figure 4.2. In general, for $n\pi < L \leq (n+1)\pi$ the attractor contains $2n+3$ equilibrium points. The flow on the attractor can be described completely. In particular, for all $L > 0$ the flow $\phi(L, 0, 1)$ on the attractor is conjugated to a simple ODE (see [105]).

We now turn our attention back to the fourth order equation with $\gamma \in (0, \frac{1}{8}]$. Theorem 4.1 states that there exists a semi-conjugacy between the flow on the attractor of the fourth order equation and the corresponding flow for the second order equation with the same number of stationary solutions. This follows immediately from [105, Theorems 1.2 & 2.1], since our problem obeys the conditions required for the analysis presented there:

- The semi-flows $\phi(L, \gamma, 1)$ have compact global attractors.
- The equilibrium solutions are given by the bifurcation diagram of Figure 4.1. The zero solution undergoes generic supercritical pitchfork bifurcations, and the equilibria $u \equiv \pm 1$ are stable.
- There exists a Lyapunov functional $J_L[u]$ (given by (4.3)).

We remark that the theorem implies that the dynamics on the attractor are at least those of the second order equation. When we denote the solution on the k-th bifurcating branch by u_k, then there exists a connecting orbit going from u_k to u_l if and only if $k < l$ (hence $J_L(u_k) < J_L(u_l)$ for $k < l$, which can also be derived directly from [124]). The semi-conjugacy does not completely determine the flow on the attractor (as a conjugacy would), since it is unknown whether the problem has the Morse-Smale property. The following lemma shows that away from the bifurcation points the equilibrium points are hyperbolic. Thus, the information which is lacking in order be able to check the Morse-Smale property is a proof of the transversality of the intersection between unstable and stable manifolds of the equilibria (for the second order equation this follows from the lap number theorem [4, 82, 101]).

Lemma 4.4 *The nontrivial equilibrium solutions are hyperbolic.*

The proof of this lemma can be found in Section 4.10.

Again, the results in this section hold for a more general class of potentials $F(u)$. Analogous results also hold for the Navier boundary conditions ($u(t, 0) = u_{xx}(t, 0) = 0$ and

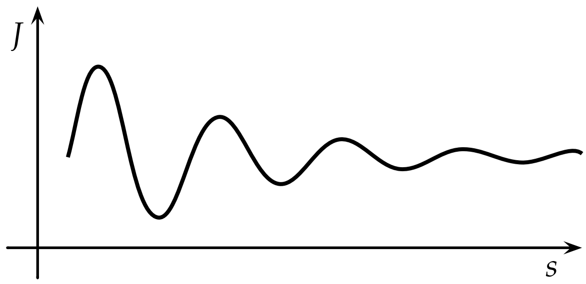

Figure 4.3: Sketch of the dependence of J on the interval length s for a gluing function close to a saddle-focus equilibrium.

$u(t,L) = u_{xx}(t,L) = 0$), and for the mixed case of Navier boundary conditions on one boundary and Neumann boundary conditions on the other boundary.

4.3 Homoclinic and heteroclinic minimisers

We start our investigation of Equation (4.1) with the Neumann boundary conditions $u_x(t,0) = u_{xxx}(t,0) = 0$ and $u_x(t,L) = u_{xxx}(t,L) = 0$ in the case that the equilibrium points are saddle-foci. Extending the solutions to $x \in \mathbb{R}$ by reflecting in $x = 0$ and $x = L$, one may regard equilibrium solutions u of (4.1) as a closed curves in (u, u_x)-plane by drawing the (u, u_x)-curve over one period. In Chapter 3 it was proved that, when we puncture the (u, u_x)-plane in $(\pm 1, 0)$, for *all* homotopy classes of closed curves in $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ there exist associated minimisers for J[4]. These minimisers lie on the energy level $E = 0$, where the energy is defined by (4.7). The periodic minimisers give rise to minimisers of J_L with Neumann boundary conditions, but the interval length is dictated by the homotopy type and thus they occur only for certain interval lengths L. Roughly speaking, when L is sufficiently large, the numbers $L \approx S_0 + nT_0 + m\omega_0$, $n, m \in \mathbb{N}$, occur as interval lengths, where S_0, T_0 and ω_0 are constants depending only on γ, β and F. The integer m can be written as $m = \sum_{i=1}^{n} m_i$, $m_i \in \mathbb{N}$ and for every n-tuple $(m_1, \ldots, m_n)$ there exists at least one minimiser with interval length $L \approx S_0 + nT_0 + m\omega_0$. We will prove that for values of L in between one can also find minimisers. Such minimisers do not necessarily lie on $E = 0$.

Let us briefly explain the idea. Trying to fit two pieces of solution together one uses a gluing function which lives in a small neighbourhood of the equilibrium point. In Figure 4.3 the dependence of the action J on the interval length s (on which the gluing takes place) is depicted for a saddle-focus equilibrium. The local minima and maxima correspond to solutions with energy $E = 0$. The minima have been found previously in Chapter 3, i.e., stable solutions are found for discrete values of the interval length. The intermediate solutions, although not local minima of the curve, can still be (local) minima of the action for *fixed* s. The gluing procedure can be made rigorous under transversality assumptions, see [36, 90] and Section 4.8. One may compare Figure 4.3 to the numerically obtained picture in Figure 4.14, but one should keep in mind that in Figure 4.14 only antisymmetric solutions are considered. In the absence of a transversality assumption, we follow a different approach.

In order to construct attracting sets which contain stable equilibria we will use the

[4]For *most* homotopy classes when the evenness assumption on F is dropped.

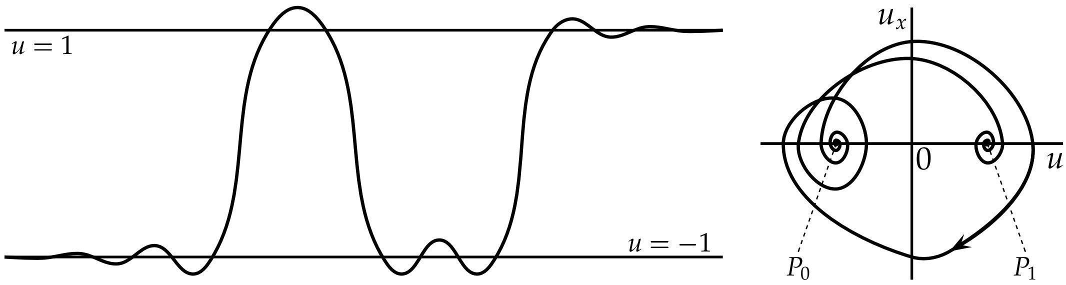

Figure 4.4: A heteroclinic solution with homotopy type $g = (2, 4)$. On the right the projection $\Gamma(u)$ of the orbit onto the (u, u_x)-plane has been depicted (schematically).

heteroclinic and homoclinic minimisers that were found in [89]. Let us first summarise the results of [89]. Consider the punctured plane $\mathcal{P} = \mathbb{R}^2 \setminus \{P_0, P_1\}$, where $P_0 = (-1, 0)$ and $P_1 = (+1, 0)$. Let u be a heteroclinic or homoclinic solution of (4.1) and let $\Gamma(u) = (u, u_x) : \mathbb{R} \to \mathcal{P}$ with $\Gamma(u(x))|_{x=\pm\infty} \in \{P_0, P_1\}$, and define its homotopy type as follows. As x goes from $x = -\infty$ to $x = \infty$, Γ can intersect the lines $L_- = \{(u, u_x) \in \mathcal{P} \,|\, u = -1\}$ and $L_+ = \{(u, u') \in \mathcal{P} \,|\, u = +1\}$. The number of consecutive intersections of L_- and L_+ is always even. We do not count the intersections of $L_\pm$ at start and finish. In between one obtains a finite sequence of even numbers denoted by $g = (g_1, \dots, g_k)$, which we call the *homotopy type* of Γ (see Figure 4.4 for an example). Note that given the homotopy type g one still has the freedom of choosing the initial point to be either P_0 or P_1. Whether Γ terminates at P_0 or P_1 then depends on the length of g.

If $F(u) = \frac{1}{4}(u^2 - 1)^2$ it follows from the results discussed in Section 4.2 that for $\frac{\gamma}{\beta^2} \le \frac{1}{8}$ the only minimisers are the constant solutions $u \equiv \pm 1$ and two heteroclinic connections with trivial homotopy type. On the contrary, for $\frac{\gamma}{\beta^2} > \frac{1}{8}$ it is proved in [89] that for any homotopy type g of any length there exists a 'geodesic' $\Gamma(u)$. In other words by minimising $J[u] = \int_{\mathbb{R}} j(u)$ over functions u for which the associated curve $\Gamma(u)$ has homotopy type g, a minimiser is found in every homotopy class[5]. The minimisation is carried out in classes of functions defined via the homotopy type, and there classes are denoted by $M(g, P_\nu)$, where $P_\nu \in \{P_0, P_1\}$ (i.e. $\nu \in \{0, 1\}$ and $(u, u_x)(-\infty) = P_\nu$ for all $u \in M(g, P_\nu)$). To be precise, let $\chi_0(x)$ be a smooth function such that $\chi_0(x) = -1$ for $x \le -1$ and $\chi_0(x) = 1$ for $x \ge 1$. Let $\chi_1(x) \equiv -1$, and let $\chi_i = \chi_{i \bmod 2}$ for $i \ge 2$. Then we define for all $m \ge 0$ and any $g \in \mathbb{N}^m$ (see [89]):

Definition 4.5 *A function u is in $M(g, P_\nu)$ if $u - (-1)^\nu \chi_m \in H^2(\mathbb{R})$ and if there exist non-empty subsets $\{A_i\}_{i=0}^{m+1}$ of $\mathbb{R}$ such that*

1. $u^{-1}(\pm 1) = \bigcup_{i=0}^{m+1} A_i$;
2. $\#A_i = g_i$ *for* $i = 1, \dots, m$;
3. $\max A_i < \min A_{i+1}$ *for* $i = 0, \dots, m$;
4. $u(x) = (-1)^{\nu+i+1}$ *for all* $x \in A_i$;
5. $\{\max A_0\} \cup \left(\bigcup_{i=1}^{m} A_i\right) \cup \{\min A_{m+1}\}$ *consists of transverse crossings of* ± 1.

Under these conditions $M(g, P_\nu)$ is an open set in $(-1)^\nu \chi + H^2(\mathbb{R})$. The function class with $m = 0$ is denoted by $M((0), P_\nu)$. We will use the notation $|g| = m$ if $g \in 2\mathbb{N}^m$, and drop the implicit dependence of $\chi_{|g|}$ on $|g|$ from the notation.

[5]This result is actually proved for general even potentials F under the condition that $\frac{\gamma}{\beta^2} > \frac{1}{4F''(\pm 1)}$.

Define

$$J(\boldsymbol{g}, P_\nu) = \inf_{u \in M(\boldsymbol{g}, P_\nu)} J[u],$$

where in this case the domain of integration is the entire real line. Finally, the set of global minimisers of J over the function class $M(\boldsymbol{g}, P_\nu)$ is denoted by

$$CM(\boldsymbol{g}, P_\nu) = \{u \in M(\boldsymbol{g}, P_\nu) \,|\, J[u] = J(\boldsymbol{g}, P_\nu)\}.$$

Since $M(\boldsymbol{g}, P_\nu)$ is an open set, minimisers $u \in CM(\boldsymbol{g}, P_\nu)$ satisfy the Euler-Lagrange equation

$$-\gamma u_{xxxx} + \beta u_{xx} + F'(u) = 0. \tag{4.8}$$

In [89] the following theorem is proved:

Theorem 4.6 *Let $F \in C^2(\mathbb{R})$ satisfy* (4.2) *and grow super-quadratically. Suppose that $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. Then*

(a) *if F is even: $J(\boldsymbol{g}, P_\nu)$ is attained for any $\boldsymbol{g}$.*

(b) *if F is not even: there exists a universal constant $N_0(F, \gamma, \beta) \in \mathbb{N}$ such that $J(\boldsymbol{g}, P_\nu)$ is attained for any $\boldsymbol{g} = (g_1, \dots g_m)$ with $g_i \in \{2\} \cup \{n \geq N_0\}$ for all $i = 1, \dots, m$.*

The homotopy types $\boldsymbol{g}$ selected in the above theorem are called *admissible* types. In the following we will always assume that F satisfies the assumptions in the above theorem, that $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$, and that $\boldsymbol{g}$ is an admissible homotopy type.

It has been proved in [89] that all minimisers obtained in Theorem 4.6 are *normalised*, i.e., all crossings of ± 1 are transverse, and between two consecutive crossings of ± 1 the function is either monotone or has exactly one local extremum.

As was already pointed out, in order to find stable solutions with respect to the Neumann boundary conditions we need to consider certain types of homoclinic connections found in [89]. Of particular interest are the symmetric types with an odd number of entries, i.e. $\boldsymbol{g} = (g_1, \dots, g_{2n+1})$ with $g_i = g_{2n+2-i}$. It follows from the minimising property that the curves Γ (and thus also the functions u) inherit the symmetry in $\boldsymbol{g}$, i.e., the functions u are symmetric with respect to the line $u_x = 0$ (cf. Lemma 3.35). To be precise, given a minimiser u there exists a point $x = x_0$ such that $u(x_0 + x) = u(x_0 - x)$. Since the minimisers are invariant under translations, one can choose a representative u such that $x_0 = 0$, and in particular we have $u_x(0) = u_{xxx}(0) = 0$. For the functions $u_- = u|_{\mathbb{R}^-}$ and $u_+ = u|_{\mathbb{R}^+}$ one can define the *restricted* homotopy type as before by counting the number of intersections of $\Gamma(u)$ with L_- and L_+. Thus $\boldsymbol{g}(u_-) = (g_1, \dots, g_n, \frac{g_{n+1}}{2})$ and $\boldsymbol{g}(u_+) = (\frac{g_{n+1}}{2}, g_{n+2}, \dots, g_{2n+1})$. Restricting to functions over $\mathbb{R}^+$ we still have the freedom of choosing the endpoint to be either P_0 or P_1. Define for all (restricted) homotopy types $\boldsymbol{g} = (g_1, \dots g_m)$ with $\boldsymbol{g}_1 \in \mathbb{N}$ and $g_i \in 2\mathbb{N}$ for $i = 2, \dots, m$,

$$M_{\mathbb{R}^+}(\boldsymbol{g}, P_\nu) = \{u \in (-1)^{\nu+1} + H^2(\mathbb{R}^+) \,|\, u_x(0) = 0,\ \boldsymbol{g}(u) = (\boldsymbol{g})\}.$$

Lemma 4.7 *The infima $J_{\mathbb{R}^+}(\boldsymbol{g}, P_\nu) = \inf_{u \in M_{\mathbb{R}^+}(\boldsymbol{g}, P_\nu)} J_{\mathbb{R}^+}[u]$ are precisely attained by $u_+ = u|_{\mathbb{R}^+}$ with $u \in CM(\boldsymbol{g}^{-1}\boldsymbol{g}, P_\nu)$, where $\boldsymbol{g}^{-1}\boldsymbol{g} = (g_m, \dots, g_2, 2g_1, g_2, \dots, g_m)$ (under the same assumptions as in Theorem 4.6).*

The minimisers of $J_{\mathbb{R}^+}(\boldsymbol{g}, P_\nu)$ in $M_{\mathbb{R}^+}(\boldsymbol{g}, P_\nu)$ are denoted by $CM_{\mathbb{R}^+}(\boldsymbol{g}, P_\nu)$. For periodic solutions one can set up the same construction (see Chapter 3). The homotopy type is now

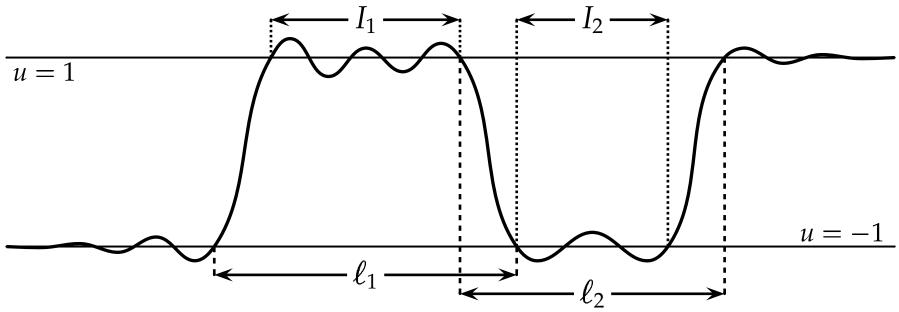

Figure 4.5: The intervals I_i and ℓ_i are indicated for a function of type $g = (6, 4)$.

determined over one period. The function classes and sets of global minimisers are denoted by $M_{\text{per}}(g, P_\nu)$ and $CM_{\text{per}}(g, P_\nu)$ respectively, and $J_{\text{per}}(g, P_\nu)$ is attained under the same assumptions as in Theorem 4.6.

4.4 A priori estimates

For the class of homoclinic and heteroclinic connections that were found in Theorem 4.6 we prove certain a priori estimates concerning their asymptotic behaviour. We assume throughout this section that for F even or F not even, the homotopy types are admissible (see Theorem 4.6). Also assume that $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$.

For easy notation we lift the translation invariance of minimisers of J by defining $CM_*(g, P_\nu) = CM(g, P_\nu)/\mathbb{R}$, represented by functions $u \in CM(g, P_\nu)$ with the property that $u(0) = (-1)^\nu$ and such that $(-1)^\nu u(x) < 1$ for all $x < 0$ (this corresponds to taking $\min(A_1) = 0$). For a minimiser $u \in CM_*(g, P_\nu)$ recall that the sets A_i represent the successive crossings of $(-1)^{\nu+i+1}$, $i = 1, .., |g|$ and define (see also Figure 4.5)

$$I_i \stackrel{\text{def}}{=} [\min A_i, \max A_i] \quad \text{and} \quad \ell_i \stackrel{\text{def}}{=} [\max A_{i-1}, \min A_{i+1}].$$

The a priori bounds on minimisers $u \in CM(g, P_\nu)$ obtained in this section will immediately carry over to minimisers on the half line on account of Lemma 4.7.

Lemma 4.8 *There exist constants C_1, C_2, $C_3 > 0$ such that for any admissible homotopy type g and any $u \in CM_*(g, P_\nu)$ it holds that*

$$\|u\|_{W^{1,\infty}(\mathbb{R})} \leq C_1,$$

and

$$\|u - (-1)^{\nu+i+1}\|_{W^{1,\infty}(\ell_i)} \geq C_2 e^{-C_3 g_i}, \qquad \text{for } i = 1, 2, \ldots, |g|,$$

where $\ell_i = [\max A_{i-1}, \min A_{i+1}]$.

Before proceeding with the proof of this lemma we first introduce the notion of covering spaces in the present context (see also Chapter 3). The fundamental group of $\mathcal{P} = \mathbb{R}^2 \setminus \{P_0, P_1\}$ is isomorphic to the free group on two generators e_1 and e_2, which represent loops (traversed clockwise) around $P_0 = (-1, 0)$ and $P_1 = (1, 0)$ respectively with basepoint $(0, 0)$. Since $\mathcal{P}$ represents the phase-plane, the curves corresponding to functions u only traverse the loops in the clockwise direction. Note that $\mathcal{P}$ is homotopic to a bouquet of two circles $X = S_1 \vee S_1$. The universal covering of X, denoted by $\widetilde{X}$, can be represented

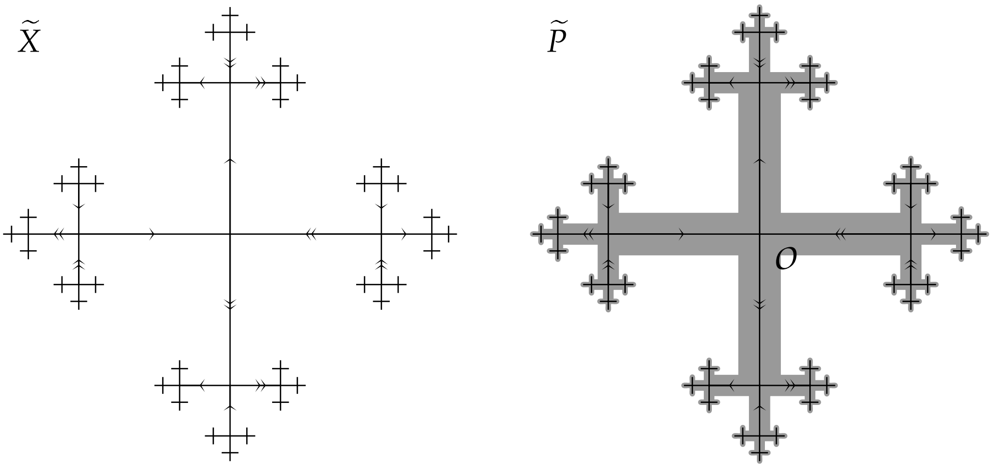

Figure 4.6: The universal covering $\widetilde{X}$ of $X = S_1 \vee S_1$ is a tree. The universal covering $\widetilde{\mathcal{P}}$ of $\mathcal{P}$ is a thickened version of $\widetilde{X}$. Its origin is denoted by O. The single and double arrows indicate the two different generators e_1 and e_2 which can only be traversed in one direction.

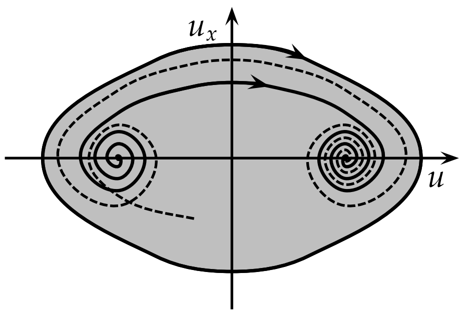

Figure 4.7: All minimisers in any class are bounded in the (u, u_x) plane from the outside by $u \in CM_{\text{per}}(2, 2)$ and from the inside by $u \in CM((0), P_\nu)$ (only $u \in CM((0), P_1)$ is depicted here). The dotted curve represents (part of) a minimiser.

by an infinite tree whose edges cover either e_1 or e_2 in X, see Figure 4.6. The universal covering of $\mathcal{P}$ denoted by $\pi : \widetilde{\mathcal{P}} \to \mathcal{P}$ can then be viewed as a thickened version of $\widetilde{X}$ so that $\widetilde{\mathcal{P}}$ is homeomorphic to an open disk in $\mathbb{R}^2$. The origin of $\widetilde{\mathcal{P}}$ will be denoted by O. Of course every point in $\mathcal{P}$ has many lifts. To be able to fix notation we distinguish a particular lift π^{-1} of the line $\{(0, u_x) \,|\, u_x \in \mathbb{R}\} \subset \mathcal{P}$ by requiring that $\pi^{-1}((0, 0)) = O$ and continuous extension. Denote $\pi^{-1}(\{(0, u_x)\})$ by $\mathcal{N} \subset \widetilde{\mathcal{P}}$.

We now turn to the proof of Lemma 4.8.

Proof. The first estimate (the outer bound) is proved in Theorem 3.27. It follows from the fact that all minimisers are bounded in the (u, u_x)-plane by a minimiser of class $M_{\text{per}}(2, 2)$ (see Figure 4.7). We will show that the second estimate in Lemma 4.8 comes from a similar argument where minimisers of class $M((0), P_\nu)$ take the role of inner bounds. The proof is completely analogous to the first estimate when we lift the problem to the covering space $\widetilde{\mathcal{P}}$. The idea is that all minimisers lie 'outside' the simple heteroclinic minimisers of type $\boldsymbol{g} = ((0), P_\nu)$, i.e., they spiral towards P_ν slower than these simple minimisers.

Let $u \in CM_*(\boldsymbol{g}, P_\nu)$ with $\boldsymbol{g} \neq (0)$. The idea now is to compare different lifts of $\Gamma(u)$ to $\widetilde{\mathcal{P}}$ with lifts of minimisers in $CM_*((0), P_\nu)$. Fix the index i to be any of the numbers $1, \ldots, |\boldsymbol{g}|$. Choose $u_0 \in CM_*((0), P_0)$ if $\nu + i$ is odd, and $u_0 \in CM_*((0), P_1)$ if $\nu + i$ is even.

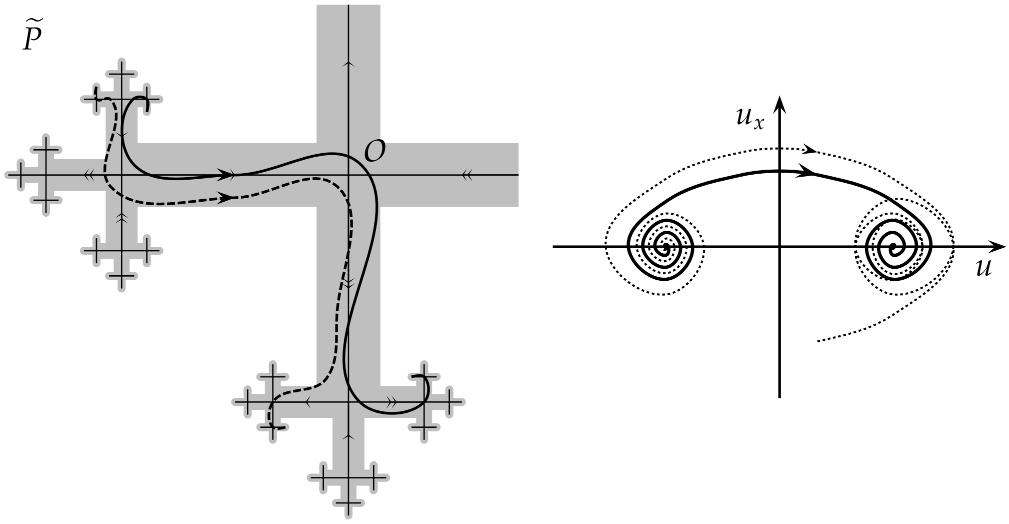

Figure 4.8: On the left: the lifts of the simple heteroclinic $\pi^{-1}(\Gamma(u_0))$ of class $g = ((0), P_1)$ and, as an example, a minimiser $\pi^{-1}(\Gamma(u))$ of class $g = ((4,2), P_0)$. On the right: the heteroclinic class $g = ((0), P_1)$ and, as an example, (part of) a minimiser of class $g = ((6), P_1)$.

Set $x_0 \stackrel{\text{def}}{=} \max\{x < \min(A_i) \,|\, u(x) = 0\}$. Now lift $\Gamma(u_0)$ and $\Gamma(u)$ to $\mathcal{P}$ requiring that both $\pi^{-1}(\Gamma(u_0(0))) \in \mathcal{N}$ and $\pi^{-1}(\Gamma(u(x_0))) \in \mathcal{N}$ (where $\mathcal{N} \subset \widetilde{\mathcal{P}}$).

We claim that the lifts $\pi^{-1}(\Gamma(u_0))$ and $\pi^{-1}(\Gamma(u))$ intersect at most once. Indeed, suppose they intersect twice in say y_0 and y_1, then their action J between y_0 and y_1 is equal, since they are both minimisers. This implies that one can replace u_0 by u between y_0 and y_1, and in this way one obtains another minimiser of the same homotopy type. Since all minimisers satisfy (4.8), this contradicts the uniqueness of the initial value problem, which proves our claim. In fact the same argument shows that, for $i = 1$, and $i = |g|$, the lifts $\pi^{-1}(\Gamma(u_0))$ and $\pi^{-1}(\Gamma(u))$ do not intersect at all.

For the remaining indices i we assert that if $\pi^{-1}(\Gamma(u_0))$ and $\pi^{-1}(\Gamma(u))$ intersect, then they do not cross. That is, if the curves have a point in common (intersect), then this intersection can be removed by an arbitrarily small perturbation (the intersection is tangent). Indeed, if the curves would cross, then there would be a second intersection point contradicting the statement above. This is most easily seen from the left picture in Figure 4.8 since both limits of $\pi^{-1}(\Gamma(u))$ as $x \to \pm\infty$ lie on the same side of $\pi^{-1}(\Gamma(u_0))$. It also follows that $\pi^{-1}(\Gamma(u))$ lies on the 'outside' of $\pi^{-1}(\Gamma(u_0))$, that is to say, on ℓ_i the curve $\Gamma(u)$ spirals around P_0 or P_1 outside the spiral of $\Gamma(u_0)$ (see Figure 4.8; right).

Finally, the elements in the set $CM_*((0), P_0)$ are ordered by their derivatives at the origin $u'(0)$ (since two minimisers cannot intersect in $\widetilde{\mathcal{P}}$). Besides, $CM_*((0), P_0)$ turns out to be compact (see Lemma 4.13). Hence there exists a smallest and a largest element of $CM_*((0), P_0)$ (measured in terms of $u'(0)$). The smallest element $\Gamma(u_0)$ spirals exponentially towards $P_{0,1}$ as $x \to \pm\infty$. A similar argument holds for $CM_*((0), P_1)$ (especially because these are the same functions with inverted x). Since all other minimisers spiral outside these minimal elements the second (exponential) estimate of the lemma follows. □

Another way to prove Lemma 4.8 is to construct annuli as covering spaces as was done in Chapter 3.

Remark 4.9 The proof also shows that the tails of any homoclinic or heteroclinic minimiser cannot spiral towards the equilibrium point faster than some fixed exponential rate. •

In [89] the *Uniform Separation Property* was introduced. This property is closely related to the question which types are admissible. Here the following result from [89] is used:

Lemma 4.10 *There exist a constant $C_4 > 0$ such that for any admissible homotopy type g and any $u \in CM_*(g, P_\nu)$ it holds that*

$$\|u - (-1)^{\nu+i}\|_{L^\infty(I_i)} \geq C_4,$$

where $I_i = [\min A_i, \max A_i]$

We now deduce a bound on the length of the interval between the tails.

Lemma 4.11 *There exists a constant $\delta_1 > 0$ so that for any admissible homotopy type g and any $\delta \leq \delta_1$ there exists constants $T_\delta^- < 0$ and $T_\delta^+ > 0$ such that for any $u \in CM_*(g, P_\nu)$*

$$\|u - (-1)^{\nu+1}\|_{W^{1,\infty}(-\infty, T_\delta^-)} < \delta, \qquad \|u - (-1)^{\nu+|g|}\|_{W^{1,\infty}(T_\delta^+, \infty)} < \delta.$$

Proof. First of all we analyse the tails. We choose $\delta_1 > 0$ so small that the local theory near the equilibrium points from Section 4 in [89] applies for all $\delta < \delta_1$. According to the local theory there exists a $0 < \delta_2 < \delta$ such that if a point $x_1 \in (-\infty, \min A_1)$ in the left tail of u is such that $|u(x_0) - (-1)^{\nu+1}| < \delta_2$ and $|u'(x_0)| < \delta_2$, then $\|u - (-1)^{\nu+1}\|_{W^{1,\infty}}(-\infty, x_1) < \delta$. This expresses the fact that $\Gamma(u)$ spirals towards P_ν as $x \to -\infty$. Of course a similar statement holds for the right tail.

Now choose $\kappa = \min\{\delta_2, C_4, C_2 e^{-C_3 \max_{1\leq i \leq |g|} g_i}\}$, where C_2, C_3 and C_4 are defined in Lemmas 4.8 and 4.10. We are going to estimate the measure of

$$K_\kappa \stackrel{\text{def}}{=} \{x \in \mathbb{R} \,|\, \text{dist}_{\mathbb{R}^2}\big((u(x), u_x(x)), \{P_0, P_1\}\big) < \kappa\},$$

or rather of its complement K_κ^c. By Lemmas 4.8 and 4.10 the interval $[\min A_1, \max A_m]$ is contained in K_κ^c if κ is sufficiently small. We assert that there is a constant $C > 0$ such that

$$J[u|_{K_\kappa^c}] \geq C|K_\kappa^c|\kappa^2.$$

Namely, considering $u \geq 0$ and $u < 0$ separately, we obtain that, for some $C > 0$, the inequality $j(u) \geq C\kappa^2$ holds pointwise for all $x \in K_\kappa^c$ (since F has non-degenerate equilibria). Since $J[u|_{K_\kappa^c}] < J(g, P_\nu)$ it follows that $|K_\kappa^c|$ is smaller than $\frac{J(g,P_\nu)}{C\kappa^2}$. Hence, choosing $|T_{\delta_1}^\pm| = \frac{J(g,P_\nu)}{C\kappa^2}$ we have proved the lemma. □

Our next aim is to obtain compactness of the set of minimisers. To proceed we need to convert to functions on a finite interval. The restriction of the minimisers in $CM_*(g, P_\nu)$ to $[T_\delta^-, T_\delta^+]$ is denoted by $CM_*^T(g, P_\nu)$. Let $H_*^2(T_\delta^-, T_\delta^+) = \{u \in H^2(T_\delta^-, T_\delta^+) \,|\, u(0) = (-1)^\nu\}$, then $CM_*^T(g, P_\nu) \subset H_*^2(T_\delta^-, T_\delta^+)$. Functions in $CM_*^T(g, P_\nu)$ can be mapped back to $CM_*(g, P_\nu)$ as follows. Define the map $E_0 : CM_*^T(g, P_\nu) \to CM_*(g, P_\nu)$:

$$E_0[u] = \begin{cases} \alpha\big(x - T_\delta^-, (u(T_\delta^-), u_x(T_\delta^-))\big) & x \in (-\infty, T_\delta^-], \\ u(x) & x \in [T_\delta^-, T_\delta^+], \\ \omega\big(x - T_\delta^+, (u(T_\delta^+), u_x(T_\delta^+))\big) & x \in [T_\delta^+, \infty), \end{cases} \tag{4.9}$$

where α and ω are unique minimisers of an appropriate functional, i.e., α is the *unique* minimiser (see e.g. [89]) for J over functions φ in $(-1)^{\nu+1} + H^2(-\infty, 0)$ for which $(\varphi(0), \varphi_x(0)) = (u(T_\delta^-), u_x(T_\delta^-))$. A similar definition holds for $\omega \in (-1)^{\nu+|g|} + H^2(0, \infty)$.

The map E_0 is well-defined for all $u \in H^2_*(T^-_\delta, T^+_\delta)$ for which $\mathrm{dist}_{\mathbb{R}^2}\big((u,u^+)(T^\pm_\delta), \{P_0, P_1\}\big)$ is sufficiently small (see e.g. [36, 89]), say $\mathrm{dist}_{\mathbb{R}^2}\big((u,u^+)(T^\pm_\delta), \{P_0, P_1\}\big) < \delta_3$.

We now fix
$$\delta = \delta_0 \stackrel{\text{def}}{=} \frac{1}{2}\min\{\delta_1, \delta_3\},$$
where δ_1 is defined in Lemma 4.11. Also fix $T^\pm \stackrel{\text{def}}{=} T^\pm_{\delta_0}$ (see Lemma 4.11). The set
$$V_\epsilon(\boldsymbol{g}, P_\nu) = \{u \in H^2_*(T^-, T^+) \,|\, \mathrm{dist}_{H^2}(u, CM^T_*(\boldsymbol{g}, P_\nu)) \le \epsilon\},$$
is a bounded neighbourhood of $CM^T_*(\boldsymbol{g}, P_\nu)$. For every $u \in V_\epsilon$ there exists $v \in CM^T_*(\boldsymbol{g}, P_\nu)$ such that $\|u - v\|_{H^2} \le \epsilon$ and thus $\|u - v\|_{W^{1,\infty}} \le \tilde{C}\epsilon$, where $\tilde{C}$ is the Sobolev embedding constant. When $\tilde{C}\epsilon < \delta_0$ then the map E_0 is well-defined on V_ϵ. If we choose
$$\epsilon \le \epsilon_0 \stackrel{\text{def}}{=} \min\{\delta_0, C_4, C_2 e^{-C_3 \max_{1\le i\le |g|} g_i}\}/\tilde{C},$$
then by Lemmas 4.8 and 4.10 the set $U_\epsilon = E_0[V_\epsilon]$ is contained $M_*(\boldsymbol{g}, P_\nu)$. Fix $\epsilon = \epsilon_0$ and write $V(\boldsymbol{g}, P_\nu) \stackrel{\text{def}}{=} V_{\epsilon_0}(\boldsymbol{g}, P_\nu)$. Of course, when necessary one can choose smaller values of ϵ and δ.

Corollary 4.12 *The map E_0 is well-defined for all $u \in V(\boldsymbol{g}, P_\nu)$ and the sets $U \stackrel{\text{def}}{=} E_0[V(\boldsymbol{g}, P_\nu)]$ are subsets of $M_*(\boldsymbol{g}, P_\nu)$.*

One now obtains the following compactness result.

Lemma 4.13 *For any admissible homotopy type $\boldsymbol{g}$ the set $CM_*(\boldsymbol{g}, P_\nu)$ is compact.*

Proof. The set $CM_*(\boldsymbol{g}, P_\nu) \subset (-1)^\nu\chi + H^2(\mathbb{R})$ is closed and bounded (follows from Lemmas 4.8 and 4.11). It remains to show that $CM_*(\boldsymbol{g}, P_\nu)$ is precompact. Let $\{u_n\}_{n=1}^\infty \subset CM_*(\boldsymbol{g}, P_\nu)$, then by Lemma 4.11 we have that
$$\mathrm{dist}_{\mathbb{R}^2}\big((u_n, u_{n,x})(x), \{P_0, P_1\}\big) \le \delta \quad \text{for } x \in [T^-, T^+]^c.$$
Define the functional $J^T \stackrel{\text{def}}{=} J \circ E_0$ on the bounded sets V. Since the functions u_n are minimisers it holds that $dJ^T[u_n] = dJ \circ E_0[u_n] = 0$, where the restriction of u_n to $[T^-, T^+]$ is again denoted by u_n. This yields the relation $0 = u_n + K[u_n]$, where K is a compact operator (cf. [90, Theorem 3.2]). For the sequence $\{u_n\}$ this implies that (possibly along a subsequence) u_n converges in $H^2(T^-, T^+)$ to some function u. Let us denote the tails of u_n on the intervals $(-\infty, T^-]$ and $[T^+, \infty)$ by α_n and ω_n respectively. Since δ_0 is sufficiently small and all α_n and ω_n satisfy Equation (4.8) it follows from the local theory near the equilibria that the tails α_n and ω_n also converge to $E_0[u]$ in $H^2(-\infty, T^-]$ and $H^2[T^+, \infty)$ respectively. Indeed, F has non-degenerate equilibria and thus $(F'(u_1) - F'(u_2))(u_1 - u_2) \ge \frac{1}{2}F''(\pm 1)(u_1 - u_2)^2$ for u_1 and u_2 sufficiently close to ± 1. Hence we obtain, using the differential equation, for some small $C > 0$
$$\begin{aligned}\gamma \int_{-\infty}^{T^-} |\alpha_{n,xx} - \alpha_{m,xx}|^2 + \beta \int_{-\infty}^{T^-} |\alpha_{n,x} - \alpha_{m,x}|^2 + C \int_{-\infty}^{T^-} |\alpha_n - \alpha_m|^2 \le \\ -\gamma(\alpha_{n,xxx} - \alpha_{m,xxx})(\alpha_n - \alpha_m)(T^-) + \gamma(\alpha_{n,xx} - \alpha_{m,xx})(\alpha_{n,x} - \alpha_{m,x})(T^-) \\ -\beta(\alpha_{n,x} - \alpha_{m,x})(\alpha_n - \alpha_m)(T^-).\end{aligned}$$
The right-hand side tends to 0 as $n, m \to \infty$, since $\alpha_n(-T)$ and $\alpha_{n,x}(-T)$ converge, and $\alpha_{n,xx}(-T)$ and $\alpha_{n,xxx}(-T)$ are bounded (this follows from regularity arguments). Therefore the sequence $\{u_n\}$ converges strongly, possibly along a subsequence, in $\chi + H^2(\mathbb{R})$, which concludes the proof. $\square$

For J^T we can derive the following geometric properties.

Lemma 4.14 *The set of all minimisers of J^T in $V(g,P_\nu)$ is given by $CM_*^T(g,P_\nu)$. Moreover, there exist constants $C_0 = C_0(g,F,\gamma,\beta) > 0$ such that $J^T|_{\partial V} \geq J(g,P_\nu) + C_0$.*

Proof. By definition $U = E_0[V]$ and thus $\inf_V J^T = \inf_U J \geq J(g,P_\nu)$. For $u \in CM_*^T(g,P_\nu) \subset V$ it follows that $J^T[u] = J(g,P_\nu)$ and therefore $\inf_V J^T = J(g,P_\nu)$. Clearly, if $J^T[u] = J(g,P_\nu)$ for some $u \in V$ then $E_0[u] \in CM_*(g,P_\nu)$ which proves the first claim.

Suppose there exists no constants C_0 such that $J^T|_{\partial V} \geq J(g,P_\nu) + C_0$. Then one can find a sequence $u_n \in \partial V$ such that $J^T[u_n] \to J(g,P_\nu)$. By Ekeland's variational principle [64] there exists a slightly different sequence $\tilde{u}_n$ with $\|\tilde{u}_n - u_n\|_{H^2(T^-,T^+)} \to 0$ as $n \to \infty$, such that $dJ^T[\tilde{u}_n] \to 0$, and $J^T[\tilde{u}_n] \leq J^T[u_n]$.

Since V is bounded it follows that there exists a subsequence, again denoted by $\tilde{u}_n$, such that $\tilde{u}_n \rightharpoonup u$ in $H^2(T^-,T^+)$ and $u_n \to u$ in $W^{1,\infty}(T^-,T^+)$. By the weak lower-semicontinuity of J we obtain the estimate $J^T[u] \leq J(g,P_\nu)$.

From the fact that $dJ^T[\tilde{u}_n] \to 0$ it follows, arguing as in the proof of Lemma 4.13, that $\tilde{u}_n \to u$ strongly in $H^2(T^-,T^+)$, hence $u_n \to u$, implying that $u \in \partial V$, and $E_0[u] \in M_*(g,P_\nu)$. From the definition of $J(g,P_\nu)$ it follows that $J^T[u] \geq J(g,P_\nu)$. Together with the reversed inequality which was already obtained, this implies that $u \in \partial V$ is a minimiser, a contradiction. $\square$

Remark 4.15 The constant C_0 in the above lemma depends on the homotopy type g. In Section 4.7 we will prove that when we the neighbourhood $V(g,P_\nu)$ is defined in a different way, C_0 can be chosen independent of g for a large class of homotopy types g. •

4.5 Stable equilibrium solutions

The a priori properties of minimisers can be used now to construct stable equilibria for Equation (4.1) via a minimisation procedure partly based on techniques used in [36] and [90]. Our first goal is to construct stable equilibria for (4.1) that satisfy the Neumann boundary conditions.

We split two symmetric homoclinics and glue the two halves together by matching their tails (see Figure 4.9). The length of the plateau thus formed in the middle can be arbitrarily long. Since our initial homoclinic minimisers are not necessarily isolated we have to perform a careful gluing procedure in special subsets V of the function space, so that the infimum of J on ∂V is strictly larger than infimum of J on V, and hence the minimum is attained in the interior of V.

Another way to express 'splitting' of symmetric homoclinic minimisers is to take minimisers from $CM_{\mathbb{R}^\pm}(g,P_\nu)$. Minimisers in $CM_{\mathbb{R}^\pm}(g,P_\nu)$ are obtained from minimisers in $CM(g^{-1}g,P_\nu)$ in the following way. Normalise functions in $CM(g^{-1}g,P_\nu)$ by setting $u(0) = 0$ at the unique point of even symmetry. The sets $CM_{\mathbb{R}^-}(g,P_\nu)$ and $CM_{\mathbb{R}^+}(g,P_\nu)$ are then obtained by restricting to the intervals $(-\infty,0]$ and $[0,\infty)$ respectively. For functions in $CM(g^{-1}g,P_\nu)$ that are normalised as described above, we now have that the conclusions of Lemma 4.11 hold for $|x| > T = (T^+ - T^-)/2$. Define $CM^T_{\mathbb{R}^-}(g,P_\nu)$ and $CM^T_{\mathbb{R}^+}(g,P_\nu)$ as the restrictions of functions in $CM_{\mathbb{R}^-}(g,P_\nu)$ and $CM_{\mathbb{R}^+}(g,P_\nu)$ to the intervals $[-T,0]$ and $[0,T]$ respectively. Let $H^2_n(0,T) = \{u \in H^2(0,T)\,|\,u_x(0) = 0\}$ and $H^2_n(-T,0) = \{u \in$

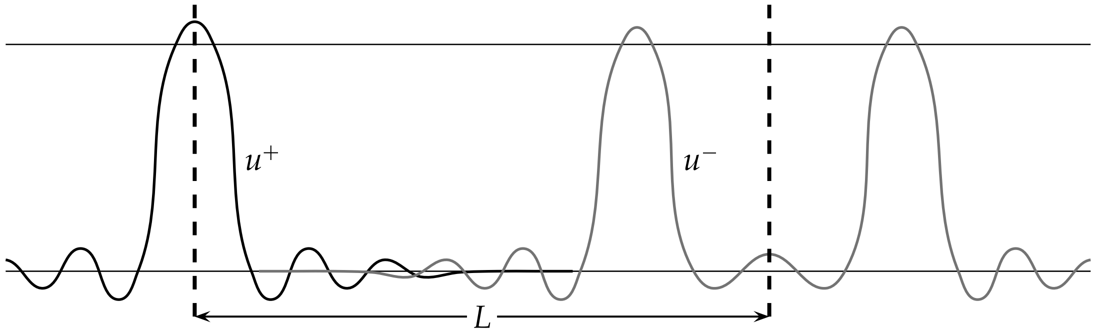

Figure 4.9: Two symmetric homoclinic minimisers which have to be glued together to produce a stable stationary solution on the interval $[0, L]$ satisfying Neumann boundary conditions.

$H^2(-T, 0) \,|\, u_x(0) = 0\}$, then $CM^T_{\mathbb{R}^-}(\boldsymbol{g}, P_\gamma) \subset H^2_n(-T, 0)$ and $CM^T_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma) \subset H^2_n(0, T)$. As in the previous section we can define the map $E_0 : CM^T_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma) \to CM_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma)$:

$$E_0^+[u] = \begin{cases} u(x) & x \in [0, T] \\ \omega\big(x - T, (u(T), u_x(T))\big) & x \in [T, \infty). \end{cases}$$

By the same token we define the map $E_0^- : CM^T_{\mathbb{R}^-}(\boldsymbol{g}, P_\gamma) \to CM_{\mathbb{R}^-}(\boldsymbol{g}, P_\gamma)$. The functionals $J_{\mathbb{R}^-} \circ E_0^-$ and $J_{\mathbb{R}^+} \circ E_0^+$ are well-defined on $CM^T_{\mathbb{R}^-}(\boldsymbol{g}, P_\gamma)$ and $CM^T_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma)$ respectively. As in the previous section we can define ϵ-neighbourhoods of $CM^T_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma) \subset H^2_n(0, T^+)$ and $CM^T_{\mathbb{R}^-}(\boldsymbol{g}, P_\gamma) \subset H^2_n(0, T^-)$, which we indicate by $V^+(\boldsymbol{g}^+)$ and $V^-(\boldsymbol{g}^-)$ respectively. The functionals $J^\pm_T$ are well-defined on these neighbourhoods if ϵ is small enough, say $\epsilon \le \epsilon_0(\boldsymbol{g})$ (see Corollary 4.12). The following is an immediate consequence of Lemma 4.14.

Lemma 4.16 *The set of all minimisers of J^+_T over V^+ is given by $CM^T_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma)$. Moreover, there exist constants $C_0 = C_0(\boldsymbol{g}, F, \gamma, \beta) > 0$ such that $J_{\mathbb{R}^+} \circ E_0^+|_{\partial V^+(\boldsymbol{g}, P_\gamma)} \ge J_{\mathbb{R}^+}(\boldsymbol{g}, P_\gamma) + C_0$. The same statement holds for $J_{\mathbb{R}^-} \circ E_0^-$.*

We now use Lemma 4.16 to construct neighbourhoods $V \subset H^2_N(0, L)$ with the property that $\inf_{\partial V} J > \inf_V J$, where

$$H^2_N(0, L) \stackrel{\text{def}}{=} \{u \in H^2(0, L) \,|\, u_x(0) = u_x(L) = 0\}.$$

In order to do so we again invoke the local theory near the equilibrium points (see Theorems 4.1 and 4.2 in [89]). Take $\bar{y} = (y_1, y_2)$ and $\bar{z} = (z_1, z_2)$, with both $|\bar{y} - (1, 0)| < \delta_1$ and $|\bar{z} - (1, 0)| < \delta_1$ and δ_1 sufficiently small (in fact one can take the same value as in Lemma 4.11). Then the boundary value problem for Equation (4.8) on an interval of length s with left and right boundary conditions given by $(u, u')(0) = \bar{y}$ and $(u, u')(s) = \bar{z}$ has a unique global minimiser if s is larger than some constant, say $s > S_0 = S_0(F, \gamma, \beta, \delta_1)$. This minimiser is denoted by $g(x, \bar{y}, \bar{z}, s)$. A similar construction is carried out for $\bar{y}$ and $\bar{z}$ close to $(-1, 0)$.

Let $\boldsymbol{g}^-$ and $\boldsymbol{g}^+$ be two admissible homotopy types, i.e. $\boldsymbol{g}^\pm = (g_1^\pm, .., g^\pm_{|\boldsymbol{g}^\pm|})$, with $g_1^\pm \in \mathbb{N}$ and $g_i^\pm \in 2\mathbb{N}$ for $i = 2, .., |\boldsymbol{g}^\pm|$. Define the map $E_2^s : CM^T_{\mathbb{R}^+}(\boldsymbol{g}^+, P_\gamma) \times CM^T_{\mathbb{R}^-}(\boldsymbol{g}^-, P_\gamma) \to H^2_N(0, 2T + s)$ as follows:

$$E_2^s[u^+, u^-] = \begin{cases} u^+(x) & x \in [0, T] \\ g(x - T, (u^+(T), u_x^+(T)), (u^-(-T), u_x^-(-T)), s) & x \in [T, T + s] \\ u^-(x - 2T - s) & x \in [T + s, 2T + s] \end{cases}$$

Arguing as in Section 4.4, since $\delta_0 \leq \frac{1}{2}\delta_1$ it follows that when $\epsilon = \min\{\epsilon_0(g^+), \epsilon_0(g^-)\}$, the functional $J_s^T \stackrel{\text{def}}{=} J_{2T+s} \circ E_2^s : V^+(g^+) \times V^-(g^-) \to \mathbb{R}$ is well-defined for any $s > S_0$.

The estimate of Lemma 4.16 carries over to the current situation.

Lemma 4.17 *There exist constants S_1, $C_0(g^-)$, and $C_0(g^+)$ such that*

$$\inf_{\partial(V^+(g^+)\times V^-(g^-))} J_s^T \geq \inf_{V^+(g^+)\times V^-(g^-)} J_s^T + \frac{1}{2}\min\{C_0(g^+), C_0(g^-)\}$$

for all $s \geq S_1$.

Proof. For any pair $(u^+, u^-) \in V^+(g^+) \times V^-(g^-)$ we have that

$$\begin{aligned} J_s^T[u^+, u^-] &= \textstyle\int_0^T j(u^+) + \int_0^s j(g) + \int_{-T}^0 j(u^-) \\ &= J_{\mathbb{R}^+} \circ E_0^+[u^+] - \textstyle\int_0^\infty j(\omega) + \int_0^s j(g) - \int_{-\infty}^0 j(\alpha) + J_{\mathbb{R}^-} \circ E_0^-[u^-] \\ &= J_{\mathbb{R}^+} \circ E_0^+[u^+] + J_{\mathbb{R}^-} \circ E_0^-[u^-] + A(s), \end{aligned}$$

where $A(s) = -\int_0^\infty j(\omega) + \int_0^s j(g) - \int_{-\infty}^0 j(\alpha)$. The behaviour of $A(s)$ is governed by the linear flow near a saddle-focus and we find that $A(s) = O(e^{-c_0 s})$ for $s \to \infty$, where $c_0 = c_0(F, \gamma, \beta) > 0$. Indeed, $A(s) = \int_0^{s/2}[j(g) - j(\omega)] - \int_{s/2}^\infty j(\omega) + \int_{-s/2}^0 [j(g(x+s)) - j(\alpha)] - \int_{-\infty}^{-s/2} j(\alpha)$, and each integral decays exponentially in s. For the second and fourth term this follows from the linearisation of the flow near the non-degenerate equilibrium point. Besides, for the first term we obtain, in a similar manner as in the proof of Lemma 4.13, that $\|\omega - g\|_{H^2(0,s/2)}$ is controlled by boundary terms and hence is of order $O(e^{-c_1 s})$ for some $c_1 > 0$. It then follows that $\int_0^{s/2} j(g) - j(\omega) = O(e^{-c_2 s})$ for some $c_2 > 0$, since ω and g are close to the (non-degenerate) equilibrium point. An analogous argument deals with the term $\int_{-s/2}^0 j(g(x+s)) - j(\alpha)$.

We choose $S_1 \geq S_0$ such that $A(s) \leq \frac{1}{4}\min\{C_0(g^+), C_0(g^-)\}$ for all $s \geq S_1$. Applying Lemma 4.16 now finishes the proof. □

The information of Lemma 4.17 can be used to find minimisers for J_s^T in $V^+(g^+) \times V^-(g^-)$ for all $s \geq S_1$. Indeed, let $(u_n^+, u_n^-) \in V^+(g^+) \times V^-(g^-)$ be a minimising sequence for J_s^T, for $s \geq S_1$ fixed. Then $\|u_n^+\|_{H_n^2(0,T)} + \|u_n^-\|_{H_n^2(-T,0)}$ is bounded and thus $(u_n^+, u_n^-) \rightharpoonup (u^+, u^-) \in H_n^2(0,T) \times H_n^2(-T,0)$. In exactly the same way as in the proof of Lemma 4.14 one obtains that in fact $(u_n^+, u_n^-) \to (u^+, u^-)$ strongly in $H_n^2(0,T) \times H_n^2(-T,0)$. It follows that $(u^+, u^-) \in V^+(g^+) \times V^-(g^-)$, and since J_s^T is weakly lower-semicontinuous we derive that (u^+, u^-) is a minimiser of J_s^T on $V^+(g^+) \times V^-(g^-)$. The fact that the sets $V^+(g^+) \times V^-(g^-)$ contain minimisers for J_s^T does not necessarily imply that the functions $E_2^s[u^+, u^-]$ are solutions of Equation (4.8). However, since the minimisers (u^+, u^-) lie in the interior of $V^+(g^+) \times V^-(g^-)$ one can prove that $E_2^s[u^+, u^-]$ are local minimisers for J and hence solutions of (4.8).

Lemma 4.18 *Let (u^+, u^-) be a minimiser of J_T^s in $V^+(g^+) \times V^-(g^-)$. For all $\phi \in H_N^2(0, 2T+s)$ with $\|\phi\|_{H^2}$ sufficiently small it holds that $J_{2T+s}[E_2^s[u^+, u^-] + \phi] \geq J_{2T+s}[E_2^s[u^+, u^-]]$. Moreover, the function $v = E_2^s[u^+, u^-]$ satisfies Equation (4.8) with the Neumann boundary conditions $u_x(0) = u_{xxx}(0) = 0$ and $u_x(2T+s) = u_{xxx}(2T+s) = 0$.*

Proof. Since the minimiser $u = E_2^s[u^+, u^-]$ lies in $\mathrm{int}(V^+(g^+) \times V^-(g^-))$ one can find small open neighbourhoods $W^+ \subset V^+(g^+)$ and $W^- \subset V^-(g^-)$ of u^+ and u^- respectively such that $J_s^T[u^+ + \phi^+, u^- + \phi^-] \geq J_s^T[u^+, u^-]$ for all $(u^+ + \phi^+, u^- + \phi^-) \in W^+ \times W^-$.

Let $W \subset H^2_N(0, 2T+s)$ be a small neighbourhood of $u = E^s_2[u^+, u^-]$, i.e., $v \in W$ can be written as $v = u + \phi$, with $\phi \in H^2_N(0, 2T+s)$ and $\|\phi\|_{H^2}$ small. If the neighbourhood W is small enough then $\phi^- = \phi|_{[T+s,2T+s]} \in W^-$ and $\phi^+ = \phi|_{[0,T]} \in W^+$. The part in the middle, $\phi|_{[T,T+s]}$, is denoted by ϕ^0. We can write $v + \phi^0 = v + \hat{\phi}^0 + (\phi^0 - \hat{\phi}^0)$, where $v + \hat{\phi}^0$ is the unique minimiser of $J_{[T,T+s]}$ over functions with boundary conditions at $x = T$ and $x = T+s$ equal to $(u^+ + \phi^+, u^+_x + \phi^+_x)(T)$ and $(u^- + \phi^-, u^-_x + \phi^-_x)(T+s)$ respectively, i.e., functions of the form $v|_{[T,T+s]} + \psi_0$ with $\psi^0 \in H^2_0(T, T+s)$.

We now have that

$$
\begin{aligned}
J_{2T+s}[E^s_2[u^+, u^-] + \phi] &= \int_0^T j(u^+ + \phi^+) + \int_T^{T+s} j(v + \phi^0) + \int_{T+s}^{2T+s} j(u^- + \phi^-) \\
&\geq \int_0^T j(u^+ + \phi^+) + \int_T^{T+s} j(v + \hat{\phi}^0) + \int_{T+s}^{2T+s} j(u^- + \phi^-) \\
&= J_{2T+s}[E^s_2[u^+ + \phi^+, u^- + \phi^-]] = J^T_s[u^+ + \phi^+, u^- + \phi^-] \\
&\geq J^T_s[u^+, u^-] = J_{2T+s}[E^s_2[u^+, u^-]].
\end{aligned}
$$

This proves the first claim. From the fact that $u = E^s_2[u^+, u^-]$ is a local minimiser of J_{2T+s} one easily deduces that u satisfies Equation (4.8) and the Neumann boundary conditions. $\square$

The next step is to construct proper attracting neighbourhoods in $H^2_N[0, 2T+s]$ for Equation (4.1) that contain the equilibria $E^s_2[u^+, u^-]$. Let $\phi \in B_r(0) \subset H^2_0(T, T+s)$ and consider the triples $(u^+, u^-, \phi) \in V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-) \times B_r(0)$. Define the map $\mathcal{F}^s : V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-) \times B_r(0) \to H^2_N(0, 2T+s)$ as follows: $\mathcal{F}^s(u^+, u^-, \phi) = E^s_2[u^+, u^-] + \tilde{\phi}$, where $\tilde{\phi} \in H^2_0(0, 2T+s)$ is the extension by zero of ϕ. Set $Y \stackrel{\text{def}}{=} \mathcal{F}^s\big(V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-) \times B_r(0)\big)$. We want to show that $\inf_{\partial Y} J_{2T+s} > \inf_Y J_{2T+s}$, and from Lemma 4.17 we see that the remaining problematic boundary of Y is $V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-) \times \partial B_r(0)$. However, if we for example choose r large enough, then this problem is overcome and

$$
J_{2T+s}[u] = J[E^s_2[u^+, u^-] + \tilde{\phi}] \geq \inf_{V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)} J^T_s + \tilde{C}_0 \qquad \text{for all } u \in \partial Y,
$$

for some $\tilde{C}_0 \in \big(0, \frac{1}{2}\min\{C_0(\boldsymbol{g}^-), C_0(\boldsymbol{g}^+)\}\big)$.

Let $\mathcal{S}$ be a the set of minimisers of J^T_s in $V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)$. As before $u \in Y$ is in $\mathcal{S}$ if and only if there is a pair (u^+, u^-) which minimises J^T_s on $V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)$ with $u = \mathcal{F}^s(u^+, u^-, 0)$. We will now show that $\mathcal{S}$ is stable.

Let $\eta < \epsilon = \min\{\epsilon_0(\boldsymbol{g}^-), \epsilon_0(\boldsymbol{g}^+)\}$, then $B_\eta(\mathcal{S}) = \{u \in H^2(0, 2T+s) \,|\, \operatorname{dist}_{H^2}(u, \mathcal{S}) < \eta\}$ is contained in Y (for r large enough).

As before, we find that

$$
a \stackrel{\text{def}}{=} \frac{1}{2}\Big(\inf_{\partial B_\eta(\mathcal{S})} J_{2T+s} - \inf_{B_\eta(\mathcal{S})} J_{2T+s} \Big) > 0.
$$

Define $N^a_\epsilon = J^a_{2T+s} \cap B_\eta(\mathcal{S})$, where J^a_{2T+s} is the sub-level set

$$
J^a_{2T+s} = \{u \in H^2_N(0, 2T+s) \,|\, J_{2T+s}[u] \leq \inf_{B_\eta(\mathcal{S})} J_{2T+s} + a\}.
$$

It follows that $J_{2T+s}|_{\partial N^a_\epsilon} = a$. Since Equation (4.1) is the L^2-gradient flow equation of J, the quantity $J_{2T+s}[u(t,x)]$ decreases in t, and thus for initial data $u(0,x) = u_0(x) \in N^a_\epsilon$ it holds that $u(t,x) \in N^a_\epsilon$ for all $t > 0$. This proves that $\mathcal{S}$ is a stable set for Equation (4.1). Since $s > S_1$ is arbitrary and this construction can be carried out for all admissible homotopy types $\boldsymbol{g}^+$ and $\boldsymbol{g}^-$, we obtain the following theorem (Theorem 4.3 in the introduction).

Theorem 4.19 *Let* $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$*. Then for any* $n \in \mathbb{N}$ *there exists a constant* $L_n > 0$ *such that for all* $L \geq L_n$ *Equation* (4.1) *with the Neumann boundary conditions has at least* n *disjoint sets of stable equilibria (in the sense of Definition 4.2).*

4.6 Different boundary conditions

Theorem 4.3 states that Equation (4.1) has an arbitrary number of stable equilibria provided that the interval length L is large enough. In the previous section we proved this in the case of Neumann boundary conditions. The result remains unchanged for various other types of boundary conditions.

In the case of the Neumann boundary conditions the stable solutions are constructed using minimisers defined on the half-spaces $\mathbb{R}^+$ and $\mathbb{R}^-$, which satisfy the Neumann boundary conditions at $x = 0$. These minimisers are derived from the homoclinic minimisers found in [89].

Now consider Equation (4.1) with the so-called *Navier* boundary conditions: $u(t, 0) = u(t, L) = 0$, $u_{xx}(t, 0) = u_{xx}(t, L) = 0$. In order to construct stable equilibria we need to find minimisers on the half-spaces $\mathbb{R}^+$ and $\mathbb{R}^-$ which satisfy the boundary conditions $u(0) = u_{xx}(0) = 0$. If the potential F is even such minimisers can be derived from the results in [89]. Indeed, consider heteroclinic minimisers with homotopy type $(g_m, .., g_1, g_1, .., g_m)$. From Chapter 3 and [89] it then follows that such minimisers are odd with respect to a unique point of odd symmetry. Due to translation invariance we can choose this point to be $x = 0$. The restriction such a minimiser to the intervals $\mathbb{R}^+$ and $\mathbb{R}^-$ now satisfies the boundary conditions $u(0) = u_{xx}(0) = 0$. From this point on the construction of stable equilibria is identical to the construction carried out in the previous section. The statement of Theorem 4.3 for the case of the Navier boundary conditions remains unchanged. Although this construction can only be carried out when F is even, the result also holds when F is not even, as we will shortly see.

Another set of boundary conditions that can be considered, are *Dirichlet* boundary conditions. General Dirichlet boundary conditions for (4.1) are $(u(t, 0), u_x(t, 0)) = \bar{y} = (y_1, y_2)$ and $(u(t, L), u_x(t, L)) = \bar{z} = (z_1, z_2)$. The minimisers on the half-spaces $\mathbb{R}^+$ and $\mathbb{R}^-$ needed for the construction of stable equilibria cannot be found via the results in [89]. To obtain such minimisers on for example $\mathbb{R}^+$, we minimise $J_{\mathbb{R}^+}[u]$ over functions u for which the induced curve $\Gamma(u)$ starts at $\bar{y}$ and terminates at P_1 (or P_0), and which has a certain homotopy type $\boldsymbol{g}$. The homotopy $\boldsymbol{g}$ is defined as before by counting the number of consecutive crossings of the lines $u = -1$ and $u = 1$ excluding the intersections in the tail. This leads to the homotopy vector $\boldsymbol{g} = (g_1, .., g_m)$, with $g_1 \in \mathbb{N}$ and $g_i \in 2\mathbb{N}$ for $i = 2, \ldots, m$. The function classes of a given homotopy $\boldsymbol{g}$ and initial point $\bar{y}$ are denoted by $M_{\mathbb{R}^\pm}(\boldsymbol{g}, \bar{y})$. The potential F is not assumed to be even here. As in [89] (see also Theorem 4.6) there exists a universal constant $N_0(\bar{y})$ such that, for homotopy types $\boldsymbol{g}$ with $g_i \geq N_0$ or $g_i = 2$, the infima of $J_{\mathbb{R}^\pm}$ over $M_{\mathbb{R}^\pm}(\boldsymbol{g}, \bar{y})$ are attained. These minimisers are again the building blocks for constructing stable solutions of the Dirichlet problem. Consequently, the statement of Theorem 4.3 also holds for the Dirichlet boundary conditions.

Let us now come back to the Navier boundary conditions when the potential F is not even. In this case the minimisers on the half-spaces $\mathbb{R}^\pm$, needed for the construction of stable solutions, are found in function classes in the space $\{u \in H^2(\mathbb{R}^+) \mid u(0) = 0\}$. It follows from the variational principle that minimisers satisfy the second boundary condition $u_{xx}(0) = 0$.

The various boundary conditions discussed above are not the only possibilities. For example, one can also treat non-homogeneous Neumann and non-homogeneous Navier boundary conditions. Furthermore, one can consider various types of mixed boundary conditions. The bottom line is that as long as one considers boundary conditions for which Equation (4.8) has a variational principle, then the method in this chapter applies and a variant of Theorem 4.3 can be obtained.

4.7 Estimating the number of equilibria

Some of the estimates obtained in Sections 4.4 and 4.5 can be made uniform with respect to the homotopy type $\boldsymbol{g}$. With such uniform estimates one can obtain a lower bound on the number of stable solutions of Equation (4.1) as a function of L. The crucial ingredient in this context is the constant introduced in Lemma 4.14:

$$C_0 = \inf_{\partial V} J[u] - \inf_{V} J[u].$$

We recall from Section 4.4 that fixing γ, β and F, one has that ϵ_0 only depends $\max_{1\leq i\leq|\boldsymbol{g}|} g_i$. The following lemma is a uniform analogue of Lemma 4.14 and shows that, with an appropriate choice of the neighbourhood V the constant C_0 also depends on $\max_{1\leq i\leq|\boldsymbol{g}|} g_i$ only.

Lemma 4.20 *For all $N_* \in \mathbb{N}$ there exists positive constants C_0, D_1 and D_2 such that for any admissible homotopy type $\boldsymbol{g}$ with $g_i \leq 2N_*$ for all $i = 1, 2, \ldots |\boldsymbol{g}|$, there exists a bounded neighbourhood $V(\boldsymbol{g}, P_\nu) \subset H^2_*(T^-, T^+)$ of $CM^T_*(\boldsymbol{g}, P_\nu)$ with $|T^\pm| \leq D_1 + D_2|\boldsymbol{g}|$, such that $E_0[V(\boldsymbol{g}, P_\nu)] \subset M_*(\boldsymbol{g}, P_\nu)$ and $\inf_{\partial V} J \circ E_0 - J(\boldsymbol{g}, P_\nu) > C_0$.*

It should be clear that we need to restrict the magnitude of g_i to get such a uniform estimate, since the higher g_i the closer $CM_*(\boldsymbol{g}, P_\nu)$ gets to the boundary of the class $M_*(\boldsymbol{g}, P_\nu)$, i.e., the more oscillations around one of the equilibrium points the closer the function approaches the equilibrium. Note however that the length $|\boldsymbol{g}|$ of the homotopy type is arbitrary. This is made possible by an appropriate choice of $V(\boldsymbol{g}, P_\nu)$, which will be discussed later on.

Before we prove the lemma we will first explain how the lemma can be used to count the number of equilibria (or attracting sets) as $L \to \infty$. Our goal is to derive the exponential lower bound on the number of stable equilibria as a function of L, mentioned in Equation (4.4). Choosing $V(\boldsymbol{g}, P_\nu)$ as in Lemma 4.20 it follows from the proof of Lemma 4.17 that S_1 depends on N_* only (since C_0 depends on N_* only). We now fix $N_* > 1$ and only consider $\boldsymbol{g}$ with $g_i \leq 2N_*$.

One can now construct stable solutions of (4.1) as in Section 4.5 by using building blocks $(u^+, u^-) \in V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)$ for which $g_i^-, g_j^+ \leq 2N_*$. The solutions are defined on intervals of length $L = T(\boldsymbol{g}^+) + T(\boldsymbol{g}^-) + s$ with $s \geq S_1$. Since $s \geq S_1$ can be chosen arbitrarily, a stable solution of such type then exist for all interval lengths $L \geq T(\boldsymbol{g}^+) + T(\boldsymbol{g}^-) + S_1$. Since $T(\boldsymbol{g}^\pm) \leq D_1 + D_2|\boldsymbol{g}^\pm|$ by Lemma 4.20, a stable solution thus exist for all interval lengths $L \geq 2D_1 + D_2(|\boldsymbol{g}^+| + |\boldsymbol{g}^-|) + S_1$. Hence we obtain a stable solutions on an interval of length L for every pair $(\boldsymbol{g}^+,\boldsymbol{g}^-)$ with $g_i^-, g_j^+ \le 2N_*$ such that $|\boldsymbol{g}^+| + |\boldsymbol{g}^-| \le (L - S_1 - 2D_1)/D_2$. The number of such pairs to grows as $(N_*)^{(L-S_1-2D_1)/D_2}$, that is, exponentially in L. This establishes Equation (4.4).

To prove Lemma 4.20 we first recall the Uniform Separation Property from [89] (see also Lemma 4.10) which holds for all admissible types $\boldsymbol{g}$:

> *Uniform Separation Property*: There exists a $\tilde{\delta} > 0$ and an $\tilde{\varepsilon} > 0$ such that for all admissible homotopy types $\boldsymbol{g}$ and all $u \in M(\boldsymbol{g}, P_\nu)$ with $J[u] \le J(\boldsymbol{g}, P_\nu) + \tilde{\delta}$ we have $|u(x) - (-1)^{\nu+i}| > \tilde{\varepsilon}$ for all $x \in I_i$, $i = 0, \ldots, |\boldsymbol{g}| + 1$.

Although in [89] $\tilde{\varepsilon}$ depends on $\boldsymbol{g}$ and the Uniform Separation Property is only used for so-called normalised functions, the constant ϵ can in fact be chosen independent of $\boldsymbol{g}$ and in absence of normalisation.

The justification of the construction of the neighbourhoods V needed in Lemma 4.20 is quite technical. First define

$$W_\epsilon \stackrel{\text{def}}{=} \{u \in M_*(\boldsymbol{g}, P_\nu) \,|\, \mathrm{dist}_{\mathbb{R}^2}\big(\Gamma(u|_{I_{\mathrm{core}}}), P_i\big) > \epsilon \text{ for } i = 0, 1\},$$

where $I_{\mathrm{core}} = [\max A_0, \min A_{|\boldsymbol{g}|}]$ is the *core* interval. Next define

$$U_{\epsilon,\delta} \stackrel{\text{def}}{=} \{u \in W_\epsilon \,|\, J[u] < J(\boldsymbol{g}, P_\nu) + \delta\}.$$

By Lemmas 4.8 and 4.10 the set W_ϵ is a neighbourhood of $CM_*(\boldsymbol{g}, P_\nu)$ for ϵ small enough and all $\boldsymbol{g}$ with $g_i \le 2N_*$. By the Uniform Separation Property we have $\overline{U}_{\epsilon,\delta} \subset M_*(\boldsymbol{g}, P_\nu)$ for δ small enough.

In order to reduce to function on a finite interval, define

$$\mathcal{U}_\eta^{T^\pm} \stackrel{\text{def}}{=} \Big\{u \in H^2_*(T^-, T^+) \;\Big|\; \mathrm{dist}_{\mathbb{R}^2}\big(\Gamma(u(T^-)), P_\nu\big) < \eta,\ \mathrm{dist}_{\mathbb{R}^2}\big(\Gamma(u(T^+)), P_{\nu+|\boldsymbol{g}|-1 \bmod 2}\big) < \eta\Big\},$$

where η is chosen so small that E_0 (see Section 4.4) is well-defined on $\mathcal{U}_\eta^{T^\pm}$. In what follows η is fixed. The following lemma shows that $U_{\epsilon,\delta} \subset \mathcal{U}_\eta^{T^\pm}$ for $T^\pm$ large enough.

Lemma 4.21 *There exist constants $\tilde{\delta}(\eta) > 0$, $\tilde{T}(\eta) > 0$ such that for any $\delta < \tilde{\delta}$ and any $\boldsymbol{g}$ with $g_i \le 2N_*$ for all $i = 1, 2, \ldots |\boldsymbol{g}|$ (and η and ϵ small enough) it holds that when $u \in U_{\epsilon,\delta}$ then $u \in \mathcal{U}_\eta^{T^\pm}$, with $T^- = C^-(\boldsymbol{g}) - \tilde{T}$ and $T^+ = C^+(\boldsymbol{g}) + \tilde{T}$, where the constants $C^\pm(\boldsymbol{g})$ can be chosen such that $C^\pm < \tilde{C}|\boldsymbol{g}|$ for some $\tilde{C}$ independent of $\boldsymbol{g}$ and η.*

Proof. The functions u in $U_{\epsilon,\delta}$ are uniformly bounded in $W^{1,\infty}$. Indeed, a function $u \in M_*(\boldsymbol{g}, P_\nu)$ with large $W^{1,\infty}$-norm can be easily modified to a function $\tilde{u} \in M_*(\boldsymbol{g}, P_\nu)$ with $J[\tilde{u}] < J[u] - C$ for some $C > \delta$ (the appropriate estimates can be found for example in [89, Lemma 5.1]). This contradiction shows that such u (with large $W^{1,\infty}$-norm) are not in $U_{\epsilon,\delta}$.

It follows from a test function argument (cf. [89, Section 4]) that there exists a constant $C > 0$, independent of g_i, such that $J[u|_{\ell_i}] < C$, and thus $J[u|_{I_{\mathrm{core}}}] \le C|\boldsymbol{g}|$. Since $u \in W_\epsilon$, i.e., $\Gamma(u)$ stays away from the equilibrium points $(\pm 1, 0)$, this implies that $|I_{\mathrm{core}}| \le \tilde{C}|\boldsymbol{g}|$ for some $\tilde{C} = \tilde{C}(\tilde{\varepsilon}) > 0$.

After taking care of the core interval, we need to estimate the tails. The action of the tails is also uniformly bounded by a test function argument. For δ smaller than $\tilde{\delta}$ (defined in the Uniform Separation Property above) this implies that the norm $\|u - (-1)^\nu\|_{H^2(-\infty,\max(A_0))}$ of the left tail is uniformly bounded (and similarly for the right tail).

Taking $\tilde{T} = \tilde{T}(\tilde{\eta})$ large enough there exists a point $x_1 \in [\max(A_0) - \tilde{T}, \max(A_0)]$ such that $\Gamma(u(x_1)) \in B_{\tilde{\eta}}(P_\nu)$. From, again, a test function argument and the local behaviour

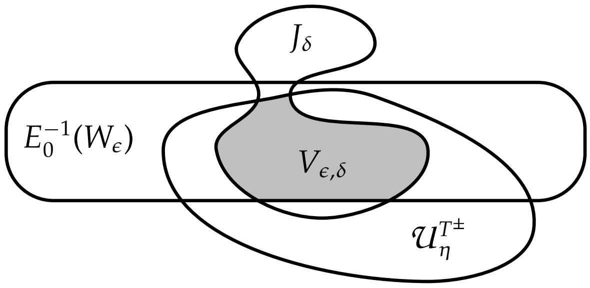

Figure 4.10: The boundary of $V_{\epsilon,\delta}$ a priori consists of three parts, since it is the intersection of $\mathcal{U}_\eta^{T^\pm}$, the sub-level set J_δ and the set $E_0^{-1}(W_\epsilon)$. When δ is sufficiently small then the (appropriate part of) the sub-level set is contained in $\mathcal{U}_\eta^{T^\pm}$.

near the equilibrium it follows that for $\tilde{\eta}$ small enough $J[u|_{(-\infty,x_1)}] \leq c_1\tilde{\eta}^2 + \delta$ for some $c_1 > 0$. On the other hand, in order for $\Gamma(u)$ to go from $\partial B_{\eta/2}(P_\nu)$ to $\partial B_\eta(P_\nu)$, it costs at least an amount $c(\eta) > 0$ of action. Take $\tilde{\eta} < \eta/2$ and moreover choose $\tilde{\eta} = \tilde{\eta}(\eta)$ and $\delta = \delta(\eta)$ so small that $c_1\tilde{\eta}^2 + \delta < c(\eta)$. This ensures that $\Gamma(u(x)) \in B_\eta(P_\nu)$ for all $x \leq x_1$ (and $x_1 \in [\max(A_0) - \tilde{T}, \max(A_0)]$). Taking $T^\pm = \tilde{C}|g| + \tilde{T}$ we obtain that $u \in \mathcal{U}_\eta^{T^\pm}$. □

Finally, we pick up the proof of Lemma 4.20. Let $\tilde{\delta}(\eta)$ and $T^\pm$ be as in Lemma 4.21. We next define the neighbourhoods V needed in Lemma 4.20:

$$V_{\epsilon,\delta}(g, P_\nu) \stackrel{\text{def}}{=} \{u \in \mathcal{U}_\eta^{T^\pm} \mid E_0[u] \in W_\epsilon \text{ and } J \circ E_0[u] < J(g, P_\nu) + \delta\}.$$

This is a bounded neighbourhood of $CM_*^T(g, P)$. Moreover, the construction of V is such that ∂V consists of three parts, i.e., any $u \in \partial V$ satisfies one of the following possibilities (see also Figure 4.10):

- $J \circ E_0[u] = J(g, P_\nu) + \delta$;
- $\Gamma(u(T^\pm)) \in \partial B_\eta(P_\nu)$;
- $E_0[u] \in \partial W_\epsilon$.

The first possibility is no problem, since we in fact want to show that $\inf_{\partial V} J \circ E_0 - \inf_V J \circ E_0$ is bounded away from zero (uniformly in g). The second possibility is excluded by choosing $\delta \leq \tilde{\delta}(\frac{\eta}{2})$ so that $u \in \mathcal{U}_{\eta/2}^{T^\pm}$ by Lemma 4.21. The third possibility is dealt with in the next lemma, which states that for such u we have $J \circ E_0[u] \geq J(g, P_\nu) + \tilde{C}_0$ for some $\tilde{C}_0 > 0$ if δ and ϵ are sufficiently small. Taking $C_0 = \min\{\tilde{C}_0, \delta\}$ finishes the proof of Lemma 4.20.

The following lemma deals with the third of the three possibilities above.

Lemma 4.22 *There exist constants $\tilde{C}_0$ and ϵ_0 such that for δ sufficiently small and any g with $g_i \leq 2N_*$ for all $i = 1, 2, \ldots |g|$ it holds that when $u \in V_{\epsilon_0,\delta}$ and $E_0[u] \in \partial W_{\epsilon_0}$, then $J \circ E_0[u] \geq J(g, P_\nu) + \tilde{C}_0$.*

Proof. Assume by contradiction that such $\tilde{C}_0$ and ϵ_0 do not exist. Thus, for all δ_0 and ϵ_0 there exist functions $u_n \in \overline{V}_{\epsilon_0,\delta_0}(g^n, P_{\nu_n})$ and $u_n \in \partial W_{\epsilon_0}(g^n, P_{\nu_n})$ such that $J[u_n] - J(g^n, P_{\nu_n}) \to 0$ as $n \to \infty$. We will choose δ_0 and ϵ_0 later on.

By taking a subsequence we may take ν_n constant, say $\nu_n = 0$, and we will drop P_ν from our notation. Let $x_n \in I^n_{\text{core}}$ be points such that

$$\operatorname{dist}_{\mathbb{R}^2}\big(\Gamma(u_n(x_n)), ((-1)^{k_n}, 0)\big) = \epsilon_0.$$

Again taking a subsequence, we may assume that k_n is constant in the previous expression, to fix ideas say $k_n = 0$ for all n (the case $k_n = 1$ is analogous).

We now want to locate the points x_n, and for this purpose we define the following sets (see also Figure 4.5 for the definition of ℓ_i and I_i):

$$S_i \stackrel{\text{def}}{=} \begin{cases} \ell_i & \text{if } i \text{ is odd,} \\ I_i & \text{if } i \text{ is even,} \end{cases} \qquad \text{for } i = 1 \ldots |g| - 1,$$

and

$$S_{|g|} \stackrel{\text{def}}{=} \begin{cases} \ell_{|g|} & \text{if } |g| \text{ is odd,} \\ [\max A_{|g|}, \min A_{|g|+1}] & \text{if } |g| \text{ is even.} \end{cases}$$

These sets cover the core interval, i.e. $I_{\text{core}} = \cup_{i=1}^{|g|} S_i$. The points x_n are in at least one of these sets S_i, say S_{i_n}. Taking a subsequence we may assume that one of the following three cases holds:

1. $1 < i_n < |g|$ for all n;
2. $i_n = 1$ for all n;
3. $i_n = |g|$ for all n.

We will exclude each of these three possibilities by choosing ϵ_0 and δ_0 small enough.

We start with Case 1. Taking a subsequence one may assume that i_n either is odd for all n, or even for all n. In the latter case we easily reach a contradiction by choosing $\epsilon_0 < \tilde{\epsilon}$ and $\delta_0 < \tilde{\delta}$, where $\tilde{\epsilon}$ and $\tilde{\delta}$ are defined in the Uniform Separation Property above.

We now deal with the case that i_n is odd for all n, which is somewhat more complicated. Taking a subsequence we can assume that $g^n_{i_n}$ is constant, say $g^n_{i_n} = \tilde{g} \in 2\mathbb{N}$. Shift all u_n so that $x_n = 0$ for all n. We now take another subsequence such that $g^n_{i_n-1}$ and $g^n_{i_n+1}$ are independent of n as well, say $g^n_{i_n-1} = \tilde{g}_l$ and $g^n_{i_n-1} = \tilde{g}_r$.

Let $I_n \stackrel{\text{def}}{=} [\max(A_{i_n-2}), \min(A_{i_n+2})]$. The functions u_n are uniformly bounded in $W^{1,\infty}$, as discussed in the proof of Lemma 4.21. By a test function argument it follows that $J[u_n|_{I_n}]$ is bounded, which in turn (since $u \in \overline{W}_{\epsilon_0}$) implies that $|I_n|$ and $\|u_n\|_{H^2(I_n)}$ are bounded.

Take a weak limit of u_n (along a subsequence) in H^2_{loc} which converges to v weakly in H^2_{loc} and strongly in $W^{1,\infty}_{\text{loc}}$. We have that $\text{dist}_{\mathbb{R}^2}\big(\Gamma(v(0)), (1,0)\big) = \epsilon_0$. The intervals I_n and S_{i_n} converge to intervals I_v and S_v respectively. It holds that $v(x) = 1$ on ∂I_v, and $v(x) = -1$ on ∂S_v. Besides, $v(x)$ has on I_v subsequently $\tilde{g}_r$ crossings of -1, then $\tilde{g}$ crossings of $+1$ (in fact these crossings occur in S_v), and finally $\tilde{g}_l$ crossings of -1.

Moreover, it is not too difficult to conclude that $v|_{I_v}$ is a minimiser of J in the sense of Definition 3.4, i.e., among function with the same boundary conditions (i.e., matching to $(v, v')|_{\partial I_v}$) and the same number of crossings of ± 1, where the interval length is arbitrary. However, such minimisers satisfy the result of Lemma 4.8 on the interval S_v, i.e., $\|v - 1\|_{W^{1,\infty}(S_v)} > c_1 e^{-2c_2 N_*}$ for some c_1, $c_2 > 0$. We now take $\epsilon_0 < c_1 e^{-2c_2 N_*}$ to reach a contradiction, i.e. contradicting the fact that $\text{dist}_{\mathbb{R}^2}\big(\Gamma(v(0)), (1,0)\big) = \epsilon_0$. Hence, the possibility in Case 1 is excluded.

In Case 2 a very similar argument holds. Namely, arguing along the same lines we now define $I_n = [T^-, \min(A_3)]$ (or $[T^-, T^+]$ if $|g| = 1$). We again find a weak limit v and $v|_{I_v}$ is a minimiser of $J \circ E_0$ in the same sense as above, i.e., $E_0[v]|_{(-\infty, \max(I_v))}$ is a minimiser of J among function with the same boundary conditions (instead of a left boundary conditions one takes functions in $-1+H^2$) and the same number of crossings of ± 1. A contradiction is reached as in the previous case.

Case 3 is completely analogous to Case 2, except that we now use Remark 4.9 to reach a contradiction.

Having reached a contradiction in all three cases, we have proved the lemma. $\square$

4.8 The bifurcation

In this section we analyse the bifurcation that occurs at $\frac{\gamma}{\beta^2}=\frac{1}{8}$. In particular, for $\frac{\gamma}{\beta^2}$ slightly larger than $\frac{1}{8}$ we will completely describe the set of stationary solutions for all $L>0$. Without loss of generality we set $\beta=1$:

$$-\gamma u_{xxxx}+u_{xx}+u-u^3=0, \tag{4.10a}$$

$$u_x(0)=u_{xxx}(0)=u_x(L)=u_{xxx}(L)=0. \tag{4.10b}$$

We stress that the bifurcation analysis in the present section is the only part of this chapter where we need transversality information.

4.8.1 The finite dimensional reduction

As discussed in Section 4.2, for $\gamma=\frac{1}{8}$ the bifurcation diagram is as depicted in Figure 4.1. The results of Chapter 2, which are used in Section 4.2, can also be applied to $\gamma>\frac{1}{8}$. One obtains the following: the only solutions of (4.10a) with $\|u\|_\infty \le \frac{4\gamma+1}{12\gamma}$ (any $\gamma>0$) are $u\equiv 0$ and a one parameter family of periodic solutions, symmetric with respect to their extrema and antisymmetric with respect to their zeros. This family of periodic solutions can be parametrised by the energy or by the period. Denote this continuous family, including $u\equiv 0$, by $\mathcal{F}_\gamma$. These solutions of (4.10) form the skeleton of the bifurcation diagram.

The additional solutions that appear in the bifurcation diagram for γ slightly larger than $\frac{1}{8}$ are all in a small neighbourhood of the heteroclinic cycle. We denote the unique monotonically increasing heteroclinic solutions at $\gamma=\frac{1}{8}$ by u_0, and we divide out the translational invariance by fixing $u_0(0)=0$. Let the heteroclinic cycle in phase space be

$$\Delta=\{(\pm 1,0,0,0)\}\cup\{\pm(u_0(x),u_0'(x),u_0''(x),u_0'''(x))\,|\,x\in\mathbb{R}\},$$

and define $B_\epsilon(\Delta)$ to be the ϵ-neighbourhood of Δ in $\mathbb{R}^4$.

Lemma 4.23 *There exists a constant $\epsilon_0>0$ such that for all $0<\epsilon<\epsilon_0$ there exists a $\delta_0=\delta_0(\epsilon)>0$ such that for all $\frac{1}{8}<\gamma<\frac{1}{8}+\delta_0$ any bounded solution of (4.10a) is either an element of $\mathcal{F}_\gamma$ or its orbit is entirely contained in $B_\epsilon(\Delta)$.*

Proof. Suppose by contradiction that the assertion does not hold. Then there exists an $\epsilon>0$ and sequence $\gamma_n\downarrow\frac{1}{8}$ with corresponding bounded solutions u_n of (4.10a), such that $u_n\notin\mathcal{F}_{\gamma_n}$ and $(u_n,u_n',u_n'',u_n''')(x_n)\notin B_\epsilon(\Delta)$ for some $x_n\in\mathbb{R}$.

After translation we may assume that $x_n=0$ for all $n\in\mathbb{N}$. Since bounded solutions of (4.10a) are uniformly bounded in $W^{3,\infty}$ there exists a subsequence, again denoted by u_n, which converges in C^3_{loc} on compact sets to some limit function u. This function u is a bounded solution of (4.10a) for $\gamma=\frac{1}{8}$. Since $(u,u',u'',u''')(0)\notin B_\epsilon(\Delta)$ we have that u is one of the solutions in $\mathcal{F}_{\frac{1}{8}}$ (this follows from the complete classification of bounded solutions at $\gamma=\frac{1}{8}$). Therefore $\mathcal{E}[u]\in(0,\frac{1}{4}]$ and $\|u\|_\infty<1$. In particular

$\|u\|_\infty < \frac{4\gamma_n+1}{12\gamma_n}$ for n sufficiently large. We now assert that $\|u_n\|_\infty \to \|u\|_\infty$, which implies that $u_n \in \mathcal{F}_{\gamma_n}$ for n sufficiently large, a contradiction. Indeed, we show that $u_n \to u$ in phase space, i.e. orbital convergence, which implies that $\|u_n\|_\infty \to \|u\|_\infty$. First notice that $\mathcal{E}[u_n] \to \mathcal{E}[u]$, since this holds for $x = 0$. Let $B_\epsilon(u)$ be the ϵ-neighbourhood of $\{(u(x), u'(x), u''(x), u'''(x)) \,|\, x \in \mathbb{R}\}$. Suppose now, by contradiction, that there exists a constant $\eta > 0$ such that $\mathrm{dist}_{\mathbb{R}^4}\big((u_n, u_n', u_n'', u_n''')(x_n), B_\epsilon(u)\big) > \eta$ for some points $x_n \in \mathbb{R}$. As before, taking a subsequence, we obtain that $u_n(x + x_n)$ converges in C^3 on compact sets to some limit function v. Again, v is a bounded solution of (4.10a) for $\gamma = \frac{1}{8}$ and $\mathrm{dist}_{\mathbb{R}^4}\big((v, v', v'', v''')(0), B_\epsilon(u)\big) \geq \eta$. On the other hand it follows that $\mathcal{E}[v] = \lim_{n\to\infty} \mathcal{E}[u_n] = \mathcal{E}[u]$. Since there is only one bounded solution of (4.10a) with $\gamma = \frac{1}{8}$ in each energy level $\mathcal{E} \in (0, \frac{1}{4}]$ we conclude that $u \equiv v$ modulo translation, a contradiction. □

For $\gamma = \frac{1}{8}$ the heteroclinic orbit is the unique, transversal intersection of $W^u(-1)$ and $W^s(+1)$. For γ slightly larger than $\frac{1}{8}$ this transversal intersection persists. This enables us to glue the two heteroclinics (going from -1 to $+1$ and back) together to form multitransition solutions. In particular we can find, for γ sufficiently close to $\frac{1}{8}$, all solutions of (4.10) in a neighbourhood of the heteroclinic cycle. This method has already been successfully applied in [90] to show that there is a countable infinity of heteroclinic solutions. Besides, in [131] the stability of multiple-pulse solutions converging to a saddle-focus was studied via a reduction to a finite-dimensional center manifold (when the pulses are far apart). Here we will use the transversality to find *all solutions of* (4.10) *and their index*.

Let u_0 be the unique monotonically increasing heteroclinic solution of (4.10a) at $\gamma = \frac{1}{8}$. The transversality implies that $d^2J[u_0]$ is an invertible operator on $H^2_c(\mathbb{R}) = \{u \in H^2(\mathbb{R}) \,|\, u(0) = 0\}$, where we have made the usual identification $(H^2)^* = H^2$. Moreover, since u_0 is a non-degenerate minimum of J one has $(d^2J[u_0]v, v) \geq C_0\|v\|^2$ for some $C_0 > 0$ and all $v \in H^2_c(\mathbb{R})$. As in Section 4.4 we consider the restriction of u_0 to a large finite interval $[-T, T]$. The tails can be recovered by an application of the extension map E_0 defined in (4.9). Note that E_0 also depends on γ. Taking T large enough this extension map $E_0^\gamma[u]$ is well-defined in a small neighbourhood of u_0 in $H^2_c(-T, T)$ for γ close to $\frac{1}{8}$. A perturbation argument shows that there exists a $C_1 > 0$ such that $(d^2(J \circ E_0^\gamma)[u]v, v) \geq C_1\|v\|^2$ for all u in a small η-neighbourhood $U_\eta(u_0) \subset H^2_c(-T, T)$ of u_0, all $v \in H^2_c(-T, T)$ and for all γ sufficiently close to $\frac{1}{8}$ and T sufficiently large.

To glue transitions from -1 to $+1$ and vice versa together, we introduce several gluing functions, as in Section 4.5. Write $\vec{u}$ for the pair (u, u'). For $\bar{y} = (y_1, y_2)$ and $\bar{z} = (z_1, z_2)$ close to $(\pm 1, 0)$ and for large s we define $g_l(x, \bar{y}, s)$, $g_r(x, \bar{y}, s)$ and $g(x, \bar{y}, \bar{z}, s)$ as the unique local solutions of (4.10a) near the equilibrium points $u = \pm 1$, such that

$$\begin{aligned} g_l'(0, \bar{y}, s) = 0, \quad g_l'''(0, \bar{y}, s) = 0 \quad &\text{and} \quad \vec{g}_l(s, \bar{y}, s) = \bar{y}; \\ \vec{g}_r(0, \bar{y}, s) = \bar{y} \quad &\text{and} \quad g_r'(s, \bar{y}, s) = 0, \quad g_r'''(s, \bar{z}, s) = 0; \\ \vec{g}(0, \bar{y}, \bar{z}, s) = \bar{y} \quad &\text{and} \quad \vec{g}(s, \bar{y}, \bar{z}, s) = \bar{z}. \end{aligned}$$

Here we have implicitly assumed that it will be clear from the context whether these solutions are close to $+1$ or close to -1. The functions g are the unique solutions of the boundary value problem which lie entirely in a small neighbourhood of the equilibrium point in phase space. On the other hand, in function space they are the unique global minimisers of the corresponding variational problem, and the unique critical points a neighbourhood of ± 1 in H^2. By symmetry one has $g_l(x,(y_1,y_2),s) = g(x+s,(y_1,-y_2),(y_1,y_2),2s)$, and similarly for g_r. Note that the solutions g also depend on γ.

We now glue n transitions together. Let $S_k = (2k+1)T + \sum_{i=0}^{k} s_i$, and define for $n \geq 1$ the gluing maps $E_n^\gamma = E_n^\gamma[u_1,\ldots,u_n;s_0,\ldots,s_n]$ as

$$E_n^\gamma = \begin{cases} g_l\big(t,\vec{u}_0(-T),s_0\big) & \text{for } t\in[0,S_0-T] \\ u_1(t-S_0) & \text{for } t\in[S_0-T,S_0+T] \\ g\big(t-S_0-T,\vec{u}_1(T),\vec{u}_2(-T),s_1\big) & \text{for } t\in[S_0+T,S_1-T] \\ u_2(t-S_1) & \text{for } t\in[S_1-T,S_1+T] \\ \vdots & \\ g\big(t-S_{n-2}-T,\vec{u}_{n-1}(T),\vec{u}_n(-T),s_{n-1}\big) & \text{for } t\in[S_{n-2}+T,S_{n-1}-T] \\ u_n(t-S_{n-1}) & \text{for } t\in[S_{n-1}-T,S_{n-1}+T] \\ g_r\big(t-S_{n-1}+T,\vec{u}_n(T),s_n\big) & \text{for } t\in[S_{n-1}+T,S_n-T]. \end{cases}$$

This gluing function is well-defined for $(u_1,\ldots,u_n)$ in a product neighbourhood $V_\eta = U_\eta(u_0)\times U_\eta(-u_0)\times\cdots\times U_\eta((-1)^{n-1}u_0)$ in $\big(H_c^2(-T,T)\big)^n$, and for $s_0,\ldots,s_n$ large enough. Note that $E_1^\gamma[u] \to E_0^\gamma[u]$ as $s_0,s_1\to\infty$. Similarly $E_2^\gamma[u_1,u_2]$ tends to a concatenation of $E_0^\gamma[u_1]$ and $E_0^\gamma[u_2]$ as $s_0,s_1,s_2\to\infty$, etcetera.

Introduce the notation $\boldsymbol{u} = (u_1,\ldots,u_n)$ and $\boldsymbol{s} = (s_0,\ldots,s_n)$. For fixed $\boldsymbol{s}$ we can find the unique critical point of $J_L\circ E_n^\gamma$ in the product neighbourhood V_η. This is easily seen by using the following fixed point argument. Consider the iteration (with $\mathbf{1}_n$ the unit matrix in $\mathbb{R}^n$)

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k - \big(d^2(J\circ E_0^\gamma)[u_0]\,\mathbf{1}_n\big)^{-1} d_{\boldsymbol{u}}(J_L\circ E_n^\gamma)[\boldsymbol{u}_k;\boldsymbol{s}].$$

This is a contraction on V_η for η sufficiently small (say $0<\eta\leq\eta_0$) and $|\gamma-\frac{1}{8}|<\delta_1(\eta)$ and $\min(\boldsymbol{s}) \stackrel{\text{def}}{=} \min_{0\leq i\leq n} s_i > \sigma(\eta)$. Here $\delta_1(\eta)$ and $\sigma(\eta)$ are positive constants which, as a function of η, are non-decreasing and non-increasing respectively. For an explicit calculation of the derivative $d_u(J_L\circ E_n^\gamma)$ we refer to [90]. The contraction thus has a unique fixed point $\boldsymbol{z}(\boldsymbol{s})$ which depends smoothly on $\boldsymbol{s}$ for $\min(\boldsymbol{s})>\sigma(\eta)$. Since $(d^2(J\circ E_0^\gamma)[u]v,v)\geq C_1\|v\|^2$ it follows that $\boldsymbol{z}(\boldsymbol{s})$ is the minimiser of $J_L\circ E_n^\gamma$ on V_η. We substitute this vector into the action and obtain

$$\mathcal{K}_n(\boldsymbol{s}) \stackrel{\text{def}}{=} J_L\circ E_n^\gamma[\boldsymbol{z}(\boldsymbol{s});\boldsymbol{s}].$$

The variational problem has thus been reduced to a finite dimensional setting. Solutions of (4.10) correspond to critical points of $\mathcal{K}_n(\boldsymbol{s})$ under the constraint $\sum_{i=0}^n s_i = L-2nT$.

Lemma 4.24 *Let $\eta\leq\eta_0$, let $\gamma\in(\frac{1}{8},\frac{1}{8}+\delta_1(\eta))$ and let $\boldsymbol{s}$ with $\min(\boldsymbol{s})>\sigma(\eta)$ be a critical point of $\mathcal{K}_n$ under the constraint $\sum_{i=0}^n s_i = L-2nT$. Then $E_n^\gamma[\boldsymbol{z}(\boldsymbol{s});\boldsymbol{s}]$ is a solution of (4.10). The index of the critical point $\boldsymbol{s}$ (under the constraint) is equal to the index of the solution $E_n^\gamma[\boldsymbol{z}(\boldsymbol{s});\boldsymbol{s}]$.*

Proof. It is immediately clear that $u = E_n^\gamma[\boldsymbol{z}(\boldsymbol{s}),\boldsymbol{s}]$ is a piecewise solution of the differential equation. We assert that these pieces connect nicely to a solution on the whole interval. Let v be a function in H_N^2 in a small neighbourhood of u. Then v has precisely n zeros, say at $x_1,\ldots,x_n$. Let $v_i(x-x_i) = v(x)|_{[x_i-T,x_i+T]}$ and $t_0 = x_1-T$ and $t_i = x_{i+1}-x_i-2T$, $1\leq i\leq n-1$ and $t_n = L-x_n-T$. Then v can be written as $v = E_n^\gamma[v_1,\ldots,v_n;t_0,\ldots,t_n]+\sum_{i=0}^n\phi_i$ with $\phi_i\in H_0^2(\tau_i,\tau_i+t_i)$ for $1\leq i\leq n-1$ where $\tau_i = 2iT+\sum_{k=0}^{i-1}t_k$, and $\phi_0\in H_{n0}^2(0,t_0)$ and $\phi_n\in H_{0n}^2(L-t_n,L)$. Here $H_{n0}^2(0,t_0) = \{u\in H^2(0,t_0)\,|\,u'(0)=u(t_0)=u'(t_0)=0\}$, and H_{0n}^2 is defined similarly. This shows that all variations in H_N^2 are covered by the decomposition of the variational method, hence u is a solution on the whole interval $[0, L]$. The statement about the index follows from the fact that both $g(\cdot, \bar{y}, \bar{z}, s_i)$ and $\boldsymbol{z}(\boldsymbol{s})$ are non-degenerate minimisers, thus the unstable directions only come from variations in s_i. □

The previous lemma describes all solutions in a small neighbourhood $B_\epsilon(\Delta)$ of the heteroclinic cycle.

Lemma 4.25 *Let $\eta \leq \eta_0$. There exists a constants $\epsilon_1(\eta)$ such that when u is a solution of* (4.10) *for $\gamma \in (\frac{1}{8}, \frac{1}{8} + \delta_1(\eta))$ with u entirely contained in $B_{\epsilon_1}(\Delta)$, then for some $n \geq 1$ it holds that $u = E_n(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{s}))$, where $\boldsymbol{s}$ is a critical point of $\mathcal{K}_n$ under the constraint $\sum_{i=0}^n s_i = L - 2nT$ and $\min(\boldsymbol{s}) > \sigma(\eta)$.*

Proof. Let u be a solution of (4.10) which lies entirely in $B_\epsilon(\Delta)$. Since $u_0'(0) \neq 0$, it follows that for ϵ sufficiently small u has a finite number of zeros, say at $x_1, \ldots x_n$. Let $s_0 = x_1 - T$ and $s_i = x_{i+1} - x_i - 2T$, $1 \leq i \leq n-1$ and $s_n = L - x_n - T$. Let $u_i(x - x_i) = u(x)|_{[x_i - T, x_i + T]}$, and $\psi_i(x - x_i - T) = u(x)|_{[\tau_i, \tau_i + s_i]}$, where $\tau_i = 2iT + \sum_{k=0}^{i-1} s_k$. The orbit of u passes close to the equilibrium points ± 1. If ϵ is small enough then the distance between two zeros is larger than $2T + \sigma(\eta)$, hence $s_i > \sigma(\eta)$.

First, we infer that $\psi_i = g(\cdot, \vec{u}_i(T), \vec{u}_{i+1}(-T), s_i)$ since ψ_i is entirely contained in some small neighbourhood of the equilibrium point ± 1, and g is the unique local solution of the corresponding boundary value problem. Second, for ϵ sufficiently small $u_i \in U_\eta(u_0)$, hence $\boldsymbol{u} \in V_\eta$. Since $\boldsymbol{z}$ are the unique critical points in V_η we have that $\boldsymbol{u} = \boldsymbol{z}(\boldsymbol{s})$ and thus $u = E_n^\gamma[\boldsymbol{z}(\boldsymbol{s}); \boldsymbol{s}]$. Finally, since u is a critical point in $H_N^2(0, L)$ it follows that $\boldsymbol{s}$ must be a critical point of $\mathcal{K}_n$ under the constraint $\sum_{i=0}^n s_i = L - 2nT$. Therefore u is obtained from a critical point of $\mathcal{K}_n$. □

It follows from the above lemma that $\epsilon_1(\eta)$ can be chosen to be a non-decreasing function of η. Hence for $\epsilon < \epsilon_1(\eta_0)$ there exists an $\eta_1(\epsilon) < \eta_0$ such that $\epsilon_1(\eta_1(\epsilon)) < \epsilon$. Combining Lemmas 4.23–4.25 now implies the following theorem:

Theorem 4.26 *Let $\epsilon < \epsilon_2 \stackrel{\text{def}}{=} \min\{\epsilon_0, \epsilon_1(\eta_0)\}$, and let $\delta_2(\epsilon) \stackrel{\text{def}}{=} \min\{\delta_0(\epsilon), \delta_1(\eta_1(\epsilon))\}$. When u is a solution of* (4.10) *for $\gamma \in (\frac{1}{8}, \frac{1}{8} + \delta_2(\epsilon))$ and $u \notin \mathcal{F}_\gamma$, then u is entirely contained in $B_\epsilon(\Delta)$ and u corresponds to a critical point $\boldsymbol{s}$ of $\mathcal{K}$ with $\min(\boldsymbol{s}) > \sigma(\eta_1(\epsilon))$.*

For $\gamma \leq \frac{1}{8}$ the functions $\mathcal{K}_n$ can also be defined, but their only critical points are the symmetric sequences $(s_0, 2s_0, 2s_0, \ldots, 2s_0, s_0)$, corresponding to the simple periodic solutions in $\mathcal{F}_\gamma$. For γ slightly larger than $\frac{1}{8}$, Theorem 4.26 implies that the additional solutions appearing in the bifurcation are *completely* determined by the *bifurcation function* $\mathcal{K}(\boldsymbol{s})$. Part of the bifurcation diagram is still formed by the solutions in $\mathcal{F}_\gamma$. The solutions corresponding to critical points of $\mathcal{K}_n$ will fit exactly onto those in $\mathcal{F}_\gamma$, and they form all of the remainder of the bifurcation diagram.

In the following we fix $\epsilon < \epsilon_2$, write $\sigma = \sigma(\eta_1(\epsilon))$, and assume that $0 < \gamma - \frac{1}{8} < \delta_2(\epsilon)$.

4.8.2 Analysis of the bifurcation function

What remains is to determine the critical points of the bifurcation function $\mathcal{K}_n$ for all $n \geq 1$. For easy notation we denote the $n+1$ gluing functions by $g_0, g_1, \ldots, g_{n-1}, g_n$. Recall that, by symmetry, one has $g_l(x, (y_1, y_2), s) = g(x + s, (y_1, -y_2), (y_1, y_2), 2s)$ and similarly for g_r, so that all g_i can be dealt with on the same footing (taking care to correctly transform the variables). In the following we will only discuss those g_i which live in a neighbourhood of $+1$, the other case being completely analogous. Calculating the partial derivatives one obtains that

$$\frac{\partial \mathcal{K}_n(\mathbf{s})}{\partial s_i} = \mathcal{E}[g_i(\cdot, z(\mathbf{s}), s_i)],$$

where $\mathcal{E}$ is the energy, see (4.7). This follows from an explicit calculation, see e.g. [90]. To investigate the partial derivatives we use the following characterisation due to Buffuni and Séré [36]. When $\gamma > \frac{1}{8}$ then the equilibria ± 1 are saddle-foci. Shift the equilibrium point to the origin and choose coordinates $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ such that the local stable and unstable manifolds are given by $W^s_{\text{loc}} = \{(\xi_1, \xi_2, 0, 0) \,|\, \xi_1, \xi_2 \text{ small}\}$ and $W^u_{\text{loc}} = \{(0, 0, \xi_3, \xi_4) \,|\, \xi_3, \xi_4 \text{ small}\}$. Denote $\xi_s = (\xi_1, \xi_2)$ and $\xi_u = (\xi_3, \xi_4)$. In a small neighbourhood $B_4(\delta) = \{|\xi_s| < \delta, |\xi_u| < \delta\}$ of the origin. the flow is given by

$$\xi' = \begin{pmatrix} -\lambda & -\omega & 0 & 0 \\ \omega & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -\omega \\ 0 & 0 & \omega & \lambda \end{pmatrix} \xi + f(\xi), \tag{4.11}$$

where $f(0,0) = 0$, $f'(0,0) = 0$, $f_u(\xi_s, 0) = 0$ and $f_s(0, \xi_u) = 0$. The parameters $\lambda > 0$ and $\omega > 0$ are the real and imaginary part of the eigenvalues of the linearised problem respectively. An important observation, to which we will come back later, is that $\lambda \to 2$ and $\omega \to 0$ as $\gamma \downarrow \frac{1}{8}$. Introduce polar coordinates (r_s, θ_s) and (r_u, θ_u): $x_1 = r_s \cos\theta_s$, $x_2 = r_s \sin\theta_s$, and $x_3 = r_u \cos\theta_u$, $x_4 = r_u \sin\theta_u$. Write the gluing function $g_i(x, z(\mathbf{s}), s_i)$ in these polar coordinates: $(r_s, \theta_s, r_u, \theta_u)(x; \mathbf{s})$. One obtains the following characterisation [36, Lemma A.2] of the energy

$$\begin{aligned} \mathcal{E}_g[s_i; \mathbf{s}] &\stackrel{\text{def}}{=} \mathcal{E}[g_i(\cdot, z(\mathbf{s}), s_i] = \frac{\partial \mathcal{K}_n(\mathbf{s})}{\partial s_i} \\ &= \sqrt{\lambda^2 + \omega^2}\, |\rho(s_i; \mathbf{s})|^2 \cos(\varphi(s_i; \mathbf{s})) + O(|\rho(s_i; \mathbf{s})|^3), \end{aligned} \tag{4.12a}$$

where

$$\rho(s_i; \mathbf{s}) = e^{-\lambda s_i/2} \sqrt{r_s(0; \mathbf{s}) r_u(s_i; \mathbf{s})}\, (1 + O(\delta)), \tag{4.12b}$$

$$\varphi(s_i; \mathbf{s}) = \omega s_i + \theta_s(0; \mathbf{s}) - \theta_u(s_i; \mathbf{s}) - \mu + O(\delta). \tag{4.12c}$$

Here μ is a constant which tends to 0 as $\gamma \to \frac{1}{8}$. The terms $O(\delta)$ and $O(|\rho(s_i; \mathbf{s})|^3)$ are due to the nonlinear influences near the equilibrium point, i.e., they represent the higher order terms in (4.11).

We first analyse the values of $r_s(0; \mathbf{s})$, $\theta_s(0; \mathbf{s})$, $r_u(s_i; \mathbf{s})$ and $\theta_u(s_i; \mathbf{s})$, which will turn out to depend only weakly on s_i, i.e., they are almost constant.

One should keep in mind that for γ close to $\frac{1}{8}$ we have $\omega \approx 0$ and $\lambda \approx 2$. However, the linearisation for $\gamma = \frac{1}{8}$ is not given by (4.11) with $\omega = 0$. This is caused by the change of coordinates necessary to convert to the above form. For $\gamma = \frac{1}{8}$ one can choose coordinates such that for $\zeta \in B_4(\tilde{\delta})$

$$\zeta' = \begin{pmatrix} -\lambda & -1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \zeta + f(\zeta).$$

Of course we choose T so large that $(u_0, u_0', u_0'', u_0''')(T) \in B_4(\tilde{\delta})$ and that the gluing functions g_i are entirely contained in $B_4(\tilde{\delta})$. Before making the connection between the ζ- and ξ-coordinates, we briefly look at the picture in ζ-coordinates. All orbits in W^s, and in particular the heteroclinic solution u_0, tend to the origin along the ζ_2 axis. In fact, in ζ-coordinates u_0 behaves as $\zeta_1(x)/\zeta_2(x) = O(1/x)$ for $x \to \infty$. For $\gamma = \frac{1}{8} + \delta_\gamma$, $0 < \delta_\gamma \ll 1$

the eigenvalues are $\pm 2(1 \pm i\sqrt{2\delta_\gamma} - 3\delta_\gamma + O(\delta_\gamma^{3/2}))$. And after an appropriate scaling in x we may assume that the real part $\pm\lambda$ of the eigenvalues is constant, i.e., the eigenvalues are of the form $\pm 2 \pm i\omega$ and we may take ω (or ω^2) as the parameter instead of γ. The choice of coordinates is such that W^s is always given by $\{\zeta_3 = \zeta_4 = 0\} = \{|\zeta_u| = 0\}$. As opposed to the ξ-coordinates, the ζ-coordinates are chosen to depend smoothly on ω for $\omega \downarrow 0$. The flow becomes

$$\zeta' = \begin{pmatrix} -2 & -1+O(\omega) & 0 & 0 \\ \omega^2+O(\omega^3) & -2 & 0 & 0 \\ 0 & 0 & 2 & -1+O(\omega) \\ 0 & 0 & \omega^2+O(\omega^3) & 2 \end{pmatrix} \zeta + f(\zeta).$$

The coordinate change to get from ζ to ξ is of the form $\xi_1 = \omega\zeta_1 + O(\omega^2)$ and $\xi_2 = \zeta_2$. Since $\theta = \arctan\frac{\xi_1}{\xi_2} = \arctan\frac{\omega\zeta_1 + O(\omega^2)}{\zeta_2}$ it follows that $\theta_s(0) = O(\omega/T)$ or $\theta_s(0) = \pi + O(\omega/T)$, and similarly for $\theta_u(s_i)$. For the difference $\theta_s(0) - \theta_u(s_i)$ there are now two possibilities, differing by a factor π. To determine which of these possibilities occurs, we look at situation at the bifurcation point. For $\gamma = \frac{1}{8}$ the only solutions of (4.10) are the periodic solutions in $\mathcal{F}_{\frac{1}{8}}$, and they have energy $\mathcal{E} \in (0, \frac{1}{4}]$, see Section 4.2. For large periods these solutions can also be described by the present variational gluing method. Since these solutions are symmetric they correspond to a critical point of the form $\boldsymbol{s}_* = (s_0, 2s_0, \dots, 2s_0, s_0)$ for some $s_0 > \sigma$, and $z(\boldsymbol{s}) = (u_1, -u_1, u_1, \dots)$ for some $u_1 \in H^2_c$. Hence $\mathcal{E}_g[2s_0, \boldsymbol{s}_*] \in (0, \frac{1}{4}]$. By continuity, for small ω and $s_0 > \sigma$ not too large the energy $\mathcal{E}_g[2s_0; \boldsymbol{s}_*]$ must be positive. Therefore it must hold that $\theta_s(0; \boldsymbol{s}) - \theta_u(s_i; \boldsymbol{s}) = O(\omega/T)$.

Choosing ϵ small in Theorem 4.26, the constants δ_2 and η_1 are arbitrary small and σ is arbitrary large, and it follows that we may restrict our attention to gluing functions g_i such that the point $(g_i, g_i', g_i'', g_i''')(0, z(\boldsymbol{s}), s_i)$ is arbitrary close to $(u_0, u_0', u_0'', u_0''')(T)$. Let $\delta_* = \mathrm{dist}_{\mathbb{R}^4}\big((u_0, u_0', u_0'', u_0''')(T), (1,0,0,0)\big)$. One thus has, for some constant $0 < \epsilon_2 \ll \delta_*$, that $||\zeta_s| - \delta_*| < \epsilon_2$, and $|\zeta_u| < \epsilon_2$. Hence $r_s(0) = \delta_* + O(\epsilon_2)$ and similarly $r_u(s) = \delta_* + O(\epsilon_2)$.

Having obtained estimates on $r_s(0; \boldsymbol{s})$, $\theta_s(0; \boldsymbol{s})$, $r_u(s_i; \boldsymbol{s})$ and $\theta_u(s_i; \boldsymbol{s})$, we are ready to investigate the function $\mathcal{E}_g[s_i; \boldsymbol{s}]$. We will first concentrate on solutions with one transition. We thus look for critical points of the function $\mathcal{K}_1(s_0, s_1)$ under the constraint $s_0 + s_1 = L - 2T$, i.e., zeros of $\mathcal{E}_g[s_0; \boldsymbol{s}] - \mathcal{E}_g[L - 2T - s_0; \boldsymbol{s}]$ with $\min(\boldsymbol{s}) > \sigma$, where $\boldsymbol{s} = (s_0, L - 2T - s_0)$. Since in the present case one has to think of the gluing functions g_l and g_r as half of an ordinary gluing function g, we define $s = 2s_0$ and $G(s) \stackrel{\text{def}}{=} \mathcal{E}_g[\frac{s}{2}; (\frac{s}{2}, \frac{L_0 - s}{2})] - \mathcal{E}_g[\frac{L_0 - s}{2}; (\frac{s}{2}, \frac{L_0 - s}{2})]$, where $L_0 = 2(L - 2T)$.

For L_0 not too large and ω small, there is only one solution of the equation $G(s) = 0$, since the corresponding function necessarily belongs to $\mathcal{F}_\gamma$, namely $s = L_0/2$. It is immediately clear that for any $L_0 > 2\sigma$ there is a symmetric solution corresponding to $s = L_0/2$. More generally, looking for zeros of $G(s)$ we consider the good approximation

$$G(s) \approx G_0(s) \stackrel{\text{def}}{=} \sqrt{\lambda^2 + \omega^2}\,\delta_*^2 \Big(e^{-\lambda s}\cos\omega s - e^{-\lambda(L_0 - s)}\cos\omega(L_0 - s)\Big).$$

The scaling $\tilde{s} = \omega s$ is useful as well, effectively setting $\omega = 1$ and $\lambda \to \infty$ as $\gamma \downarrow \frac{1}{8}$.

It follows that for small ω, zeros of $G_0(s)$ only occur in the neighbourhood of the lines (in the (s, L_0)-plane) $s = \frac{L_0}{2}$, and $s = \frac{(2k-1)\pi}{2\omega}$, $s < \frac{L_0}{2}$ for $k \in \mathbb{N}$, and $s = L_0 - \frac{(2k-1)\pi}{2\omega}$, $s > \frac{L_0}{2}$ for $k \in \mathbb{N}$, see Figure 4.11. The second and third case are related by symmetry. Next we

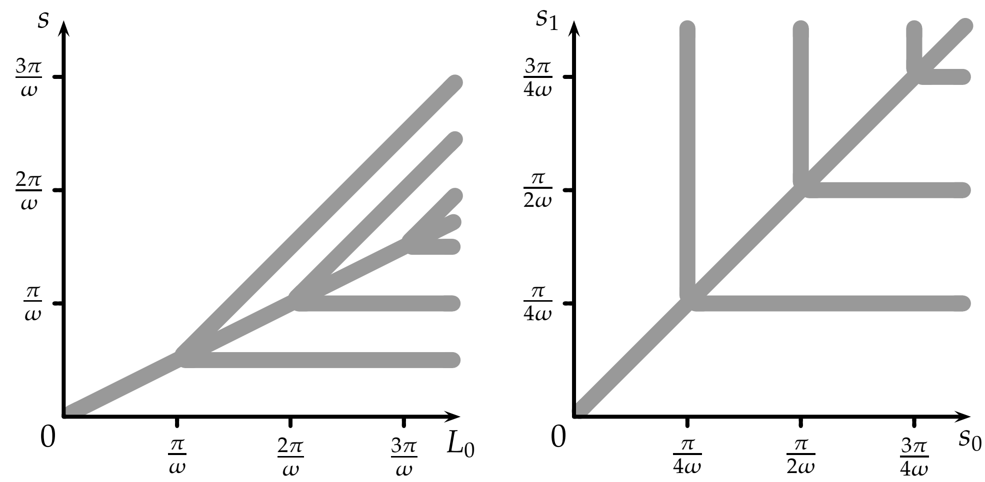

Figure 4.11: Critical points of $\mathcal{K}_1$ can only occur in the grey regions, which are shown both in the (L_0, s)-plane and in the (s_0, s_1)-plane.

consider the derivative of G_0 in the neighbourhood of these lines:

$$\begin{aligned}\frac{G_0'(s)}{\sqrt{\lambda^2+\omega^2\,\delta_*^2}} &= -\lambda e^{-\lambda s}\cos\omega s - \omega e^{-\lambda s}\sin\omega s \\ &\qquad - \lambda e^{-\lambda(L_0-s)}\cos\omega(L_0-s) - \omega e^{-\lambda(L_0-s)}\sin\omega(L_0-s).\end{aligned}$$

First, in a neighbourhood of the line $s=\frac{L_0}{2}$ it follows that $G_0'(s)\neq 0$ if (s,L_0) is away from the points $s=\frac{L_0}{2}=\frac{(2k-1)\pi}{2\omega}$, because there the first and third terms in $G_0'(s)$ are dominant ($\omega\approx 0$). This means that for fixed $L_0\not\approx\frac{(2k-1)\pi}{\omega}$ the only zero of $G_0(s)$ in a neighbourhood of $s=\frac{L_0}{2}$ is on the diagonal itself: $s=L_0$.

Second, in a neighbourhood of the line $s=\frac{(2k-1)\pi}{2\omega}$, $s<\frac{L_0}{2}$ it follows that $G_0'(s)\neq 0$ if (s,L_0) is away from the point $s=\frac{L_0}{2}=\frac{(2k-1)\pi}{2\omega}$, because there the second term in $G_0'(s)$ is dominant. This implies that for fixed $L_0>\frac{(2k-1)\pi}{\omega}$ there is exactly one zero of $G_0(s)$ in a neighbourhood of $s=\frac{(2k-1)\pi}{2\omega}$.

We conclude that, away from the special points $s=\frac{L_0}{2}=\frac{(2k-1)\pi}{2\omega}$ the zeros of $G_0(s)$ are transverse and thus depend smoothly on L_0. On the line $s=\frac{L_0}{2}$ there are bifurcation points s_* near $s=\frac{(2k-1)\pi}{2\omega}$. These points are characterised by the fact that $G_0'(s_*)=0$. Interpreting G_0 as a function of s and the parameter L_0 one calculates that at these points ($s=s_*$, $L_0=2s_*$) a forward pitchfork bifurcation takes place:

$$\frac{\partial G_0}{\partial L_0}=0,\qquad \frac{\partial^2 G_0}{\partial s^2}=0,\qquad \frac{\partial^2 G_0}{\partial L_0\partial s}\cdot\frac{\partial^3 G_0}{\partial s^3}<0. \tag{4.13}$$

Next one has to consider the difference between $G(s)$ and $G_0(s)$. We have already obtained estimates on $r_s(0;\mathbf{s})$, $\theta_s(0;\mathbf{s})$, $r_u(s_i;\mathbf{s})$ and $\theta_u(s_i;\mathbf{s})$, but we also need estimates on their derivative with respect to s_i. For this purpose we first look at $\frac{\partial g_i(0,z(s),s_i)}{\partial s_i}$. Let us consider $\bar{g}(x;s_i)=g_i(x,\bar{y},\bar{z},s_i)-1$, which is the solution of (we write $\bar{y}=(1+y_1,y_2)$ and $\bar{z}=(1+z_1,z_2)$)

$$\begin{cases} -\gamma\bar{g}''''+\bar{g}''-2\bar{g}=3\bar{g}^2+\bar{g}^3 \\ \bar{g}(0)=y_1,\ \bar{g}'(0)=y_2,\ \bar{g}(s_i)=z_1,\ \bar{g}'(s_i)=z_2. \end{cases}$$

Scaling $\tilde{x}=x/s_i$ we get for $\tilde{g}(\tilde{x})=g(x)$:

$$\begin{cases} -\gamma\frac{1}{s_i^4}\tilde{g}''''+\frac{1}{s_i^2}\tilde{g}''-2\tilde{g}=3\tilde{g}^2+\tilde{g}^3 \\ \tilde{g}(0)=y_1,\ \tilde{g}'(0)=y_2 s_i,\ \tilde{g}(1)=z_1,\ \tilde{g}'(1)=z_2 s_i. \end{cases}$$

For $\tilde{h}(\tilde{x}) = \frac{\partial \tilde{g}}{\partial s_i}$ we obtain:

$$\begin{cases} -\gamma \frac{1}{s_i^4}\tilde{h}'''' + \frac{1}{s_i^2}\tilde{h}'' - 2\tilde{h} = 6\tilde{h}\tilde{g} + 3\tilde{h}\tilde{g}^2 + \frac{4\gamma}{s_i^5}\tilde{g}'''' - \frac{2}{s_i^3}\tilde{g}'' \\ \tilde{h}(0) = 0,\ \tilde{h}'(0) = y_2,\ \tilde{h}(1) = 0,\ \tilde{h}'(1) = z_2. \end{cases}$$

And finally for $h(x) = \tilde{h}(\tilde{x}) = \frac{\partial \tilde{g}}{\partial s_i} = \frac{\partial g_i}{\partial s_i}$ one gets:

$$\begin{cases} -\gamma h'''' + h'' - 2h = 6h\bar{g} + 3h\bar{g}^2 + \frac{4\gamma}{s_i}\bar{g}'''' - \frac{2}{s_i}\bar{g}'' \\ h(0) = 0,\ h'(0) = y_2/s_i,\ h(s_i) = 0,\ h'(s_i) = z_2/s_i. \end{cases}$$

Since $\|\bar{y} - (1,0)\| < \delta$ and $\|\bar{z} - (1,0)\| < \delta$, we conclude that $\|h\|_{W^{3,\infty}(0,s_i)} = O(\delta/s_i)$. By differentiating the identity $d(J \circ E_n^\gamma)[\boldsymbol{z}(\boldsymbol{s});\boldsymbol{s}] = 0$, one finds that

$$d_u^2(J \circ E_n^\gamma)\frac{\partial \boldsymbol{z}(\boldsymbol{s})}{\partial s_i} = -\frac{\partial}{\partial s_i} d_u(J \circ E_n^\gamma) = O(\|\tfrac{\partial g_i}{\partial s_i}\|_{W^{3,\infty}(0,s_i)}).$$

Here the last equality follows from an explicit calculation of $d_u(J \circ E_n^\gamma)$. Combining this with the above estimate on $\|\frac{\partial g_i}{\partial s_i}\|_{W^{3,\infty}(0,s_i)}$, we obtain that $\frac{\partial \boldsymbol{z}(\boldsymbol{s})}{\partial s_i} = O(\delta/s_i)$, so that $\frac{\partial r_s(0;\boldsymbol{s})}{\partial s_i} = O(\delta/s_i)$, $\frac{\partial r_u(s_i;\boldsymbol{s})}{\partial s_i} = O(\delta/s_i)$, and $\frac{\partial \theta_s(0;\boldsymbol{s})}{\partial s_i} = O(\omega/s_i)$, $\frac{\partial \theta_u(s_i;\boldsymbol{s})}{\partial s_i} = O(\omega/s_i)$.

From the previous analysis it is clear that we are only interested in values of s which are larger than approximately $\frac{\pi}{2\omega}$, since for smaller s there will only be one critical point of $\mathcal{K}_1$, which is of the form $(\frac{s}{2}, \frac{s}{2})$. Since δ is small it follows that for such values of s the dominant term in (4.12c) is ωs_i, so that the zeros of $G(s)$ can again only occur near the lines $s = \frac{L_0}{2}$, and $s = \frac{(2k-1)\pi}{2\omega}$, $s < \frac{L_0}{2}$ for $k \in \mathbb{N}$, and $s = L_0 - \frac{(2k+1)\pi}{2\omega}$, $s > \frac{L_0}{2}$ for $k \in \mathbb{N}$, see Figure 4.11. To be able to carry over the analysis of $G_0'(s)$ to $G'(s)$ we need that $\frac{1}{r_s}\frac{\partial r_s}{\partial s_i} \ll \lambda$, $\frac{\partial \theta_s}{\partial s_i} \ll \omega$, which is true by the estimates above for large s_i, i.e. for ω sufficiently small. Moreover, we need estimates on the derivatives of the terms of order $O(\delta)$ in (4.12). Since these terms originate from the higher order terms in (4.11) one finds that they are of order $O(\frac{\partial \boldsymbol{z}(\boldsymbol{s})}{\partial s_i}) = O(\delta/s_i)$. Therefore these terms are dominated by ω, for small δ and $s_i > \frac{\pi}{4\omega}$. Hence, as for $G_0'(s)$, we conclude that, away from the special points $s = \frac{L_0}{2} = \frac{(2k-1)\pi}{2\omega}$, the zeros are unique (near the fore-mentioned lines) and depend continuously on L_0.

The analysis of the bifurcation points also carries over from $G_0(s)$ to $G(s)$, since estimates on the higher order derivatives are found in a similar manner as before: $\frac{\partial^2 \boldsymbol{z}(\boldsymbol{s})}{\partial s_i^2} = O(\delta/s_i^2)$ and $\frac{\partial^3 \boldsymbol{z}(\boldsymbol{s})}{\partial s_i^3} = O(\delta/s_i^3)$. Thus, at the bifurcation points s_*, characterised by $G'(s_*) = 0$, the inequality of (4.13) holds (for G instead of G_0), while the equalities follow from the symmetry.

Finally, the index of a critical point $(\frac{s}{2}, \frac{L_0 - s}{2})$ is easily calculated: it is 1 if $G'(s) < 0$, and it is 0 if $G'(s) > 0$. More explicitly, the index is 0 if either $s = \frac{L_0}{2}$ and $s \in \big(\frac{(4k-3)\pi}{2\omega} + \epsilon, \frac{(4k-1)\pi}{2\omega} - \epsilon\big)$, $k \in \mathbb{N}$, or $s \approx \frac{(4k-1)\pi}{2\omega}$, $s < \frac{L_0}{2} - \epsilon$ or $s \approx L_0 - \frac{(4k-1)\pi}{2\omega}$, $s > \frac{L_0}{2} + \epsilon$ for $k \in \mathbb{N}$. Here ϵ is some small positive number which tends to 0 as $\omega \to 0$. On the complementary (parts of) branches the index of the critical point is 1. The points where the index changes are of course precisely the bifurcation points. Because all this is much easier to understand from a picture, Figure 4.12 shows all solutions (and their index) on the first branch (consisting of solutions with one zero/transition) of the bifurcation diagram for γ slightly larger than $\frac{1}{8}$.

We now turn our attention to the solutions with more transitions/zeros. To find critical points one needs to solve $\frac{\partial \mathcal{K}_n}{\partial s_i} = \frac{\partial \mathcal{K}_n}{\partial s_j}$ for all $0 \leq i, j \leq n$. Since all partial derivatives are of

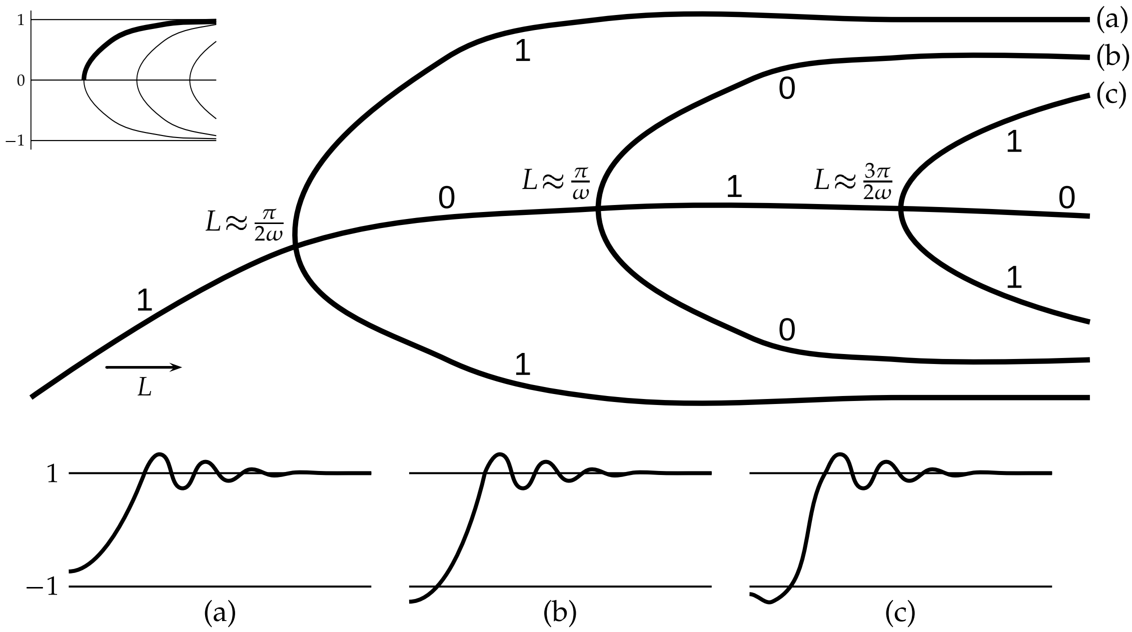

Figure 4.12: A blow-up of the first branch of the bifurcation diagram for γ slightly larger than $\frac{1}{8}$. The branch consists of solutions of (4.10) with one zero (at which it has positive slope). The profile of solutions on different parts of the branch are depicted below (for large L). The index of the solution branches is also shown.

the form (4.12) the analysis of the case $n = 1$ can be repeated for $n \geq 2$. To make notation easier we define $\tilde{s}_0 = 2s_0$ and $\tilde{s}_n = 2s_n$ and subsequently drop the tildes from the notation. The critical points of $\mathcal{K}_n$ can only occur near the diagonal $\{s_0 = s_1 = \cdots = s_n\}$, and, for any permutation τ, any $0 \leq m \leq n-1$ and any sequence $\{k_i\}_{i=0}^m \subset \mathbb{N}$ with $k_i \leq k_{i+1}$, near the line

$$\big\{s_{\tau(i)} = \tfrac{(2k_i-1)\pi}{2\omega},\, 0 \leq i \leq m\big\} \cap \big\{s_{\tau(m+1)} = \cdots = s_{\tau(n)} \geq \tfrac{(2k_m-1)\pi}{2\omega}\big\}. \tag{4.14}$$

In words this means that some (but not all) of the s_i are fixed at an odd multiple of $\frac{\pi}{2\omega}$, while the remaining s_i are all equal and larger than the maximum of the fixed s_i. This gives the complete bifurcation diagram; for fixed L one needs to restrict to $\sum_{i=0}^n s_i - \frac{s_0+s_n}{2} = L - 2nT$.

We are solving the $(n+1)$ equations $f_i \stackrel{\text{def}}{=} \frac{\partial \mathcal{K}_n}{\partial s_i} - \frac{\partial \mathcal{K}_n}{\partial s_{i+1}} = 0$, $0 \leq i \leq n-1$, and $f_n \stackrel{\text{def}}{=} \sum_{j=0}^n s_j - (L_0 - 2nT) = 0$. To conclude uniqueness (and continuous dependence) of the solutions of these equations, one needs $\det\big(\frac{\partial f_i}{\partial s_j}\big) \neq 0$. A direct calculation shows that

$$\det\Big(\frac{\partial f_i}{\partial s_j}\Big) = \sum_{i=0}^{n} \prod_{\substack{0\leq j\leq n\\ j\neq i}} \frac{\partial \mathcal{E}_g[s_j; \boldsymbol{s}]}{\partial s_j} + \text{ other terms},$$

where all other terms are small compared to the first term if δ, ω and $\frac{1}{\min(s)}$ are sufficiently small. As in the case $n = 1$ discussed above, good bookkeeping reveals the dominant term(s) in this expression when $(\boldsymbol{s}, L_0)$ is not close one of the exceptional points, and one concludes that $\det\big(\frac{\partial f_i}{\partial s_j}\big) \neq 0$. The exceptional points are the points where two or more of the lines, which were defined above, meet.

The index of the critical points is equal to the number of negative eigenvalues of the $(n \times n)$-matrix $\frac{\partial^2 \mathcal{K}_n(s_0,s_1,\ldots,s_{n-1},L-\sum_{k=0}^{n-1} s_k)}{\partial s_i \partial s_j}$. Since $\frac{\partial \mathcal{K}_n}{\partial s_i} = \mathcal{E}_g[s_i; \boldsymbol{s}] - \mathcal{E}_g[L - \sum_{k=0}^{n-1} s_k; \boldsymbol{s}]$ and $\mathcal{E}_g[s_i; \boldsymbol{s}]$

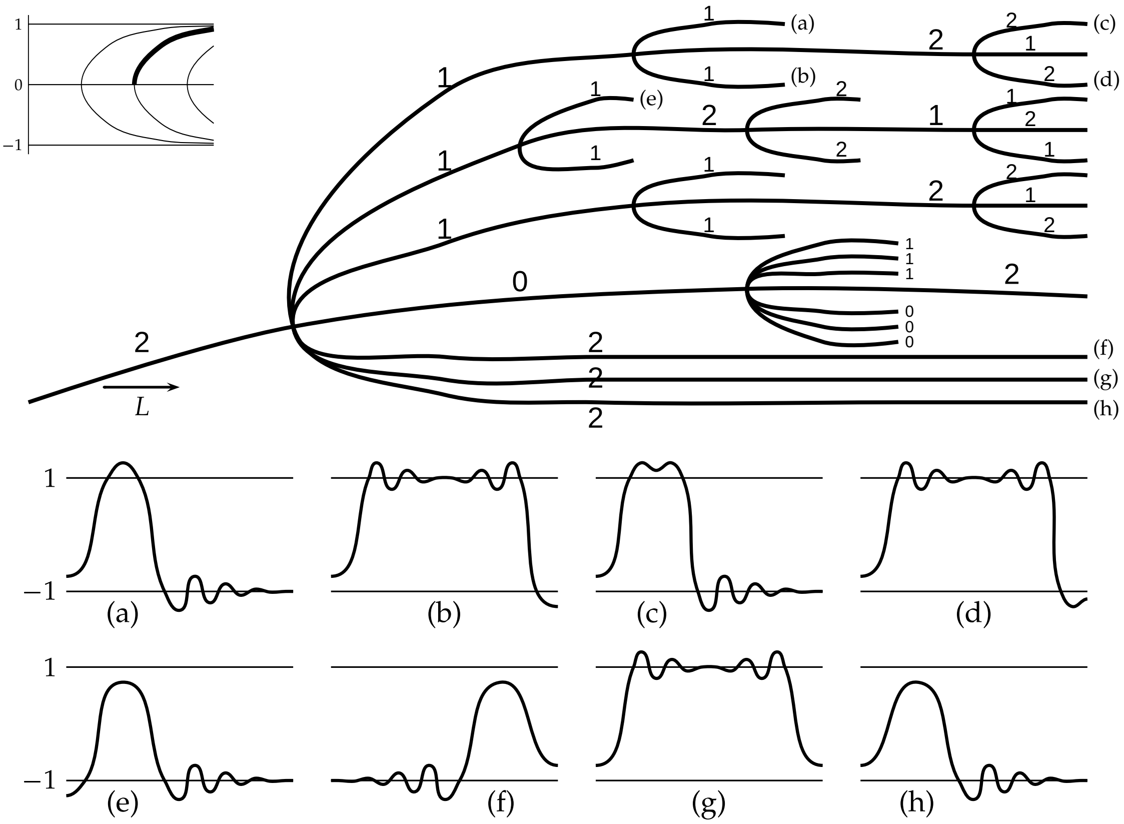

Figure 4.13: A blow-up of part of the second branch of the bifurcation diagram for γ slightly larger than $\frac{1}{8}$. The branch consists of solutions of (4.10) with two zeros. The profile of solutions on different parts of the branch are depicted below (for large L). The index of the solution branches is also shown.

is well approximated by $F(s_i) \stackrel{\text{def}}{=} Ce^{-\lambda s_i}\cos\omega s_i$, we get

$$\left(\frac{\partial^2 \mathcal{K}_n}{\partial s_i \partial s_j}\right) = \begin{pmatrix} F'(s_0)+F'(s_n) & F'(s_n) & \cdots & F'(s_n) \\ F'(s_n) & F'(s_1)+F'(s_n) & \cdots & F'(s_n) \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ F'(s_n) & F'(s_n) & \cdots & F'(s_{n-1})+F'(s_n) \end{pmatrix} + \text{small terms.}$$

On the diagonal $\{s_0 = s_1 = \cdots = s_n\}$ this reduces to

$$\left(\frac{\partial^2 \mathcal{K}_n}{\partial s_i \partial s_j}\right) \approx F'(s_0) \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}.$$

Since the matrix is positive definite, the index of the critical point $(\frac{s}{2}, s, s, \ldots, s, \frac{s}{2})$ is 0 if $s \in \left(\frac{(4k-3)\pi}{2\omega} + \epsilon, \frac{(4k-1)\pi}{2\omega} - \epsilon\right)$, $k \in \mathbb{N}$ with $\epsilon > 0$ small. On the complementary part of the diagonal the index is n.

Working out the number of negative eigenvalues on the other branches of solutions we get the following. Near the line (4.14) and away from the bifurcation points the index of the critical point is equal to the number $\#\{0 \le i \le m \,|\, k_i \text{ is odd}\}$ raised by $n - m - 1$ if $s_{\tau(m+1)} = \cdots = s_{\tau(n)} \in \left(\frac{(4j-1)\pi}{2\omega} + \epsilon, \frac{(4j-3)\pi}{2\omega} - \epsilon\right)$ for some $j \in \mathbb{N}$.

A full examination of the bifurcation points for $n \ge 2$ is beyond the scope of the current investigation. We remark that a (numerical) analysis for the model function $F_i = Ce^{-\lambda s_j}\cos\omega s_j$ (instead of $\mathcal{E}_g[s_i; \mathbf{s}]$) already gives a lot of insight. Walking along one of the curves of solutions near the lines (4.14), branches bifurcate in the neighbourhood of points where all s_i are equal to an odd multiple of $\frac{\pi}{2\omega}$. The number of bifurcating branches is $(n-m)(n-m-1)$, which can be explained as follows. The jump in the index along the primary curve is $n-m-1.$, while there is an $(n-m)$-fold symmetry which is broken upon bifurcation. We refer to [14, 66] for rigorous results on the multiplicity of bifurcating branches in the presence of symmetries. However, keep in mind that the symmetry is usually broken upon returning to $\mathcal{E}_g[s_i; \mathbf{s}]$ instead of the model function F_i. As an illustration part of the branch of solutions of (4.10) for $n = 2$ (i.e., with two transitions/zeros) is shown in Figure 4.13.

4.9 Numerical results

The analysis in this chapter describes properties of the attractor for $\beta > 0$. It is not difficult to see that the results also hold for $\beta = 0$ (the estimates needed are slightly more involved). A natural question is to ask what can be said about the parameter region $\beta < 0$. To answer that question, fairly detailed information is needed about the stationary solutions of the problem, and this information has so far been lacking. Although an overall picture is still missing, recent progress has been made in the investigation of periodic stationary solutions. In Chapter 6 the existence of many families of periodic solutions with energy $E = 0$ is proved by a shooting method, some existing for all $\beta \in (-\infty, \sqrt{8})$, others existing only in a finite parameter range. The variational structure is used in Chapters 7 and 8 to prove the existence of many periodic solutions with the use of a Twist map.

In this section we shall briefly discuss some numerical results on what happens to the stationary solutions found for $\beta \geq 0$ when β becomes negative. We focus on stationary solutions of the Equation (4.5), which for $\beta < 0$ corresponds to the well-known Swift-Hohenberg equation [137]. Without loss of generality we fix $\gamma = 1$ throughout this section. The numerical calculations were performed using the continuation program AUTO [57]. Part of these calculations were also presented in [19], but there the emphasis was on heteroclinic solutions instead of solutions on a finite interval.

We have investigated the branch of solutions which bifurcates at the first bifurcation point from the trivial solution $u \equiv 0$, and for simplicity we have restricted our attention to solutions which are anti-symmetric with respect to the zero at $x = \frac{L}{2}$. There are many bifurcations which break this symmetry, but we want to focus on the simplest possible case. As characteristic parameters for the solutions we have chosen the interval length L, the action J, and the energy E.

In Figure 4.14 the (J, L)-diagram of this branch is shown for $\beta = -2$. This picture is representative for the whole range $-2 \leq \beta < \sqrt{8}$. One sees the bifurcation from the trivial solution $u \equiv 0$ (the straight line in the figure) and as one follows the branch it converges in an oscillating manner to a limiting value of J as $L \to \infty$. The branch is also represented in the (J, E)-plane, where we see that it spirals to a point on the line $E = 0$.

The solutions for five points on this branch for $\beta = -2$ are presented in Figure 4.15. The first four points are chosen in the energy level $E = 0$, which are special points in the sense that the action J is extremal at these points. This is due to the fact that when variations in the interval length are taken into consideration, then the extrema of J all lie in the energy level $E = 0$ (e.g. see Chapter 3). The existence of the infinite number of solutions with $E = 0$ on this branch can been proved by a shooting method (see Chapter 6 and [118]) in the parameter regime $0 \leq \beta < \sqrt{8}$. Half of these, namely the solutions for

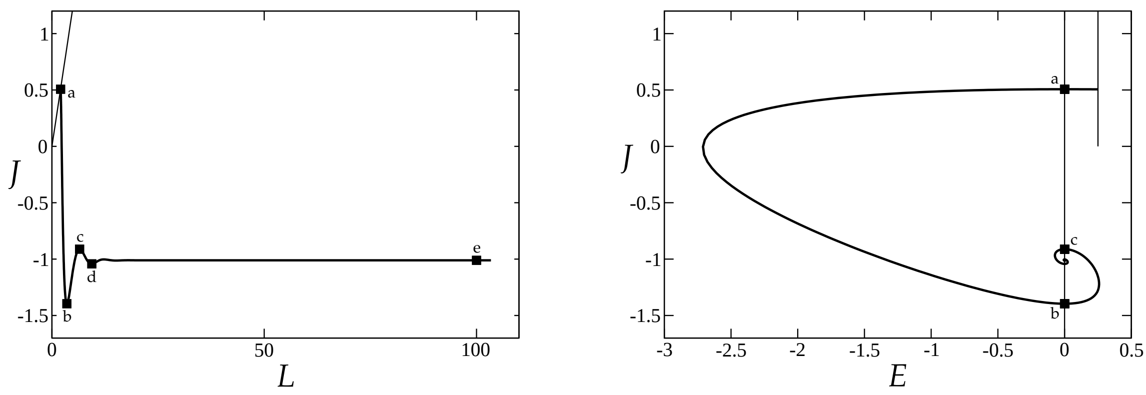

Figure 4.14: The (J, L)- and (J, E)-diagram of the simplest branch at $\beta = -2$. The little squares indicate the solutions shown in Figure 4.15 (in the (J, E)-diagram only the first three of the squares are indicated for clarity).

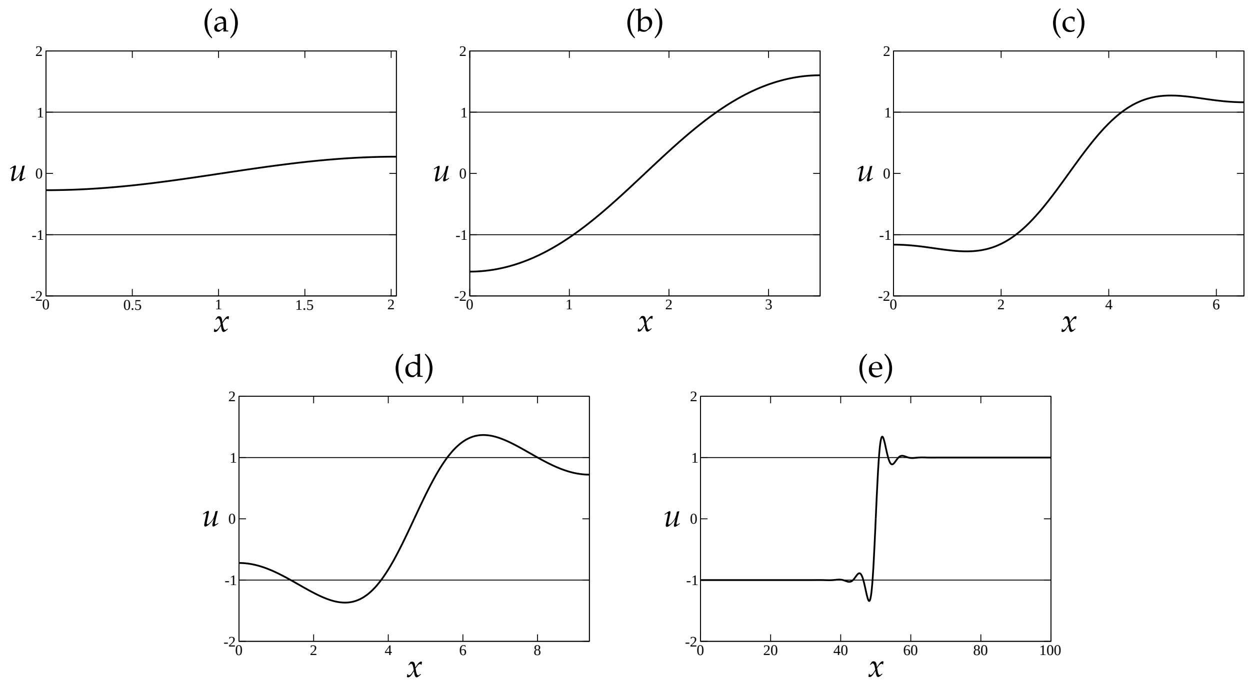

Figure 4.15: The five functions for the parameter values indicated by the little squares in Figure 4.14 ($\beta = -2$). The functions in figures (a)-(d) have energy $E = 0$.

which J attains a minimum, can also be found by a variational method (see Chapter 3). However, these methods fail for $\beta < 0$ and we will come back to this later on.

We see that the four solutions with $E = 0$ are clearly distinguished by the number and relative position of their minima and maxima (relative to the lines/solutions $u = \pm 1$). This is a general phenomenon: all solutions with $E = 0$ on this branch are clearly distinct. The fifth solution shown has interval length $L = 100$ and its energy is, for all practical purposes, indistinguishable from 0. The branch of solutions thus converges to a heteroclinic orbit, in fact a very simple one

The solutions on this branch are not all stable (they form the central branch in Figure 4.12), but they are stable in the class of antisymmetric functions (e.g., taking Navier boundary conditions on the left and Neumann boundary conditions on the right).

In Figure 4.16 the first four functions are continued in the parameter β, where we have fixed the energy level $E = 0$. The first three branches (Figure 4.16a) exist for all $\beta \in (-\infty, \sqrt{8})$. These solutions have been proved to exist in Chapter 6 and [120], and in the limit $\beta \to \sqrt{8}$ they converge to a simple heteroclinic solution (cf. [120]). We remark that there are many other branches of solutions with zero energy which start at $\beta = \sqrt{8}$, bifurcating from the heteroclinic orbit, and extending all the way to $\beta \to -\infty$.

Another feature is that the action J becomes negative along two of the branches. This is responsible for the breakdown of the variational method of [89] in the parameter regime $\beta < 0$. It seems likely that the variational method could give results as long as any monotone lap between two extrema has positive action J. This ceases to be true at the first zero in Figure 4.16a, i.e., at approximately $\beta = -0.92$. We conjecture that this ($\beta \approx -0.92$) is in fact the highest value of β for which there exists a monotone lap with action less than or equal to 0 (cf. [46, 106]). Let us remark that the minimisation technique in homotopy type classes used in [88, 89] and in this chapter, relies on the positivity of the Lagrangian L to apply cut-and-paste techniques, i.e., it only works for $\beta \geq 0$. Nevertheless it seems that one might be able to extend this to $\beta < 0$ as long as $\beta > -0.92$, since for $\beta > -0.92$ every function corresponding to a loop in the configuration plane $\mathcal{P}$ has positive action. Therefore, cutting out a loop always lowers the action. As mentioned before, it is only natural that our analysis cannot be performed globally, i.e. for all $\beta < 0$, because it is observed that most of the solutions found in this chapter cease to exist when β becomes sufficiently negative (while the equilibria are still saddle-foci). An example of this phenomenon is discussed next.

The fourth branch (Figure 4.16b) does not exists for all $\beta \in (-\infty, \sqrt{8})$ but folds back at approximately $\beta = -2.06$. The lower part of the branch converges again to the simple heteroclinic solution as $\beta \to \sqrt{8}$, while the upper part converges to three copies of this heteroclinic (one increasing from -1 to $+1$ and two decreasing from $+1$ to -1), which move further and further apart as $\beta \to \sqrt{8}$. Again, there are many other branches which behave in this way. They all have a different value of β where they fold back. The infinite number of solutions with $E = 0$ on the branch in Figure 4.14 all behave in this way, except for the first three (Figure 4.16a). As mentioned before, existence of these solutions has been proved for $\beta \in [0, \sqrt{8})$, while the fact that the folding point is different for each branch makes it difficult to extend these results to negative β. The fact that the pair of solutions (of index 0 and 1) is able to coalesce and disappear (at $\beta \approx -2.06$) can be understood from the Morse-type analysis in Chapter 8.

In Figure 4.17 three solutions on the folding branch of Figure 4.16b are shown and we clearly see the difference between the two functions for $\beta = -1$ on the lower and the upper part of the branch. In Figure 4.17c we see how the upper part of the branch converges to three copies of a heteroclinic orbit.

Note that apart from the two types of branches shown in Figure 4.16 there exists a third type of branch. This type exist on a finite interval $(\beta_0, \sqrt{8})$, where $\beta_0 \leq -\sqrt{8}$ is different for each branch (but contrary to the folding branches, there is an expression for β_0). As $\beta \to \beta_0$ the solutions on such a branch converge to one of the homogeneous states $u \equiv \pm 1$. We refer to Chapter 6 for an extensive description of this type of branches.

In Figure 4.18 the continuation of the heteroclinic of Figure 4.15e is shown (of course, in reality it is a solution on a finite, but large, interval). Existence of this heteroclinic

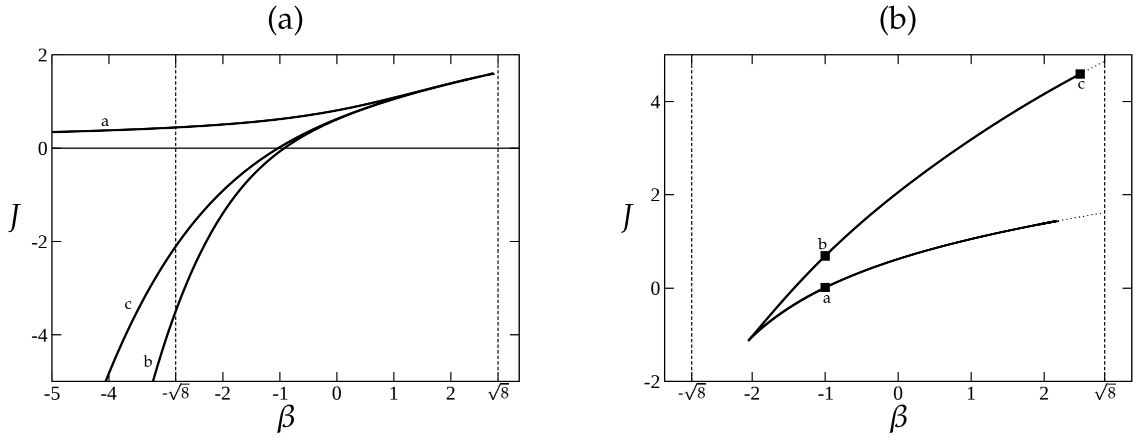

Figure 4.16: On the left the continuation in β of the solutions in Figures 4.15a-c (with corresponding labels) lying in the energy level $E = 0$. On the right the continuation of the solution in Figure 4.15d. The functions corresponding to the little squares are shown in Figure 4.17. The extrapolation to $\beta = \sqrt{8}$ is shown by a dotted line. The region where the equilibrium points $u = \pm 1$ are saddle-foci is bounded by the two vertical dashed lines.

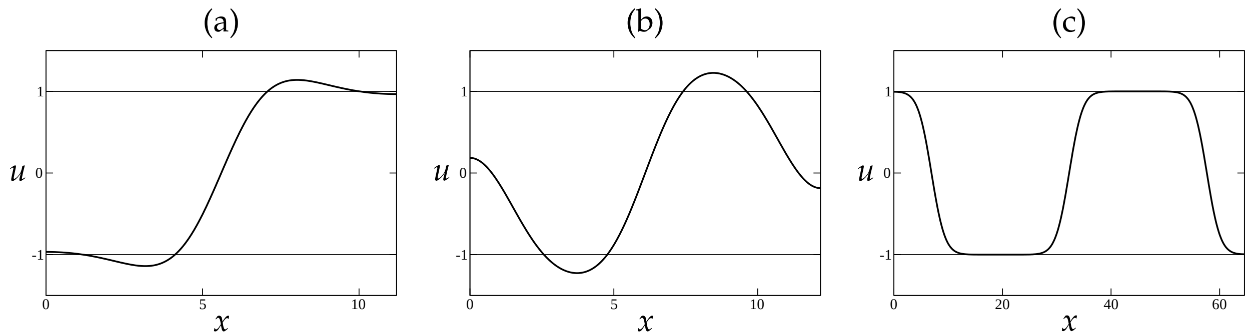

Figure 4.17: The three solutions which are indicated by little squares in Figure 4.16b: a) the solution for $\beta = -1$ with lower action J; b) the solution for $\beta = -1$ with higher J; c) the solution on the tip of the upper part of the branch.

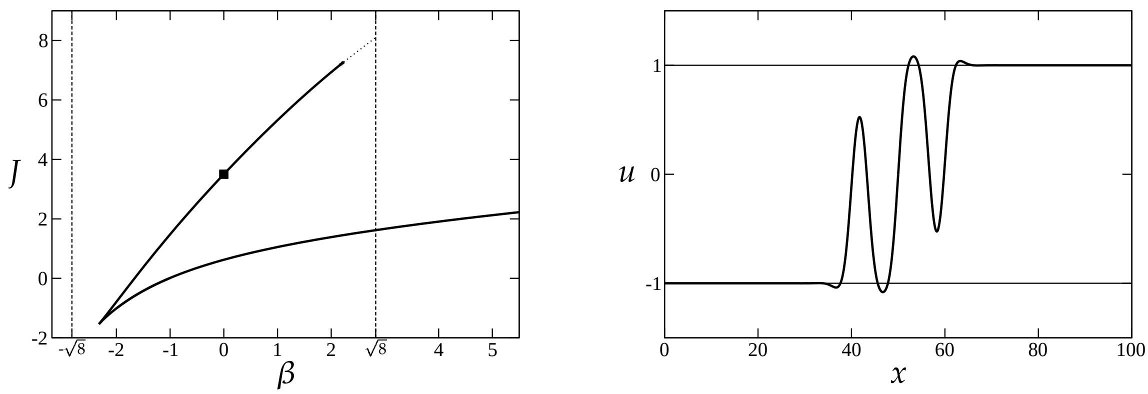

Figure 4.18: The continuation diagram of the simple heteroclinic solution. The solution for $\beta = 0$ on the upper part of the branch is shown on the right. The solution in Figure 4.15e (for $\beta = -2$) lies on the lower part of the branch.

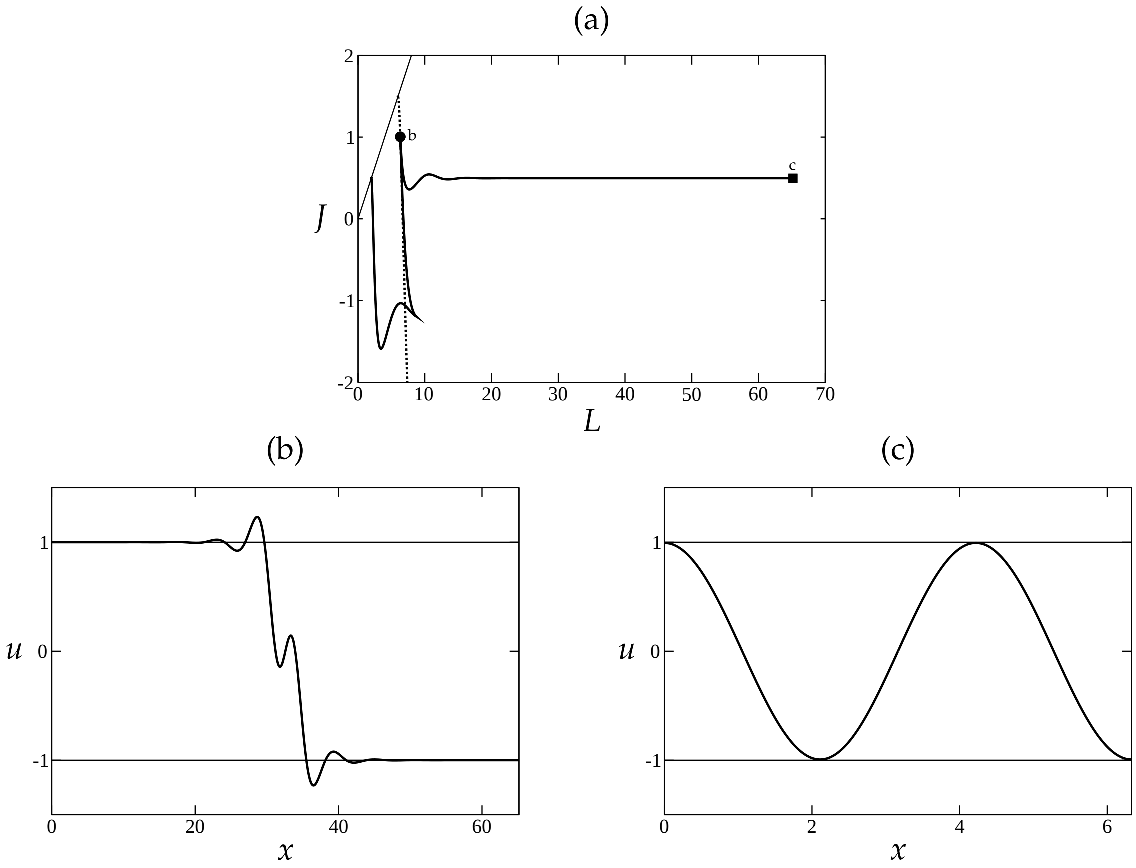

Figure 4.19: At the top the (J,L)-diagram for $\beta = -2.1$. The solutions indicated by the little square and the little are depicted below, in (b) and (c) respectively (the fact that the maximum of this function lies on the line $u = 1$ is purely coincidental). The little circle denotes the bifurcation point.

solution has been proved for all $\beta \in [0, \infty)$ [117, 118, 124]. For $\beta \in [\sqrt{8}, \infty)$ it has been shown to be unique (see Chapter 2 and [96]). The branch of heteroclinic solutions folds in the same way as the solution in Figure 4.16b, and this is characteristic for all branches of heteroclinic solutions: they all have a folding point $\beta_* \in (-\sqrt{8}, 0)$. Existence of many families of heteroclinic solutions has been proved in [89, 118] for $\beta \in [0, \sqrt{8})$ and the folding behaviour has been extensively investigated numerically [19]. Similar behaviour was observed for homoclinic solutions [35]. In [141] the folding behaviour of the branches of heteroclinic solutions is discussed in the context of homoclinic snaking.

The branch in Figure 4.14 contains an infinite number of solutions with $E = 0$. Since the fourth of these solutions ceases to exist for β smaller than about -2.06, it is interesting to see what happens to the branch of solutions for $\beta < -2.06$. In Figure 4.19 the branch is depicted for $\beta = -2.1$, and we indeed see a very different behaviour. The branch starts of in the same manner as for $\beta = -2$, but then suddenly goes in a different direction, and finally tends to a heteroclinic orbit (Figure 4.19b) which differs significantly from the limiting heteroclinic for $\beta = -2$. In the (J,L)-diagram the second branch bifurcating from $u \equiv 0$ is also shown (dotted line). It is just a (triple) multiple of the first branch, and it crosses the first branch at a bifurcation point, denoted by a circle. The solution at this bifurcation point is shown in Figure 4.19c.

To get a better understanding of the difference between $\beta = -2$ and $\beta = -2.1$, we

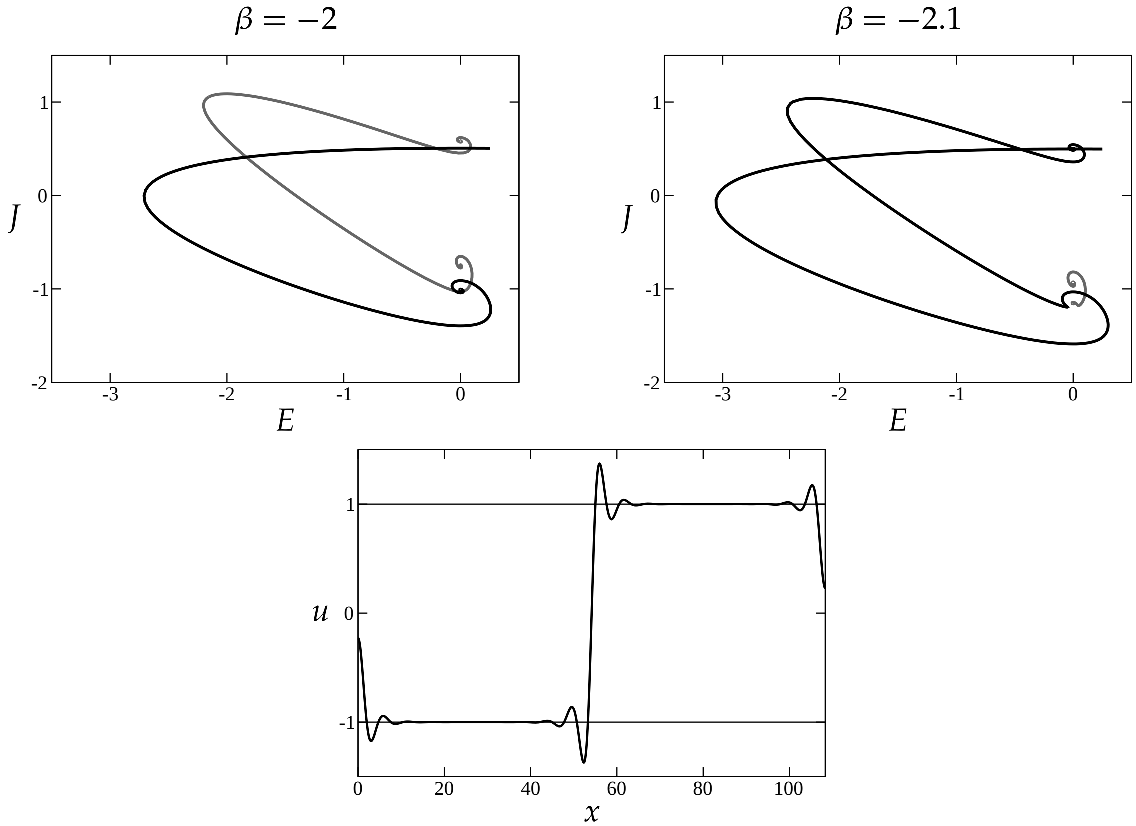

Figure 4.20: At the top the (J, E)-diagrams for $\beta = -2$ (left) and $\beta = -2.1$ (right); in black the branch coming from the first bifurcation point, and in grey a different branch. One of the limits of the grey branch is depicted below.

have in Figure 4.20 compared the (J, E) diagrams for these two values. In black we see, as usual, the branch bifurcating from the first bifurcation point of the trivial solution. In grey another branch of solutions is shown; one of the limits of this branch, the limit which is (almost) the same for both values of β, is also depicted in Figure 4.20. This limit consists of a simple heteroclinic orbit with half a homoclinic orbit on each side. The heteroclinic and homoclinic pieces move further and further apart as the interval length grows.

The other limit of the grey branch is different in both cases: for $\beta = -2$ it resembles the function in Figure 4.19c, whereas for $\beta = -2.1$ it resembles the function in Figure 4.15e. It should be clear from these pictures that a bifurcation, where a connection of branches is exchanged, has occurred between $\beta = -2$ and $\beta = -2.1$, most probably at $\beta \approx -2.06$, the folding point in Figure 4.16b.

The phenomenon described above repeats itself as β decreases. On the left in Figure 4.21 the branch for $\beta = -2.3$ is depicted. One sees that the limiting heteroclinic now has two central oscillations compared to one central oscillation for $\beta = -2.1$.

When $\beta \leq -\sqrt{8}$ the equilibrium points $u = \pm 1$ are centers and heteroclinic solutions no longer exist. It is therefore natural to ask what happens in this parameter range. On the right in Figure 4.21 the branch is shown for $\beta = -3$. We see that the process discussed above has (presumably) repeated itself an infinite number of times. The branch does not converge to a heteroclinic solution but continues to generate more and more oscillations around 0. Notice that we have no clear information about the index, and an overview of the attractor is still very much lacking.

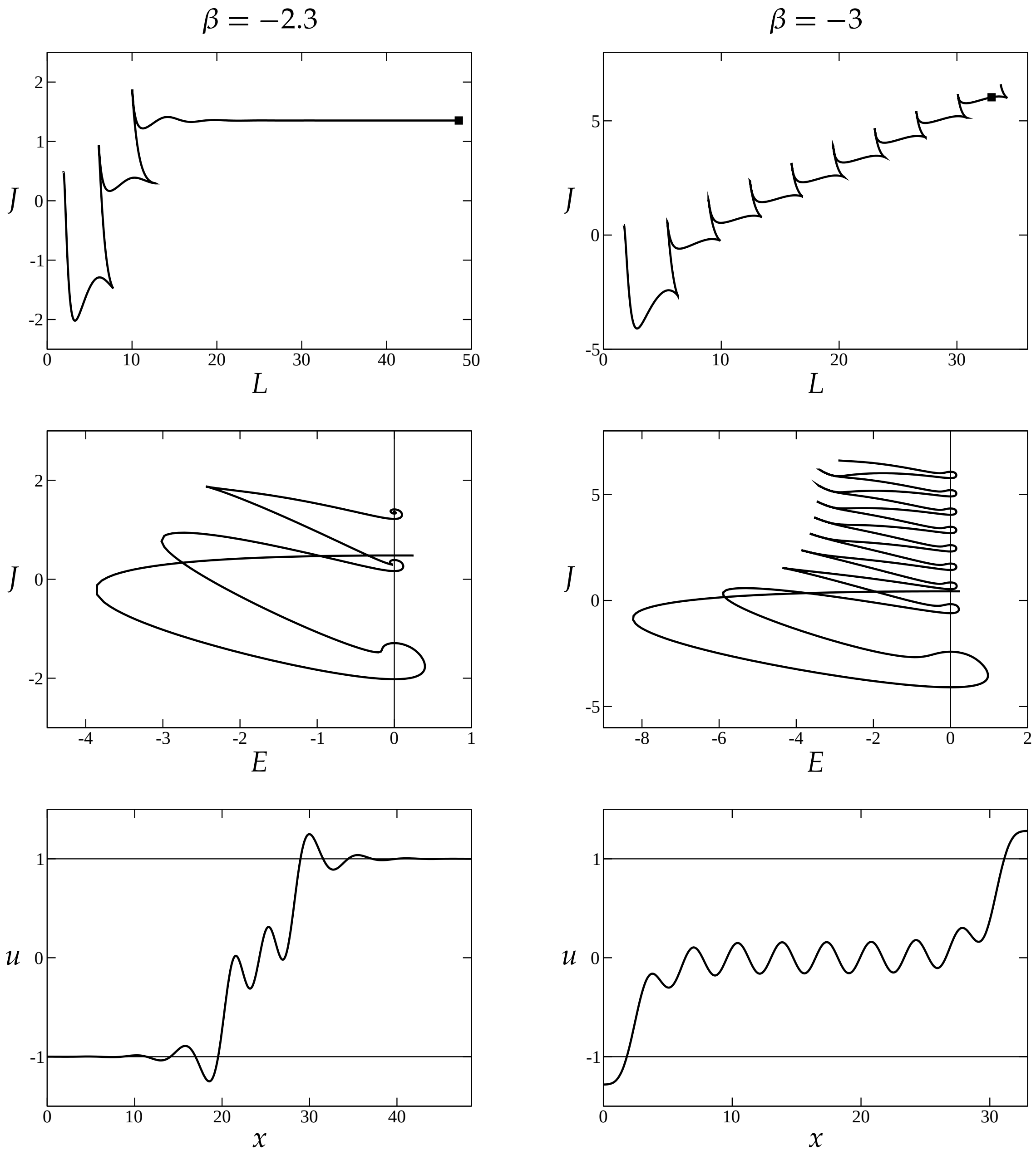

Figure 4.21: On the left the (J, L)- and (J, E) diagram for $\beta = -2.3$ and below the limiting heteroclinic solution. On the right the same diagrams for $\beta = -3$ and below a characteristic solution on the branch.

We stress that we have only tried to describe one interesting phenomenon which occurs when β becomes negative. There are of course many other branches of solutions which we have not shown here. It is clear that this parameter regime has to be subjected to a deeper investigation, both from the point of view of stationary solutions, as well as regarding the behaviour of the parabolic equation. For example, an interesting question is whether for all $\beta < 0$ the number of stable stationary states grows to infinity as $L \to \infty$, or whether on the contrary this number is bounded for sufficiently large $|\beta|$.

4.10 Hyperbolicity

We show that the non-trivial solutions of (4.5), for $\frac{\gamma}{\beta^2} \le \frac{1}{8}$, are hyperbolic for all $L > 0$ (Lemma 4.4). It is clear that the homogeneous states $u = \pm 1$ are hyperbolic and stable for all $L > 0$, and that $u = 0$ is hyperbolic except at the bifurcation points $L = nL_0 = n\pi\sqrt{\frac{2\gamma}{\sqrt{\beta^2+4\gamma}-\beta}}$, $n \in \mathbb{N}$. For convenience we set $\gamma = 1$ in (4.5) and we also shift the interval, i.e.,

$$\begin{cases} u''''(x) - \beta u''(x) - u(x) + u^3(x) = 0 & \text{for } x \in (-\frac{L}{2}, \frac{L}{2}) \\ u'(\pm\frac{L}{2}) = u'''(\pm\frac{L}{2}) = 0. \end{cases} \tag{4.15}$$

The condition $\beta \ge \sqrt{8}$ corresponds to $\frac{\gamma}{\beta^2} \le \frac{1}{8}$. As before we will extend solutions of (4.15) to all of $\mathbb{R}$.

As is depicted in Figure 4.1 we have the following lemma, which follows from the results in Chapter 2:

Lemma 4.27 *For all $\beta \ge \sqrt{8}$, problem* (4.15) *has, apart from the homogeneous solutions $u = 0$ and $u = \pm 1$, exactly the following solutions: for any $n \ge 1$ and all $L > nL_0(\beta)$ there are two solutions, $\pm u_n(x; L)$ with precisely n zeros on $(-\frac{L}{2}, \frac{L}{2})$. It holds that $u_n(x; L) = u_1(x + \frac{n-1}{2n}L; L/n)$, i.e., $u_n(x; L)$ is obtained by extending the solution $u_1(x; L/n)$. The functions $u_1(x; L)$ have the property that $u_1'(x; L) > 0$ and $|u_1(x; L)| < 1$ for all $L > L_0$ and all $x \in (-\frac{L}{2}, \frac{L}{2})$. Besides, u_1 is antisymmetric with respect to its zero, and $u_1''(x) < 0$ for all $x \in (0, \frac{L}{2}]$. Finally, $u_1(x; L)$ is a continuous function of L and $u_1'(0; L)$ is strictly increasing in L.*

To show that all these solutions are hyperbolic we need to investigate the 0-eigenvalue problem

$$\begin{cases} v''''(x) - \beta v''(x) + (3u_n^2(x; L) - 1)v(x) = 0 & \text{for } x \in (-\frac{L}{2}, \frac{L}{2}) \\ v'(\pm\frac{L}{2}) = v'''(\pm\frac{L}{2}) = 0. \end{cases} \tag{4.16}$$

To tackle this problem we use the following Comparison Lemma (see also Lemma 2.4 and [35]):

Lemma 4.28 (Comparison Lemma) *Let $|u(x)| < 1$ for all $x \in \mathbb{R}$ and let $\beta \ge \sqrt{8}$. Suppose that v_1 and v_2 are bounded functions on $\mathbb{R}$, both satisfying*

$$v_i'''' - \beta v_i'' + (3u^2(x) - 1)v_i = 0 \qquad \text{for all } x \in \mathbb{R},\ i = 1, 2, \tag{4.17}$$

and

$$\begin{aligned} (v_1 - v_2)(0) &\ge 0, & (v_1 - v_2)''(0) - \lambda(v_1 - v_2)(0) &\ge 0, \\ (v_1 - v_2)'(0) &\ge 0, & (v_1 - v_2)'''(0) - \lambda(v_1 - v_2)'(0) &\ge 0, \end{aligned}$$

where $\lambda = \frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 - 2}$. Then $v_1 \equiv v_2$.

The proof of this lemma is along the same lines as in Lemma 2.4 (see Section 2.2), and we will not repeat it here. The core of the argument is the splitting of Equation (4.17) into

$$\begin{cases} v'' - \lambda v &= w, \\ w'' - \mu w &= (3 - 3u(x)^2)v, \end{cases}$$

where $\mu = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 - 2}$, and observing that the right-hand sides are increasing (in w and v).

Let us start with the simplest case, namely that u_1 is hyperbolic, i.e., u_1 does not have a 0-eigenvalue. The following lemma shows that there is no symmetric 0-eigenfunction.

Lemma 4.29 *For $n = 1$ there is no symmetric nontrivial solution of problem* (4.16).

Proof. By contradiction, suppose that $v(x)$ is a symmetric nontrivial solution of problem (4.16). Notice that $u_1(x)$ and $v(x)$ are bounded functions on $\mathbb{R}$. If $v(0) = 0$, then the Comparison Lemma (comparing with the function $\tilde{v} \equiv 0$), implies that $v \equiv 0$. One thus finds that $v(0) \neq 0$. Taking $v_1(x) = \frac{u_1'(0)}{v(0)} v(x)$ and $v_2(x) = u_1'(x)$, we find that

$$(v_1 - v_2)(0) = 0, \quad (v_1 - v_2)'(0) = 0, \quad (v_1 - v_2)'''(0) = 0.$$

Since either $(v_1 - v_2)''(0) \geq 0$ or $(v_2 - v_1)''(0) > 0$ holds, we now find from the Comparison Lemma that $v_1 \equiv v_2$. However, $v_1'(\frac{L}{2}) = 0$ whereas $v_2'(\frac{L}{2}) = u_1''(\frac{L}{2}) < 0$ (see Lemma 4.27), a contradiction. □

The next lemma is slightly more involved. It shows that there is no antisymmetric 0-eigenfunction of problem (4.16).

Lemma 4.30 *For $n = 1$ there is no antisymmetric nontrivial solution of problem* (4.16).

Proof. By contradiction, suppose that $v(x)$ is an antisymmetric nontrivial solution of problem (4.16) for $L = L_*$. We first introduce the notation $z(x; \alpha, \eta)$ for the solution of

$$\begin{cases} z'''' - \beta z'' - z + z^3 = 0 \\ z(0) = 0, \quad z'(0) = \alpha, \quad z''(0) = 0, \quad z'''(0) - \lambda z'(0) = \eta, \end{cases}$$

where, as before, $\lambda = \frac{\beta}{2} - \sqrt{\left(\frac{\beta}{2}\right)^2 - 2}$. Now define two families of solutions. First, consider the family

$$w(x; s) \stackrel{\text{def}}{=} z\big(x; u_1'(0) + v'(0)s, u_1'''(0) - \lambda u_1'(0) + [v'''(0) - \lambda v'(0)]s\big).$$

One has $\frac{\partial w}{\partial s}(x; 0) = v(x)$. Since $v'(\frac{L_*}{2}) = v'''(\frac{L_*}{2}) = 0$, this implies that

$$\begin{aligned} \tfrac{\partial z'}{\partial \alpha}(\tfrac{L_*}{2}; 0, 0)\, v'(0) \ &+\ \tfrac{\partial z'}{\partial \eta}(\tfrac{L_*}{2}; 0, 0)[v'''(0) - \lambda v'(0)] \ = \ 0, \\ \tfrac{\partial z'''}{\partial \alpha}(\tfrac{L_*}{2}; 0, 0)\, v'(0) \ &+\ \tfrac{\partial z'''}{\partial \eta}(\tfrac{L_*}{2}; 0, 0)[v'''(0) - \lambda v'(0)] \ = \ 0. \end{aligned}$$

Viewing these as linear equations in $v'(0)$ and $v'''(0) - \lambda v'(0)$, one concludes that

$$\left(\frac{\partial z'}{\partial \alpha}\frac{\partial z'''}{\partial \eta} - \frac{\partial z'}{\partial \eta}\frac{\partial z'''}{\partial \alpha}\right)(\tfrac{L_*}{2}; 0, 0) = 0. \tag{4.18}$$

Second, consider the family $u_1(x; L)$. Clearly $u_1(x; L) = z(x; \alpha(L), \eta(L))$ for certain continuous functions $\alpha(L)$ and $\eta(L)$. We know from Lemma 4.27 that $\alpha(L)$ is strictly increasing, and hence invertible. We may therefore write

$$y(x; \alpha) \stackrel{\text{def}}{=} z\big(x; \alpha, \eta(L(\alpha))\big) = u_1(x; L(\alpha)),$$

and from the Implicit Function Theorem we obtain

$$\frac{dL}{d\alpha}(\alpha(L_*)) = -\frac{\left(\frac{\partial z'}{\partial \alpha}\frac{\partial z'''}{\partial \eta} - \frac{\partial z'}{\partial \eta}\frac{\partial z'''}{\partial \alpha}\right)(\frac{L_*}{2};0,0)}{\left(\frac{\partial z'}{\partial x}\frac{\partial z'''}{\partial \eta} - \frac{\partial z'}{\partial \eta}\frac{\partial z'''}{\partial x}\right)(\frac{L_*}{2};0,0)}.$$

Here the denominator is nonzero (this follows from the analysis in Section 2.8), whereas the numerator is zero by (4.18), and hence $\frac{dL}{d\alpha}(\alpha(L_*)) = 0$. Combining this with the fact that $y'(L(\alpha),\alpha) = u_1'(L(\alpha);L(\alpha)) = 0$ and similarly $y'''(L(\alpha);\alpha) = 0$, we infer that $\frac{\partial y'}{\partial \alpha}(\frac{L_*}{2};\alpha(L_*)) = 0$ and $\frac{\partial y'''}{\partial \alpha}(\frac{L_*}{2};\alpha(L_*)) = 0$. It follows that $\tilde{v}(x) \equiv \frac{\partial y}{\partial \alpha}(x;\alpha(L_*))$ satisfies (4.16) for $L = L_*$, and thus $\tilde{v}$ is an antisymmetric 0-eigenfunction of $u_1(x;L_*)$.

Notice that $\tilde{v}(x) > 0$ for $x \in (0,\frac{L_*}{2})$. Indeed, $\tilde{v}(x) \geq 0$ for $x \in (0,\frac{L_*}{2})$ since $u_1(x;L) > u_1(x;\tilde{L})$ for all $x \in (0,\frac{\tilde{L}}{2})$ and $L > \tilde{L}$ (see e.g. the proof of Lemma 2.35). If $\tilde{v}(x_0) = 0$ for some $x_0 \in (0,\frac{L_*}{2})$ then also $\tilde{v}'(x_0) = 0$, and it follows from the Comparison Lemma that $\tilde{v} \equiv 0$, but $\tilde{v}'(0) > 0$, a contradiction.

Finally, using integration by parts and the differential equation (4.15), we obtain that

$$0 = \int_0^{\frac{L_*}{2}} u_1(\tilde{v}'''' - A\tilde{v}'' + (3u_1^2 - 1)\tilde{v}) = \int_0^{\frac{L_*}{2}} 2u_1^3\tilde{v},$$

which contradicts the fact that $u_1 > 0$ and $\tilde{v} > 0$ on $(0,\frac{L_*}{2})$. □

Combing these two lemmas we obtain:

Lemma 4.31 *For $n = 1$ there is no nontrivial solution of problem* (4.16).

Proof. By contradiction, suppose that $w(x)$ is a nontrivial solution. Then either $v(x) = w(x) + w(-x)$ or $\tilde{v}(x) = w(x) - w(-x)$ is a nontrivial solution of (4.16). However, the existence of a symmetric (i.e. $v(x) = v(-x)$) 0-eigenfunction is excluded by Lemma 4.29, while an antisymmetric (i.e. $\tilde{v}(x) = -\tilde{v}(-x)$) 0-eigenfunction is excluded by Lemma 4.30. □

We now proceed by induction. Let $n > 1$ and suppose that u_k has no 0-eigenvalue for $k < n$. In the following it is proved that then u_n has no 0-eigenvalue. As in Lemma 4.31 we may assume, without loss of generality, that an eigenfunction is either symmetric or antisymmetric. We split up the argument into four cases.

1. If n is even then there is no antisymmetric 0-eigenfunction.
2. If n is odd then there is no symmetric 0-eigenfunction.
3. If n is even then there is no symmetric 0-eigenfunction.
4. If n is odd then there is no antisymmetric 0-eigenfunction.

Proving these four statements finishes the proof. Case 1 and 2 follow from the Comparison Lemma as in Lemma 4.29. Next we deal with Case 3. Suppose, by contradiction, that $v(x)$ is a symmetric 0-eigenfunction of $u_n(x;L)$. Note that $u_n(x-\frac{L}{2};L) = u_{n/2}(x;\frac{L}{2})$. Obviously $v(x-\frac{L}{2})$ is therefore a 0-eigenfunction of $u_{n/2}(x;\frac{L}{2})$, contradicting the induction hypothesis.

As for Case 4, let $n \geq 3$ be odd and assume, by contradiction, that $v(x)$ is an antisymmetric 0-eigenfunction of $u_n(x;L)$. The function u_n has an extremum at $x = \frac{L}{2n} \stackrel{\text{def}}{=} a$, and it is symmetric with respect to this extremum. One distinguishes two cases: $v(2a) = 0$ and $v(2a) \neq 0$.

In the case $v(2a) = 0$, define $v_1(x) = v(x+2a)$ and $v_2(x) = -v(-x+2a)$, then

$$(v_1 - v_2)(0) = 0, \quad (v_1 - v_2)'(0) = 0, \quad (v_1 - v_2)'''(0) = 0.$$

As in Lemma 4.29 the Comparison Lemma implies that $v_1 \equiv v_2$, i.e., v is antisymmetric with respect to $x = 2a$. Since v is antisymmetric with respect to $x = 0$ and $x = 2a$, the boundary conditions $v'(na) = v'''(na) = 0$ (with n odd) imply that $v'(a) = v'''(a) = 0$. Therefore $v(x)$ is a 0-eigenfunction of $u_1(x; \frac{L}{n})$ on $(-a, a)$, contradicting the induction hypothesis.

If $v(2a) \neq 0$, define $w(x) = v(x + 2a) - v(x - 2a)$. Since $v(x)$ is antisymmetric, $w(x)$ is symmetric. We now apply the Comparison Lemma to $v_1(x) = \frac{u_n'(0)}{w(0)} w(x)$ and $v_2(x) = u'(x)$, which yields

$$v(x + 2a) - v(x - 2a) = cu'(x), \qquad \text{where } c = \tfrac{2v(2a)}{u'(0)} \neq 0. \tag{4.19}$$

Successively substituting $x = 2ka$ in (4.19) for $k = 1, 2, \ldots$ gives that $v(2ka) = (-1)^{k-1} k v(2a)$, for $k = 0, 1, 2, \ldots$ (since $u'(2ka) = (-1)^k u'(0)$). This is impossible since v is symmetric with respect to $x = na$, and a contradiction has been reached.

Chapter 5

Travelling waves

5.1 Introduction

Fourth order parabolic equations of the form

$$u_t = -\gamma u_{xxxx} + u_{xx} + f(u), \qquad \gamma > 0, \tag{5.1}$$

where $x \in \mathbb{R}$, $t > 0$, occur in many physical models such as the theory of phase-transitions [53], nonlinear optics [1], shallow water waves [35], etcetera. Usually the potential $F(u) = \int f(s)\,ds$ has at least two local maxima (stable states), and one local minimum (unstable state)[1]. A prototypical example is $f_a(u) = (u + a)(1 - u^2)$ with $-1 < a < 1$.

For a thorough understanding of Equation (5.1), the stationary problem is of great importance. An extensive literature on this subject exists (see e.g. [3, 21, 35, 88, 89, 90, 112, 116, 117, 118]). Typically, depending on the parameter γ, the stationary problem displays a multitude of periodic, homoclinic, and heteroclinic solutions. The stationary equation is Hamiltonian, which restricts the possible connections between the equilibrium points. As an example we mention that when the maximum of F is attained in two points, e.g. $F(u) = -\frac{1}{4}(u^2 - 1)^2$, a solution connecting these maxima exists for all $\gamma > 0$. One could regard this solution as a standing wave. The heteroclinic solution is unique (modulo the obvious symmetries) for small values of γ, say $\gamma \leq \gamma_1(f)$ (see Chapter 2 and [96]). On the other hand, for large γ, say $\gamma > \gamma_2(f)$, there is a multitude of (multi-bump) solutions connecting the two maxima [89, 90, 118]. This is due to the fact that as γ crosses the critical value $\gamma = \gamma_2(f)$, the eigenvalues of the linearised stationary equation around the two maxima of F become complex.

In the special case $f(u) = u - u^3$, corresponding to $F(u) = -\frac{1}{4}(u^2 - 1)^2$, it holds that $\gamma_1(f) = \gamma_2(f) = \frac{1}{8}$. Although in many simple cases equality holds, generally there will be a gap between $\gamma_1(f)$ and $\gamma_2(f)$. The critical value γ_1 is not necessarily small, and a lower bound on γ_1 can in general be explicitly determined (see Chapter 2 for more details).

For the time-dependent problem travelling fronts of the form $u(x,t) = U(x + ct)$, connecting extrema of the potential F, play a prominent role in most models. Results on travelling waves for Equation (5.1) have previously been obtained in [34], where nonlinearities of the form $f(u) = f_a(u) = (u + a)(1 - u^2)$, $a \approx 0$, are studied using transversality arguments and perturbing near a standing wave. Moreover, in [2] singular perturbations techniques were applied near $\gamma = 0$. In both cases travelling waves between local maxima (stable states) are studied. A recent work [130] deals with singular perturbations techniques for travelling waves connecting an unstable and a stable state; the stability of these waves for very small γ is also established. Furthermore, in the context of singular perturbation theory, travelling waves for higher order parabolic equations have been

[1] Sometimes the potential is denoted by $-F$ so that the stable states correspond to local minima.

studied in [72].

The objective of this chapter is to obtain existence results for a large range of parameter values. We therefore study travelling waves of (5.1) via topological arguments rather than perturbation methods. To illustrate the underlying ideas of the method, let us consider the related second order parabolic equation, i.e. $\gamma = 0$. Such equations arise as models in for example population genetics and combustion theory [11]. In the special case where $f(u) = f_a(u)$, Equation (5.1) with $\gamma = 0$ admits a travelling wave solution $u(x,t) = \tanh\left(\frac{x+a\sqrt{2}t}{\sqrt{2}}\right)$. This travelling wave connects the two stable homogeneous states $u = -1$ and $u = +1$. The literature on this problem is extensive and we will not attempt to give a complete list. However, a few key references are of importance for explaining the similarities of the second and fourth order problems. In the case $\gamma = 0$ the equation for travelling waves $u(x,t) = U(x+ct)$ is given by $cU' = U'' + f(U)$. A phase-plane analysis for both $0 < c \ll 1$ and $c \gg 1$ shows two topologically different phase portraits, from which the conclusion may be drawn that a global bifurcation has to take place for some intermediate c-value(s). In this way a wave speed c_0 can be found for which a travelling wave exists which connects the two local maxima of F. In this context we mention the work by Fife and McLeod [68] based on an analytic approach, and Conley's more topological approach [48].

From the second order problem we learn that for the present problem it is sensible to look for topologically different phase portraits (in $\mathbb{R}^4$) for small and large values of c. A big part of our analysis will be to do just that.

In order to simplify the exposition of the main results we reformulate (5.1) as

$$u_t = -u_{xxxx} + \beta u_{xx} + f(u), \tag{5.2}$$

via the rescaling $x \mapsto \gamma^{\frac{1}{4}} x$, with $\beta = \frac{1}{\sqrt{\gamma}}$. Notice that Equation (5.2) also has meaning for $\beta \leq 0$.

Let us start now with the hypotheses on the nonlinearity:

$$(\mathrm{H}_0) \quad \begin{cases} \bullet\ F'(u) = f(u) \in C^1(\mathbb{R}); \\ \bullet\ f(u) = 0 \Leftrightarrow u \in \{-1, -a, 1\} \text{ for some } a \in (-1,1), \text{ and } f'(\pm 1) \neq 0,\ f'(-a) \neq 0; \\ \bullet\ F(-1) < F(+1); \\ \bullet\ F(u) \to -\infty \text{ as } u \to \pm\infty; \\ \bullet\ \text{for some } M > 0 \text{ it holds that } f'(u) \leq M \text{ for all } u \in \mathbb{R}.^2 \end{cases}$$

Of course, the prototypical example $f_a(u) = (u+a)(1-u^2)$ satisfies (H_0) for $a > 0$. We remark that the third condition excludes the existence of a standing wave which connects two different equilibria. The last condition is a technical one, which we use to obtain certain a priori bounds. Without loss of generality we set

$$F(u) = \int_1^u f(s)ds,$$

so that $F(1) = 0$.

Denote the wave speed by c, and, searching for a travelling wave, we set $u(x,t) = U(x+ct)$, which, switching to lower case again, reduces (5.2) to the ordinary differential

[2] Note that $f'(u)$ may be unbounded from below.

equation

$$cu' = -u'''' + \beta u'' + f(u). \tag{5.3}$$

An important ingredient of our analysis is a conserved quantity for (5.3) when $c = 0$, which is a Lyapunov function when $c \neq 0$. Define

$$\mathcal{E}(u, u', u'', u''') \stackrel{\text{def}}{=} -u'u''' + \frac{1}{2}u''^2 + \frac{\beta}{2}u'^2 + F(u). \tag{5.4}$$

Multiplying (5.3) by u' we find that

$$\mathcal{E}'(u, u', u'', u''') = cu'^2, \tag{5.5}$$

so that $\mathcal{E}$, which will be referred to as the *energy* of the solution, is increasing along orbits if $c > 0$, constant if $c = 0$, and decreasing if $c < 0$. When we are looking for a solution of (5.3) connecting $u = -1$ to $u = 1$, we see that we can restrict our attention to $c > 0$.

The first theorem deals with the connection between the two stable states $u = -1$ and $u = +1$. This connection is non-generic with respect to the wave speed c. Noting that $F(u) \leq 0$ for all $u \in \mathbb{R}$ if f satisfies hypothesis (H_0), we define

$$\sigma(f) \stackrel{\text{def}}{=} \min_{-1<u<-a} \frac{-F(u)}{2f(u)^2}. \tag{5.6}$$

Theorem 5.1 *Let f satisfy hypothesis* (H_0) *and let $\beta > \frac{1}{\sqrt{\sigma(f)}}$. Then, for some wave speed $c = c_0(f) > 0$, there exists a travelling wave solution of* (5.2) *connecting $u = -1$ to $u = +1$.*

The analogous condition on γ for Equation (5.1) reads $0 < \gamma < \sigma(f)$. We remark that the theorem also holds when $f'(-a) = 0$, since the (non)degeneracy of $u = -a$ does not play any role for connections between -1 and $+1$.

At the minimum in (5.6) the equality $\frac{-F(u)}{2f(u)^2} = \frac{-1}{4f'(u)}$ holds. We easily derive that for our model nonlinearity f_a we have $\sigma(f_a) > \frac{1}{8(1-a)}$ for all $0 < a < 1$. Although this estimate is sharp for $a \to 0$, it is not sharp at all for larger values of a.

For general nonlinearities $f(u)$ satisfying (H_0), a lower bound on σ is

$$\sigma \geq \min\left\{ \frac{-1}{4f'(u)} \;\middle|\; u \in (-1, -a) \text{ and } f'(u) < 0 \right\}. \tag{5.7}$$

This estimate is often easier to compute than σ itself, but it is in general a rather blunt estimate. Finally, we remark that the critical value σ is also encountered in the study of homoclinic orbits for $c = 0$ (see [116, Theorem B]). This originates from the similarity of that problem with the proof of Lemma 5.20, which is in fact the only instance in our analysis where γ is required to be smaller than σ.

We do not obtain much insight in the shape of the travelling wave from Theorem 5.1. Because Theorem 5.1 does not give information about the wave speed, it is not known whether the connected equilibrium points are approached monotonically or in an oscillatory manner. The linearised equation around the equilibrium points leads to the following characteristic equation for the eigenvalues: $c\lambda = -\lambda^4 + \beta\lambda + f'(\pm 1)$. A few conclusions can be drawn from analysing this equation. It follows that for $\beta \geq \sqrt{-4f'(1)}$ the travelling wave tends to $+1$ monotonically as $x \to \infty$. Besides, for $\beta \leq \sqrt{-4f'(-1)}$ the travelling wave tends to -1 in an oscillatory way as $x \to -\infty$. For other cases the behaviour in the limits depends on the (a priori unknown) value of c.

The travelling wave solution found in Theorem 5.1 connects the two maxima of F. Theorem 5.1 can be extended to potentials F having many local extrema, i.e. $f(u)$ having many zeros. In that case we find a travelling wave connecting the global maximum and the second largest local maximum of F. The other conditions on F remain the same, but we also need the sign condition $f(u)u < 0$ for large values of $|u|$. The definition of σ in this case is (setting $\max_{u\in\mathbb{R}} F(u) = 0$):

$$\sigma(f) \stackrel{\text{def}}{=} \inf\Big\{ \frac{-F(u)}{2f(u)^2} \;\Big|\; u \in \mathbb{R} \text{ and } f(u)f'(u) > 0 \Big\}.$$

The travelling wave solution found in Theorem 5.1 connects the two stable states. The following theorems deal with travelling waves connecting the unstable state $u = -a$ to one of the stable states $u = \pm 1$. These theorems also apply to the parameter regime where $\beta \geq 0$, but for these parameter values we need an additional condition on f:

$$(\mathrm{H}_1) \quad f \text{ satisfies } (\mathrm{H}_0) \quad \text{and} \quad \lim_{|u|\to\infty} \frac{f(u)}{u} = -\infty.$$

Theorem 5.2 *Let $\beta \in \mathbb{R}$ and let f satisfy hypothesis* (H_0) *if $\beta < 0$ and* (H_1) *if $\beta \geq 0$.*[3] *Then for every $c > 0$ there exists a travelling wave solution of* (5.2) *connecting $u = -a$ to $u = -1$.*

The limiting behaviour of the travelling waves can be determined from the characteristic equations. For $\beta \geq \sqrt{-4f'(-1)}$ the solution tends to -1 monotonically for $x \to \infty$ regardless of the speed c. On the other hand, for $\beta < \sqrt{-4f'(-1)}$ the limit behaviour is oscillatory for small c and monotonic for large c. The limit behaviour near $u = -a$ as $x \to -\infty$ is more complicated. For small c the behaviour is generically oscillatory, while for large c the solutions generically tends to $-a$ monotonically. We do not know whether the behaviour is indeed generic. However, for $\beta > \sqrt{12f'(-a)}$ there is an intermediate range of c-values for which the travelling wave certainly tends to $-a$ monotonically.

For general potentials F this result applies to any pair of consecutive non-degenerate extrema u_- (a minimum) and u_+ (a maximum), for which the interval $\big(F(u_-), F(u_+)\big)$ contains no critical values and either u_- or u_+ is the only critical point at level $F(u_\pm)$. The other conditions on F remain the same. The method of proof of Theorem 5.2 requires only one of the two extrema -1 or $-a$ to be non-degenerate.

The next theorem deals with the case of travelling waves from $-a$ to $+1$.

Theorem 5.3 *Let $\beta \in \mathbb{R}$ and let f satisfy hypothesis* (H_0) *if $\beta < 0$ and* (H_1) *if $\beta \geq 0$. Then there exists a constant $c^*(f) > 0$, such that for every $c > c^*$ there exists a travelling wave solution of* (5.2) *connecting $u = -a$ to $u = +1$.*

Theorem 5.3 extends to general potentials, giving travelling waves between any pair of consecutive non-degenerate extrema u_- (a minimum) and u_+ (a maximum), provided the local minimum $\tilde{u}_-$ on the other side of u_+, if it exists, satisfies $F(\tilde{u}_-) > F(u_-)$. Of course, if the opposite inequality holds then one can exchange u_- and $\tilde{u}_-$. If equality holds, i.e. $F(\tilde{u}_-) = F(u_-)$, then one obtains for every $c > c^*$ a travelling wave connecting either u_- or $\tilde{u}_-$ to u_+. Again, the other conditions on F remain the same.

In certain cases one obtains information about the constant c^* in Theorem 5.3. In that case the situation is very much analogous to the second order equation.

[3]The result also holds when $F(-1) = F(+1)$.

Corollary 5.4 *Let f satisfy hypothesis* (H_0) *and let* $\beta > \frac{1}{\sqrt{\sigma(f)}}$. *Then there exists a* $c^*(f) > 0$, *such that* c^* *is the largest speed for which there exists a travelling wave solution of* (5.2) *connecting* $u = -1$ *to* $u = +1$. *Moreover, for all* $c > c^*$ *there exists travelling wave solution of* (5.2) *connecting* $u = -a$ *to* $u = +1$.

Finally, we discuss nonlinearities with different behaviour for $u \to \pm\infty$. Assume that f has *two* zeros and satisfies

$$(\mathrm{H}_2)\quad \begin{cases} \bullet\ F'(u) = f(u) \in C^1(\mathbb{R}); \\ \bullet\ f(u) = 0 \Leftrightarrow u \in \{0,1\}, \text{ and } f'(0) \neq 0,\ f'(1) \neq 0; \\ \bullet\ \text{for some } D < 0 \text{ it holds that } F(u) > F(1) \text{ for all } u < D; \\ \bullet\ F(u) \to -\infty \text{ as } u \to \infty; \\ \bullet\ \text{if } \beta \geq 0, \text{ then } \lim_{|u|\to\infty} \frac{f(u)}{u} = -\infty. \end{cases}$$

A typical example is $f(u) = u(1-u)$. The following theorem is analogous to Theorem 5.2.

Theorem 5.5 *Let* $\beta \in \mathbb{R}$ *and let f satisfy hypothesis* (H_2). *Then for every* $c > 0$ *there exists a travelling wave solution of* (5.2) *connecting* $u = 0$ *to* $u = 1$.

This last theorem is just an example of how the methods in this chapter can also be applied when $F(u)$ does not tend to $-\infty$ as $u \to \pm\infty$. The theorem holds under weaker conditions, but we leave this to the interested reader.

Of the results in this chapter, the proof of Theorem 5.1 is by far the most involved. This is caused by the fact that connections between local maxima are non-generic with respect to the wave speed c. Hence, part of the problem is to determine the wave speed c. The idea behind the proof is that one can detect a change in the phase portrait (in $\mathbb{R}^4$) of Equation (5.3) as c goes from small values to large values. In particular, looking for a travelling wave which connects -1 to $+1$, we investigate the global behaviour of the orbits in the stable manifold $W^s(1)$ of the equilibrium point $u = +1$.

The analysis for $c > 0$ large is based on a continuation argument deforming the nonlinearity $f(u)$ into a function which is linear on some interval containing $u = 1$. For $c > 0$ small the analysis is much more involved. A crucial step is that for $c = 0$ all orbits in $W^s(1)$ are unbounded. A first result in this direction was already proved in Chapter 2. There it was shown that, for γ not too large, the bounded stationary solutions of (5.1) correspond exactly to the bounded stationary solutions of the second order equation ($\gamma = 0$). This excludes the existence of bounded orbits in $W^s(1)$. However, since the analysis comprises *all* bounded solutions, this result is limited to a restricted parameter regime. In particular, the equilibrium points $u = \pm 1$ need to be real saddles. In the present situation we want to exclude bounded solutions in the stable manifold of $u = 1$, i.e., we can restrict the analysis to the energy level $\mathcal{E} = 0$. This allows us to cover a larger range of β-values, to be precise: $\beta > \frac{1}{\sqrt{\sigma(f)}}$. This parameter regime includes cases where both equilibrium points $u = \pm 1$ are saddle-foci. To give an example, for our model nonlinearity $f_a = (u+a)(1-u^2)$ with $0 < a < 1$ the result from Chapter 2 holds for $\beta \geq \sqrt{8(1+a)}$. The equilibrium points $u = 1$ and $u = -1$ become saddle-foci for $\beta < \sqrt{8(1+a)}$ and $\beta < \sqrt{8(1-a)}$ respectively. One may compare this to the estimate $\sigma(f_a) > \frac{1}{8(1-a)}$. Notice that this estimate, although sharp for $a \to 0$, is very blunt for a close to 1.

For the description of unbounded orbits we use a modified Poincaré transformation which we believe is of independent interest. We investigate the unbounded orbits, and we will show that, in an appropriate compactification of the phase space, these orbits must converge to a unique periodic orbit lying at infinity in the phase space. The analysis at infinity largely relies on a global analysis of bounded and unbounded solutions of the family of equations

$$u'''' + u^s = 0 \quad \text{with the convention that} \quad u^s = |u|^{s-1}u,\ s \geq 1.$$

This equation is invariant under the scaling $u(t) \mapsto \kappa u(\kappa^{\frac{s-1}{4}} t)$ for all $\kappa > 0$. The analysis of this equation is in particular used in the proof of finite time blow-up of unbounded solutions, and, more importantly, to determine the behaviour of unbounded orbits for $0 \leq c \ll 1$.

From this analysis we conclude that the phase portrait for c positive but small is different from the phase portrait for c large, which in turn is used to prove the existence of a connection between -1 and $+1$ for some intermediate wave speed c_0.

The organisation of the chapter is as follows. We start with some a priori bounds in Section 5.2. In Section 5.3 we give the proof of Theorem 5.1, and in the Sections 5.4 to 5.6 the details of this proof are filled in. In particular, in Section 5.4 we perform an analysis of the flow 'at infinity'. Sections 5.5 and 5.6 deal with the analysis of the orbits in $W^s(1)$ for small c and large c respectively. Section 5.7 discusses the existence of travelling waves connecting $u = -a$ to $u = \pm 1$; Theorems 5.2 to 5.5 are proved here. We conclude with some remarks on open problems in Section 5.8.

5.2 A priori estimates

We establish a priori bounds on the wave speed c and the profile u for any travelling wave connecting -1 and $+1$. The bound on the wave speed c holds for all $\beta \in \mathbb{R}$.

Lemma 5.6 *Let f satisfy hypothesis* (H_0) *and let $\beta \in \mathbb{R}$. There exists a constant c_0, depending only on β, $F(-1)$, $F(-a)$, and the upper bound M for $f'(u)$, such that when $c > 0$ is a speed for which there exists a travelling wave solution of* (5.3) *connecting -1 to $+1$, then $c \leq c_0$.*

Proof. Suppose u is a solution of (5.3) connecting -1 to $+1$. Integrating (5.5), we have

$$-F(-1) = F(1) - F(-1) = c \int_{-\infty}^{\infty} u'^2. \tag{5.8}$$

Multiplying (5.3) by u'' and integrating (by parts) we obtain

$$\int_{-\infty}^{\infty} u'''^2 + \beta \int_{-\infty}^{\infty} u''^2 = \int_{-\infty}^{\infty} (f(u))' u' = \int_{-\infty}^{\infty} f'(u) u'^2 \leq M \int_{-\infty}^{\infty} u'^2 = M \frac{-F(-1)}{c}. \tag{5.9}$$

Let $u_1 \in (-a, 1)$ be defined by

$$F(u_1) = \frac{F(-a) + F(-1)}{2}.$$

There must be points t_0, $t_1 \in \mathbb{R}$, $t_0 < t_1$, such that $u(t_0) = -a$, $u(t_1) = u_1$ and $u(t) \in [-a, u_1]$ for $t \in [t_0, t_1]$. The length of this interval is estimated from below by

$$(u_1 + a)^2 = \left(\int_{t_0}^{t_1} u'(t) dt \right)^2 \leq (t_1 - t_0)^2 \int_{t_0}^{t_1} u'(t)^2 dt \leq (t_1 - t_0)^2 \frac{-F(-1)}{c}.$$

On the one hand, because the energy $\mathcal{E}$ increases along orbits, we have

$$
\begin{aligned}
\int_{t_0}^{t_1} &\big(-u'''(t)u'(t) + \frac{1}{2}u''(t)^2 + \frac{\beta}{2}u'(t)^2\big)dt \\
&\geq \int_{t_0}^{t_1} \big(F(-1) - F(u(t))\big)\,dt \\
&\geq (F(-1) - F(u_1))(t_1 - t_0) = \frac{F(-1) - F(-a)}{2}(t_1 - t_0) \\
&\geq \frac{F(-1) - F(-a)}{2}(u_1 + a)\sqrt{\frac{c}{-F(-1)}}. \qquad (5.10)
\end{aligned}
$$

We now first restrict to the case that $\beta > 0$, and come back to the other case later on. Using (5.8) and (5.9), we obtain the estimate

$$
\begin{aligned}
\int_{t_0}^{t_1} &\big(-u'''(t)u'(t) + \frac{1}{2}u''(t)^2 + \frac{\beta}{2}u'(t)^2\big)dt \\
&\leq \int_{t_0}^{t_1} \Big(\frac{1}{2}\big(u'''(t)^2 + u''(t)^2\big) + \frac{1+\beta}{2}u'(t)^2\Big)dt \\
&\leq \big(M\max\{\frac{1}{\beta}, 1\} + 1 + \beta\big)\frac{-F(-1)}{2c}. \qquad (5.11)
\end{aligned}
$$

By combining (5.10) and (5.11) we obtain

$$
\frac{F(-1) - F(-a)}{2}(u_1 + a)\sqrt{\frac{c}{-F(-1)}} \leq \big(M\max\{\frac{1}{\beta}, 1\} + 1 + \beta\big)\frac{-F(-1)}{2c}.
$$

Since also

$$
\frac{F(-1) - F(-a)}{2} = F(u_1) - F(-a) \leq \frac{M}{2}(u_1 + a)^2,
$$

it follows that

$$
c \leq M^{\frac{1}{3}}\big(M\max\{\frac{1}{\beta}, 1\} + 1 + \beta\big)^{\frac{2}{3}}\frac{-F(-1)}{F(-1) - F(-a)}.
$$

This completes the proof of the lemma for the case that $\beta > 0$.

We now deal with the case $\beta \leq 0$. The first part of estimate (5.11) is replaced by

$$
\begin{aligned}
\int_{t_0}^{t_1} &\big(-u'''(t)u'(t) + \frac{1}{2}u''(t)^2 + \frac{\beta}{2}u'(t)^2\big)dt \\
&\leq \int_{-\infty}^{\infty} \Big(\frac{1}{2}u'''(t)^2 + \frac{1}{2}u''(t)^2 + \frac{1}{2}u'(t)^2\Big)dt \\
&= \int_{-\infty}^{\infty} \Big(u'''(t)^2 + \beta u''(t)^2 + \big(\frac{1}{2} - \beta\big)u''(t)^2 - \frac{1}{2}u'''(t)^2 + \frac{1}{2}u'(t)^2\Big)dt \\
&\leq \int_{-\infty}^{\infty} \Big(u'''(t)^2 + \beta u''(t)^2 + \frac{4\beta^2 - 4\beta + 5}{8}u'(t)^2\Big)dt,
\end{aligned}
$$

where we have used that $\int_{-\infty}^{\infty} u''^2 \leq \lambda \int_{-\infty}^{\infty} u'''^2 + \frac{1}{4\lambda}\int_{-\infty}^{\infty} u'^2$ for all $\lambda > 0$. The remainder of the proof is the same as above. □

The L^∞-bound on the profile u holds for $\beta > 0$, or equivalently, for all $\gamma > 0$.

Lemma 5.7 *Let f satisfy hypothesis* (H$_0$) *and let $\beta > 0$. There exists a constant C_1, depending only on β, $F(-1)$, $F(-a)$, and the upper bound M for $f'(u)$, such that when u is, for some $c > 0$, a travelling wave solution of* (5.3) *connecting -1 to $+1$, then $F(u) \geq C_1$.*

Proof. We may suppose that there is a connection u with range not contained in the bounded interval $\{u \in \mathbb{R} \,|\, F(u) \geq F(-a)\}$, otherwise we already have our desired uniform bound. Therefore, without loss of generality we may assume that

$$F(u(0)) = \min_{t\in\mathbb{R}} F(u(t)) < F(-a). \tag{5.12}$$

We consider the case where $u(0) < -1$ (the case $u(0) > 1$ is completely analogous). Since

$$\mathcal{E}(u, u', u'', u''')(t) \in \big(F(-1), F(1)\big) = \big(F(-1), 0\big) \quad \text{for all } t \in \mathbb{R}, \tag{5.13}$$

we clearly have that

$$u(0) < -1, \quad u'(0) = 0, \quad 0 < \sqrt{2\big(F(-1) - F(u(0))\big)} < u''(0) < \sqrt{-2F(u(0))}.$$

We now consider two cases: $u'''(0) \geq 0$ and $u'''(0) < 0$. We start with the latter case. Since $u(t)$ tends to an equilibrium point as $t \to -\infty$, there exists a $t_1 < 0$ such that $u'''(t) < 0$ for $t_1 < t < 0$ and $u'''(t_1) = 0$. Equation (5.5) implies that

$$-u'(t)u'''(t) + F(u(t)) - F(u(0)) = -\frac{1}{2}\big(u''(t)^2 - u''(0)^2\big) - \frac{\beta}{2}u'(t)^2 + c\int_0^t u'(s)^2 ds. \tag{5.14}$$

By (5.12) we know that $F(u(t_1)) \geq F(u(0))$, so that

$$\frac{1}{2}\big(u''(t_1)^2 - u''(0)^2\big) + \frac{\beta}{2}u'(t_1)^2 \leq -c\int_{t_1}^0 u'(s)^2 ds.$$

Since $u''(t)$ decreases on $(t_1, 0)$ and β is positive, this implies that $c < 0$, a contradiction.

We now deal with the case that $u'''(0) \geq 0$. Since $u''''(0) > 0$ by the differential equation, and since $u(t)$ tends to an equilibrium point as $t \to \infty$, there exists a $t_2 > 0$ such that $u'''(t) > 0$ for $0 < t < t_2$ and $u'''(t_2) = 0$. By (5.12) we know that $F(u(t_2)) \geq F(u(0))$. Since $\beta > 0$, it follows from (5.14) and the fact that $u''(t)$ increases on $(0, t_2)$, that

$$\frac{\beta}{2}u'(t_2)^2 \leq c\int_0^{t_2} u'(s)^2 ds \leq c\int_{-\infty}^{\infty} u'(s)^2 ds \leq -F(-1). \tag{5.15}$$

Furthermore, from the fact that $u''(t)$ increases on $(0, t_2)$ we infer that

$$u''(0)t \leq u'(t) \leq u'(t_2) \qquad \text{for } t \in [0, t_2]. \tag{5.16}$$

On the one hand it follows from (5.15) and (5.16) that $\frac{\beta}{2}u'(t_2)^2 \leq c\int_0^{t_2} u'(s)^2 ds \leq cu'(t_2)^2 t_2$, hence

$$t_2 \geq \frac{\beta}{2c}. \tag{5.17}$$

On the other hand it follows from (5.15) and (5.16) that $-F(-1) \geq c\int_0^{t_2} u'(s)^2 ds \geq \frac{1}{3}ct_2^3 u''(0)^2$. Combining with (5.17) we thus obtain that

$$u''(0)^2 \leq \frac{-24c^2F(-1)}{\beta^3}.$$

This gives a bound on $u''(0)^2$, because it follows from Lemma 5.6 that the wave speed c is bounded above by a constant $c_0\big(\beta, M, F(-a), F(-1)\big)$.

Finally, by (5.12) and (5.13) we have

$$F(u(t)) \geq F(u(0)) \geq F(-1) - \frac{1}{2}u''(0)^2 \qquad \text{for all } t \in \mathbb{R}.$$

This completes the proof of Lemma 5.7. □

5.3 Proof of Theorem 5.1

In this section we give the proof of Theorem 5.1. Some of the major steps, which require a quite involved analysis, are only stated as a proposition in this section and are proved in subsequent sections.

We first use the a priori bounds of Section 5.2 to reduce our analysis to nonlinearities $f(u)$ of the form $f(u) = -u^3 + g(u)$, where $g(u)$ has compact support. The advantage of such nonlinearities is that they behave nicely as $u \to \pm\infty$, and it will thus be possible to analyse the flow near/at infinity.

Let $f(u)$ satisfy hypothesis (H_0). Lemma 5.7 implies that there exists a constant C_0 such that any travelling wave solution u connecting -1 to $+1$ satisfies $\|u\|_\infty < C_0$. Define the cut-off function $\phi \in C_0^\infty$ with $0 \le \phi \le 1$, $\phi(y) = 1$ for $|y| \le C_0$, and $\phi(y) = 0$ for $|y| > C_0 + 1$. We now consider the modified nonlinearity $\tilde{f}(u) = \phi(u)f(u) - u^3(1 - \phi(u))$. Lemma 5.7 ensures that u is a travelling wave solution for nonlinearity $f(u)$ if and only if u is a travelling wave solution for nonlinearity $\tilde{f}(u)$. Besides, $\sigma(f) = \sigma(\tilde{f})$. This shows that we may restrict our analysis to nonlinearities $f(u)$ such that

$$f(u) = -u^3 + g(u) \quad \textit{with } g \textit{ compactly supported, and } f \textit{ satisfies hypothesis } (\mathrm{H}_0). \tag{5.18}$$

The purpose of the reduction to nonlinearities f which satisfy (5.18) is that it makes it possible to analyse the orbits which are unbounded. An important property of unbounded solutions, which we will need in the following, is formulated in the next lemma.

Lemma 5.8 *Let f satisfy hypothesis* (5.18) *and let $\beta, c \in \mathbb{R}$. Then any unbounded solution of* (5.3) *blows up in finite time.*

This lemma is proved in Section 5.4.5, Theorem 5.18b, and is based on the analysis of the flow near/at infinity.

As already discussed in the introduction, denote the wave speed by c. For finding a travelling wave we set $u(x,t) = U(x+ct)$, which reduces (5.1) to the ordinary differential equation (5.3). Written as a four-dimensional system, (5.3) becomes

$$u' = v; \quad v' = w; \quad w' = z; \quad z' = \beta w - cv + f(u). \tag{5.19}$$

The equilibria of this system are $(u,v,w,z) = (-1,0,0,0)$, $(u,v,w,z) = (-a,0,0,0)$ and $(u,v,w,z) = (1,0,0,0)$ (for short: $u = -1$, $u = -a$ and $u = 1$). To prove Theorem 5.1 we look for a $c \neq 0$ and a corresponding heteroclinic orbit of (5.19) connecting $u = -1$ to $u = 1$. Linearising around $u = \pm 1$ we find that, irrespective of c, both $u = -1$ and $u = 1$ have two-dimensional stable and unstable manifolds, denoted by $W^s(\pm 1)$ and $W^u(\pm 1)$. Generically $W^s(1)$ and $W^u(-1)$ will not intersect but varying c we expect to pick up a non-empty intersection.

We recall that the *energy* is defined as

$$\mathcal{E}(u,v,w,z) \stackrel{\text{def}}{=} -vz + \frac{1}{2}w^2 + \frac{\beta}{2}v^2 + F(u),$$

where the potential $F(u) = \int_1^u f(s)ds$ is depicted in Figure 5.1. Since we are looking for a solution of (5.3) which connects $u = -1$ to $u = 1$, we see from (5.5) that we can restrict our attention to $c > 0$. The energy $\mathcal{E}$ thus increases along orbits.

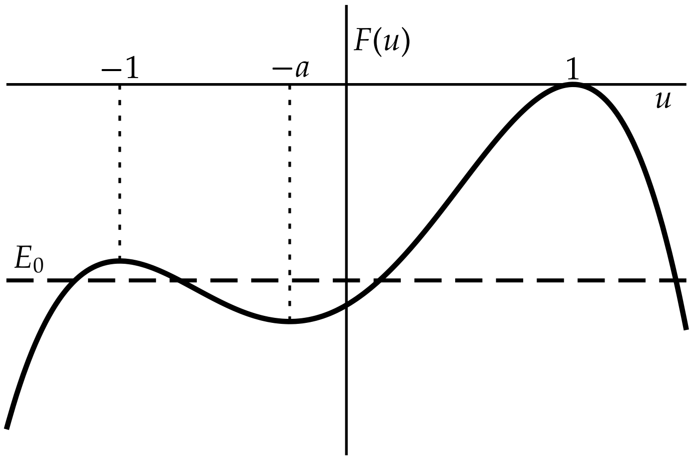

Figure 5.1: The potential $F(u)$ and the energy level E_0 separating $u = -a$ from $u = \pm 1$.

To separate the equilibrium point $u = -a$ from $u = \pm 1$, we choose an energy level E_0 such that (see also Figure 5.1)

$$F(-a) < E_0 < F(-1) < 0,$$

and we define the set

$$K \stackrel{\text{def}}{=} \{(u, v, w, z) \in \mathbb{R}^4 \mid \mathcal{E}(u, v, w, z) \geq E_0\}. \tag{5.20}$$

This allows us to formulate the following lemma:

Lemma 5.9 *Let f satisfy hypothesis* (5.18) *and let $\beta \in \mathbb{R}$. If $c > 0$ is such that $W^s(1) \cap W^u(-1) = \varnothing$, then every orbit in $W^s(1)$ enters K through its boundary δK and $\hat{\Gamma} = W^s(1) \cap \delta K$ is a simple closed curve. The set of positive c for which this property holds is open and $\hat{\Gamma}$ varies continuously with c.*

Proof. In view of (5.5) the intersection of $W^s(1)$ and δK must be transverse. Assume that $W^s(1) \cap W^u(-1) = \varnothing$. We need to show that every orbit in $W^s(1)$ can be traced back to δK, for then there is bijection between $W^s(1) \cap \delta K$ and a smooth simple closed curve in $W^s_{\text{loc}}(1)$ winding around $u = 1$ (in $W^s_{\text{loc}}(1)$). Arguing by contradiction we assume that there is an orbit in $W^s(1)$ which is completely contained in K. Let $u(t)$ be a solution representing this orbit. Then $u(t)$ exists on some maximal time interval $(t_{\min}, \infty)$. Since $u(t)$ has energy larger than E_0, it follows from (5.5) and (5.20) that

$$\int_{t_{\min}}^{\infty} u'^2 \leq \frac{F(1) - E_0}{c} = \frac{-E_0}{c}, \tag{5.21}$$

so that $u(t)$ remains bounded on $(t_{\min}, \infty)$ if $t_{\min}$ is finite. Therefore $t_{\min} = -\infty$ and, by Lemma 5.8, $u(t)$ is bounded. It follows from standard arguments that the orbit converges to a limit as $t \to -\infty$. Because $u = -1$ is the only equilibrium in K with energy less than the energy of $u = 1$, we infer that $u(t) \in W^u(-1)$. This contradicts the assumption that $W^s(1) \cap W^u(-1) = \varnothing$. The second statement is an immediate consequence of the (topological) transversality of $W^s(1) \cap \delta K$. □

It now suffices to show that there is a $c > 0$ for which the assumption of Lemma 5.9 fails. Again arguing by contradiction, we assume that Lemma 5.9 applies to all $c > 0$ and search for a topological obstruction. This requires a description of δK that allows us to

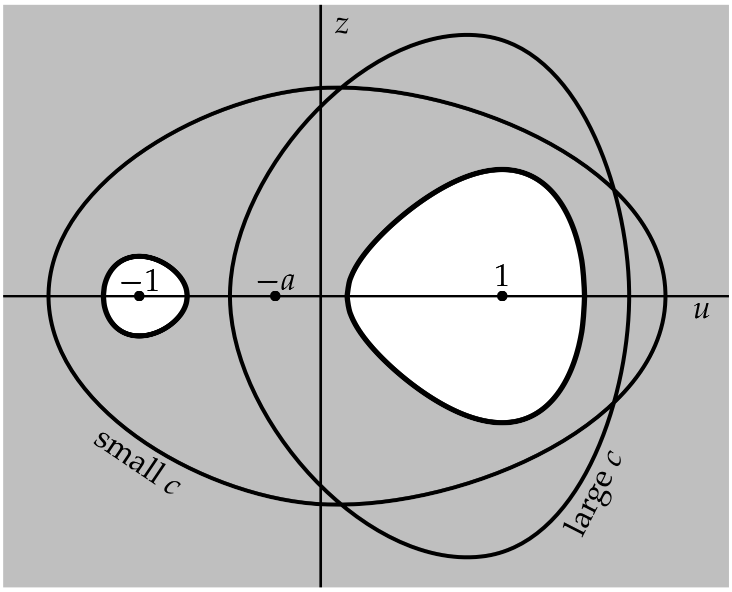

Figure 5.2: The projection (in grey) of δK onto the (u,z)-plane. The closed curves which form the boundary of the grey area are given by Equation (5.22). The other two curves depict Γ (i.e., the projection of $W^s(1) \cap \delta K$ onto the (u,z)-plane) for small c and large c.

form a global picture of this set. To this end we write δK as (with $\beta > 0$)

$$\delta K = \Big\{ (u,v,w,z) \in \mathbb{R}^4 \;\Big|\; \frac{\beta}{2}\big(v - \frac{1}{\beta}z\big)^2 + \frac{1}{2}w^2 = E_0 - F(u) + \frac{1}{2\beta}z^2 \Big\}.$$

In Figure 5.2 we have plotted the projection of δK onto the (u,z)-plane. For (u,z) lying inside one of the two closed curves (see Figure 5.2) defined by

$$E_0 - F(u) + \frac{1}{2\beta}z^2 = 0, \tag{5.22}$$

every (u,v,w,z) belongs to K, hence there are no points in δK with (u,z) lying inside these two closed curves. For (u,z) lying outside the two closed curves we have that (u,v,w,z) is in K if (v,w) is outside the ellipse defined by $\frac{\beta}{2}\big(v - \frac{1}{\beta}z\big)^2 + \frac{1}{2}w^2 = 0$. We conclude that the projection of δK onto the (u,z)-plane is the region outside the two closed curves defined by (5.22), see Figure 5.2.

The projection of δK onto the (u,z)-plane maps $\hat{\Gamma} = W^s(1) \cap \delta K$, which by assumption exists for all $c > 0$, to a closed but not necessarily simple curve Γ in the (u,z)-plane for which the winding numbers[4] $n(\Gamma,-1)$ and $n(\Gamma,1)$ around $(u,z) = (-1,0)$ and $(u,z) = (1,0)$ respectively, are well-defined and independent of c (by continuity). However, the following proposition establishes the configuration depicted in Figure 5.2, contradicting the assumption that $W^s(1) \cap W^u(-1) = \varnothing$ for all $c > 0$, and thereby completing the proof of Theorem 5.1.

Proposition 5.10 *Let f satisfy hypothesis* (5.18).

(a) *Let $\beta > \frac{1}{\sqrt{\sigma(f)}}$. Then there exists a $c_* > 0$ such that $n(\Gamma,-1) = 1$ and $n(\Gamma,1) = 1$ for all $0 < c < c_*$.*

(b) *Let $\beta \in \mathbb{R}$. Then there exists a $c^* > 0$ such that $n(\Gamma,-1) = 0$ and $n(\Gamma,1) = 1$ for all $c > c^*$.*

[4]We may choose the orientation of the simple closed curve in $W^s_{\text{loc}}(1)$ winding around $u = 1$ in such a way that its projection onto the (u,z) plane has winding number equal to $+1$.

Part (a) of Proposition 5.10 will be proved in Theorem 5.22 in Section 5.5, while Part (b) is proved in Section 5.6, Theorem 5.24.

5.4 Classification of unbounded solutions

In this section we investigate the behaviour of unbounded solutions, or in other words, we analyse the flow at infinity. This analysis is relevant both for the proof of finite time blow-up of unbounded solutions, and to determine the behaviour of unbounded orbits for $0 \leq c \ll 1$. We have argued in Section 5.3 that we may restrict our attention to nonlinearities of the form $f(u) = -u^3 + g(u)$, where $g(u)$ has compact support. It turns out that the flow for large u is governed by the *reduced* equation $u'''' + u^3 = 0$, i.e., only the highest order derivative and the highest order term in the nonlinearity play a role at infinity. In the following sections we investigate the reduced equation, and in Section 5.4.5 we come back to the full equation.

5.4.1 A modified Poincaré transformation

We analyse the reduced equation

$$u'''' + u^s = 0 \quad \text{with the convention that} \quad u^s = |u|^{s-1}u,\ s \geq 1, \tag{5.23}$$

and we use this notational convention throughout. Written as a system, (5.23) reads

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = -x_1^s, \tag{5.24}$$

where x_1, x_2, x_3 and x_4 correspond to u, u', u'' and u'''. Note that for this system the energy (or Hamiltonian)

$$H(x_1, x_2, x_3, x_4) \stackrel{\text{def}}{=} -x_2x_4 + \frac{x_3^2}{2} - \frac{|x_1|^{s+1}}{s+1}$$

is a conserved quantity.

Introduce five new dependent variables X_1, X_2, X_3, X_4 and $X_5 > 0$ by setting

$$x_i = \frac{X_i}{X_5^{a_i}} \qquad (i = 1,2,3,4), \tag{5.25}$$

where the exponents a_i are to be chosen shortly. Unbounded orbits of (5.24) will correspond to orbits in the new variables with X_5 approaching zero. By substituting (5.25) in (5.24) we obtain the equations

$$\begin{aligned} X_5X_1' - a_1X_1X_5' &= X_2X_5^{1+a_1-a_2}; && (5.26a)\\ X_5X_2' - a_2X_2X_5' &= X_3X_5^{1+a_2-a_3}; && (5.26b)\\ X_5X_3' - a_3X_3X_5' &= X_4X_5^{1+a_3-a_4}; && (5.26c)\\ X_5X_4' - a_4X_4X_5' &= -X_1^sX_5^{1+a_4-sa_1}, && (5.26d) \end{aligned}$$

with a fifth equation pending. We choose the exponents in such a way that all the exponents in the right-hand sides of (5.26) are the same, i.e,

$$b \stackrel{\text{def}}{=} 1 + a_1 - a_2 = 1 + a_2 - a_3 = 1 + a_3 - a_4 = 1 + a_4 - sa_1.$$

Solving for a_1, a_2, a_3, a_4 and b we find

$$a_1 = 4\lambda; \quad a_2 = (s+3)\lambda; \quad a_3 = (2s+2)\lambda; \quad a_4 = (3s+1)\lambda; \quad b = 1-(s-1)\lambda, \tag{5.27}$$

where λ is still free and, for the moment, positive. We close system (5.26) by imposing as a fifth equation

$$X_1^s X_1' + X_2 X_2' + X_3 X_3' + X_4 X_4' = 0. \tag{5.28}$$

If we multiply (5.26a-5.26d) by X_1^s, X_2, X_3 and X_4 respectively, and add up the resulting equations, we obtain

$$PX_5' = -\frac{1}{\lambda} Q X_5^b. \tag{5.29}$$

Here we have set

$$P \overset{\text{def}}{=} 4|X_1|^{s+1} + (3+s)X_2^2 + (2+2s)X_3^2 + (1+3s)X_4^2, \tag{5.30}$$

which is non-negative, and

$$Q \overset{\text{def}}{=} X_1^s(X_2 - X_4) + X_3(X_2 + X_4).$$

Introducing a new independent variable, we write

$$\dot{X}_5 = PX_5^{(s-1)\lambda} X_5' = -\frac{1}{\lambda} Q X_5, \tag{5.31}$$

where the dot denotes derivation with respect to this new independent variable from which the old one may be recovered by integration. Thus, combining (5.31) and (5.26), we arrive at the system

$$\dot{X}_1 = X_2 P - 4X_1 Q; \tag{5.32a}$$
$$\dot{X}_2 = X_3 P - (3+s)X_2 Q; \tag{5.32b}$$
$$\dot{X}_3 = X_4 P - (2+2s)X_3 Q; \tag{5.32c}$$
$$\dot{X}_4 = -X_1^s P - (1+3s)X_4 Q. \tag{5.32d}$$

Note that X_5 has been decoupled from the equations. By construction (the choice of (5.28)) the system (5.32) leaves the surfaces

$$\Sigma \overset{\text{def}}{=} \left\{ (X_1, X_2, X_3, X_4) \;\middle|\; \frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0 \right\} \cong S^3 \tag{5.33}$$

invariant for all $C_0 > 0$. The free parameter λ only appears in (5.31) and may be discarded.

The Poincaré transformation (5.25) is used here to blow up the flow near 'infinity'. As will be explained in Section 5.4.4 this is equivalent to blowing up the flow near the equilibrium point $u = 0$. This blowing-up technique is frequently used in the study of flows in the neighbourhood of non-hyperbolic equilibrium points (see e.g. [60, 61, 110]). The transformation defined by (5.25) and (5.33) is a variant of the standard Poincaré transformation, which has $a_1 = a_2 = a_3 = a_4 = 1$ and imposes as fifth equation that $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2$ be constant, so that the transformed problem is situated on the Poincaré sphere. The modification presented above, in particular the choice of exponents, is needed to obtain a non-trivial vector field at infinity from which we may derive the qualitative properties of the flow of the system (5.24) near infinity. The values of the exponents are derived from the invariance of (5.23) under the scaling $u(t) \mapsto \kappa u(\kappa^{\frac{s-1}{4}} t)$.

In Equation (5.28) we have chosen not to include a term $X_5 X_5'$ and to modify the exponent of X_1. This simplifies the new vector field and allows the decoupling of the $\dot{X}_5$-equation. Note that instead of a Poincaré sphere we now have a Poincaré cylinder Π, namely the topological product of the deformed sphere Σ and the positive X_5-axis:

$$\Pi \overset{\text{def}}{=} \{(X_1, X_2, X_3, X_4, X_5) \,|\, (X_1, X_2, X_3, X_4) \in \Sigma,\ X_5 \geq 0\} \cong S^3 \times [0, \infty).$$

The flow of (5.24) is completely determined by the flow of (5.32) on Σ. Therefore we have a reduction from dimension 4 for (5.24) to dimension 3 for (5.32). The roles of $X_5 = 0$ and $X_5 = \infty$ can be reversed by changing from positive to negative λ at the expense of a minus sign in (5.31).

Remark 5.11 The choice of $C_0 > 0$ in (5.33) is arbitrary, because the flows on all spheres Σ are C^1-conjugated (modulo the introduction of the new independent variable in Equation (5.31)). This is in fact the very idea of Poincaré transformations, namely that we divide out the invariance of (5.23) and focus on the resulting flow. From a more abstract point of view one can construct a flow on the quotient manifold $\big(\mathbb{R}^4 \setminus \{0\}\big)/\mathbb{R}^+ \cong S^3$ via the scaling invariance $u(t) \mapsto \kappa u(\kappa^{\frac{s-1}{4}} t)$ ($\mathbb{R}^+$-action), see [109] for more details. Our construction involves explicit choices of coordinates, for which the flows, by general theory, are all related by conjugation.

To be explicit, let X_i and Y_i be two sets of Poincaré coordinates, i.e.,

$$x_i = \frac{X_i}{X_5^{a_i}} = \frac{Y_i}{Y_5^{a_i}} \qquad \text{for } i = 1,2,3,4,$$

with constraints

$$\frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0, \tag{5.34a}$$

$$\frac{|Y_1|^{s+1}}{s+1} + \frac{Y_2^2}{2} + \frac{Y_3^2}{2} + \frac{Y_4^2}{2} = C_1. \tag{5.34b}$$

When we define $\mu = \frac{X_5}{Y_5}$, then the two sets of coordinates are related by

$$X_5 = \mu Y_5 \qquad \text{and} \qquad X_i = \mu^{a_i} Y_i \quad \text{for } i = 1,2,3,4. \tag{5.35}$$

Substituting this into (5.34a) we obtain

$$G(Y_1, Y_2, Y_3, Y_4, \mu) \equiv \mu^{(s+1)a_1} \frac{|Y_1|^{s+1}}{s+1} + \mu^{2a_2} \frac{Y_2^2}{2} + \mu^{2a_3} \frac{Y_3^2}{2} + \mu^{2a_4} \frac{Y_4^2}{2} = C_0.$$

Since $\frac{\partial G}{\partial \mu} > 0$ for all Y_i that obey (5.34b), it follows from the Implicit Function Theorem that $\mu(Y_1, Y_2, Y_3, Y_4)$ is a differentiable function. It is now easily seen from (5.35) that X_i and Y_i are related by a C^1-conjugacy. Therefore, we may choose the constant C_0 according to our liking to obtain a description of the flow that is most suitable to our needs. •

5.4.2 The flow at infinity

For the analysis of (5.32) we first observe the following.

Lemma 5.12 *System* (5.32) *has no stationary points on* Σ *for any* $C_0 > 0$.

Proof. Since $X_1 = X_2 = X_3 = X_4 = 0$ is excluded we have that P, defined by (5.30), is positive. Equating the right-hand sides of (5.32) to zero and considering the resulting equations as linear equations in P and Q, it follows that we can only have solutions if every determinant of every pair of two equations vanishes. This would give for instance that

$$\begin{aligned} 0 &\leq (2+2s)X_3^2 = (3+s)X_2X_4; \\ 0 &\leq 4|X_1|^{s+1} = -(1+3s)X_2X_4. \end{aligned}$$

We conclude that $X_2X_4 = 0$ and with any of the $X_i = 0$ the others follow immediately. □

We next use the conserved quantity to obtain a further reduction from dimension 3 to dimension 2 for the limit sets of orbits of (5.26) which approach infinity ($X_5 \to 0$) or the origin ($X_5 \to \infty$). In the new variables the Hamiltonian is

$$H = \Big(-X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1} \Big) X_5^{-4\lambda(s+1)}.$$

Denote the first factor of H by H_0:

$$H_0 \stackrel{\text{def}}{=} -X_2X_4 + \frac{X_3^2}{2} - \frac{|X_1|^{s+1}}{s+1}. \tag{5.36}$$

Since H is a conserved quantity, we conclude that for $\lambda > 0$

$$X_5 \to 0 \quad \Leftrightarrow \quad H_0 \to 0. \tag{5.37}$$

For the classification of unbounded orbits we have to analyse the flow restricted to the invariant set given by

$$\begin{aligned} T &\stackrel{\text{def}}{=} \{(X_1, X_2, X_3, X_4) \in \Sigma \mid H_0 = 0\} \\ &= \Big\{ (X_1, X_2, X_3, X_4) \ \Big| \ \frac{|X_1|^{s+1}}{s+1} + \frac{X_2^2}{2} + \frac{X_3^2}{2} + \frac{X_4^2}{2} = C_0,\ \frac{X_3^2}{2} = X_2X_4 + \frac{|X_1|^{s+1}}{s+1} \Big\}. \end{aligned}$$

This set is a topological torus as can be seen by setting

$$X_1 = \xi_1; \quad X_2 = \frac{\xi_2 + \xi_4}{\sqrt{2}}; \quad X_3 = \xi_3; \quad X_4 = \frac{\xi_2 - \xi_4}{\sqrt{2}}, \tag{5.38}$$

so that, in terms of the ξ-variables,

$$T = \Big\{ (\xi_1, \xi_2, \xi_3, \xi_4) \ \Big| \ \frac{2}{s+1}|\xi_1|^{s+1} + \xi_2^2 = \xi_3^2 + \xi_4^2 = C_0 \Big\} \cong S^1 \times S^1. \tag{5.39}$$

Clearly we have that T is the product of two topological circles, one in the (ξ_1, ξ_2)-plane, the other in the (ξ_3, ξ_4)-plane.

Lemma 5.13 *Let $s \geq 1$ and fix the constant $C_0 > 0$. Then there exist precisely two periodic orbits Λ_- and Λ_+ of* (5.32) *on the torus T.*

Proof. The proof is based on the observation that the coefficient Q in (5.31), which after transforming by (5.38) reads

$$Q = \sqrt{2}(\xi_1^s \xi_4 + \xi_2 \xi_3), \tag{5.40}$$

plays a double role. Obviously it determines which parts of infinity attract solutions towards $X_5 = 0$, in forward and in backward time. We begin by showing that Q can also be seen as minus the divergence of the vector field restricted to the invariant torus T. From (5.32) and (5.38) we derive

$$\dot{\xi}_1 = \frac{\xi_2 + \xi_4}{\sqrt{2}} P - 4\xi_1 Q; \tag{5.41a}$$

$$\dot{\xi}_2 = \frac{\xi_3 - \xi_1^s}{\sqrt{2}} P - ((2+2s)\xi_2 + (1-s)\xi_4) Q; \tag{5.41b}$$

$$\dot{\xi}_3 = \frac{\xi_2 - \xi_4}{\sqrt{2}} P - (2+2s)\xi_3 Q; \tag{5.41c}$$

$$\dot{\xi}_4 = \frac{\xi_3 + \xi_1^s}{\sqrt{2}} P - ((1-s)\xi_2 + (2+2s)\xi_4) Q. \tag{5.41d}$$

We parametrise T by 'polar coordinates'

$$\xi_1 = f_1(\phi); \quad \xi_2 = g_1(\phi); \quad \xi_3 = f_2(\theta); \quad \xi_4 = g_2(\theta), \tag{5.42}$$

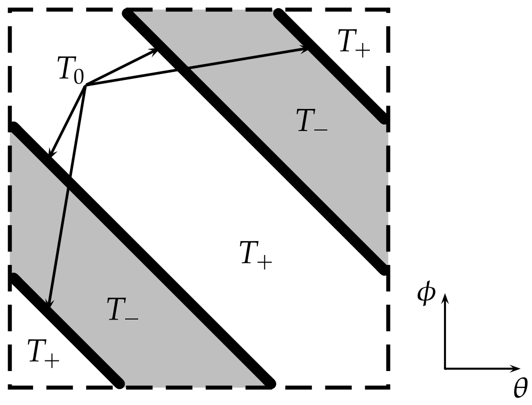

Figure 5.3: A fundamental domain of the torus, in which T_-, T_+ and T_0 are indicated (schematically).

satisfying

$$f_1' = -g_1; \quad g_1' = f_1^s; \quad f_2' = -g_2; \quad g_2' = f_2. \tag{5.43}$$

Note that when $C_0 = 1$ and $s = 1$ we just have

$$\xi_1 = \cos\phi; \quad \xi_2 = \sin\phi; \quad \xi_3 = \cos\theta; \quad \xi_4 = \sin\theta.$$

From (5.41a), (5.41c), (5.42) and (5.43) we derive that on T the flow is given by: .

$$\dot\phi = \frac{P}{\sqrt{2}}(-1 - \frac{g_2}{g_1}) + 4Q\frac{f_1}{g_1} \equiv w_1(\phi,\theta), \tag{5.44a}$$

$$\dot\theta = \frac{P}{\sqrt{2}}(1 - \frac{g_1}{g_2}) + 2(s+1)Q\frac{f_2}{g_2} \equiv w_2(\phi,\theta), \tag{5.44b}$$

where in terms of f_1, g_1, f_2, g_2,

$$P = 4(s+1)C_0 + 2(1-s)g_1g_2, \qquad \text{and} \qquad Q = \sqrt{2}(f_1^s g_2 + f_2 g_1).$$

The functions w_1 and w_2, defined in (5.44), appear to have singularities, but using (5.39) they can be written as

$$\begin{aligned} w_1(\phi,\theta) &= \sqrt{2}\big[-2(s+1)C_0 - (s+3)g_1g_2 + (s-1)g_2^2 + 4f_1f_2\big], \\ w_2(\phi,\theta) &= \sqrt{2}\big[2(s+1)C_0 - (3s+1)g_1g_2 + (s-1)g_1^2 + 2(s+1)f_1^s f_2\big]. \end{aligned}$$

Taking the divergence of the vector field w we obtain (using (5.43),

$$\nabla\cdot w = \frac{\partial w_1}{\partial\phi} + \frac{\partial w_2}{\partial\theta} = \sqrt{2}(-5-3s)(f_1^s g_2 + f_2 g_1) = -(3s+5)Q.$$

Next, we split T into

$$T_+ = \{(X_1, X_2, X_3, X_4)\,|\,Q > 0\} \quad \text{and} \quad T_- = \{(X_1, X_2, X_3, X_4)\,|\,Q < 0\}.$$

These two sets share the boundary

$$T_0 = \{(X_1, X_2, X_3, X_4)\,|\,Q = 0\},$$

which, in view of (5.39) and (5.40), consists of two topological circles, which both wind once around the two homotopically distinct simple loops on the torus (see Figure 5.3). We will show in Lemma 5.14 that, when C_0 is chosen properly, an orbit can only pass through T_0 from T_- to T_+. It then follows from the negativity of $\nabla\cdot w$ in T_+ and the winding properties of T_0 on T, that T_+ contains precisely one periodic orbit. The same statement holds for T_- with respect to the backward flow on T.

To be precise, we deduce from (5.42), (5.43) and (5.39) that we may choose $\xi_3 = f_2(\theta) = \sqrt{C_0}\cos\theta$. Define the set $S \stackrel{\text{def}}{=} \{(\theta,\phi) \in T \,|\, \theta = \frac{\pi}{2}\}$, and it follows that

$$\dot{\theta}\Big|_S = \sqrt{2}[2(s+1)C_0 - (3s+1)\sqrt{C_0}g_1 + (s-1)g_1^2].$$

Since $|g_1| \le \sqrt{C_0}$, it is easy to check that $\dot{\theta}|_S \ge 0$, and equality only holds at the point where $g_1 = \sqrt{C_0}$. By continuity arguments the orbit through this point also crosses S in the direction of increasing θ. Thus S is a global section for the flow on T. Moreover, the return map is well-defined, since there is no point in T for which the forward orbit is contained in $T \setminus S$. Indeed, such a forward orbit would either be contained in T_- or eventually be in T_+, because T_+ is positively invariant and orbits can only pass through T_0 from T_- to T_+. In the absence of equilibrium points (Lemma 5.12) its ω-limit set would be a periodic orbit. However, there would have to be an equilibrium point inside this periodic orbit, contradicting Lemma 5.12. Hence the return map is well-defined. The intersection $S \cap (T_+ \cup T_0)$ consists of the line segment $\{(\theta,\phi) \in T \,|\, \theta = \frac{\pi}{2}, f_1(\phi) \ge 0\}$. The return map maps this line segment into itself, which implies the existence of a periodic orbit in T_+. Similarly there exists a periodic orbit in T_-. The return map is contracting in T_+ and expanding in T_-, since the divergence of the vector field is negative in T_+ and positive in T_-. This proves the uniqueness of the two period orbits and shows that all other orbits on the torus T have Λ_- as α-limit set and Λ_+ as ω-limit set.

We remark that the same conclusion can be reached by combining the Poincaré-Bendixson theorem for flows on the torus with Morse theory for Morse-Smale flows.

Finally, note that although the preceding proof needs C_0 to have a particular value (see Lemma 5.14 and Equation (5.47)), the statement in Lemma 5.13 is true for any choice of $C_0 > 0$ (see Remark 5.11).

Another observation is that the linear case $s = 1$ may be treated by direct computation, i.e., by transforming the general solution of the then linear equation (5.23) to the X-variables. □

We still have to show that an orbit can only pass through T_0 from T_- to T_+.

Lemma 5.14 *Let $s > 1$. There exists a $C_0 > 0$ such that orbits on T can only pass through T_0 in the direction from T_- to T_+.*

Proof. We deduce from (5.40) and (5.41) that

$$\dot{Q}\Big|_{Q=0} = P\Big(|\xi_1|^{2s} + \xi_2^2 + \xi_3^2 + \xi_4^2 + (s|\xi_1|^{s-1} - 1)(\xi_2 + \xi_4)\xi_4\Big). \tag{5.45}$$

Notice that for $s = 1$, P is positive on T (see (5.30)), thus $\dot{Q}|_{Q=0} > 0$ on T. For $s > 1$ we define R as the second factor on the right-hand side of (5.45) and simplify it using the expression (5.39) for T:

$$\begin{aligned} R &\stackrel{\text{def}}{=} |\xi_1|^{2s} + \xi_2^2 + \xi_3^2 + \xi_4^2 + (s|\xi_1|^{s-1} - 1)(\xi_2 + \xi_4)\xi_4 \\ &= 2C_0 + |\xi_1|^{2s} - \frac{2}{s+1}|\xi_1|^{s+1} - (1 - s|\xi_1|^{s-1})(\xi_2 + \xi_4)\xi_4. \end{aligned} \tag{5.46}$$

From (5.39) we infer that

$$(\xi_2 + \xi_4)\xi_4 \le ((C_0 - \frac{2}{s+1}|\xi_1|^{s+1})^{\frac{1}{2}} + C_0^{\frac{1}{2}})C_0^{\frac{1}{2}} = C_0(1 + (1 - \frac{2}{C_0(s+1)}|\xi_1|^{s+1})^{\frac{1}{2}}).$$

Fix
$$C_0 = \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}, \tag{5.47}$$
and set
$$|\xi_1| = x\left(\frac{1}{s}\right)^{\frac{1}{s-1}}, \text{ where } 0 \le x \le 1.$$
It follows that
$$\begin{aligned} R &\geq \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}\Big(2 + \frac{s+1}{2s}x^{2s} - x^{s+1} - (1-x^{s-1})(1+(1-x^{s+1})^{\frac{1}{2}})\Big) \\ &= \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}\Big(1 + \frac{s+1}{2s}x^{2s} - x^{s+1} + x^{s-1} - (1-x^{s-1})(1-x^{s+1})^{\frac{1}{2}}\Big) \\ &= \frac{2}{s+1}\Big(\frac{1}{s}\Big)^{\frac{s+1}{s-1}}\Big((1-x^{s+1})^{\frac{1}{2}}\big((1-x^{s+1})^{\frac{1}{2}} - (1-x^{s-1})^{\frac{1}{2}}\big) + x^{s-1} + \frac{s+1}{2}x^{2s}\Big). \end{aligned}$$
Since $0 \le x \le 1$ we see that $R > 0$ unless $x = 0$. Looking at (5.46) we infer that R can only be zero if $\xi_1 = \xi_3 = 0$ and $\xi_2 = \xi_4 = \pm\sqrt{C_0}$, or, in terms of the X_i, if $X_1 = X_3 = X_4 = 0$. By continuity arguments it follows that also in these two points the orbits go from T_- to T_+. Thus, with the particular choice of C_0 given by (5.47) we have indeed that T_+ is positively invariant and T_- is negatively invariant. □

Having proved the existence of precisely two periodic orbits, Λ_- and Λ_+, on the torus T, we analyse some of their properties.

Lemma 5.15 *The three non-trivial Floquet multipliers of Λ_+ are contained in the interval $(0,1)$, and the three non-trivial Floquet multipliers of Λ_- are contained in the interval $(1,\infty)$.*

Proof. Restricted to T the nontrivial Floquet multiplier of Λ_+ equals (see e.g. [125, p. 198])
$$\exp\Big(\oint_{\Lambda_+} \nabla\cdot w\Big) = \exp\Big(\oint_{\Lambda_+} -(3s+5)Q\Big).$$
Since Q is uniformly positive on Λ_+, this Floquet multiplier lies in the interval $(0,1)$. Close to the periodic orbit Λ_+ we choose ϕ, θ, X_5 and H_0 as coordinates on the Poincaré cylinder Π, where H_0 given by (5.36). Since $H = H_0X_5^{-4\lambda(s+1)}$ is a conserved quantity on Π, it follows from (5.31) that
$$\dot{H}_0 = -4(s+1)Q\,H_0.$$
Together with (5.31) this implies that the other Floquet multipliers are
$$\exp\Big(\oint_{\Lambda_+} -4(s+1)Q\Big) \quad \text{and} \quad \exp\Big(\oint_{\Lambda_+} -\frac{1}{\lambda}Q\Big),$$
which are in $(0,1)$ as before. Thus Λ_+ is exponentially stable. The statement for Λ_- is obtained by time reversal. □

Lemma 5.16 *Every orbit (other than $\Lambda_\pm$) on the sphere Σ, has Λ_- as α-limit set and Λ_+ as ω-limit set.*

Proof. We have already dealt with the flow on the torus T in Lemma 5.13. Orbits of the flow on the complement $\Sigma \setminus T$ of the torus T on the sphere Σ, correspond to solutions with non-zero Hamiltonian H. Since X_5 does not appear in (5.31), the motion on Σ is independent of X_5. Let $X_5 \neq 0$, then the dynamics of X_5 are governed by (5.32), and the motion takes place in the part of the Poincaré cylinder Π that corresponds to the finite part of phase space in the x-variables. In other words, orbits of the flow on the set $\Sigma \setminus T$ correspond to solutions of (5.24) with non-zero Hamiltonian.

Since $H = H_0 X_5^{-4\lambda(s+1)}$ and H_0 is bounded on Σ (because Σ is compact), it follows that for such orbits X_5 remains bounded, i.e., in x-variables the solution stays away from the origin. Thus orbits in $\Sigma \setminus T$ are bounded in the X-variables and hence have nonempty invariant α- and ω-limit sets. We have to show that these limit sets can only be the two periodic orbits Λ_- and Λ_+ provided by Lemma 5.13. To this end it suffices to show that all solutions of (5.23) with $H \neq 0$ are unbounded in forward and backward time, i.e., $X_5 \to 0$ along a sequence of points in forward and backward time.

Postponing the proof of the unboundedness of solutions with $H \neq 0$, we first show how unboundedness in backward and forward time implies that Λ_- and Λ_+ are the α- and ω-limit sets. By (5.37) $X_5 \to 0$ implies that also $H_0 \to 0$. An unbounded orbit thus comes arbitrary close to the torus T. We choose an open tubular neighbourhood Λ_-^ε of Λ_- in T, such that $Q < 0$ in Λ_-^ε. Clearly all orbits starting in $T \setminus \Lambda_-^\varepsilon$ tend to Λ_+ in forward time. Note that $T_0 \cup T_+ \subset T \setminus \Lambda_-^\varepsilon$. By compactness of T and since Λ_+ is asymptotically stable (see Lemma 5.15), there exists an open neighbourhood T^ε of $T \setminus \Lambda_-^\varepsilon$ in Π such that all orbits starting in T^ε tend to Λ_+ in forward time. Since an orbit which comes close to $X_5 = 0$ (and thus close to T), can only do so with non-negative Q, it enters T^ε and hence tends to Λ_+. The statement for Λ_- follows by time reversal.

We still have to prove that any solution of (5.23) with non-zero Hamiltonian is unbounded in forward and backward time. We recall that solutions with $H \neq 0$ stay away from the origin. If an orbit would be bounded in backward or forward time, then its (nonempty) α- or ω-limit set would consisted of bounded orbits, i.e., orbits which are bounded for all time. However, this is not possible, because it has been proved in [96] that (5.23) admits no bounded solutions except $u \equiv 0$. Here we present a different proof of the fact that (5.23) admits no bounded solutions except $u \equiv 0$, because we need to extend this result to more general situations (see Remark 5.17).

Assume, by contradiction, that $u \not\equiv 0$ is a bounded solution of (5.23). First observe that if u tends to a limit as $t \to \pm\infty$, then this limit can only be 0. It follows that u attains at least one positive maximum or one negative minimum. Switching from u to $-u$ if necessary, we may suppose that u attains a positive maximum at t_0:

$$u(t_0) > 0, \quad u'(t_0) = 0, \quad u''(t_0) \leq 0.$$

Changing from t to $-t$ if necessary, we may assume that $u'''(t_0) \leq 0$ and apply an oscillation argument from Section 2.4, which we repeat here for the sake of completeness. There exists a $t^* > t_0$ such that $u'''(t) < 0$ for $t_0 < t < t^*$ and $u'''(t^*) = 0$. Using the fact that,

$$H = -u'u''' + \frac{1}{2}u''^2 - \frac{1}{s+1}|u|^{s+1}$$

is constant, it follows that $u(t^*) < -u(t_0)$ and that the next minimum must occur at $t_1 > t^*$ with $u(t_1) < u(t^*) < -u(t_0)$ and both $u''(t_1)$ and $u'''(t_1)$ positive. Repeating this argument we obtain a sequence $t_1 < t_2 < t_3 < \ldots$, in which $u(t)$ has non-degenerate extrema with $|u(t_1)| < |u(t_2)| < |u(t_3)| < \ldots$. By assumption these extrema remain bounded, say $\lim_{i\to\infty} |u(t_i)| = a \in \mathbb{R}^+$, and the derivatives are bounded as well. A compactness argument now shows that there must be a solution $\tilde{u}$ of (5.24) in the ω-limit set of u with

$$\tilde{u}(t_0) = a, \quad \tilde{u}'(t_0) = 0, \quad \tilde{u}''(t_0) < 0, \quad \text{and} \quad \tilde{u}'''(t_0) \le 0 \qquad \text{at some } t_0 \in \mathbb{R},$$

and such that $|\tilde{u}(t)| \le a$ for all $t \in \mathbb{R}$. However, when we apply the above argument to $\tilde{u}$ we obtain that $\tilde{u} < -a$ at the first minimum to the right of t_0, a contradiction. This completes the proof of Lemma 5.16. □

Remark 5.17 The oscillation argument above will be applied several times in this chapter to differential equations that differ from the present one. It holds that any solution of (5.3) with $c = 0$ and $\beta \ge 0$ which does not have its range contained in

$$\{u \in \mathbb{R} \,|\, F(u) \ge F(-a)\},$$

oscillates towards infinity either in forward or in backward time in exactly the way described above (the additional second order term does not cause any difficulties). For more details we refer to Section 2.4. •

5.4.3 The reduced system in the linear limit

We have shown in the previous section that for any $s \ge 1$ the flow of (5.23) is basically governed by two periodic orbits at infinity. For the linear equation ($s = 1$) this was already observed (in a broader setting) by Palis [110]. The analysis thus shows that the behaviour for all $s > 1$ is largely analogous to the linear equation. In this section we make some observations about the limit $s \downarrow 1$.

Let us rewrite this system as

$$\dot{X} = V(X;s), \quad X = (X_1, X_2, X_3, X_4). \tag{5.48}$$

Then the vector field $V(\cdot, s)$ is continuously differentiable for every $s \ge 1$ and the first order partial derivatives are bounded on compact sets, uniformly in $s \ge 1$. We do not have that $V(\cdot, s) \to V(\cdot, 1)$ in C^1_{loc} because of the term X_1^s appearing in V, but we do have that $V(\cdot, s) \to V(\cdot, 1)$ uniformly on compact sets. Therefore the orbits of (5.48) with $s > 1$, which are bounded uniformly in s in view of (5.33), converge to orbits of (5.48) with $s = 1$ as $s \to 1$. More precisely, the solution map

$$(\tau, \xi, s) \to X(\tau; \xi, s),$$

where $X(\tau; \xi, s)$ is the solution $X(\tau)$ of (5.48) with $X(0) = \xi$, is continuous on $\mathbb{R} \times \mathbb{R}^4 \times [1, \infty)$. In particular, this implies that the two periodic orbits Λ_- and Λ_+ depend continuously on s for $s \in [1, \infty)$.

In the limit case $s = 1$ the two periodic orbits on

$$T = \{(\xi_1, \xi_2, \xi_3, \xi_4) \,|\, \xi_1^2 + \xi_2^2 = \xi_3^2 + \xi_4^2 = C_0\}$$

are given by

$$\xi_1\xi_3 - \xi_2\xi_4 = 0, \tag{5.49}$$

or in terms of (5.42), by $\phi + \theta = \pm\frac{\pi}{2}$. This can be seen from a second conservation law that exists in the linear case: multiplying $u'''' + u = 0$ by u''' we infer that $\frac{1}{2}u'''^2 + uu'' - \frac{1}{2}u'^2$ is constant. In particular, after transforming to the X-variables,

$$\frac{1}{2}X_4^2 + X_1X_3 - \frac{1}{2}X_2^2 = 0$$

is invariant, whence (5.49), which defines two circles on the torus T.

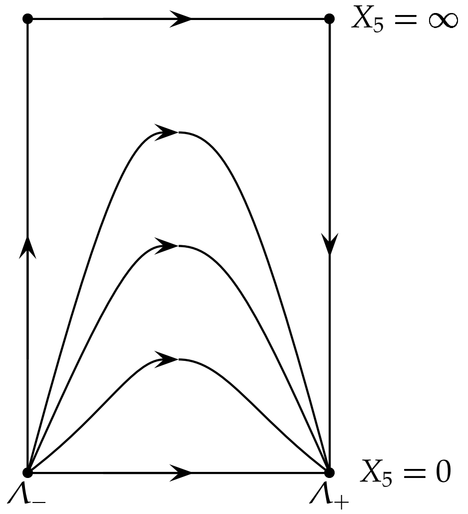

Figure 5.4: A schematic view of the flow on the Poincaré cylinder Π for the equation $u'''' + u^s = 0$. The role of $X_5 = 0$ and $X_5 = \infty$ is reversed when λ is negative.

5.4.4 Small solutions

We observed in Section 5.4.1 that the role of $X_5 = 0$ and $X_5 = \infty$ may be reversed. This is a direct consequence of the scaling invariance of (5.23). Therefore we may also use (5.25) for the analysis of small solutions to (5.23). The situation is depicted schematically in Figure 5.4. We simply apply (5.25) with a negative λ so that $X_5 \to 0$ corresponds to $u \to 0$. This only changes the sign in the Equation (5.31) for X_5 and means that the orbit Λ_+ now lies in the part of $X_5 = 0$ which repels solutions with $X_5 > 0$. Hence the stable manifold of Λ_+ is contained in $\Pi \cap \{X_5 = 0\}$. The unstable manifold of Λ_+ is given by the direct product $\Lambda_+ \times \{X_5 \,|\, X_5 > 0\}$ and has dimension 2. In the original variables it is the unstable manifold of $u = 0$ if $s = 1$ and the center-unstable manifold if $s > 1$. Likewise, the stable manifold of Λ_- is $\Lambda_- \times \{X_5 \,|\, X_5 > 0\}$, i.e., the direct product of Λ_- and the positive X_5-axis. As we have seen in Section 5.4.3, the limit $s \to 1$ is well behaved in the X-variables.

We will use this analysis of the behaviour near the equilibrium point $u = 0$ in Section 5.5 to perform a continuous deformation of the stable manifold for $s = 1$ to the center-stable manifold for $s > 1$. We remark that, based on the similarity of the linear and nonlinear problem, the equilibrium point $u = 0$ of (5.23) for $s > 1$ can be considered as the nonlinear equivalent of a saddle-focus.

5.4.5 The full system

Applying the Poincaré transformation (5.25) with exponents (5.27) to the full differential equation (5.3), or more generally, to

$$x_1' = x_2; \quad x_2' = x_3; \quad x_3' = x_4; \quad x_4' = \Phi(x_1, x_2, x_3, x_4),$$

we arrive at

$$
\begin{aligned}
\dot{X}_1 &= X_2 P - 4X_1 Q\,, && (5.50a)\\
\dot{X}_2 &= X_3 P - (3+s)X_2 Q\,, && (5.50b)\\
\dot{X}_3 &= X_4 P - (2+2s)X_3 Q\,, && (5.50c)\\
\dot{X}_4 &= \Psi P - (1+3s)X_4 Q\,, && (5.50d)\\
\dot{X}_5 &= \quad -\frac{1}{\lambda}X_5 Q\,, && (5.50e)
\end{aligned}
$$

where

$$Q = X_1^s X_2 + X_4\Psi + X_3(X_2 + X_4), \tag{5.51}$$

and

$$\Psi = X_5^{4\lambda s}\Phi\Big(\frac{X_1}{X_5^{4\lambda}}, \frac{X_2}{X_5^{(3+s)\lambda}}, \frac{X_3}{X_5^{(2+2s)\lambda}}. \frac{X_4}{X_5^{(1+3s)\lambda}}\Big),$$

In the case of (5.3) we have

$$\Phi(x_1, x_2, x_3) = \beta x_3 - cx_2 + f(x_1),$$

where $f(x_1) = -x_1^3 + g(x_1)$ with $g(x_1)$ compactly supported. With $s = 3$ and $\lambda = \frac{1}{2}$ we thus obtain

$$\Psi = -X_1^3 + \beta X_3 X_5^2 - cX_2 X_5^3 + g\Big(\frac{X_1}{X_5^2}\Big)X_5^6. \tag{5.52}$$

The last term in (5.52) is C^2 and has its derivatives up to second order vanishing in $X_5 = 0$. The extra terms are thus at least quadratic in X_5 for small X_5. Therefore the local analysis near $X_5 = 0$ and in particular the Floquet multipliers of $\Lambda_\pm$ in the previous section are completely unaffected. The flow on the sphere Σ at infinity is identical to the flow for the reduced equation (5.24). Only the flow on $\Pi \setminus \Sigma$ is different. Note that in this analysis it is essential that the exponent s is larger than 1. We have the following theorem (compare Lemmas 5.13, 5.15 and 5.16).

Theorem 5.18 *Let f satisfy hypothesis* (5.18) *and let $\beta, c \in \mathbb{R}$.*

(a) *The stable periodic orbit Λ_+ of* (5.32) *is an asymptotically stable periodic orbit of the full system* (5.50) *with non-trivial Floquet multipliers in* $(0, 1)$. *Every solution of* (5.3) *which is unbounded in forward time corresponds to a solution of* (5.50) *having Λ_+ as ω-limit set. A similar statement holds for solutions unbounded in backward time and Λ_-.*

(b) *Unbounded solutions of* (5.3) *blow up oscillatorily in finite time.*

(c) *If $c \neq 0$ the energy $\mathcal{E}$ also blows up.*

Proof. By Lemma 5.16 all solutions of (5.50) which lie in the invariant set $\Pi \cap \{X_5 = 0\} \setminus \Lambda_-$ tend to Λ_+ in forward time. Reminiscent of the proof of Lemma 5.16 we choose a small negatively invariant open tubular neighbourhood Λ_-^ε of Λ_- in Π. By compactness of $\Pi \cap \{X_5 = 0\}$ there exists an open neighbourhood Σ^ε of $\Pi \cap \{X_5 = 0\} \setminus \Lambda_-^\varepsilon$ in Π such that all orbits with starting point in Σ^ε tend to Λ_+ in forward time. Clearly every unbounded solution of (5.3) enters Σ^ε and thus tends to Λ_+.

For Part (b) we observe that the exponent b in (5.29) is smaller than 1 so that in the old time variable X_5 can only go to zero in finite time. Finally we have that the energy $\mathcal{E}$ can only remain bounded if its derivative is integrable. For $c \neq 0$ this implies that u' is square integrable (see (5.5)) and thus u itself is (locally) bounded, which prohibits finite time blow-up, a contradiction. □

Remark 5.19 Theorem 5.18 establishes that large solutions of (5.3) are really described by oscillating solutions of $u'''' + u^3 = 0$. Thus large solutions do not "see" the other terms in (5.3) as they oscillate away to infinity. This is not only true for perturbations of the form $-u^3 + g(u)$ with g compactly supported and smooth, but also for global lower order perturbations. For such lower order perturbations Theorem 5.18 applies as well. •

5.5 The winding number for small speeds

In this section we prove Part (a) of Proposition 5.10. Before we can prove this theorem we first need a description of the global behaviour of $W^s(1)$ for $c = 0$. In the following lemma we show that for $\beta > \frac{1}{\sqrt{\sigma(f)}}$ all orbits in the stable manifold $W^s(1)$ are unbounded, and, after transforming to the X-variables of Section 5.4, they all have Λ_- as α-limit set. Because all the non-trivial Floquet multipliers of Λ_- lie in $(1, \infty)$ (see Theorem 5.18a), this remains true for $c > 0$ sufficiently small.

Lemma 5.20 *Let f satisfy hypothesis* (5.18)*, let $\beta > \frac{1}{\sqrt{\sigma(f)}}$ and $c = 0$. Then $W^s(1)$ consists of unbounded orbits only, all of which connect Λ_- to $u = 1$.*

Proof. The proof is a combination of arguments also used in [112]. Any bounded solution must have its range in the set

$$V = \{u \in \mathbb{R} \,|\, F(u) \geq F(-a)\},$$

because a solution reaching outside this interval oscillates away towards infinity, as mentioned in Remark 5.17. Besides, any bounded solution must have at least one minimum below the line $u = -a$, again basically by the same oscillation argument as in the proof of Lemma (5.15). We now assume, arguing by contradiction, that u is a bounded orbit in $W^s(1)$. We will show that the range of u is not contained in V, so that u is in fact unbounded. It then follows from Theorem 5.18 that u tends to Λ_- as $t \to -\infty$.

Thus, suppose that u is a bounded solution in $W^s(1)$. Changing from t to $-t$ if necessary we have that in such a minimum (using the fact that $\mathcal{E}(u, u', u'', u''') = 0$)

$$u(t_0) \leq -a, \quad u'(t_0) = 0, \quad u''(t_0) = \sqrt{-2F(u(t_0))} > 0, \quad u'''(t_0) \geq 0. \tag{5.53}$$

We will show that $u(t)$ increases to a value outside V for $t > t_0$, which immediately leads to a contradiction.

Define an auxiliary function

$$G(t) \stackrel{\text{def}}{=} u''(t) - \sqrt{-2F(u(t))}.$$

The following line of reasoning is depicted in Figure 5.5. First, $G(t_0) = 0$ and we show that $G(t) > 0$ in a right neighbourhood of t_0. It is seen from the condition on β and the observation that $f(u) > 0$ on $(-\infty, -1) \cup (-a, 1)$, that

$$f(u) > -\sqrt{-\frac{\beta^2}{2}F(u)} \qquad \text{for } u < 1. \tag{5.54}$$

If $u'''(t_0) > 0$, then clearly $G'(t_0) > 0$, whereas when $u'''(t_0) = 0$ then $G'(t_0) = 0$, and (since $u'(t_0) = 0$)

$$G''(t_0) = u''''(t_0) + \frac{f(u(t_0))}{\sqrt{-2F(u(t_0))}} u''(t_0) = \beta\sqrt{-2F(u(t_0))} + 2f(u(t_0)) > 0$$

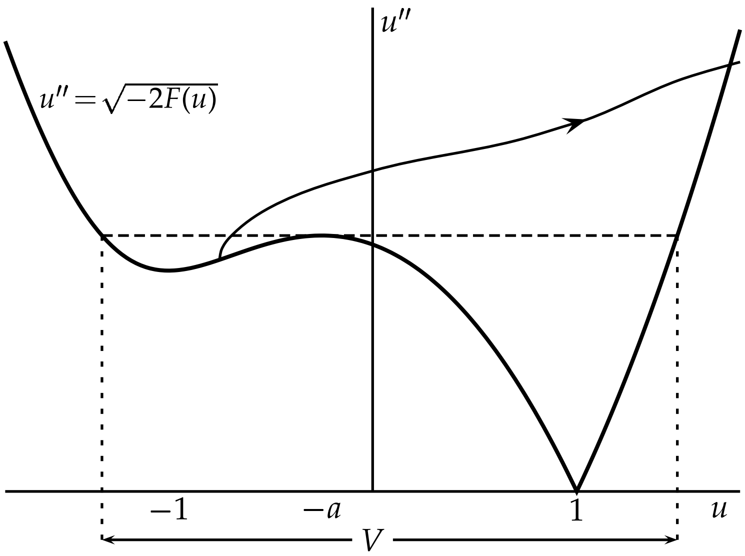

Figure 5.5: The (u,u'')-plane with the curve $u''=\sqrt{-2F(u)}$. We have sketched the orbit of u for $t\geq t_0$, which is discussed in the proof of Lemma 5.20. We have also indicated the set V, in which every bounded solution has its range.

by the differential equation, and (5.53) and (5.54). Thus $G(t)>0$ in a right neighbourhood of t_0.

Second, we show that $G(t)>0$ as long as $u(t)<1$. We define $t_1>t_0$ as the first maximum of $u(t)$ and $t_2>t_0$ as the first point where $G(t_2)=0$ (a priori, both t_1 and t_2 may be ∞). Then $t_2<t_1$ since $u''(t)>0$ as long as $G(t)>0$. It now follows from the expression (5.4) for the energy and by (5.54) that

$$\begin{aligned} G'(t) &= u'''(t)+\frac{f(u(t))}{\sqrt{-2F(u(t))}}u'(t)\\ &= \frac{\frac{1}{2}u''^2(t)+F(u(t))}{u'(t)}+\left(\frac{\beta}{2}+\frac{f(u(t))}{\sqrt{-2F(u(t))}}\right)u'(t)\\ &> 0, \end{aligned}$$

as long as $G(t)>0$ and $u(t)<1$. Since $G(t)>0$ in a right neighbourhood of t_0 this implies that $G(t)>0$ and $G'(t)>0$ as long as $u(t)<1$, and thus $u(t_2)\geq 1$.

Finally, we define $t_3>t_0$ as the first point where $u(t)=-a$. It is now immediate that $t_3<t_2$. By the energy expression we have that $u'''(t)>0$ as long as $G(t)>0$, thus $u''(t_2)>u''(t_3)>\sqrt{-2F(-a)}$. Combining the inequalities $u(t_2)\geq 1$ and $F(u(t_2))=-\frac{1}{2}u''^2(t_2)<F(-a)$, we infer that $u(t_2)$ lies outside V, so that u is unbounded. By Theorem 5.18 all unbounded orbits converge to Λ_-. □

Remark 5.21 Because all the non-trivial Floquet multipliers of Λ_- lie in $(1,\infty)$ (see Theorem 5.18a), Lemma 5.20 remains true for $c>0$ sufficiently small. •

The following Theorem is equivalent to Proposition 5.10a. We recall that K is defined in (5.20), and that its boundary δK is a level set of the energy.

Theorem 5.22 *Let f satisfy hypothesis* (5.18) *and let $\beta>\frac{1}{\sqrt{\sigma(f)}}$. For $F(-a)<E_0<F(-1)$ let K be defined by* (5.20) *and let $W^s(1)$ be the stable manifold of the equilibrium $u=1$. Then, provided $c>0$ is sufficiently small, $W^s(1)\cap\delta K$ is a topological circle. Its projection*

Γ on the (u, u''')-plane winds exactly once around a disk containing both closed curves defined by $E_0 - F(u) + \frac{1}{2\beta}u'''^2 = 0$ (see also Figure 5.2), i.e., $n(\Gamma, -1) = n(\Gamma, 1) = 1$.

Proof. Our strategy is to deform $f(u)$ in several steps to the pure cubic $-u^3$ and let β go to zero. We have to do this in such a way that for each intermediate f the conclusion of Lemma 5.20 remains valid. All orbits in the stable manifold $W^s(1)$ thus tend to Λ_- in backward time, and this remains true during the entire deformation process. At the end of the deformation process we arrive at the reduced equation $u'''' + u^3 = 0$. We then use the analysis performed in Section 5.4 to find a precise description of the orbits in $W^s(1)$. Finally, we obtain the results of Theorem 5.22 for the original equation (5.3) via continuation arguments.

Recall that $f(u) = -u^3 + g(u)$ with g having compact support, say $g(u) = 0$ for all $|u| \geq C_0$. Taking C_0 sufficiently large, define the cut-off function $\phi \in C_0^\infty$ with $0 \leq \phi \leq 1$, $\phi(y) = 1$ for $|y| \leq C_0$, and $\phi(y) = 0$ for $|y| > C_0 + 1$.

Step 1. First deform $f(u)$ to a function which changes sign at $u = 1$ only. Let

$$f_\lambda(u) = f(u) - \lambda(u-1)\phi(u).$$

For λ large enough, say $\lambda > \lambda_0$, the function $f_\lambda(u)$ has a zero at $u = 1$ only.

Lemma 5.23 *Let $\beta > \frac{1}{\sqrt{\sigma(f)}}$ and replace $f(u)$ by $f_\lambda(u)$. Then for all $\lambda \in [0, \lambda_0]$ the stable manifold $W^s(1)$ consists of unbounded orbits only, all of which connect Λ_- to $u = 1$.*

Proof. Let $\lambda_1 = \inf\{\lambda \,|\, f_\lambda(u) > 0 \text{ for all } u < 1\}$. For any $\lambda < \lambda_1$ the argument is exactly the same as in the proof of Lemma 5.20, where we use the following generalised definition of σ:

$$\sigma(f_\lambda) = \min\left\{ \frac{-F(u)}{2f(u)^2} \;\middle|\; u < 1 \text{ and } f(u) < 0 \right\}.$$

Note that $\sigma(f_\lambda) \leq \sigma(f_0)$ for $0 < \lambda < \lambda_1$, since $f_\lambda(u)$ and $-F_\lambda(u)$ are increasing in λ for all $u < 1$. For $\lambda \geq \lambda_1$ the result also holds, but by a different and less restrictive oscillation argument, which applies to any $f(u)$ with a single zero at which it goes from positive to negative, and all $\beta \geq 0$. We already used this in the proof of Lemma 5.16; the argument showing that every solution $u \not\equiv 1$ oscillates towards infinity is almost identical (for $\beta \geq 0$ the second order term does not cause any difficulties). This completes the proof of the lemma. □

Continuing with the proof of Theorem 5.22, we change f to $f^1 \stackrel{\text{def}}{=} f_{\lambda_0}$ by letting λ go from 0 to λ_0. This leaves the local structure near $X_5 = 0$, and in particular near Λ_-, unaffected (see Section 5.4.5).

Step 2. We change $f^1(u) = -u^3 + g^1(u)$ with $g^1(u) = g(u) - \lambda_0(u-1)\phi$ to $f^2(u) \stackrel{\text{def}}{=} -u^3(1-\phi) - (u-1)\phi$. Using the deformation functions

$$f_\lambda(u) = -u^3(1-\phi(u)) + (1-\lambda)(-u^3\phi(u) + g^1(u)) - \lambda(u-1)\phi(u),$$

we let λ go from 0 to 1, thus continuously deforming f^1 into f^2. All orbits in $W^s(1)$ are still unbounded and tend to Λ_- as $t \to -\infty$ during this deformation, since $f_\lambda(u)$ has a single zero at which it goes from positive to negative (see the proof of Lemma 5.23).

Step 3. It is now easy to shift the zero to the origin. Define

$$f_\lambda(u) = -u^3(1-\phi(u)) - (u-(1-\lambda))\phi(u).$$

Letting λ change from 0 to 1 deforms f^2 into $f^3 \stackrel{\text{def}}{=} -u^3(1-\phi) - u\phi$. Since we have shifted the origin we now have $W^s(0)$ instead of $W^s(1)$. All orbits in $W^s(0)$ are still unbounded and tend to Λ_- as $t \to -\infty$.

Step 4. Next we let β go to zero. The stable manifold $W^s(0)$ changes smoothly and the local structure near Λ_- again remains unaffected because β only appears in terms quadratic in X_5. For $\beta = 0$ we have arrived at the equation

$$u'''' - f^3(u) = 0, \quad \text{with } f^3(u) = -u^3(1-\phi) - u\phi.$$

Step 5. We change f^3 using a family of functions

$$f_s(u) = -u^3(1-\phi) - u^s\phi.$$

Letting s increase from $s = 1$ to $s = 3$ we obtain a function $f^4(u) \stackrel{\text{def}}{=} u^3$. We note (see Section 5.4.4) that for $s > 1$ the manifold W is the center-stable manifold of 0. Here we use Section 5.4.3 to conclude that in this process W changes continuously, with the orbits in manifold $W = W^{cs}(0)$ still tending to Λ_- in backward time.

By Sections 5.4.1 and 5.4.4 we have that, after going through Steps 1–5, W is the product of Λ_- and the X_5-axis, i.e. $W = \Lambda_- \times \{X_5 \,|\, X_5 > 0\}$. In view of the non-trivial Floquet multipliers of Λ_- being in $(1, \infty)$, it holds that for any small $\varepsilon > 0$ there exists a negatively invariant tubular neighbourhood Λ_-^ε of Λ_- in Π with

$$\Lambda_-^\varepsilon \subset \{X = (X_1, X_2, X_3, X_4, X_5) \in \Pi \,|\, d(X, \Lambda_-) < \varepsilon\}.$$

We can choose this neighbourhood such that

$$\overline{\Lambda_-^\varepsilon} \cap \{X_5 = \varepsilon\} \;=\; \{(X_1, X_2, X_3, X_4) \in \Lambda_-,\, X_5 = \varepsilon\}. \tag{5.55}$$

Besides, we can choose Λ_ε such that the flow for our final equation $u'''' + u^3$ is transverse to $\delta\Lambda_-^\varepsilon$. Moreover, for $\varepsilon > 0$ sufficiently small, we can choose Λ_ε such the flow is transverse to $\delta\Lambda_-^\varepsilon$ for every intermediate $f(u)$ and β in the deformation process of Steps 1–5 above, hence also for the original equation (5.3) with $c = 0$.

For any given $r > 0$ we can choose $\varepsilon > 0$ so small that the projection Γ_ε of $W \cap \delta\Lambda_-^\varepsilon$ on the (x_1, x_4)-plane (or, equivalently, on the (u, u''')-plane) is a curve with minimal distance to the origin at least r. To see this, we observe that the solution of (5.23) represented by Λ_- cannot have a point where $u = u''' = 0$, for in such a point also $u'' = 0$ in view of the energy $\mathcal{E}$ being zero. This would contradict the fact that $Q < 0$ on Λ_-. Thus in the X-variables Λ_- is uniformly bounded away from $(X_1, X_4) = (0, 0)$, so that for any $r > 0$ we can find an $\varepsilon > 0$ such that the projection of Λ_-^ε on the (u, u''')-plane has a distance larger than r from the origin. Therefore, the winding numbers around $u = \pm 1$ of the projection Γ_ε of $W \cap \delta\Lambda_-^\varepsilon$ on the (u, u''')-plane are well-defined for ε sufficiently small.

It follows from (5.55) that for our final equation $u'''' + u^3 = 0$ we have

$$W \cap \delta\Lambda_-^\varepsilon = \{(X_1, X_2, X_3, X_4, X_5) \,|\, (X_1, X_2, X_3, X_4) \in \Lambda_-,\, X_5 = \varepsilon\},$$

so that, choosing r large, $n(\Gamma_\varepsilon, -1) = n(\Gamma_\varepsilon, 1) = 1$. By continuity the winding numbers of Γ_ε do not change if we reverse Steps 1–5, and again by continuity arguments and Remark 5.21 this remains true for $c > 0$ sufficiently small.

Finally, for our original equation (5.3) we know that, tracing back orbits in $W^s(1)$ until they hit $\delta\Lambda_-^\varepsilon$, their energy $\mathcal{E}$ remains close to 0, provided we keep $c > 0$ sufficiently small. Thus $W^s(1) \cap \delta K$ is contained in Λ_-^ε for small $c > 0$. Following $W^s(1) \cap \delta\Lambda_-^\varepsilon$ backwards along the flow to $W^s(1) \cap \delta K$ (which is a transverse intersection for $c > 0$), we see that the winding numbers $n(\Gamma, \pm 1)$ of the projection of $W^s(1) \cap \delta K$ are also 1. This completes the proof of Theorem 5.22. $\square$

5.6 The winding number for large speeds

In this section we prove Part (b) of Proposition 5.10:

Theorem 5.24 *Let f satisfy hypothesis* (5.18) *and let $\beta \in \mathbb{R}$. For $c > 0$ sufficiently large the intersection of the stable manifold $W^s(1)$ of $u = 1$ and the boundary δK of K is a smooth simple closed curve, which projects on a closed curve Γ in the (u, z)-plane with $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$.*

Proof. We first prove the theorem for a deformation of $f(u)$. We choose the nonlinearity $\tilde{f}(u)$ to satisfy

$$\tilde{f}(u) = f'(1)(u-1) \quad \text{in a neighbourhood } B_\varepsilon(1) \text{ of } u = 1.$$

For this deformed nonlinearity $\tilde{f}$ we compute the energy $\tilde{\mathcal{E}}$ on a closed curve in $\tilde{W} = W^s(1)$ winding once around $u = 1$ with u-values contained in $B_\varepsilon(1)$. The equation is now linear near $u = 1$, and the characteristic equation

$$-\mu^4 + \beta\mu^2 + f'(1) = c\mu$$

has two eigenvalues $-\mu_1$ and $-\mu_2$ with negative real part (recall that $f'(1) < 0$). For $c > 0$ large enough μ_1 and μ_2 are real, and asymptotically

$$\mu_1 \sim c^{\frac{1}{3}} \quad \text{and} \quad \mu_2 \sim \frac{-f'(1)}{c} \qquad \text{as } c \to \infty. \tag{5.56}$$

Since the equation is linear, $\tilde{W}$ is given by (for c large enough)

$$\tilde{W} = \{(u, v, w, z) \,|\, u = u(t) = 1 + A_1 e^{-\mu_1 t} + A_2 e^{-\mu_2 t},\, v = u'(t),\, w = u''(t),\, z = u'''(t)\} \tag{5.57}$$

We may choose a curve $S_1 \subset \tilde{W}$ around $u = 1$ parametrised by $\phi \in [0, 2\pi)$, by taking $t = 0$ and $A_1 = r\cos\phi$, $A_2 = r\sin\phi$ in (5.57) for some fixed $r > 0$. The projection of S_1 on the (u, u''')-plane is given by

$$\{(u, z) \,|\, u = 1 + r(\cos\phi + \sin\phi),\, z = -r(\mu_1^3\cos\phi + \mu_2^3\sin\phi),\, 0 \le \phi < 2\pi\}.$$

The energy on S_1 is given by

$$\begin{aligned} -\mathcal{E} &= \int_0^\infty cu'(t)^2 dt = c\int_0^\infty (A_1\mu_1 e^{-\mu_1 t} + A_2\mu_2 e^{-\mu_2 t})^2 dt \\ &= c\Big(\frac{A_1^2\mu_1}{2} + \frac{2A_1A_2\mu_1\mu_2}{\mu_1+\mu_2} + \frac{A_2^2\mu_2}{2}\Big) = c\mu_2\Big(\frac{A_1^2\mu_1}{2\mu_2} + \frac{2A_1A_2\mu_1}{\mu_1+\mu_2} + \frac{A_2^2}{2}\Big). \end{aligned} \tag{5.58}$$

Using (5.56) and estimating (5.58) from below we have, for c sufficiently large,

$$\mathcal{E} \le \frac{f'(1)}{4} r^2 < 0 \quad \text{on } S_1.$$

Thus, choosing an energy level $0 > \tilde{E}_0 > \frac{f'(1)}{4} r^2$, we have that S_1 lies in the complement of K. Let $\tilde{S} = \tilde{W} \cap \delta\tilde{K}$. Then $\tilde{S}$ lies inside S_1 and is obtained by tracing solutions in (5.57) of the linear equation forwards in time (starting on S_1) until they enter $\tilde{K}$. It follows that S_1 and $\tilde{S}$ wind around $u = 1$ in $\tilde{W}$ exactly once and therefore its projection $\tilde{\Gamma}$ on the (u, z)-plane winds once around $(u, z) = (1, 0)$.

The calculations above only involve u-values between $1 - r\sqrt{2}$ and $1 + r\sqrt{2}$ so we may change the definition of $\tilde{f}(u)$ outside this range. In particular, taking r small, we may choose $\tilde{f}(u)$ such that $\tilde{F}(u)$ has a minimum $\tilde{F}(-a) < \tilde{E}_0$ and a maximum $\tilde{F}(-1) \in (\tilde{E}_0, \tilde{F}(1))$, with $-1 < -a < 1 - r\sqrt{2}$. Clearly $\tilde{\Gamma}$ does not wind around the point $(u, z) = (-1, 0)$.

We continuously deform $\tilde{f}$ to f and $\tilde{E}_0$ to E_0, keeping the above configuration, and taking c large enough as to stay within a class of nonlinearities for which there does not exist a connection between $u = -1$ and $u = 1$ (see Lemma 5.6). By continuity we still have that $n(\Gamma, -1) = 0$ and $n(\Gamma, 1) = 1$. □

5.7 Travelling waves connecting an unstable to a stable state

In this section we focus on travelling waves that connect the unstable state $u = -a$ to one of the two stable states $u = \pm 1$. As in the proof of Theorem 5.1 in Section 5.3 we begin by reducing to nonlinearities f which satisfy (5.18).

To obtain the necessary bound for $\beta > 0$ we fix $c > 0$ and simply follow the argument in the proof of Lemma 5.7 with $F(-1)$ replaced by $F(-a)$ (for connections from $-a$ to $+1$), or by $F(-1) - F(-a)$ (for connections from $-a$ to -1).

By different methods it is also possible to prove a priori bounds in the case that $\beta \leq 0$. Applying a result by T. Gallay [71] to the present context we obtain the following. Let f satisfy (H_1), i.e. $\lim_{|u|\to\infty} \frac{f(u)}{u} = -\infty$, and fix $c > 0$. Then for any $\beta \in \mathbb{R}$ there exists a constant C_0 such that any travelling wave solution $u(t,x) = U(x + ct)$ of (5.1) satisfies $\|u\|_\infty \leq C_0$. The constant C_0 only depends on β and $m \stackrel{\text{def}}{=} \sup\{|u| : \frac{f(u)}{u} \geq -D_\beta\}$, where $D_\beta > 0$ is a constant which depends on β only.

The idea is to consider $\Phi_y(t) = \int_{-\infty}^{\infty} h_y(x) u^2(t,x) dx$, where $h_y(x) = \frac{1}{1+(x-y)^2}$. Using the differential equation (5.1) one obtains an estimate of the form $\frac{d\Phi_y}{dt} \leq A_0 - \Phi_y$ for some constant A_0 independent of y and t (A_0 only depends on β and m), hence $\Phi_y(t) \leq A_0 + \Phi_y(0)e^{-t}$. Defining $\Psi(t) = \sup_{y\in\mathbb{R}} \Phi_y(t)$ one derives that for travelling waves Ψ is independent of t, hence $\Psi \leq A_0$. Combining with the fact that $\int_{-\infty}^{\infty} (\frac{du}{dx})^2 dx = \frac{F(\pm 1) - F(-a)}{c}$, one then obtains an L^∞-bound on u.[5]

Thus, for every $c > 0$ there exists a constant $C_0 > 0$ such that any solution of (5.3) connecting $-a$ to ± 1 satisfies $\|u\| < C_0$. This a priori estimate implies that we may replace f by $\tilde{f}(u) = \phi(u) f(u) - u^3(1 - \phi(u))$, where the cut-off function $\phi \in C_0^\infty$ is such that $0 \leq \phi \leq 1$, $\phi(y) = 1$ for $|y| \leq C_0$, and $\phi(y) = 0$ for $|y| > C_0 + 1$. As in Section 5.3 it holds that u is a travelling wave solution with speed c for nonlinearity $f(u)$ if and only if u is a travelling wave solution with speed c for nonlinearity $\tilde{f}(u)$.

The above argument shows that, looking for travelling waves, we may as well assume that f satisfies (5.18). The next theorem thus proves Theorem 5.2.

Theorem 5.25 *Let f satisfy hypothesis* (5.18) *and let $\beta \in \mathbb{R}$. For every $c > 0$ there exists a solution of* (5.3) *connecting $u = -a$ to $u = -1$.*

[5] With a little bit more effort the estimate can be made uniform in c.

Proof. For all $c > 0$ we have that the three equilibria are hyperbolic and

$$\dim W^s(\pm 1) = \dim W^u(\pm 1) = 2, \quad \dim W^u(-a) = 3, \quad \dim W^s(-a) = 1.$$

Travelling wave solutions connecting $u = -a$ and $u = -1$ correspond to a nonempty intersection of $W^u(-a)$ and $W^s(-1)$. Recall that

$$\mathcal{E}(u,u',u'',u''') = -u'u''' + \frac{1}{2}u''^2 + \frac{\beta}{2}u'^2 + F(u), \quad \text{where } F(u) = \int_1^u f(s)ds,$$

satisfies (5.5). We take $F(-1) < E_1 < F(1)$ and consider the set

$$\tilde{K} = \{(u,v,w,z) \,|\, \mathcal{E}(u,v,w,z) = -vz + \frac{1}{2}w^2 + \frac{\beta}{2}v^2 + F(u) \leq E_1\}.$$

Now suppose that for some $c > 0$ the theorem is false. Then all orbits in $W^u(-a)$ have to leave $\tilde{K}$ through $\delta\tilde{K}$, because an orbit with bounded energy has no other choice than to converge to an equilibrium, see the proof Lemma 5.9, and $u = -1$, the only equilibrium in $\tilde{K}$ with energy larger than $\mathcal{E}(-a,0,0,0)$, is excluded by assumption. Thus we have that the intersection of $W^u(-a)$ and $\delta\tilde{K}$ is homeomorphic to a 2-sphere S^2.

For the moment we consider the case that $\beta > 0$. Since $\delta\tilde{K}$ is given by

$$\beta(v - \frac{z}{\beta})^2 + w^2 = 2E_1 - 2F(u) + \frac{z^2}{\beta}, \tag{5.59}$$

we may deform it smoothly into

$$\{(u,v,w,z) \,|\, u^2 + z^2 = 1 + v^2 + w^2\},$$

which defines a 3-manifold homeomorphic to $\mathbb{R}^2 \times S^1$. As deformations we use

$$(\lambda\beta + 1 - \lambda)(v - \lambda\frac{z}{\beta})^2 + w^2 = G(u,\lambda) + (1 - \lambda + \frac{\lambda}{\beta})z^2,$$

with λ running from 1 to 0, and $G(u,1) = 2E_1 - 2F(u)$ and $G(u,0) = -1 + u^2$. Singularities can only appear in points on these manifolds where $\frac{dG}{du} = v = w = z = 0$ and can thus be avoided by the choice of E_1.

It follows that $\delta\tilde{K}$ is homeomorphic to $\mathbb{R}^2 \times S^1$, or, equivalently, to the open solid torus. The intersection $W^u(-a) \cap \delta\tilde{K}$, being homeomorphic to S^2, divides $\delta\tilde{K}$ into two components, one bounded and homeomorphic to an open ball in $\mathbb{R}^3$, the other unbounded. This division is in fact not completely straightforward. One needs to lift (a neighbourhood of) $W^u(-a) \cap \delta\tilde{K}$ to the universal covering space $\mathbb{R}^3$ of $\tilde{K}$ and show that the unbounded part of the complement of the countable union of lifts is path-connected. Using the fact that the intersection $W^u(-a) \cap \delta\tilde{K}$ is induced by a flow, one can invoke the generalised Schoenflies theorem (see [30, Theorem 19.11]) to conclude that one lift of $W^u(-a) \cap \delta\tilde{K}$ divides $\mathbb{R}^3$ into an unbounded and a bounded component, which is homeomorphic to an open ball. Besides, the bounded components of the countable infinity of lifts can be contracted to points. The unbounded component (the complement of the countable union of bounded components) is thus homeomorphic to $\mathbb{R}^3 \setminus \mathbb{Z}$, hence path-connected[6].

Now consider the piecewise smooth 3-manifold formed by the disjoint union of the point $(-a,0,0,0)$ and $W^u(-a) \cap \tilde{K}$ and the bounded component of $\delta\tilde{K} \setminus (W^u(-a) \cap \delta\tilde{K})$.

[6] We gratefully acknowledge several discussions with H. Geiges. He showed us that, via the Jordan-Brouwer separation theorem and an inductive Mayer-Vietoris argument, the division of $\delta\tilde{K}$ into two components can also be derived without using the extra information provided by the flow.

This 3-manifold is homeomorphic to two closed three-dimensional balls sharing an S^2, namely $W^u(-a) \cap \delta\tilde{K}$, as boundary and is therefore homeomorphic to an S^3. By the Jordan-Brouwer theorem this 3-manifold divides $\mathbb{R}^4$ to two components, one bounded, the other unbounded. We notice that the bounded component is negatively invariant. Clearly both components contain exactly one of the two orbits which together form the stable manifold $W^s(-a)$. Now consider the orbit in $W^s(-a)$ contained in the bounded component (which is negatively invariant). Since its energy is bounded we may, again by the argument in the proof of Lemma 5.9, conclude that, tracing it backwards, it must go to an equilibrium with energy less than the energy of $u = -a$. Since such an equilibrium does not exist, we have arrived at a contradiction.

The cases $\beta < 0$ and $\beta = 0$ are similar, the only changes being that we deform $\delta\tilde{K}$, given by (5.59), to $u^2 + v^2 = 1 + z^2 + w^2$ if $\beta < 0$, and that for $\beta = 0$ we rewrite $\delta\tilde{K}$ as $-2vz + w^2 = 2E_1 - 2F(u)$, which deforms into $-2vz + w^2 = -1 + u^2$ or $\frac{1}{2}(v+z)^2 + u^2 = \frac{1}{2}(v-z)^2 + w^2 + 1$. This completes the proof of the theorem. □

Remark 5.26 In the proof of Theorem 5.25 above we have used the non-degeneracy of the equilibrium point $u = -a$, while $u = -1$ may degenerate (i.e. $f'(-1) = 0$). The theorem also holds when $u = -a$ is degenerate and $u = -1$ is non-degenerate; in this case the argument in the proof of Theorem 5.27 below can be used. If $F(-1) = F(1)$ one also applies the proof of Theorem 5.27, see Remark 5.28. •

Next we prove Theorem 5.3. Let

$$c^* \overset{\text{def}}{=} \inf\{\tilde{c} > 0 \,|\, \text{there is no connection from } -1 \text{ to } +1 \text{ for } c > \tilde{c}\}.$$

From Lemma 5.6 we see that c^* is well-defined, and $c^* > 0$ for $\beta > \frac{1}{\sqrt{\sigma(f)}}$ by Theorem 5.1. The argument at the beginning of this section shows that, in order to prove Theorem 5.3, we may restrict to nonlinearities f which satisfy (5.18). If $c_* > 0$, then it follows from Lemma 5.9 that for $c = c^*$ there exists a solution of (5.3) which connects -1 to $+1$. The following theorem thus proves both Theorem 5.3 and Corollary 5.4.

Theorem 5.27 *Let f satisfy hypothesis* (5.18) *and let $\beta \in \mathbb{R}$. For every $c > c^*$ there exists a solution of* (5.3) *connecting $u = -a$ to $u = 1$.*

Proof. We consider the stable manifold $W = W^s(1)$ of $u = 1$. We have shown in Theorem 5.24 that for $c > 0$ large enough the intersection of W and the boundary δK of K (defined in (5.20)) is a smooth simple closed curve which projects on a closed curve Γ in the (u,z)-plane with $n(\Gamma,-1) = 0$ and $n(\Gamma,1) = 1$. It follows from the definition of c^* and Lemma 5.9 that, by continuity, this remains true for all $c > c_*$. Now fix $c > c^*$.

Let us assume by contradiction that there is no connection between $u = -a$ and $u = 1$. The intersection between W and δK depends continuously on the energy level E as long as we do not encounter an equilibrium point. Assuming there is no connection between $u = -a$ and $u = 1$, we let E decrease from $F(-1) > E_0 > F(-a)$ to $E_2 < F(-a)$. The projection Γ in the (u,z)-plane then depends continuously on E, as do the winding numbers, so that $n(\Gamma,-1) = 0$ and $n(\Gamma,1) = 1$ for all $E_0 \le E \le E_2$. However, for the energy level E_2 we have that $(-1,0)$ and $(1,0)$ lie in the same component of the complement of the projection of δK onto the (u,z) plane. Therefore $n(\Gamma,-1) = n(\Gamma,1)$, a contradiction. □

Remark 5.28 When $F(-1) = F(+1)$ then the same method shows that there exist travelling waves connecting $u = -a$ to $u = -1$ and connecting $u = -a$ to $u = +1$ for all $c > 0$ and all $\beta \in \mathbb{R}$. Besides, as already noted in Remark 5.26, the method in the proof of Theorem 5.27 can be used to obtain an alternative proof of Theorem 5.25. •

Finally, we prove Theorem 5.5 which deals with nonlinearities with two zeros (and a different behaviour for $u \to \pm\infty$).

Theorem 5.29 *Let* $\beta \in \mathbb{R}$ *and let* f *satisfy hypothesis* (H$_2$). *For every* $c > 0$ *there exists a solution of* (5.3) *connecting* $u = 0$ *to* $u = 1$.

Proof. Since the shape of the nonlinearity differs significantly from the one considered so far, we cannot invoke Lemma 5.9 directly. Besides, we find a priori bounds via a slightly different method.

Let $D \stackrel{\text{def}}{=} \sup\{\tilde{u} < 0 \,|\, F(u) > 0 \text{ on } (-\infty, \tilde{u})\}$. Travelling wave solutions connecting 0 to 1 satisfy $u \geq D$, since it follows from (5.4) and (5.5) that u can have no extremum in the range $u < D$ (at an extremum one would have $\mathcal{E} > F(1)$, which is impossible). Therefore, we may without loss of generality replace f by any function f_1 for which $f_1(u) = f(u)$ for $u \geq D$, and $f_1(u) < 0$ for $u < D$. We choose f_1 such that $f_1(u) = u$ for $u < D - 1$.

Now that we have a bound from below, we can also obtain a bound from above. As was just explained, a connecting solution of (5.3) is also a solution of (5.3) with f_1 replaced by any f_2 for which $f_2(u) = f_1(u)$ for all $u \geq D - 1$. We choose $f_2(u) = -u^3$ for $u < D - 2$, and argue as at the beginning of this section to conclude that there exists a uniform bound $\|u\|_\infty \leq C_0$ on all travelling wave solutions. We may thus replace f_1 by a function f_3 for which $f_3(u) = f_1(u)$ for $u \leq C_0$ and $f_3(u) = -u^3$ for $u \geq C_0 + 1$. We conclude that u is a travelling wave solution with speed c for nonlinearity $f(u)$ if and only if u is a travelling wave solution with speed c for nonlinearity $f_3(u)$.

In the following we therefore assume, without loss of generality, that $f(u) = u$ for $u \leq D - 1$, and $f(u) = -u^3$ for $u \geq C_0 + 1$.

We now follow the argument in the proof of Lemma 5.9. However, we cannot use Lemma 5.8 to show that orbits in $W^s(1)$ which are completely contained in K, are bounded. Instead, we argue as follows. Suppose, by contradiction, that an orbit $u(t)$ in $W^s(1)$ is completely contained in K and is unbounded. As in the proof of Lemma 5.9 it follows from Equation (5.21) that $u(t)$ exists for all $t \in \mathbb{R}$. There are now two possibilities: either $u(t) \geq D - 1$ for all $t \in \mathbb{R}$, or there exists some $t_0 \in \mathbb{R}$ such that $u(t_0) < D - 1$. First we deal with the latter case.

Since $u(t)$ cannot attain an extremum in the range $u < D$ (see above), it follows that $u(t)$ is increasing for $t \leq t_0$. Hence $u(t)$ obeys, for $t \leq t_0$, the linear equation $cu' = -u'''' + \beta u'' + u$. Since u is unbounded as $t \to -\infty$, it follows that $u = -a_0 e^{-a_1 t} + o(1)$ for some $a_0, a_1 > 0$ as $t \to -\infty$. By substituting this into Equation (5.21) a contradiction is reached.

Next we deal with the case where $u(t) \geq D - 1$ for all $t \in \mathbb{R}$. Clearly $u(t)$ is a solution of (5.3) with f replaced by any function $\tilde{f}$ for which $\tilde{f}(u) = f(u)$ for all $u \geq D - 1$. We choose $\tilde{f}(u) = -u^3$ for $u < D - 2$, and it follows from Lemma 5.8 that u blows up in finite time, a contradiction.

Having circumvented the problem in the proof of Lemma 5.9 we conclude that for $F(0) < E_0 < F(-1)$ the intersection of the stable manifold W of $u = -1$ and the boundary

δK of K (defined in (5.20)) is a smooth simple closed curve which projects on a closed curve Γ in the (u,z)-plane with $n(\Gamma,1)=1$.

The rest of the argument is analogous to the proof of Theorem 5.27. Assuming that there is no connection between $u=0$ and $u=1$, the final contradiction is now obtained by the fact that $n(\Gamma,1)=0$ for $E_2 < F(0)$. □

5.8 Concluding remarks

The most apparent open problem concerns the range of β-values for which a travelling wave connecting -1 to $+1$ exists. For some examples it can be shown that such a travelling wave does not exist for all $\beta \in \mathbb{R}$. The more general question whether for any nonlinearity satisfying (H_1) a bound $\beta_* \in \mathbb{R}$ exists such that there are no travelling waves for $\beta < \beta_*$ remains open.

Regarding the uniqueness of the various travelling wave solutions not much is known. For large β (i.e $\gamma \approx 0$) the travelling wave connecting -1 to $+1$ may be expected to be unique (analogous to the limiting second order case). The results in [34] show that uniqueness does not hold for $f_a(u)=(u+a)(1-u^2)$ with a small when $\beta < \sqrt{8}$. Equation (5.1) with $f(u)=u-u^3$ admits an abundance of standing wave solutions for $0 \leq \beta < \sqrt{8}$. It has been proved in [34] that these solutions can be perturbed to travelling waves for $f_a(u)$ with small a and small $c=c(a)$. Since this can be done for any standing wave, an infinite family of solution curves in the (a,c)-plane passing through the origin is thus obtained.

The method used in this chapter does not give any information about the shape of the solution. For example, we would like to know for which values of β the solution is monotone. Since we do not know the value of c for which a traveling wave occurs, we in general do not even know whether the connected equilibrium points are approached monotonically or in an oscillatory manner.

Finally, the question arises to what extent the travelling wave solution is of importance to the dynamics of the PDE. It might be a limit profile for a broad class of initial conditions as is the case for the second order equation [68]. Since travelling waves connecting $u=-a$ to $u=\pm 1$ exist for large ranges of c, it would also be interesting to know which of these waves is generally encountered. In [53, 62] the wave selection mechanism has been investigated for a propagating front which is formed from localised initial data (i.e., $u+a$ is localised). Using the physically motivated assumption that the linearised equation (around $u=-a$) drives the system, it is argued that for $\beta > \sqrt{12f'(-a)}$ one of the travelling waves is selected (and the wave speed is calculated), while for $\beta < \sqrt{12f'(-a)}$ the propagating front is argued not to have a fixed profile. However, the only rigorous stability result that we know of, is of a perturbative nature [130] (i.e. β very large) and moreover it does not answer the question of the selection of the wave speed.

Chapter 6

Multi-bumps via the shooting method

6.1 Introduction

In this chapter we present new families of global branches of single and multi-bump periodic solutions of the fourth order equation

$$\frac{d^4u}{dx^4} + q\frac{d^2u}{dx^2} + u^3 - u = 0, \qquad q \in \mathbb{R}. \tag{6.1}$$

This equation arises in a variety of problems in mathematical physics and mechanics. As an important example we mention that (6.1) describes stationary solutions of the Swift-Hohenberg (SH) equation:

$$\frac{\partial U}{\partial t} = -\left(1 + \frac{\partial^2}{\partial x^2}\right)^2 U + \alpha U - U^3, \qquad \alpha > 0. \tag{6.2}$$

Equation (6.2) was first introduced by Swift & Hohenberg [137] in studies of Rayleigh-Bénard convection, and was proposed by Pomeau & Manneville [126] as a good description of cellular flows just past the onset of instability. Of particular interest in these studies was the formation of stationary periodic patterns, and the selection of their wavelengths. For further references about the SH equation we refer to the book by Collet & Eckmann [47] and the survey by Cross & Hohenberg [52]. If $\alpha > 1$, then stationary solutions U of Equation (6.2), when suitably scaled, are readily seen to be solutions of Equation (6.1). Specifically, U and u are related through

$$u(x) = \frac{1}{\sqrt{\alpha - 1}} U((\alpha - 1)^{-\frac{1}{4}} x) \qquad \text{and} \qquad q = \frac{2}{\sqrt{\alpha - 1}}. \tag{6.3}$$

It is clear from (6.3) that in this example, q only takes *positive* values. An example where q takes *negative* values, is the Extended Fisher-Kolmogorov (EFK) equation [49, 53],

$$\frac{\partial U}{\partial t} = -\gamma \frac{\partial^4 U}{\partial x^4} + \frac{\partial^2 U}{\partial x^2} + U - U^3, \qquad \gamma > 0,$$

which yields (6.1) if we set

$$u(x) = U(\gamma^{-1/4} x) \qquad \text{and} \qquad q = -\frac{1}{\sqrt{\gamma}}.$$

Equation (6.1) also arises as the Euler-Lagrange equation of variational problems involving functionals with second order gradients, such as in the study of period selection in cellular flows [126] or layering phenomena in second order materials [99, 46, 106]. We mention in particular in this context the governing equation of a strut [86, 138] with stiffness EI under an axial compression P and subjected to a load $Q(y)$:

$$EIy^{(iv)} + Py'' + Q(y) = 0.$$

Here y denotes the deflection of the strut in a direction perpendicular to its axis.

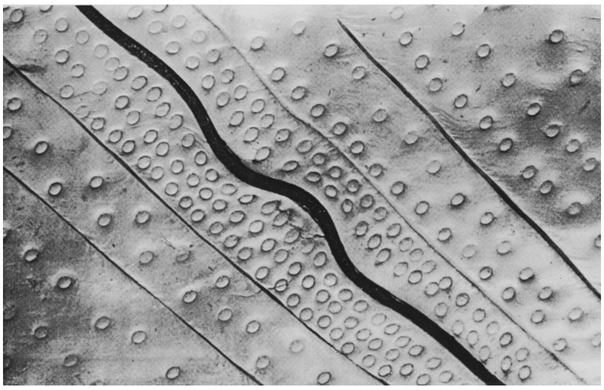

Figure 6.1: Folding of a stiff layer in a ductile material (reproduced from [129]).

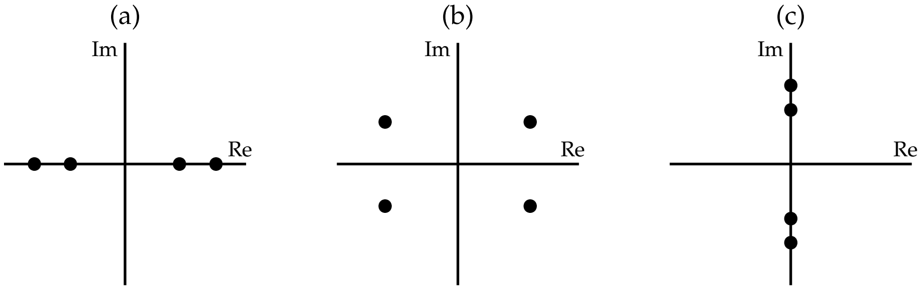

Figure 6.2: The spectrum of the linearisation around P_+ and P_-: (a) for $q \leq -\sqrt{8}$; (b) for $q \in (-\sqrt{8}, \sqrt{8})$; (c) for $q \geq \sqrt{8}$.

The investigation in this paper is part of a study of complex patterns in physics and mechanics in the description of which Equation (6.1) plays an important role. A typical example of such a pattern is the phenomenon of *localised buckling* in mechanics. In this type of buckling, the deflections are confined to a small portion of the otherwise unperturbed material. In Figure 6.1 we give an example of such a pattern due to Ramsay [129]. It shows the effect of compression on a layered material in which the layers have different stiffness. Because the stiffer layer (black, in the center) will not contract as easily as the more ductile material that surrounds it, the stiff layer deflects sideways and produces folds. Patterns are described by bounded solutions of Equation (6.1) on the real line. Thus, mathematically this study amounts to an investigation of the different types of bounded solutions Equation (6.1) possesses.

In recent years a great deal has been learnt about the structure of the set of bounded solutions of Equation (6.1) on the real line. It turns out to depend very much on the value of the parameter q. In particular, one can identify two critical values of q: $+\sqrt{8}$ and $-\sqrt{8}$. At these values the linearisation around the constant solutions $u = \pm 1$, i.e. the points $P_\pm = (\pm 1, 0, 0, 0)$ in (u, u', u'', u''') phase space, changes character, as indicated in Figure 6.2.

For $q \leq -\sqrt{8}$, the set of bounded solutions is very limited, and consists (modulo translations) of a one parameter family of single bump periodic solutions, which are even with respect to their extrema and odd with respect to their zeros, and two heteroclinic orbits or kinks, connecting P_+ and P_- (see Chapter 2 and [117, 120]); both are odd, one is strictly increasing and one is strictly decreasing. Because we shall often need to refer to it, we denote the odd increasing kink for $q = -\sqrt{8}$ by $\varphi(x)$.

As q increases beyond $-\sqrt{8}$ the set of bounded solutions becomes much richer. It has been proved that for $-\sqrt{8} < q \leq 0$ it includes a great variety of multi-bump periodic solutions, heteroclinic orbits and homoclinic orbits to P_+ and P_-, as well as chaotic solutions. For detailed results we refer to [90, 89, 118, 119, 120]. For $q > 0$ the results are more tentative and incomplete: although numerical studies for Equation (6.1) [19] and related equations [35, 43] suggest an abundance of bounded solutions in this parameter range as well, much of this still remains unproved.

The aim of this chapter is to investigate the existence and qualitative properties of *multi-bump* periodic solutions of Equation (6.1). By this we mean here solutions which have more than one critical point in each period. When q lies in a right neighbourhood of $-\sqrt{8}$, then all local extrema of the periodic solutions lie near the constant solutions $u = \pm 1$, and solutions have transitions between these uniform states. In this regime the term 'multi-bump' corresponds to the way it is commonly used in dynamical systems theory. However, as q moves away from $-\sqrt{8}$, local extrema are no longer tied to $u = \pm 1$, and it is not easy to identify the transitions. However, we shall still describe such solutions as multi-bump periodic solutions. Thus, this work extends previous results [120, 106] in which the properties of *single bump* periodic solutions were studied. In particular, we will investigate the existence and global behaviour of families of *odd* and *even*, single and multi-bump periodic solutions which bifurcate from the strictly increasing kink φ at $q = -\sqrt{8}$. Odd solutions may also be even with respect to some of their critical points. On the other hand, by even solutions we mean solutions, which are not odd with respect to any of their zeros. Below we indicate some of our findings:

1. We obtain a family of periodic solutions, bifurcating from the kink φ at $q = -\sqrt{8}$ and extending to infinity, i.e., these solutions exist for all $q > -\sqrt{8}$ (see Figure 6.3a). The family consists of a countable infinity of distinct periodic solutions. The simplest examples of these are shown in Figures 6.4-6.6. In the bifurcation diagram we graph the supremum norm $M = \|u\|_\infty$ against q.
2. In addition, another *pair* of families, both consisting of a countable infinity of distinct periodic solutions, are proven to exist for $q \in (-\sqrt{8}, 0]$. These solutions continue to exist for some, but not all, positive values of q. Numerical evidence suggests that the solutions from both families pairwise lie on *loops* in the (q, M) plane (see Figure 6.3b), of which the projection on the q-axis is of the form $(-\sqrt{8}, q^*]$, and one solution lies on the top of the loop while the other solution lies on the bottom. At $q^* > 0$ the two solutions coalesce.
3. Finally, we find a third kind of periodic solutions. These again come as a family of countable many distinct periodic solutions which bifurcate from the kink φ at $q = -\sqrt{8}$. However, this family does not extend to infinity nor do they lie on loops. Instead, our numerical results indicate that these periodic solutions bifurcate from the constant solution $u = 1$ as q tends to a critical value q_n (see Figure 6.3c) which is of the form
$$q_n = \sqrt{2}\Big(n + \frac{1}{n}\Big), \qquad n = 1, 2, \ldots. \tag{6.4}$$
Note that $q_1 = \sqrt{8}$, that $q_{n+1} > q_n$ for every $n \geq 1$, and that $q_n \to \infty$ as $n \to \infty$. For $n \geq 2$ these solutions come in pairs. Graphs of some of them are shown in Figures 6.8,

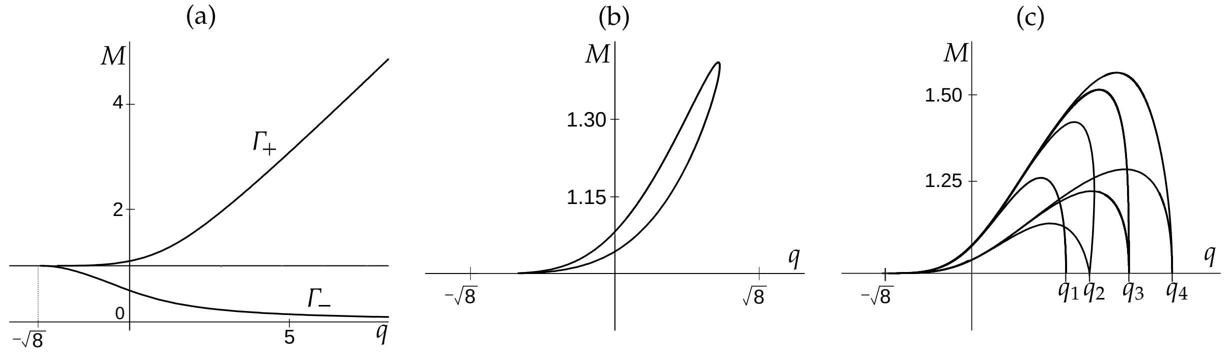

Figure 6.3: The three types of bifurcation graphs: (a) Type 1; (b) Type 2; (c) Type 3.

6.9 and 6.11. The critical values q_n arise when the moduli of the eigenvalues of the linearisation around $u = 1$ are a multiple of one another. Further details of the derivation of (6.4) are given in Section 6.5 (see also [42]).

Samples of the bifurcation curves in the (q, M) plane of these three types of solutions are shown in Figure 6.3.

Equation (6.1) admits a first integral, often referred to as the *energy*,

$$\mathcal{E}[u] \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + F(u),$$

where[1]

$$F(u) = \frac{1}{4}(u^2 - 1)^2.$$

The energy $\mathcal{E}[u]$ is constant if u is a solution. For the solutions $u(x) = \varphi(x)$ and $u(x) = 1$ it is clear that $\mathcal{E}[u] = 0$. In this chapter we focus on branches of periodic solutions which bifurcate from either $u = \varphi$ or $u = 1$ or both. This motivates us to consider solutions which have zero energy, that is for which

$$\mathcal{E}[u] = 0. \tag{6.5}$$

In constructing periodic solutions, we make extensive use of symmetry properties of solutions: if $a \in \mathbb{R}$ is a point where $u' = 0$ as well as $u''' = 0$, then thanks to the reversibility of Equation (6.1), it is easily verified that u is even with respect to a:

$$u(a - y) = u(a + y) \qquad \text{for all } \in \mathbb{R}.$$

Also, since the function F is even, it follows that if $b \in \mathbb{R}$ is a point where $u = 0$ as well as $u'' = 0$, then u is odd with respect to b:

$$u(b - y) = -u(b + y) \qquad \text{for all } \in \mathbb{R}.$$

We begin with a brief summary of previous results [120, 106] in which the existence of two families of single-bump periodic solutions u_+ and u_- was proved.

Theorem 6.1 *For every $q > -\sqrt{8}$ there exist two periodic solutions u_+ and u_- of Equation* (6.1) *such that $\mathcal{E}[u_\pm] = 0$. Both u_+ and u_- are odd with respect to their zeros and even with respect to their critical points, and*

$$M_+ = \max\{u_+(x) \,|\, x \in \mathbb{R}\} > 1 \qquad \textit{and} \qquad M_- = \max\{u_-(x) \,|\, x \in \mathbb{R}\} < 1.$$

[1]Note that in this chapter the potential $F(u)$ is defined with the opposite sign compared to Chapter 1.

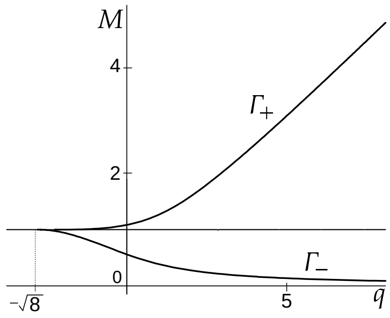

Figure 6.4: The branches Γ_+ and Γ_-.

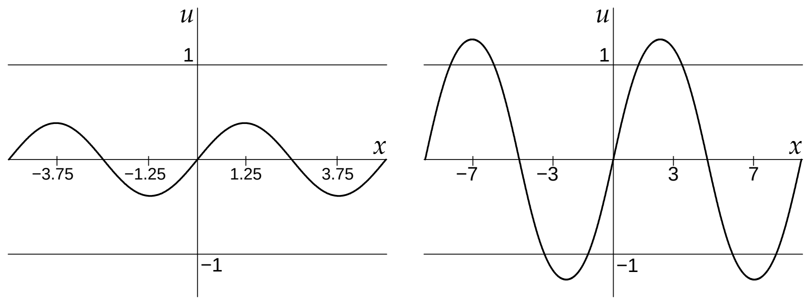

Figure 6.5: Small and large single bump periodic solutions ($q = 1$).

We denote these two families of solutions by

$$\Gamma_\pm = \{u_\pm(\cdot, q) \,|\, q > -\sqrt{8}\}.$$

Numerical computations of these branches made with AUTO [57] are shown in Figure 6.4. As will be done throughout this chapter when depicting solution branches, we set q along the horizontal axis and the supremum norm of the solution,

$$M \stackrel{\text{def}}{=} \|u\|_\infty,$$

along the vertical axis. Graphs of u_+ and u_- made with Phaseplane [65] are given in Figure 6.5.

The two families Γ_+ and Γ_- will be used to construct families of multi-bump periodic solutions of greater complexity. We shall characterise these solutions by the critical points and the critical values of their graphs. We label the positive critical points corresponding to local maxima by $\{\xi_k\}$ and the positive critical points corresponding to local minima by $\{\eta_k\}$. For *odd* solutions with $u'(0) > 0$ these points satisfy

$$0 < \xi_1 \leq \eta_1 \leq \xi_2 \leq \ldots.$$

In fact, since $-u(x)$ is a solution whenever $u(x)$ is, we will assume throughout that $u'(0) > 0$ for odd solutions. For *even*, non-constant solutions such that $\mathcal{E}[u] = 0$, we find that $u(0) \in \mathbb{R} \setminus \{-1, 1\}$. In this case u' also has infinitely many positive zeros and these satisfy

$$\begin{aligned} 0 < \xi_1 \leq \eta_1 \leq \xi_2 \leq \ldots \qquad &\text{if} \quad u''(0) > 0, \\ 0 < \eta_1 \leq \xi_2 \leq \eta_2 \leq \ldots \qquad &\text{if} \quad u''(0) < 0. \end{aligned}$$

Starting from the solutions u_+ and u_-, we use a shooting technique to obtain a countable family of odd multi-bump periodic solutions which also exists on the entire q-interval $-\sqrt{8} < q < \infty$. The solutions of this family obey the rule that all their local maxima lie

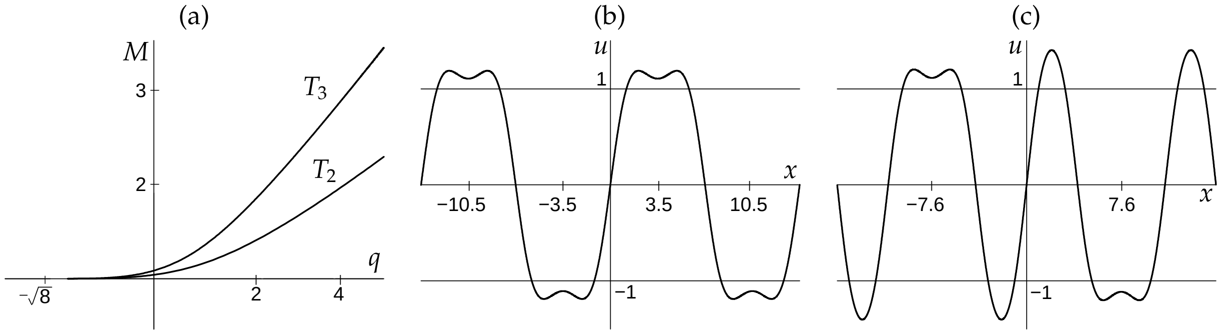

Figure 6.6: The branches T_2 and T_3, and corresponding solutions at $q = 1.5$: (b) the solution on T_2; (c) the solution on T_3.

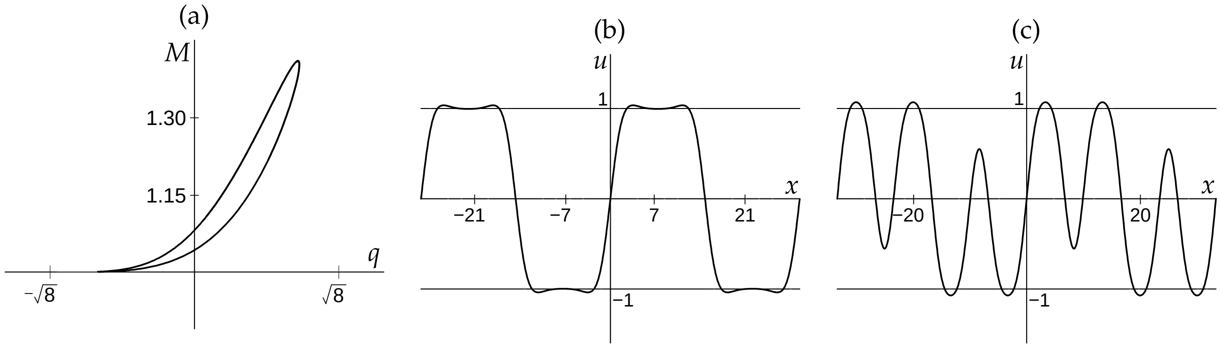

Figure 6.7: Loop-shaped branch and odd periodic solutions at $q = -\frac{1}{10}$. The solution in (b) is on the lower part of the loop.

above $u = +1$ and all their local minima lie *below* $u = -1$. However, the first point of symmetry ζ is an exception: at such a point $u(\zeta)$ lies *below* $u = -1$ if it is a *maximum*, and *above* $u = +1$ if it is a *minimum*:

$$\begin{aligned} u(\zeta) &< -1 \qquad \text{if } \zeta = \xi_k \text{ for some } k \geq 1, \\ u(\zeta) &> +1 \qquad \text{if } \zeta = \eta_k \text{ for some } k \geq 1. \end{aligned}$$

We denote by T_N the branch of odd periodic solutions of this family, of which the N^{th} critical point ζ_N is the first point of symmetry:

$$T_N = \{u(\cdot, q) \,|\, q > -\sqrt{8}, \quad u(\cdot, q) \text{ is symmetric with respect to } \zeta_N\}.$$

In Figure 6.6 we give the numerically computed branches T_2 and T_3, as well as specific solutions which lie on T_2 and T_3 at $q = 1.5$. In Section 6.3 the precise result for this family is formulated in Theorem 6.16.

In addition to these branches which exist on the entire q-interval $(-\sqrt{8}, \infty)$, we prove the existence of a second family of odd periodic solutions on $(-\sqrt{8}, 0]$. They exist in pairs, and our numerical experiments indicate that they lie on loop shaped branches, which extend well into the regime $q > 0$. An example of such a loop, together with the corresponding two solutions, is given in Figure 6.7. For this particular family, the solutions are symmetric with respect to η_1 with $-1 < u(\eta_1) < 1$. The precise description of these solutions is given in Theorems 6.17 and 6.19.

The techniques used to prove the existence of these solutions for $-\sqrt{8} < q \leq 0$ have been developed in [120, 118, 119]. In the present chapter we show how to obtain several families of single and multi-bump periodic solutions via this method. However, we do not aim at completeness, and there are many more branches of periodic solutions that can be established using this technique for $-\sqrt{8} < q \leq 0$ (see also [118, 119]) than those

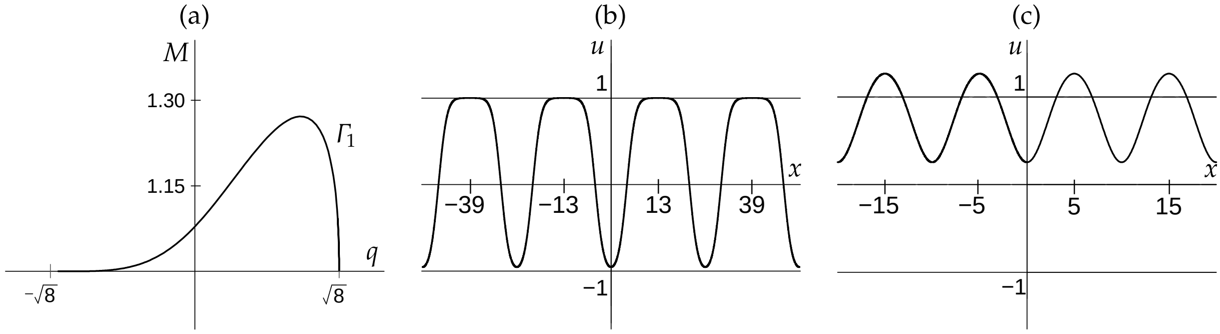

Figure 6.8: Branch of even single bump periodic solutions. The solution at $q = -2$ and $q = 2$ are depicted in (b) and (c) respectively.

presented here. We are not able to extend this existence proof to the regime $q > 0$. An essential difficulty seems to be that the solutions cease to exist at a coalescence point q^*, which is different for every branch of solutions.

A different method for proving the existence of periodic solutions with energy $\mathcal{E}[u] = 0$ in the regime $-\sqrt{8} < q < 0$ has been presented in [89]. There, a minimisation procedure is used to obtain periodic solutions both with and without symmetry with respect to a zero or an extremum. Since only the *minimisers* of an associated functional are considered, the variational method establishes (for example) the existence of only one of the two solutions in Figure 6.8.

In Sections 6.4, 6.5 and 6.6 we turn to *even* periodic solutions of Equation (6.1). Here we find a third type of branching phenomenon: solutions existing on finite q-intervals of the form $(-\sqrt{8}, q_n)$, still bifurcating from the kink φ at the lower end, and according to numerical evidence, also bifurcating from the constant solution $u = 1$ at the top end. Our results here extend those obtained in our analysis [121] of the equation

$$u^{(iv)} + c^2 u'' + e^u - 1 = 0,$$

proposed in [97] in connection with the study of travelling waves (with speed c) in suspension bridges. Concerning even single bump periodic solutions of (6.1) we prove the following existence theorem:

Theorem 6.2 *For every $q \in (-\sqrt{8}, \sqrt{8})$ there exists a periodic solution u of Equation* (6.1)*, such that $\mathcal{E}[u] = 0$, which is even with respect to all its critical points, with the property:*

$$-1 < \min\{u(x) : x \in \mathbb{R}\} < +1 < \max\{u(x) : x \in \mathbb{R}\}.$$

Two such solutions, at $q = -2$ and $q = +2$, are shown in Figure 6.8b,c, and the branch of solutions on which they lie is presented in Figure 6.8a.

In order to formulate our results about even *multi-bump* periodic solutions we need to introduce the notion of an *n-lap solution*. If u is an even periodic solution for which all the critical points are local maxima or minima, we say that u is an n-lap solution if it is symmetric with respect to its n^{th} critical point, so that its graph will have $2n$ monotone segments in one period. Recall that $q_n = \sqrt{2}(n + \frac{1}{n})$.

Theorem 6.3 *For each $n \geq 2$ there exist two families of even periodic n-lap solutions when $q \in (-\sqrt{8}, q_n)$. At the points of symmetry, ζ_n, we have*

$$u(\zeta_n) > 1 \qquad \text{for every } n \geq 1.$$

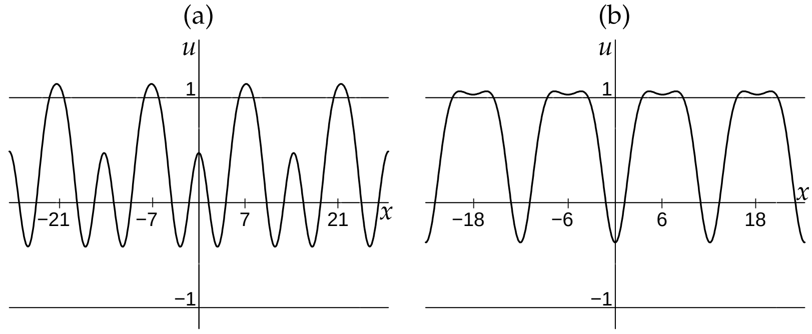

Figure 6.9: Two even periodic 2-lap solutions at $q = 2$: (a) on the upper branch Γ_{2a}; (b) on the lower branch Γ_{2b}. See Figure 6.10 for a picture of the complete branches $\Gamma_{2a,b}$.

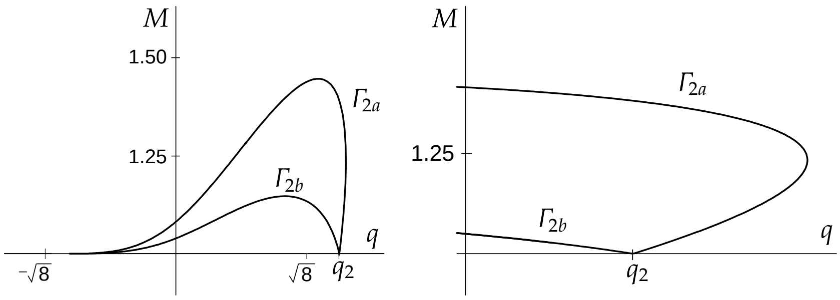

Figure 6.10: Branches of even periodic 2-lap solutions, and a blowup at q_2.

For $n = 2$ and $n = 3$, and $q \in (-\sqrt{8}, q_n)$ it is possible to show in addition that the critical values of the solutions all lie in the interval $(-1, 1)$ with the exception of the point of symmetry, and, if the solution is symmetric with respect to a minimum, its neighbouring maxima. Note that the case $n = 1$ is discussed in Theorem 6.2, and corresponding 1-lap solutions are shown in Figure 6.8.

In Figure 6.9 we present two 2-lap solutions and in Figure 6.10 we show the branches of the two solutions, as well as a blowup near the point $(q, M) = (q_2, 1)$.

In Figure 6.10, the solutions on the upper branch Γ_{2a} satisfy $u''(0) < 0$ and are symmetric with respect to ξ_2. Along the lower branch Γ_{2b} the solutions satisfy $u''(0) > 0$ and are symmetric with respect to η_1. Thus, both branches consist of 2-lap solutions. The existence of these solutions is proved in Theorem 6.33.

Corresponding results for 3-lap solutions are presented in Figures 6.11 and 6.12. Solutions on the upper branch Γ_{3a} satisfy $u''(0) > 0$ and are symmetric with respect to ξ_2. On the lower branch Γ_{3b} the solutions satisfy $u''(0) < 0$ and are symmetric with respect to η_2. Thus both branches consist of 3-lap solutions. The existence of these solutions is proved in Theorem 6.36.

It is interesting to note the difference in the local behaviour of the solution branches near the points $(q_2, 1)$ and $(q_3, 1)$ in the (q, M)-plane (see Figures 6.10 and 6.21). Although a detailed analysis of this local behaviour is beyond the scope of this chapter, we do present a local analysis of the branches Γ_{2a} and Γ_{2b} near $(q_2, 1)$. This yields the angles θ_a and θ_b between the branches Γ_{2a} and Γ_{2b} and the q axis at $(q_2, 1)$. They are given by

$$\tan\theta_a = \frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta_b = -\frac{\sqrt{2}}{3}.$$

In addition to the branches of n-lap solutions extending over $(-\sqrt{8}, q_n)$, which bifurc-

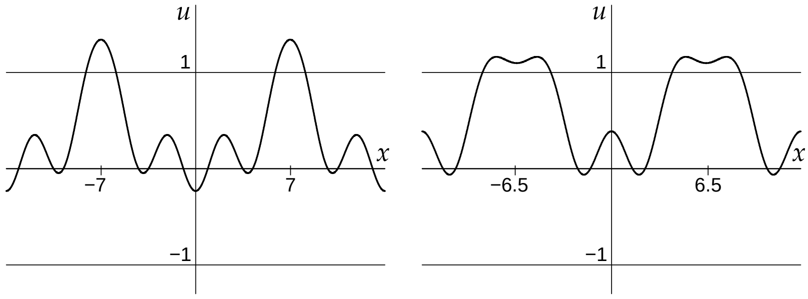

Figure 6.11: Two even periodic 3-lap solutions at $q = 2$: (a) on the upper branch Γ_{3a}; (b) on the lower branch Γ_{3b}. See Figure 6.12 for a picture of the complete branches $\Gamma_{3a,b}$.

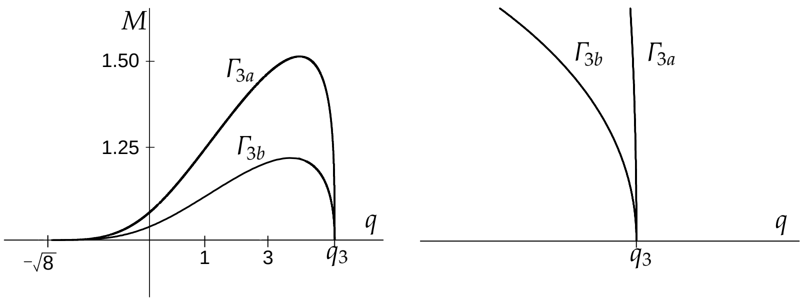

Figure 6.12: Branches of even periodic 3-lap solutions, and a blowup at q_3.

ate at $q = -\sqrt{8}$ and at $q = q_n$, it is possible to construct branches of even periodic solutions which bifurcate at $q = -\sqrt{8}$ and at $q = q_{m,n}$, where

$$q_{m,n} = \sqrt{2}\Big(\frac{n}{m} + \frac{m}{n}\Big), \quad m, n \geq 1. \tag{6.6}$$

In this chapter we do not study these solution exhaustively, but merely prove the existence of two branches of 3-lap periodic solutions which connect $q = -\sqrt{8}$ and $q = q_{2,3}$. This is done in Theorem 6.37.

As an example of the general phenomenon we present in Figures 6.13 and 6.14 solutions which lie on the branches that bifurcate at $q = q_{m,5}$ for $m = 1, 2, 3, 4$. In each case only solutions on one of the two branches bifurcating from each bifurcation point are shown. All depicted solutions are at $q = 2$. Without going into details, we observe that n is the number of monotone laps, while m is the number of laps that cross the constant solution $u = 1$ (between two points of symmetry).

The organisation of the chapter is the following. In Section 6.2 we introduce some notation and recall the important properties of critical points obtained in earlier papers [117, 118, 119, 120, 121]. In Section 6.3 we establish the existence of several families of odd periodic solutions, some of which exist for all $q > -\sqrt{8}$ and some on finite q-intervals only. In Section 6.4 we begin our analysis of even periodic solutions with a discussion of the interval $-\sqrt{8} < q < \sqrt{8}$. After a brief section on the local behaviour of solutions near $u = 1$, this study is continued in Section 6.6 for values of q in the interval $(-\sqrt{8}, q_3)$. In this interval it is possible to obtain information about the location of the critical values of the solution graphs thanks to a Comparison Lemma which is valid for $-\sqrt{8} < q < q_3$. In Section 6.7 we study the local behaviour of solution branches near q_2, and in Section 6.8 we establish of families of periodic solutions with an arbitrary large number of laps (The-

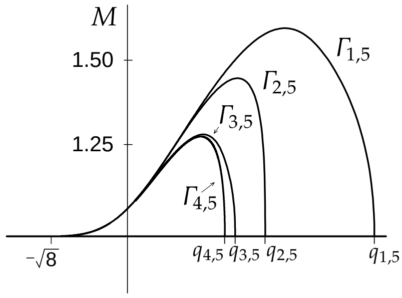

Figure 6.13: Four branches of periodic 5-lap solutions. See Figure 6.14 for the shape of the solutions.

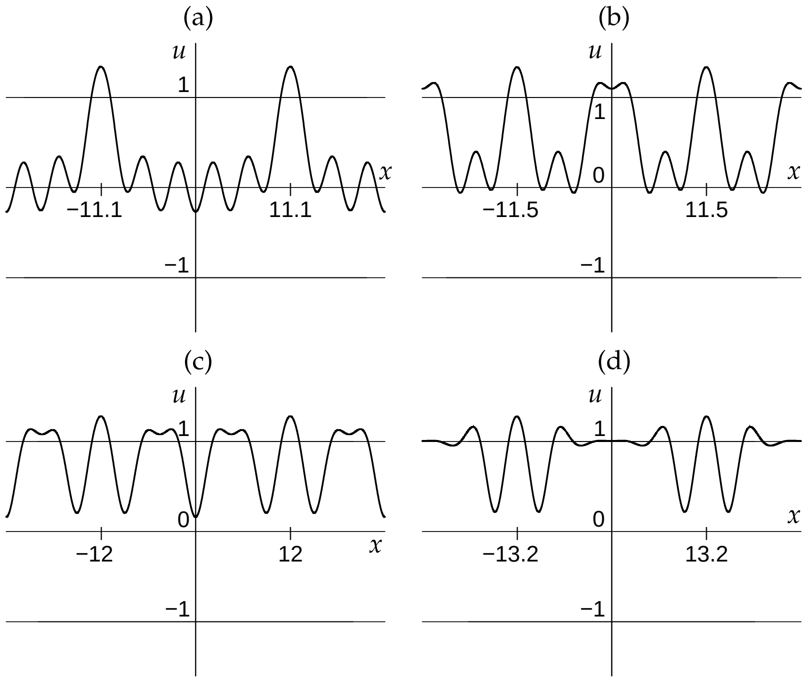

Figure 6.14: Periodic 5-lap solutions on branches (see Figure 6.13) that bifurcate at $q = q_{m,5}$ for (a) $m = 1$; (b) $m = 2$; (c) $m = 3$; (d) $m = 4$. All solutions are at $q = 2$.

orem 6.3). In Section 6.9 we conclude with the rather technical proof of the Comparison Lemma used in Section 6.6.

6.2 Critical points

To establish the existence of new families of periodic solutions we shall further develop the topological shooting method established in [120, 118, 119, 121]. For this, we begin with a summary of the key properties of critical points in Lemmas 6.4, 6.6 and 6.9. Then, in Lemma 6.12, we prove a new global result. Finally, Lemmas 6.13 and 6.14 summarise two previously obtained global results. These properties will allow us to extend our shooting method and enables us to obtain further families of solutions and more detailed information about their qualitative properties.

The solutions we discuss in this chapter will be either even or odd, and thus we shall study Equation (6.1), which we restate here for convenience:

$$u^{(iv)} + qu'' + u^3 - u = 0, \tag{6.7}$$

and supply appropriate initial conditions. When looking for *odd* solutions we impose the conditions

$$u(0) = 0, \quad u'(0) = \alpha, \quad u''(0) = 0, \quad u'''(0) = \beta, \tag{6.8a}$$

and when looking for *even* solutions we set

$$u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0. \tag{6.8b}$$

In both cases we shall assume that the first integral is zero, i.e.

$$\mathcal{E}[u] \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + \frac{1}{4}(u^2-1)^2 = 0. \tag{6.9}$$

This means that the constants α and β are related by

$$\beta \stackrel{\text{def}}{=} \beta(\alpha) = \begin{cases} -\dfrac{q\alpha}{2} - \dfrac{1}{4\alpha} \quad (\alpha \neq 0) & \text{for odd solutions,} \qquad (6.10a) \\ \pm\dfrac{1}{\sqrt{2}}|\alpha^2 - 1| & \text{for even solutions.} \qquad (6.10b) \end{cases}$$

For brevity we denote problem (6.7), (6.8a), (6.9), (6.10a) by *Problem A* and problem (6.7), (6.8b), (6.9), (6.10b) by *Problem B*. We observe that if u is a solution of Equation (6.7), then so is $-u$. Therefore, when discussing odd solutions of Problem A, we restrict our attention to solutions with a *positive* initial slope, i.e. $\alpha > 0$.

For any given $\alpha \in \mathbb{R}^+$ there exists a unique local solution of Problem A and for any given $\alpha \in \mathbb{R}$ there exists a unique local solution of Problem B. In both cases we denote it by $u(x, \alpha)$. The critical points of the solution graphs of $u(x, \alpha)$, that is the zeros of $u'(x, \alpha)$, will play a pivotal role in the construction and classification of the different families of periodic solutions. Below we summarise the most important properties of these points. They were derived in [117, 118, 119, 120, 121].

We begin with a preliminary lemma which implies that all critical points are isolated.

Lemma 6.4 ([120]) *Suppose that u is a non-constant solution of* (6.1) *such that $\mathcal{E}[u] = 0$, and that $u'(x_0) = 0$ at some $x_0 \in \mathbb{R}$.*

(a) *If $u''(x_0) = 0$ then $u(x_0) = \pm 1$ and $u'''(x_0) \neq 0$.*

(b) *If $u(x_0) = \pm 1$ then $u''(x_0) = 0$ and $u'''(x_0) \neq 0$.*

Lemma 6.4 implies that, unless u is a constant solution, we can number the critical points of the graph of $u(x, \alpha)$. We denote the positive local maxima by ξ_k and the minima by η_k with $k = 1, 2, \ldots$. At inflection points these points coincide. To start the sequences, we need to distinguish two cases:

$$(i)\ u' > 0 \text{ in } (0, \delta) \qquad \text{and} \qquad (ii)\ u' < 0 \text{ in } (0, \delta),$$

for some small $\delta > 0$.

Case (i): When $u' > 0$ in a right-neighbourhood of the origin, we define

$$\xi_1 = \sup\{x > 0 \,|\, u' > 0 \text{ on } [0, x)\}. \tag{6.11a}$$

If $u''(\xi_1) < 0$ we set

$$\eta_1 = \sup\{x > \xi_1 \,|\, u' < 0 \text{ on } (\xi_1, x)\}. \tag{6.11b}$$

When $u''(\xi_1) = 0$, and so $u(\xi_1) = 1$ by Lemma 6.4, we set

$$\eta_1 = \xi_1. \tag{6.11c}$$

Case (*ii*): When $u' < 0$ in a right-neighbourhood of the origin, we *skip* ξ_1 and define

$$\eta_1 = \sup\{x > 0 \,|\, u' < 0 \text{ on } [0, x)\}. \tag{6.11d}$$

In (6.11) we have defined the first terms in the sequences $\{\xi_k\}$ and $\{\eta_k\}$ in both cases. We can now continue formally to larger values of k. For $k \geq 2$ we define

$$\xi_k = \begin{cases} \sup\{x > \eta_{k-1} \,|\, u' > 0 \text{ on } (\eta_{k-1}, x)\} & \text{if } u' > 0 \text{ in } (\eta_{k-1}, \eta_{k-1} + \delta_1), \\ \eta_{k-1} & \text{otherwise,} \end{cases} \tag{6.12a}$$

where δ_1 is some small positive number. Similarly, we set

$$\eta_k = \begin{cases} \sup\{x > \xi_k \,|\, u' < 0 \text{ on } (\xi_k, x)\} & \text{if } u' < 0 \text{ in } (\xi_k, \xi_k + \delta_2), \\ \xi_k & \text{otherwise,} \end{cases} \tag{6.12b}$$

in which δ_2 is some small positive number. It is readily seen that

$$\xi_k \leq \eta_k \leq \xi_{k+1}, \qquad k \geq 1. \tag{6.13}$$

We will make extensive use of the following observation:

Remark 6.5 If u is a non-constant solution, then one of the inequalities in (6.13) must be strict. To see this, suppose that $\xi_k = \eta_k$. Then, because the zeros of u' are isolated by Lemma 6.4, it follows from (6.12b) that $u' > 0$ in a right-neighbourhood of ξ_k, so that u has an inflection point at ξ_k, where $u''' > 0$. Hence by (6.12a) $\eta_k < \xi_{k+1}$. On the other hand, if $\eta_k = \xi_{k+1}$, then by Lemma 6.4 and (6.12a) $u' < 0$ in a right-neighbourhood of ξ_k, and u has an inflection point at η_k, where $u''' < 0$. Therefore by (6.12b) $\xi_k < \eta_k$. In particular, this implies that $\xi_k < \xi_{k+1}$ and $\eta_k < \eta_{k+1}$ for every $k \geq 1$. •

In the following lemma we present the important continuity properties of the critical points. In particular, we emphasise that, as α changes, critical points are preserved and cannot disappear by coalescing with one another.

Lemma 6.6 ([118, 120]) *Suppose that $q > -\sqrt{8}$. For every $\alpha \in I$, where $I = \mathbb{R}^+$ in Problem A, and $I = \mathbb{R} \setminus \{-1, +1\}$ in Problem B, and for every $k \geq 1$,*

(a) *$\xi_k(\alpha) < \infty$ and $\eta_k(\alpha) < \infty$;*

(b) *$u'(\xi_k(\alpha), \alpha) = 0$ and $u'(\eta_k(\alpha), \alpha) = 0$;*

(c) *$\xi_k \in C(I)$ and $\eta_k \in C(I)$.*

Remark 6.7 In [118, 120], Lemma 6.6 has been proved for solutions of Problem A. For solutions of Problem B the proof is similar (see also [121]). •

Remark 6.8 For odd solutions we have the following result: if $q \leq -\sqrt{8}$, then there exists a unique value $\alpha_0 > 0$ for which the corresponding solution $u(x, \alpha_0)$ tends monotonically to 1 (the *kink*), so that $\xi_1(\alpha_0) = \infty$ and the sequence $\{\xi_k\}$ is not well defined [117]. •

In order to proceed with the construction of new families of periodic solutions with complex structure, we need to determine the precise local behaviour of $u(\xi_k)$ and $u(\eta_k)$ when they cross the level $u = 1$ or $u = -1$ as α changes. This is the subject of the next lemma.

Lemma 6.9 ([118, 121]) *Let $q \in \mathbb{R}$. Suppose that for some $k \geq 1$*

$$u(\xi_k) = 1 \quad \textit{and} \quad u''(\xi_k) = 0 \qquad \textit{at } \alpha = \alpha^*,$$

and for some $\delta > 0$

$$u(\xi_k(\alpha), \alpha) > 1 \quad \text{for } \alpha^* < \alpha < \alpha^* + \delta.$$

(a) *If* $u'''(\xi_k) > 0$ *at* α^**, then there exists an* $\varepsilon > 0$ *such that*

$$u(\xi_k(\alpha), \alpha) > u(\eta_k(\alpha), \alpha) > 1 \quad \text{for } \alpha^* < \alpha < \alpha^* + \varepsilon.$$

(b) *If* $u'''(\xi_k) < 0$ *at* α^**, then there exists an* $\varepsilon > 0$ *such that*

$$u(\xi_k(\alpha), \alpha) > u(\eta_{k-1}(\alpha), \alpha) > 1 \quad \text{for } \alpha^* < \alpha < \alpha^* + \varepsilon.$$

Remark 6.10 In [118], Part (a) of Lemma 6.9 was first proved for $q \le 0$, and in [121] this restriction on q was subsequently removed. The proof of Part (b) is completely analogous to that of Part (a). A similar result applies when $u(\xi_k)$ and $u(\eta_k)$ cross the line $u = +1$ from below, or when $u(\xi_k)$ and $u(\eta_k)$ cross the line $u = -1$ from above or below. •

Remark 6.11 It is clear from Lemma 6.9 that critical values cross the lines $u = \pm 1$ *in pairs*. This is a property which we shall very much exploit in Section 6.8. •

The next three lemmas give important *global* properties of solutions of Equation (6.1). The first one applies to solutions which have a critical point on the line $u = 1$ or on $u = -1$. Thus, let u be a solution of Equation (6.1), and let $a \in \mathbb{R}$ be a critical point where u has the following properties:

$$u(a) = 1, \quad u'(a) = 0, \quad u''(a) = 0 \quad \text{and} \quad u'''(a) > 0. \tag{6.14}$$

Then $u' > 0$ in a right-neighbourhood of a so that the point

$$b = \sup\{x > a \,|\, u' > 0 \text{ on } (a, x)\}$$

is well defined. By Lemma 6.6 it is also finite. We now derive some properties of u and its derivatives at b.

Lemma 6.12 *Suppose that*

$$-\sqrt{8} < q < q_3 = \sqrt{2}\Big(3 + \frac{1}{3}\Big).$$

Let u *be a solution of Equation* (6.1) *which at a point* $a \in \mathbb{R}$ *has the properties listed in* (6.14)*. Then*

$$u(b) > 1, \quad u'(b) = 0, \quad u''(b) < 0 \quad \text{and} \quad u'''(b) < 0. \tag{6.15}$$

The proof of Lemma 6.12 is given in Section 6.9. This result will play an important role in the analysis of n-lap periodic solutions given in Section 6.6.

The second lemma applies to solutions for which $0 \le u \le \frac{1}{\sqrt{3}}$ at a critical point, and yields properties of the subsequent maxima and minima. We emphasise that it is only valid for *non-positive* values of q.

Lemma 6.13 ([119]) *Suppose that* $-\sqrt{8} < q \le 0$*. Let* u *be a solution of Equation* (6.1) *on* $\mathbb{R}$ *such that* $\mathcal{E}[u] = 0$*, and let for some* $a \in \mathbb{R}$

$$0 \le u(a) \le \frac{1}{\sqrt{3}}, \quad u'(a) = 0, \quad u''(a) > 0 \quad \text{and} \quad u'''(a) \ge 0.$$

Then

$$|u| > \sqrt{2} \quad \text{whenever} \quad u' = 0 \quad \text{on } (a, \omega).$$

Here $[a, \omega)$ *is the maximal interval in* $[a, \infty)$ *on which* u *exists.*

We conclude with a universal bound for bounded solutions.

Lemma 6.14 ([119] or Lemma 2.27) *Suppose that $-\sqrt{8} < q \leq 0$. Let u be a solution of Equation* (6.1) *which is uniformly bounded on $\mathbb{R}$. Then*

$$\|u\|_\infty < \sqrt{2}.$$

6.3 Odd periodic solutions

In this section we investigate the existence and qualitative properties of odd periodic solutions of Equation (6.1) for which $\mathcal{E}[u] = 0$. In previous studies (cf. Chapter 2 and [120]) it was shown that for $q \leq -\sqrt{8}$ there are no such odd *zero energy* periodic solutions. However, for $q > -\sqrt{8}$ odd zero energy periodic solutions do exist. In fact, it was proved in [120] that as q increases from $-\sqrt{8}$, two families of odd, single bump periodic solutions emerge as the result of a bifurcation from the unique increasing kink φ which exists at $q = -\sqrt{8}$, and it was shown in [120] and [106] that they continue to exist for all $q > -\sqrt{8}$. A precise description of these results is given below in Theorem 6.15. As we shall see, these families of single bump periodic solutions will form a basis for our topological shooting arguments, which lead to the construction of multi-bump periodic solutions with a more complicated structure.

Theorem 6.15 *For every $q > -\sqrt{8}$ there exist two odd periodic solutions u_+ and u_- of Equation* (6.1) *such that $\mathcal{E}[u_\pm] = 0$, with the following properties:*

(a) $\|u_+\|_\infty > 1$ *and* $\|u_-\|_\infty < 1$.

(b) *If $u_\pm(a) = 0$ for some $a \in \mathbb{R}$, then $u_\pm(a - y) = -u_\pm(a + y)$ for $y \in \mathbb{R}$.*

(c) *If $u_\pm'(a) = 0$ for some $a \in \mathbb{R}$, then $u_\pm(a - y) = u_\pm(a + y)$ for $y \in \mathbb{R}$.*

By way of convention we choose the origin such that $u_\pm'(0) > 0$. It was also shown in [120] that, as q decreases to $-\sqrt{8}$, both families of periodic solutions tend to the unique odd increasing kink φ at $q = -\sqrt{8}$: $u_\pm(\cdot, q) \to \varphi$ as $q \to -\sqrt{8}$ uniformly on compact sets. On the other hand, as q tends to infinity, the small amplitude solutions u_- tend to zero uniformly on $\mathbb{R}$, while the amplitude of the large solutions u_+ tends to infinity. More specifically,

$$u_-(x, q) \sim \frac{1}{q\sqrt{2}} \sin(x\sqrt{q}) \qquad \text{as } q \to \infty, \tag{6.16a}$$

and

$$u_+(x, q) \sim q\, V(x\sqrt{q}) \qquad \text{as } q \to \infty, \tag{6.16b}$$

where V is an odd solution of the equation

$$v^{(iv)} + v'' + v^3 = 0,$$

which possesses the symmetry properties listed in Theorem 6.15, and

$$\max\{|V(t)| : t \in \mathbb{R}\} \in \Big(0, \frac{1}{2\sqrt{2}}\Big).$$

At present it is not known whether the solution V is unique. Thus, the convergence in (6.16b) is along sequences, and the function V may possibly depend on the choice of the sequence.

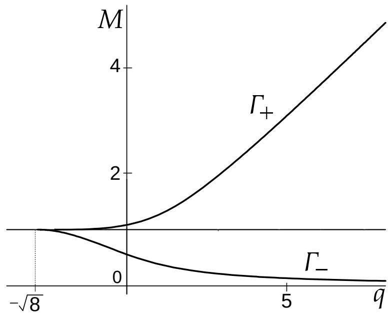

Figure 6.15: The branches Γ_+ and Γ_- of odd single bump periodic solutions.

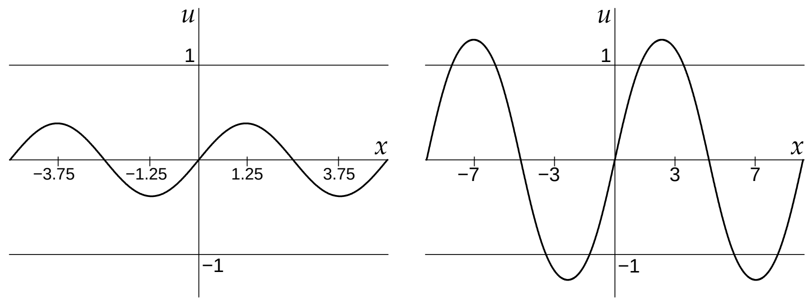

Figure 6.16: Small and large single bump periodic solutions ($q = 1$).

A numerically obtained plot of the two branches Γ_+ and Γ_- of odd single bump periodic solutions $\Gamma_\pm = \{u_\pm(\cdot, q) : q > -\sqrt{8}\}$ is presented in Figure 6.15. Along the vertical axis we put $M = \|u\|_\infty$. In Figure 6.16 we give graphs of solutions on the branches Γ_+ and Γ_- at $q = 1$. As mentioned earlier, the two families of single bump periodic solutions will be used to construct further families of periodic solutions. In the following theorem we present a family which also exists on the entire interval $-\sqrt{8} < q < \infty$. These solutions look like u_+ in that their local maxima lie *above* $u = +1$ and all the local minima lie *below* $u = -1$, with the exception of the point of symmetry $x = \zeta$. At that point these extrema lie on the 'wrong' side of the constant solution $u = +1$, in case of a maximum, or $u = -1$ in case of a minimum:

$$\begin{aligned} u(\zeta) &< -1 \qquad \text{if } \zeta = \xi_k \text{ for some } k \geq 2, \\ u(\zeta) &> +1 \qquad \text{if } \zeta = \eta_k \text{ for some } k \geq 1. \end{aligned}$$

We denote by T_N ($N \geq 2$) the branch of odd periodic solutions of this family, of which the N^{th} critical point is the first point of symmetry:

$$T_N = \{u(\cdot, q) \,|\, q > -\sqrt{8},\, u'(\zeta_N, q) = 0,\, u'''(\zeta_N, q) = 0\}.$$

The branches T_2 and T_3, as well as solutions on these branches at $q = 1.5$, are presented in Figure 6.6. The existence of this family is the content of the next theorem.

Theorem 6.16 *Let* $q > -\sqrt{8}$.

(a) *For each* $N \geq 1$ *there exists an odd periodic solution* u *of Equation* (6.1) *such that* $\mathcal{E}[u] = 0$ *and* $u'(0) > 0$*, which is symmetric with respect to* η_N*, and has the properties*

$$\begin{aligned} u(\xi_k) &> 1 \qquad \textit{for } 1 \leq k \leq N \\ u(\eta_k) &< -1 \qquad \textit{for } 1 \leq k \leq N-1 \;\; (N \geq 2) \end{aligned} \qquad \textit{and} \quad u(\eta_N) > 1.$$

(b) *For each $N \geq 2$ there exists an odd periodic solution u of Equation* (6.1)*, such that $u'(0) > 0$, which is symmetric with respect to ξ_N, and has the properties*

$$\begin{matrix} u(\xi_k) > 1 & \text{for } 1 \leq k \leq N-1 \\ u(\eta_k) < -1 & \text{for } 1 \leq k \leq N-1 \end{matrix} \qquad \text{and} \quad u(\xi_N) < -1.$$

Proof. We use an iterative type of argument, and begin by proving the existence of a periodic solution which is symmetric with respect to η_1. This is the case $N = 1$ of Part (a). Such a solution is illustrated in Figure 6.6b.

We denote the initial slopes of the zero energy odd single bump periodic solutions u_+ and u_-, constructed in Theorem 6.15, by α_+ and α_- respectively, i.e. $\alpha_\pm = u'_\pm(0)$. From the construction in [120] we know that $0 < \alpha_- < \alpha_+$. Plainly (see Figure 6.16),

$$u(\xi_k) > 1 \text{ and } u(\eta_k) < -1 \qquad \text{for } k \geq 1 \qquad \text{when } \alpha = \alpha_+, \tag{6.17a}$$

$$0 < u(\xi_k) < 1 \text{ and } -1 < u(\eta_k) < 0 \qquad \text{for } k \geq 1 \qquad \text{when } \alpha = \alpha_-, \tag{6.17b}$$

where $u(\xi_k) = u(\xi_k(\alpha), \alpha)$ and $u(\eta_k) = u(\eta_k(\alpha), \alpha)$. In view of (6.17a) we can define

$$a_1 = \inf\{\alpha > 0 \,|\, u(\xi_1) > 1 \text{ on } (\alpha, \alpha_+)\},$$

and it follows from (6.17b) that $a_1 \in (\alpha_-, \alpha_+)$. By Lemma 6.6, the location of the first critical point $\xi_1(\alpha)$ depends continuously on α, and by standard theory the solution $u(x, \alpha)$ of problem (6.7), (6.8) depends continuously on α for x in compact sets. Since $\mathcal{E}[u] = 0$, it follows from Lemma 6.4 that

$$u(\xi_1) = 1, \quad \eta_1 = \xi_1 \quad \text{and} \quad u'''(\xi_1) > 0 \qquad \text{if } \alpha = a_1. \tag{6.18}$$

By Lemma 6.9 this implies that $u(\eta_1(\alpha), \alpha) > 1$ for $\alpha \in (a_1, a_1 + \delta)$, where $\delta > 0$ is a small positive constant. Hence, we can define

$$a_1^+ = \sup\{\alpha > a_1 \,|\, u(\eta_1) > 1 \text{ on } (a_1, \alpha)\}.$$

As we saw in (6.17a), $u(\eta_1) < -1$ at α_+, so that $a_1^+ \in (a_1, \alpha_+)$. Invoking the continuity of $\eta_1(\alpha)$ and $u(x, \alpha)$, we deduce that

$$u(\eta_1) = 1, \quad \eta_1 = \xi_2 \quad \text{and} \quad u'''(\eta_1) < 0 \qquad \text{if } \alpha = a_1^+. \tag{6.19}$$

Using the continuity of $\eta_1(\alpha)$ and of u and its derivatives, we see that (6.18) and (6.19) imply that there must exist a point $\alpha_1^* \in (a_1, a_1^+)$ where $u'''(\eta_1)$ vanishes, and so

$$u'(\eta_1(\alpha_1^*), \alpha_1^*) = 0 \quad \text{and} \quad u'''(\eta_1(\alpha_1^*), \alpha_1^*) = 0.$$

This means that the solution $u(x, \alpha_1^*)$ is symmetric with respect to $\eta_1(\alpha_1^*)$. Since it is also odd with respect to the origin, we conclude that $u(x, \alpha_1^*)$ is a periodic solution with period $4\eta_1$. It is readily verified that it has the desired properties.

We continue with the construction of the periodic solution which is symmetric with respect to ξ_2. This is the case $N = 2$ of Part (b), and such a solution is illustrated in Figure 6.6c.

Because $u(\eta_1) < -1$ at α_+, we can define

$$b_1 = \inf\{\alpha < \alpha_+ \,|\, u(\eta_1) < -1 \text{ on } (\alpha, \alpha_+)\},$$

and it follows from (6.18) that $b_1 \in (a_1^+, \alpha_+)$. Like at a_1^+, we once again invoke the continuity of η_1 and u and the fact that $\mathcal{E}[u] = 0$ to conclude from Lemma 6.4 that

$$u(\eta_1) = -1, \quad \eta_1 = \xi_2 \quad \text{and} \quad u'''(\eta_1) < 0 \qquad \text{at } b_1.$$

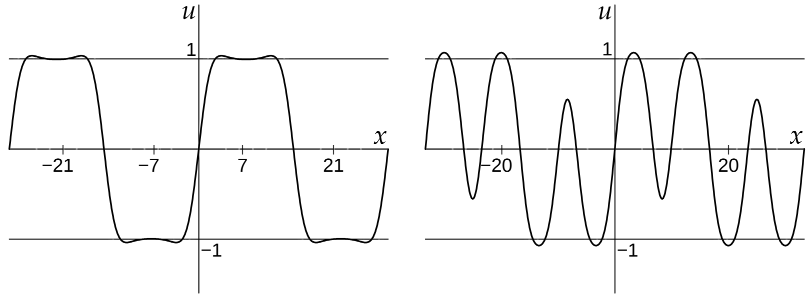

Figure 6.17: Solutions symmetric with respect to η_1 from Theorem 6.17a; $N = 1, q = -\frac{1}{10}$.

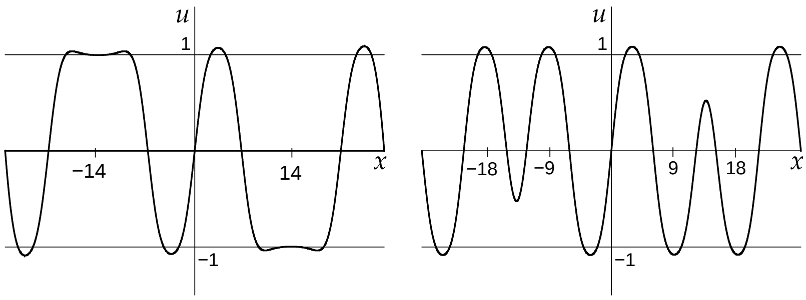

Figure 6.18: Solutions symmetric with respect to ξ_2 from Theorem 6.17b; $N = 2, q = -\frac{1}{10}$.

By a result similar to Lemma 6.9 we find that $u(\xi_2) < -1$ in an interval $(b_1, b_1 + \delta)$, where $\delta > 0$ is sufficiently small. Thus, we can define

$$b_1^+ = \sup\{\alpha > b_1 \mid u(\xi_2) < -1 \text{ on } (b_1, \alpha)\}.$$

Remembering (6.17a), we see that $b_1^+ \in (b_1, \alpha_+)$, and using the continuity properties of ξ_2 and u, we conclude that

$$u(\xi_2) = -1, \quad \xi_2 = \eta_2 \quad \text{and} \quad u'''(\xi_2) > 0 \qquad \text{at } b_1^+.$$

Another application of the continuity of ξ_2 and u and its derivatives implies the existence of a point $b_1^* \in (b_1, b_1^+)$ such that

$$u(\xi_2) < -1, \quad u'(\xi_2) = 0 \quad \text{and} \quad u'''(\xi_2) = 0 \qquad \text{at} \quad b_1^*.$$

As in the previous case, this means that $u(x, b_1^*)$ is a periodic solution with period $4\xi_2$. Recall that $b_1^* > a_1$, so that $u(\xi_1) > 1$. Thus, this solution has the desired properties.

Continuing in this manner, we successively prove the existence of all the periodic solutions listed in Theorem 6.16. □

In addition to these branches of solutions, which exist for all $q > -\sqrt{8}$, there exists a multitude of odd zero energy periodic solutions for $-\sqrt{8} < q \le 0$. In Theorems 6.17 and 6.19 we present a few of these families. They exist in pairs. Those corresponding to Theorem 6.17 are shown in Figures 6.17 and 6.18, and those obtained in Theorem 6.19 in Figure 6.20. A numerical study shows that the solutions obtained in Theorem 6.17 lie on branches which are loop shaped. The branch for Part (a) is shown in Figure 6.19.

Theorem 6.17 *Let* $-\sqrt{8} < q \le 0$.

(a) *For each* $N \ge 1$ *there exist two odd periodic solutions* u_1 *and* u_2 *of Equation* (6.1) *such that* $\mathcal{E}[u_i] = 0$ *and* $u_i'(0) > 0$ *(*$i = 1, 2$*), which are symmetric with respect to* η_N,

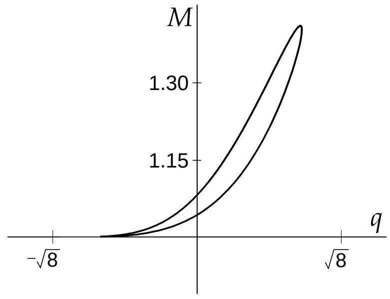

Figure 6.19: Branch of solutions as constructed in Theorem 6.17a for $N = 1$

and have the properties

$$\begin{aligned} u_1(\xi_k) > 1, \quad u_2(\xi_k) > 1 \qquad &\textit{for} \quad 1 \le k \le N \\ u_1(\eta_k) < -1, \quad u_2(\eta_k) < -1 \qquad &\textit{for} \quad 1 \le k \le N-1 \ (\textit{if } N \ge 2) \\ -1 < u_1(\eta_N) < 0 < u_2(\eta_N) < 1. & \end{aligned}$$

(b) *For each $N \ge 2$ there exist two odd periodic solutions u_1 and u_2 of Equation* (6.1)*, such that $\mathcal{E}[u_i] = 0$ and $u_i'(0) > 0$ ($i = 1, 2$), which are symmetric with respect to ξ_N, and have the properties*

$$\begin{aligned} u_1(\xi_k) > 1, \quad u_2(\xi_k) > 1 \qquad &\textit{for} \quad 1 \le k \le N-1 \\ u_1(\eta_k) < -1, \quad u_2(\eta_k) < -1 \qquad &\textit{for} \quad 1 \le k \le N-1 \\ -1 < u_1(\xi_N) < 0 < u_2(\xi_N) < 1. & \end{aligned}$$

Proof. We pick up the line of argument in the proof of Theorem 6.16, and consider the interval $[a_1, b_1]$. We recall that

$$u(\eta_1) = 1 \quad \text{at } a_1 \qquad \text{and} \qquad u(\eta_1) = -1 \quad \text{at } b_1.$$

By continuity this implies that $u(\eta_1)$ has a zero on (a_1, b_1). Let c_1 be the smallest zero of $u(\eta_1)$ on (a_1, b_1), and let

$$c_1^- = \inf\{\alpha < c_1 \,|\, u(\eta_1) < 1 \text{ on } (\alpha, c_1)\}. \tag{6.20a}$$

Similarly, let d_1 be the largest zero of $u(\eta_1)$ on (a_1, b_1), and let

$$d_1^+ = \sup\{\alpha > d_1 \,|\, u(\eta_1) > -1 \text{ on } (d_1, \alpha)\}. \tag{6.20b}$$

Plainly, $c_1^- \in [a_1^+, c_1)$ and $d_1^+ \in (d_1, b_1]$, and

$$u(\eta_1) = 1 \quad \text{at } c_1^- \qquad \text{and} \qquad u(\eta_1) = -1 \quad \text{at } d_1^+.$$

Since $u(\xi_1) > 1$ on $(a_1, b_1]$, it follows that $u'''(\eta_1) < 0$ at c_1^- as well as at d_1^+.

We deduce from Lemma 6.13 that $u'''(\eta_1) > 0$ at c_1 and at d_1. For suppose to the contrary that $u'''(\eta_1) \le 0$ at c_1 or at d_1. Then we deduce from Lemma 6.13 that $u > \sqrt{2}$ at every critical point on the interval $[-\eta_1, \eta_1]$. But $u(-\eta_1) = 0$ because u is odd, a contradiction. Thus, $u'''(\eta_1)$ changes sign on (c_1^-, c_1) and on (d_1, d_1^+), so that there exist a point $c_1^* \in (c_1^-, c_1)$ and a point $d_1^* \in (d_1, d_1^+)$ such that

$$u'''(\eta_1) = 0 \quad \text{at } c_1^* \text{ and } d_1^*.$$

Writing $u_1(x) = u(x, d_1^*)$ and $u_2(x) = u(x, c_1^*)$, we conclude that u_1 and u_2 are odd periodic solutions which are symmetric with respect to η_1, and that they have the properties

$$u_1(\xi_1) > 1, \quad u_2(\xi_1) > 1 \quad \text{and} \quad -1 < u_1(\eta_1) < 0 < u_2(\eta_1) < 1.$$

This completes the proof of the case $N = 2$ of Part (a).

Next, we construct a pair of odd periodic solutions which are symmetric with respect to ξ_2. This corresponds to the case $N = 2$ of Part (b). To this end, we define

$$a_2 = \inf\{\alpha < \alpha_+ \mid u(\xi_2) > 1 \text{ on } (\alpha, \alpha_+)\}.$$

Plainly, $b_1 < a_2 < \alpha_+$. Because $u(\xi_2) = -1$ at b_1 and $u(\xi_2) = +1$ at a_2, $u(\xi_2)$ has a zero on (b_1, a_2). Let e_1 be the smallest zero of $u(\xi_2)$ on (b_1, a_2) and f_1 the largest. Then, proceeding as in the previous case we show that $u'''(\xi_2)$ has two zeros, e_1^* and f_1^*, on (b_1^+, a_2) such that $u_1(x) = u(x, e_1^*)$ and $u_2(x) = u(x, f_1^*)$ are periodic solutions with the following properties:

$$u_i(\xi_1) > 1 \text{ and } u_i(\eta_1) < -1 \quad (i = 1, 2), \qquad \text{and} \qquad -1 < u_1(\xi_2) < 0 < u_2(\xi_2) < 1.$$

This completes the proof of the case $N = 2$ of Part (b).

For the next step we define

$$b_2 = \inf\{\alpha < \alpha_+ \mid u(\eta_2) < -1 \text{ on } (\alpha, \alpha_+)\}.$$

Since $u(\eta_2) = 1$ at a_2, it follows that $b_2 > a_2$. By repeating the arguments applied to the interval $[a_1, b_1]$ to $[a_2, b_2]$, we prove Part (a) for $N = 2$. We can continue this process indefinitely, and so successively prove all the cases in Parts (a) and (b) of Theorem 6.17. □

Remark 6.18 Using a continuity argument, we can show that the families of periodic solutions described in Theorem 6.17 continue to exist for small values of $q > 0$. This is consistent with the numerically computed bifurcation branch shown in Figure 6.19 for the case $N = 1$ of Part (a). •

In the next theorem we obtain a different family of periodic solutions. To explain the difference, let u be a periodic solution which is symmetric with respect to its n^{th} critical point ζ_n. Then solutions of Theorem 6.17 have the property that

$$|u(\zeta_k)| > 1 \quad \text{for } 1 \le k \le n-1 \qquad \text{and} \qquad |u(\zeta_n)| < 1.$$

In contrast, the solutions of Theorem 6.19 have the property that

$$|u(\zeta_k)| > 1 \quad \text{for } 1 \le k \le n-2 \qquad \text{and} \qquad |u(\zeta_{n-1})| < 1,\ |u(\zeta_n)| < 1.$$

Thus, whereas in the first family, the point of symmetry is the only critical point at which $u \in (-1, 1)$, in the second family the value of u at the point of symmetry, *as well as at the two adjacent critical points*, lie in the interval $(-1, 1)$. A pair of such solutions is shown in Figure 6.20. Since the characteristics of these solutions are not very clear when $q \le 0$, we have set $q = 1.5$.

Theorem 6.19 *Let* $-\sqrt{8} < q \le 0$.

(a) *For each* $N \ge 2$ *there exist two odd periodic solutions* u_1 *and* u_2 *of Equation* (6.1) *such that* $\mathcal{E}[u_i] = 0$ *and* $u_i'(0) > 0$ $(i = 1, 2)$, *which are symmetric with respect to* ξ_N, *and have the following properties:*

$$\left.\begin{array}{lll} u_i(\xi_k) > 1 & \textit{for} & 1 \le k \le N-1 \\ u_i(\eta_k) < -1 & \textit{for} & 1 \le k \le N-2 \ (\textit{if } N \ge 3) \end{array}\right\} \quad \textit{for } i = 1, 2,$$

and

$$0 < u_i(\xi_N) < 1 \quad \textit{for } i = 1, 2,$$
$$-1 < u_1(\eta_{N-1}) < 0 < u_2(\eta_{N-1}) < 1.$$

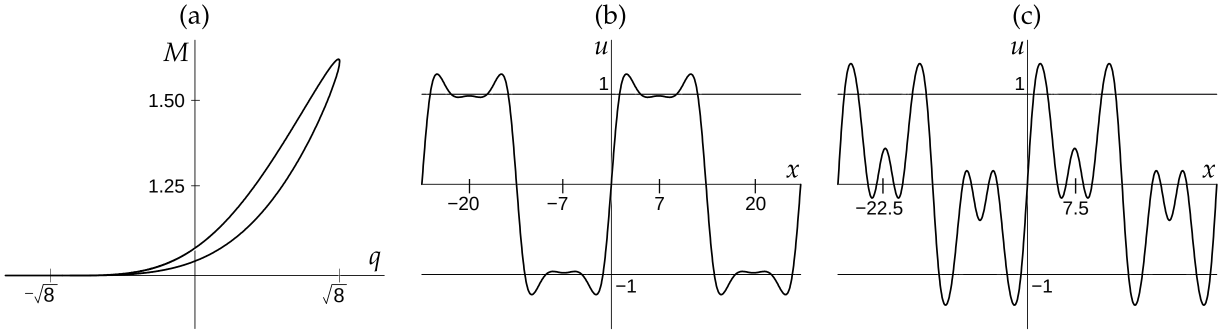

Figure 6.20: Solutions symmetric with respect to ξ_2 from Theorem 6.19a with $N = 2$. The solutions in (b) and (c) are at $q = 1.5$, the one in (b) being on the lower part of the loop.

(b) *For each* $N \geq 2$ *there exist two odd periodic solutions* u_1 *and* u_2 *of Equation* (6.1) *such that* $\mathcal{E}[u_i] = 0$ *and* $u_i'(0) > 0$ $(i = 1, 2)$, *which are symmetric with respect to* η_N, *and have the following properties:*

$$\left.\begin{array}{ll} u_i(\xi_k) > 1 & \text{for} \quad 1 \leq k \leq N-1 \\ u_i(\eta_k) < -1 & \text{for} \quad 1 \leq k \leq N-1 \end{array}\right\} \quad \text{for } i = 1, 2,$$

and

$$-1 < u_i(\eta_N) < 0 \quad \text{for } i = 1, 2,$$
$$-1 < u_1(\xi_N) < 0 < u_2(\xi_N) < 1.$$

Proof. We begin with the proof of Part (a) for $N = 2$. To that end, we return to the interval $[a_1, b_1]$ defined in the proof of Theorem 6.17, and consider the two subintervals:

$$I_1^- = [a_1, c_1] \quad \text{and} \quad I_1^+ = [d_1, b_1],$$

where c_1 and d_1 are, respectively, the smallest and the largest zero of $u(\eta_1)$ on (a_1, b_1). We observe that by Lemma 6.13,

$$u(\xi_k) > 1 \quad \text{and} \quad u(\eta_k) < -1 \qquad \text{for } k \geq 2 \quad \text{at } c_1 \text{ and } d_1. \tag{6.21}$$

We first consider the interval I_1^-. As in (6.20a) we set

$$c_1^- = \inf\{\alpha < c_1 \mid u(\eta_1) < 1 \text{ on } (\alpha, c_1)\}.$$

Then $a_1^+ \leq c_1^- < c_1$. By Lemma 6.9 $u(\xi_2) < 1$ in a right-neighbourhood of c_1^-, so that we can define

$$c_1^+ = \sup\{\alpha > c_1^- \mid u(\xi_2) < 1 \text{ on } (c_1^-, \alpha)\}.$$

It is clear that $c_1^+ \in (c_1^-, c_1)$ and that

$$u(\xi_2) = u(\eta_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \qquad \text{at } c_1^+.$$

Since $u'''(\xi_2) < 0$ at c_1^-, it follows that $u'''(\xi_2)$ has a zero $c_1^* \in (c_1^-, c_1^+)$. Thus $u_2(x) = u(x, c_1^*)$ is an even periodic solution with the properties

$$u_2(\xi_1) > 1, \quad 0 < u_2(\xi_2) < 1, \quad 0 < u_2(\eta_1) < 1 \quad \text{and} \quad u_2'''(\xi_2) = 0.$$

For the second solution we consider the interval $I_1^+ = [d_1, b_1]$. Because $u(\eta_1) = u(\xi_2) = -1$ at b_1 we can define

$$b_1^- = \sup\{\alpha > d_1 \mid u(\eta_1) > -1 \text{ on } (d_1, \alpha)\}.$$

Then $b_1^- \in (d_1, b_1]$, and we define

$$d_1^+ = \inf\{\alpha < b_1^- \mid u(\xi_2) < 1 \text{ on } (\alpha, b_1^-)\},$$

and in view of (6.21) it follows that $d_1^+ \in (d_1, b_1^-)$. Since $u(\eta_1) \le 0$ on I_1^+ and $u(\xi_2) = 1$ at d_1^+, we conclude that $u'''(\xi_2) > 0$ at d_1^+. Let

$$\tilde{\beta}_1 = \sup\{\alpha > d_1^+ \,|\, u(\xi_2) > 0 \text{ on } (d_1^+, \alpha)\}.$$

Plainly, $d_1^+ < \tilde{\beta}_1 < b_1^-$. As in the proof of Theorem 6.17, we deduce from Lemma 6.13 that $u'''(\xi_2) < 0$ at $\tilde{\beta}_1$. Therefore, $u'''(\xi_2)$ changes sign on $(d_1^+, \tilde{\beta}_1)$, and hence there exists a point $b_1^* \in (d_1^+, \tilde{\beta}_1)$ where $u'''(\xi_2)$ vanishes. This means that $u_1(x) = u(x, b_1^*)$ is a periodic solution endowed with the properties

$$u_1(\xi_1) > 1, \quad 0 < u_1(\xi_2) < 1, \quad -1 < u_1(\eta_1) < 0 \quad \text{and} \quad u_1'''(\xi_2) = 0.$$

This completes the proof of Theorem 6.19a for $N = 2$.

To prove Part (a) for $N = 3$, we repeat the above arguments for the interval $[a_2, b_2]$ defined in the proof of Theorem 6.17. For $N = 4$ we consider the corresponding interval $[a_3, b_3]$ and generally, we consider the interval $[a_{N-1}, b_{N-1}]$ for arbitrary $N \ge 2$.

For the proof of Part (b), say for $N = 2$, we consider the interval $[b_1, a_2]$. Proceeding as in the proof of Part (a) ($N = 2$) we now find solutions which are symmetric with respect to η_2 for values of α on (b_1, e_1) and (f_1, a_2), where e_1 and f_1 have been defined in the proof of Theorem 6.17. The argument, and its generalisation to higher values of N is very similar to the arguments involved in the proof of Part (a), and we shall therefore omit the details. □

6.4 Even periodic solutions: $-\sqrt{8} < q < \sqrt{8}$

In this section we establish the existence of an infinite sequence of countable families of even periodic solutions of Equation (6.1) for $q \in (-\sqrt{8}, \sqrt{8})$, distinguished by the number and location of local maxima and minima of their graphs. Thus, whereas some of the results obtained for odd solutions were only valid for $q \le 0$, the results proved in this section are also valid for *positive* values of q up to $\sqrt{8}$. In the next five sections we go even beyond this number, and show how the branches of multi-bump periodic solutions obtained in this section extend to higher values of q. As we stated in Section 6.2, we shall use a shooting technique to establish the existence of these solutions, and hence, thanks to symmetry with respect to $x = 0$, we will study the initial value problem

$$\begin{cases} u^{(iv)} + qu'' + u^3 - u = 0 \qquad \text{for } x > 0, & (6.22\text{a}) \\ u(0) = \alpha, \quad u'(0) = 0, \quad u''(0) = \beta, \quad u'''(0) = 0. & (6.22\text{b}) \end{cases}$$

Again, we only discuss solutions for which the first integral is zero, i.e.

$$\mathcal{E}[u] \stackrel{\text{def}}{=} u'u''' - \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + F(u) = 0, \tag{6.23a}$$

where

$$F(u) = \frac{1}{4}(u^2 - 1)^2 \qquad \text{and} \qquad F'(u) = f(u) = u^3 - u. \tag{6.23b}$$

This means that

$$\beta \stackrel{\text{def}}{=} \beta(\alpha) = \pm\frac{1}{\sqrt{2}}|\alpha^2 - 1|.$$

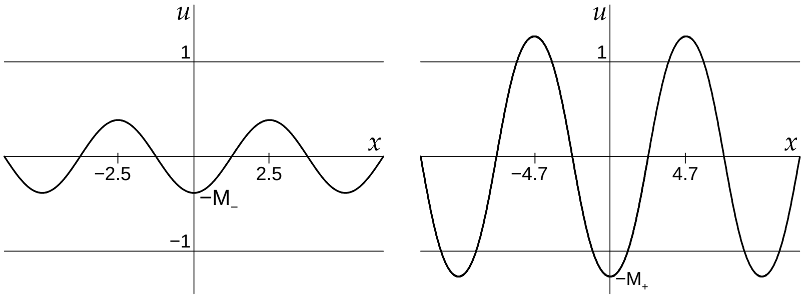

Figure 6.21: The small and the large single bump periodic solutions for Case I.

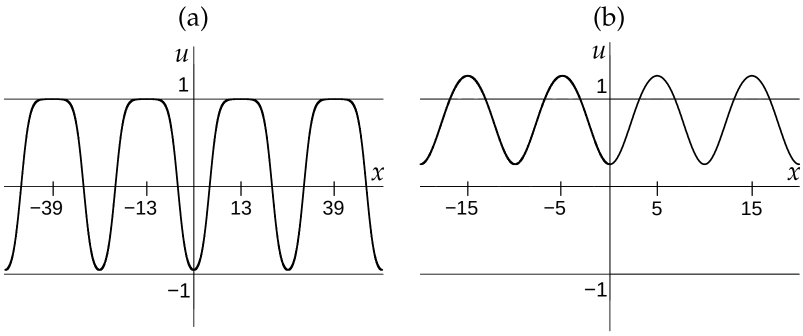

Figure 6.22: Even single bump periodic solutions for (a) $q = -2$ and (b) $q = 2$.

The cases $u''(0) > 0$ and $u''(0) < 0$ will be dealt with in succession. We refer to them by, respectively, Case I and Case II:

$$\text{Case I:} \quad u''(0) > 0 \qquad \text{and} \qquad \text{Case II:} \quad u''(0) < 0.$$

In both cases we shall denote the solution of problem (6.22) by $u(x, \alpha)$.

Note that the single bump periodic solutions of Section 6.3 become even solutions after a shift over a quarter of a period. Thus if the period of $u_\pm$ is $4\ell_\pm$, and $M_\pm = \|u_\pm\|_\infty$, then

$$\begin{aligned} u(x, -M_\pm) &= u_\pm(x - \ell_\pm) \qquad && \text{in Case I,} \\ u(x, +M_\pm) &= u_\pm(x + \ell_\pm) \qquad && \text{in Case II.} \end{aligned}$$

These solutions provide the point of departure for the shooting arguments which will yield new families of even periodic solutions with more complicated structure. For convenience we provide the graphs of $u(x, -M_\pm)$ in Figure 6.21.

We begin by establishing the existence of a new family of even, single bump periodic solutions whose maxima lie above the line $u = +1$, and whose minima lie *between* the lines $u = -1$ and $u = +1$. In Figure 6.22 we give examples of two such solutions computed at $q = -2$ and $q = 2$.

It is readily apparent that these solutions are qualitatively different from those shown in Figure 6.21. Like Γ_- and Γ_+, this new family of solutions forms a branch Γ_1 which bifurcates from the unique odd kink $\varphi(x)$ at $q = -\sqrt{8}$. However, as the bifurcation diagram in Figure 6.23 shows, in contrast to the branches Γ_- and Γ_+, which extend all the way to $q = +\infty$ (see Figure 6.4), our computations indicate that Γ_1 only extends over the *finite* q-interval $(-\sqrt{8}, \sqrt{8})$, and bifurcates at $q = \sqrt{8}$ from the constant solution $u = +1$. In Theorem 6.20 we prove that the new solutions indeed exist for every $q \in (-\sqrt{8}, \sqrt{8})$.

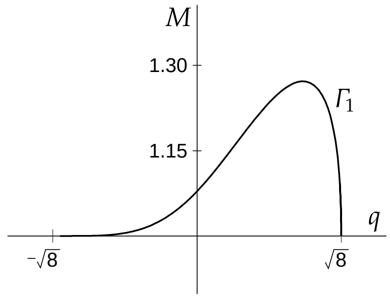

Figure 6.23: Branch of even single bump periodic solutions

Theorem 6.20 *Let $q \in (-\sqrt{8}, \sqrt{8})$. Then there exists an even, single bump periodic solution u such that $\mathcal{E}[u] = 0$, and*

$$-1 < \min\{u(x)\,|\,x \in \mathbb{R}\} < +1 < \max\{u(x)\,|\,x \in \mathbb{R}\}.$$

Proof. The proof uses ideas developed in [121]. We seek a single bump periodic solution, and we choose Case I, i.e. $u''(0) > 0$. We follow $u(\xi_1)$ as α varies, and seek a value of $\alpha \in (-1, 1)$ such that

$$u(\xi_1) > 1 \qquad \text{and} \qquad u'''(\xi_1) = 0.$$

Then, by symmetry, u is a periodic solution with half-period $L = \xi_1$ whose maxima and minima are located as indicated above:

$$\min\{u(x)\,|\,x \in \mathbb{R}\} = \alpha \in (-1, 1) \quad \text{and} \quad \max\{u(x)\,|\,x \in \mathbb{R}\} = u(\xi_1) > 1.$$

To find such a value of α, we use the auxiliary functional

$$\mathcal{H}(u) = \frac{1}{2}(u'')^2 + \frac{q}{2}(u')^2 + F(u),$$

where F has been defined in (6.23b). Let $u(x)$ be a smooth function. Then we write

$$H(x) \stackrel{\text{def}}{=} \mathcal{H}(u(x)).$$

Differentiation yields

$$H' = u''u''' + qu'u'' + f(u)u', \tag{6.24}$$

and if u is a solution of Equation (6.1), then

$$H'' = (u''')^2 + qu'u''' + f'(u)(u')^2. \tag{6.25}$$

The right-hand side of (6.25) is a second order polynomial in u''', with discriminant

$$D = \{q^2 - 4f'(u)\}(u')^2.$$

Thus, H'' will be nonnegative whenever

$$q^2 < 4(3u^2 - 1) \qquad \text{or} \qquad u^2 > \frac{q^2 + 4}{12}. \tag{6.26}$$

Define

$$\alpha_0 = \sqrt{\frac{q^2 + 4}{12}}. \tag{6.27}$$

Then $\alpha_0 \in (0, 1)$ as long as $q^2 < 8$.

Lemma 6.21 *Let $q^2 < 8$, and let $\alpha \in [\alpha_0, 1)$. Then*

$$u(\xi_1) > 1 \quad \textit{and} \quad u'''(\xi_1) < 0.$$

Proof of Lemma 6.21. Let $\alpha \in [\alpha_0, 1)$. Then $u''(0) > 0$, so that $u'(x,\alpha) > 0$ and $u(x,\alpha) > \alpha_0$ for $0 < x \le \xi_1$. This implies, by (6.25) and (6.26), that $H''(x) > 0$ for $0 < x \le \xi_1$, and so

$$H'(x) > H'(0) = 0 \qquad \text{for } x \in (0, \xi_1]. \tag{6.28}$$

Hence

$$H(\xi_1) > H(0). \tag{6.29}$$

According to the identity (6.23),

$$(u'')^2 = 2F(u) \qquad \text{if } u' = 0. \tag{6.30}$$

Therefore, at any critical point ζ of u we have

$$H(\zeta) = 2F(u(\zeta)). \tag{6.31}$$

Combining (6.29) and (6.31), we conclude that

$$F(u(\xi_1)) > F(\alpha_0).$$

Because $F' = f < 0$ on $(0,1)$ and $\alpha_0 \in (0,1)$, this implies that $u(\xi_1) > 1$. From (6.30) we deduce that $u''(\xi_1) < 0$. Because $H'(\xi_1) > 0$ by (6.28), we conclude from (6.24) that $u'''(\xi_1) < 0$, and the proof of Lemma 6.21 is complete. □

We now continue with the proof of Theorem 6.20. By Lemma 6.21, $u(\xi_1) > 1$ at $\alpha = \alpha_0$. Thus, remembering that $u(\xi_1) = M_- < 1$ when $\alpha = -M_-$, we can introduce the point

$$\alpha_1 = \inf\{\alpha < \alpha_0 \,|\, u(\xi_1) > 1 \text{ on } (\alpha, \alpha_0)\},$$

and conclude that $\alpha_1 \in (-M_-, \alpha_0)$. It follows from the continuity of $\xi_1(\alpha)$ and $u(\xi_1(\alpha), \alpha)$, established in Lemma 6.4 and Lemma 6.6, which we can apply because $\mathcal{E}[u] = 0$, that

$$u(\xi_1) = 1, \quad u'''(\xi_1) > 0 \quad \text{and} \quad \xi_1 = \eta_1 \qquad \text{at } \alpha_1. \tag{6.32}$$

The fact that $u'''(\xi_1)$ is positive at α_1 follows from Lemma 6.4 and the definition of ξ_1. Thus $u'''(\xi_1)$ has changed sign on (α_1, α_0). Since $u'''(\xi_1(\alpha), \alpha)$ depends continuously on α by Lemma 6.6, there must be a point $\alpha_1^* \in (\alpha_1, \alpha_0)$ where $u'''(\xi_1)$ vanishes, i.e. $u'''(\xi_1(\alpha_1^*), \alpha_1^*) = 0$. Remembering that $u'(\xi_1) = 0$ as well, it follows by symmetry that the function $u(x, \alpha_1^*)$ is an even, single bump periodic solution, which is symmetric with respect to ξ_1, such that $u(\xi_1) > 1$ and the period is $2\xi_1$. This completes the proof of Theorem 6.20. □

We continue with the construction of an even, 1-bump periodic solution, which is symmetric with respect to ξ_2. We have seen in (6.32) that when $\alpha = \alpha_1$, then $u = 1$ and $u''' > 0$ at ξ_1. This means that $u(\xi_2) > 1$ at α_1. We now define the point

$$\tilde{\alpha}_1 = \sup\{\alpha > -M_- \,|\, u(\xi_1) < 1 \text{ on } (-M_-, \alpha)\}.$$

Then $\tilde{\alpha}_1 \in (-M_-, \alpha_0)$, and (6.32) holds, but now at $\tilde{\alpha}_1$. Thus, $u(\xi_2) > 1$ at $\tilde{\alpha}_1$, and we define

$$\alpha_2 = \inf\{\alpha < \tilde{\alpha}_1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, \tilde{\alpha}_1)\},$$

Since $u(\xi_2, -M_-) = M_- < 1$, we can conclude again that $\alpha_2 \in (-M_-, \tilde{\alpha}_1)$. As before, $u'''(\xi_2) > 0$ at α_2. Thus, it remains to determine the sign of $u'''(\xi_2)$ when $\alpha = \tilde{\alpha}_1$. We have

$$H'(\xi_1) = 0 \quad \text{at } \tilde{\alpha}_1.$$

Because $u(x) > 1$ for $x \in (\xi_1, \xi_2]$, it follows that $H'' > 0$ on (ξ_1, ξ_2), and

$$H'(\xi_2) = u''(\xi_2)\, u'''(\xi_2) > 0 \quad \text{at } \tilde{\alpha}_1.$$

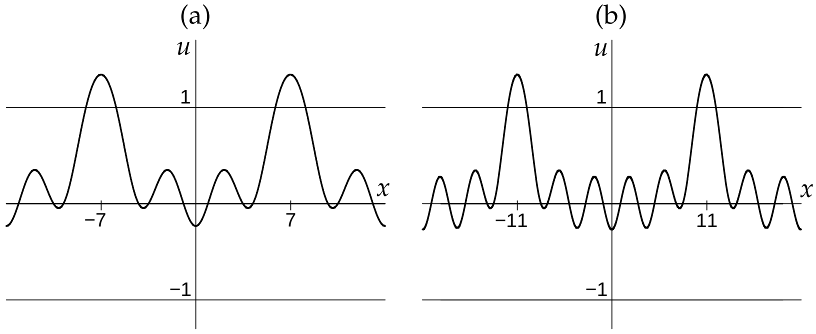

Figure 6.24: Multi-bump periodic solutions of Theorem 6.22 with $u''(0) > 0$ at $q = 2$ for (a) $N = 2$ and (b) $N = 3$.

Plainly, $u(\xi_2) > 1$, and hence, by the first integral, $u''(\xi_2) < 0$. Thus, $u'''(\xi_2) < 0$ at $\tilde{\alpha}_1$, and $u'''(\xi_2)$ has changed sign on $(\alpha_2, \tilde{\alpha}_1)$, and therefore has a zero α_2^* in this interval: $u'''(\xi_1(\alpha_2^*), \alpha_2^*) = 0$. By symmetry this yields an even periodic solution $u(x, \alpha_2^*)$ which is symmetric with respect to ξ_2, so that $u(\xi_2) > 1$, $u(\xi_1) < 1$ and the period is $2\xi_2$ (cf. Fig 6.24a).

We can now construct an N-bump periodic solution for any $N \geq 2$ by continuing the above process in an iterative manner. This yields a decreasing sequence of numbers $\{\alpha_k^*\}$ such that the solutions $u_k(x) = u(x, \alpha_k^*)$ are even and periodic with period $2\xi_k$. They have the properties

$$u_k(\xi_j) < 1 \quad \text{for } j = 1, \ldots, k-1 \qquad \text{and} \qquad u_k(\xi_k) > 1.$$

Thus we have proved:

Theorem 6.22 *Let $-\sqrt{8} < q < \sqrt{8}$. Then for any $N \geq 2$ there exists an even periodic solution u of Equation* (6.1) *such that $\mathcal{E}[u] = 0$ and $u''(0) > 0$, which is symmetric with respect to ξ_N and has the properties:*

$$u(\xi_k) < 1 \quad \textit{for } 1 \leq k \leq N-1 \qquad \textit{and} \qquad u(\xi_N) > 1. \tag{6.33}$$

For $N = 2$ and $N = 3$ such solutions are shown in Figure 6.24. Equation 6.33 means that all the local maxima of u, except the one at the point of symmetry ξ_N, lie below the line $u = 1$, but $u(\xi_N) > 1$.

Remark 6.23 The proof of Theorem 6.22 can be modified to obtain more precise information on the position of the local minima. One can prove that there are solutions which, in addition to (6.33), satisfy $-1 < u(\eta_k) < 1$ for $1 \leq k \leq N-1$. Hence these solutions obey $u(x) > -1$ for all $x \in \mathbb{R}$. This is also observed in Figure 6.24. In the following theorems (and in Sections 6.6 and 6.8) this additional property can in fact always be proved. •

In addition to the family of periodic solutions u_N described in Theorem 6.20 and Theorem 6.22, which are symmetric with respect to ξ_N for some $N \geq 1$, there exists a corresponding family of periodic solutions which are similar to u_N, but they are symmetric with respect to η_N. Such solutions are shown in Figure 6.25. The existence of this family of solutions is established in Theorem 6.24.

Theorem 6.24 *Let $-\sqrt{8} < q < \sqrt{8}$. Then for every $N \geq 1$ there exists an even periodic solution u of Equation* (6.1) *such that $\mathcal{E}[u] = 0$ and $u''(0) > 0$, which is symmetric with respect to η_N and has the properties:*

$$u(\xi_k) < 1 \quad \textit{for } 1 \leq k \leq N-1 \textit{ if } N > 2, \qquad \textit{and} \qquad u(\xi_N) > 1 \textit{ and } u(\eta_N) > 1.$$

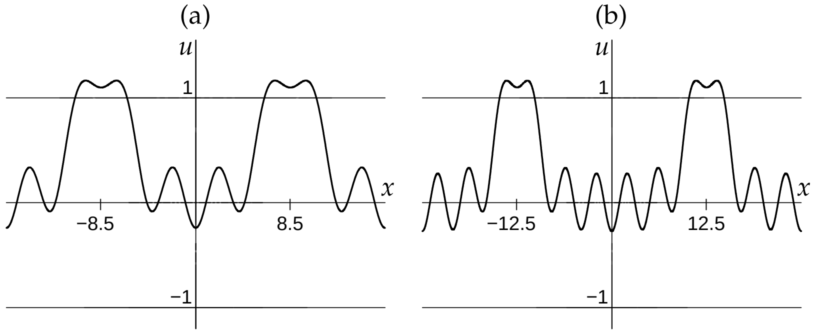

Figure 6.25: Multi-bump periodic solutions of Theorem 6.24 with $u''(0) > 0$ at $q = 2$ for (a) $N = 2$ and (b) $N = 3$.

Proof. We give the proof for $N = 2$. For $N = 1$ and for $N \geq 3$ it is similar. For further details we refer to [121]. We recall the point α_2 defined in the proof of Theorem 6.22, and in particular that

$$u(\xi_2) = 1, \quad u'''(\xi_2) > 0 \quad \text{and} \quad \xi_2 = \eta_2 \qquad \text{at } \alpha_2. \tag{6.34}$$

Therefore, by Lemma 6.9, $u(\eta_2) > 1$ for $\alpha \in (\alpha_2, \alpha_2 + \delta)$ for some small $\delta > 0$. At the point $\alpha_2^* > \alpha_2$, also defined in the proof of Theorem 6.22, the solution $u(x, \alpha_2^*)$ is symmetric with respect to ξ_2, and hence

$$u(\eta_2) = u(\eta_1) < u(\xi_1) < 1 \qquad \text{at } \alpha_2^*.$$

Thus

$$\bar{\alpha}_2 = \sup\{\alpha > \alpha_2 \,|\, u(\eta_2) > 1 \text{ on } (\alpha_2, \alpha)\}$$

is well defined, and $\bar{\alpha}_2 \in (\alpha_2, \alpha_2^*)$. We have

$$u(\xi_2) > 1, \quad u(\eta_2) = 1 \quad \text{and} \quad u'''(\eta_2) < 0 \qquad \text{at } \bar{\alpha}_2.$$

Remembering from (6.34) that $u'''(\eta_2) > 0$ at α_2, we conclude that there must be a point $\alpha_2^{**} \in (\alpha_2, \bar{\alpha}_2)$ where $u'''(\eta_2)$ vanishes, so that $u(x, \alpha_2^{**})$ is a periodic solution of Equation (6.1) with the properties listed in Theorem 6.24 for $N = 2$. □

For any $q \in (-\sqrt{8}, \sqrt{8})$, there exists yet another family $\{u_n\}$ of even periodic solutions. They are characterised by the properties:

(P1) For every $n \geq 1$, u_n is symmetric with respect to the n^{th} critical point ζ_n;

(P2) For $k = 1, \ldots, n-1$ it holds that $u_n(\zeta_k) > 1$ if ζ_k is a maximum, whereas $u_n(\zeta_k) < 1$ if ζ_k is a minimum; these inequalities are *reversed* for $k = n$.

This family was first investigated in some detail in [121], where the proof of their existence can be found. We show three solutions of this family in Figure 6.26. These solutions are interesting because of the following conjecture:

Conjecture 6.25 *For any $q \in (-\sqrt{8}, \sqrt{8})$, let u_n be a sequence of even periodic solutions with the properties* (P1) *and* (P2)*, and let $u_n(0) = \alpha_n$. Then $\alpha_n \to \alpha^*$ as $n \to \infty$, and $u(x, \alpha^*)$ is an even homoclinic solution of Equation* (6.1) *with $u(0, \alpha^*) < 1$.*

Remark 6.26 Following the construction described in [121], it is possible to obtain an infinite sequence of periodic solutions $\tilde{u}_n$ such that $\tilde{u}_n(0) > 1$ and $\tilde{u}_n''(0) < 0$. This leads one to conjecture that there exists a second homoclinic solution $\tilde{u}(x)$ of Equation (6.1), this one with $\tilde{u}(0) > 1$ (for any $q \in (-\sqrt{8}, \sqrt{8})$). •

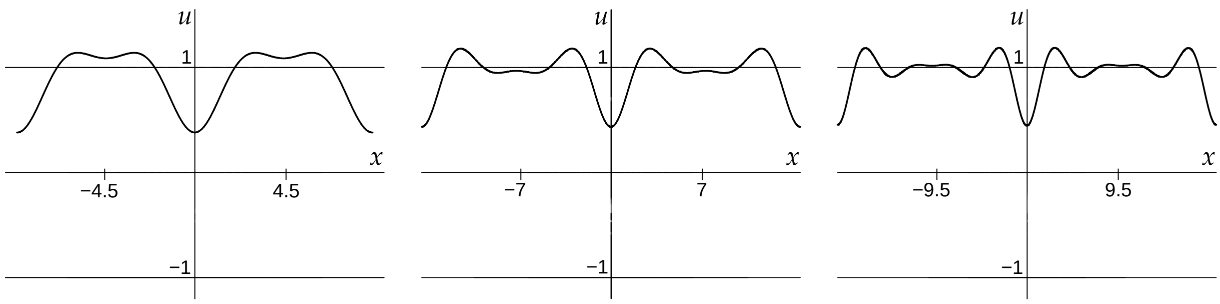

Figure 6.26: Multi-bump periodic solutions at $q = 2.5$ with properties (P1) and (P2) for $n = 2, 3, 4$. They figure in Conjecture 6.25.

We now investigate Case II, where we assume that $u''(0) < 0$, and establish results similar to those obtained in Theorems 6.20, 6.22 and 6.24 for Case I. We emphasise that since the first critical point is now a minimum, which is denoted η_1, we skip ξ_1 and number the critical points as follows:

$$0 < \eta_1 < \xi_2 < \eta_2 < \dots .$$

We begin our analysis of Case II by proving a result analogous to Lemma 6.21. Recall the definition of α_0 given in (6.27), and the fact that $0 < \alpha_0 < 1$ for $-\sqrt{8} < q < \sqrt{8}$.

Lemma 6.27 *Let $q^2 < 8$, and $u''(0) < 0$. Then there exists a point $\tilde{\alpha} \in (\alpha_0, 1)$ such that if $\alpha \in [\tilde{\alpha}, 1)$, then*

$$u(\xi_2) > 1 \quad \textit{and} \quad u'''(\xi_2) < 0 \qquad \textit{at } \alpha.$$

Proof. From a linear analysis at $u = 1$, of which the details can be found in Section 6.10, we see that

$$\xi_2(\alpha) \to \frac{3\pi}{\sqrt{q + \sqrt{8}}} \quad \text{and} \quad u(\eta_1(\alpha), \alpha) \sim 1 - (1 - \alpha)\frac{a}{b}\sinh\left(\frac{\pi a}{2b}\right) \qquad \text{as } \alpha \to 1,$$

where a and b are positive constants which are independent of α, and given in Equation (6.98) of Section 6.10. Thus, there exists an $\tilde{\alpha} \in (\alpha_0, 1)$ such that if $\alpha \in (\tilde{\alpha}, 1)$, then

$$u(x, \alpha) > \alpha_0 \quad \text{if } 0 < x \le \xi_2.$$

Hence by (6.26)

$$H''(x) > 0 \quad \text{if } 0 < x \le \xi_2.$$

Because $H'(0) = u''(0)u'''(0) = 0$, this implies that

$$H'(x) > 0 \quad \text{if } 0 < x \le \xi_2. \tag{6.35}$$

Thus $H(0) < H(\eta_1) < H(\xi_2)$, and hence by (6.31), $F(u(0)) < F(u(\eta_1)) < F(u(\xi_2))$. This means that $u(\xi_2) > 1$. We also deduce from (6.35) that $H' = u''u''' > 0$ at ξ_2. Since $u''(\xi_2) < 0$, we conclude that $u'''(\xi_2) < 0$. This completes the proof. □

Thus, for $\alpha \in (0, 1)$ sufficiently close to 1, we have $u(\xi_2) > 1$ and $u'''(\xi_2) < 0$. On the other hand, when $\alpha = M_-$, we have $u(\xi_2) < 1$. Therefore, we can define the number

$$\alpha_2 = \inf\{\alpha < 1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, 1)\}$$

and $\alpha_2 \in (M_-, \tilde{\alpha})$. Plainly, $u(\xi_2) = 1$ and $u'''(\xi_2) > 0$ at α_2, so that $u'''(\xi_2)$ must have a zero for some $\alpha_2^* \in (M_-, 1)$, which yields the first of a family of even periodic solutions with $u''(0) < 0$. As in the proof of Theorems 6.22 and 6.24, we can continue inductively and prove the existence of two families of periodic solutions (see also Figure 6.27):

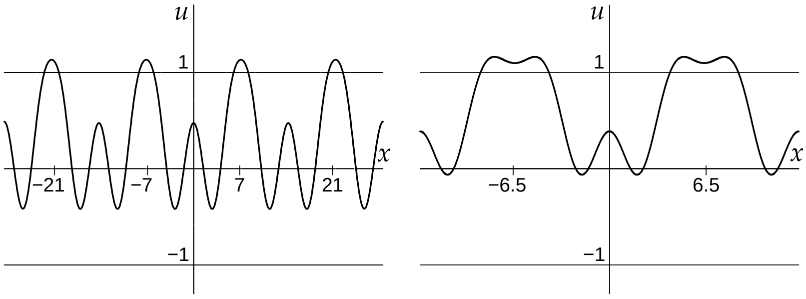

Figure 6.27: Multi-bump periodic solutions with $u''(0) < 0$ at $q = 2$: (a) $N = 2$, see Theorem 6.28; (b) $N = 3$, see Theorem 6.29.

Theorem 6.28 *Let $-\sqrt{8} < q < \sqrt{8}$. Then for any $N \geq 2$ there exists an even periodic solution u of Equation* (6.1) *such that $\mathcal{E}[u] = 0$ and $u''(0) < 0$, which is symmetric with respect to ξ_N and has the properties:*

$$u(\xi_k) < 1 \quad \text{for } 2 \leq k \leq N-1 \text{ if } N \geq 3, \qquad \text{and} \qquad u(\xi_N) > 1.$$

Theorem 6.29 *Let $-\sqrt{8} < q < \sqrt{8}$. Then for every $N \geq 2$ there exists an even periodic solution u of Equation* (6.1) *such that $\mathcal{E}[u] = 0$ and $u''(0) < 0$, which is symmetric with respect to η_N and has the properties:*

$$u(\xi_k) < 1 \quad \text{for } 2 \leq k \leq N-1 \text{ if } N \geq 3, \qquad \text{and} \qquad u(\xi_N) > 1 \text{ and } u(\eta_N) > 1.$$

6.5 Local analysis near $u = 1$

In order to extend the existence results of the previous section to the range $q \geq \sqrt{8}$, we need to develop further analytical techniques. This is because Lemma 6.21 no longer holds for $q \geq \sqrt{8}$. These techniques will rely on a detailed analysis of the local behaviour of solutions near $u = 1$. Thus, we substitute $u = 1 + \varepsilon v$ into (6.1) and require that $u(0) = 1 - \varepsilon$ After omitting the higher order terms in ε, we then obtain the linear equation

$$v^{(iv)} + qv'' + 2v = 0 \tag{6.36a}$$

and at the origin, the initial conditions become

$$v(0) = -1, \qquad v'(0) = 0, \qquad v''^2(0) = 2, \quad \text{and} \quad v'''(0) = 0. \tag{6.36b}$$

The fact that v should be even implies that v' and v''' vanish at the origin, and the assumption that the energy $\mathcal{E}$ is zero leads to the condition on v''^2. A detailed analysis of this problem is given in Section 6.10 and in [121, Appendix B]. For easy reference, we give here the main results of this analysis. As in Section 6.4, it is necessary to distinguish two cases:

$$\text{Case I: } v''(0) = \sqrt{2} \qquad \text{and} \qquad \text{Case II: } v''(0) = -\sqrt{2},$$

and we denote the solution of problem (6.36) in these two cases by $v_{\pm}(x)$, so that $v''_{\pm}(0) = \pm\sqrt{2}$.

The roots $\pm\lambda$ and $\pm\mu$ of the corresponding characteristic equation are defined by

$$\lambda = ia \quad \text{and} \quad \mu = ib, \tag{6.37a}$$

in which $a > 0$ and $b > 0$ are defined by

$$a^2 = \frac{1}{2}(q + \sqrt{q^2 - 8}) \qquad \text{and} \qquad b^2 = \frac{1}{2}(q - \sqrt{q^2 - 8}). \tag{6.37b}$$

In what follows, the values of q at which resonance occurs (cf. [77], p. 397), i.e.

$$\frac{a}{b} = \frac{n}{m}, \qquad m, n \in \mathbb{N} \quad (n \geq m), \tag{6.38}$$

will play a special role. These values are readily computed to be

$$q_{m,n} = \sqrt{2}\left(\frac{n}{m} + \frac{m}{n}\right). \tag{6.39a}$$

For convenience we write $q_n = q_{1,n}$, i.e.

$$q_n = \sqrt{2}\left(n + \frac{1}{n}\right). \tag{6.39b}$$

In the following two lemmas we state the main results about the solutions $v_\pm(x)$ of problem (6.36). In the first one we present the explicit expressions of these solutions.

Lemma 6.30 *The solutions $v_\pm(x)$ of problem* (6.36) *are given by*

$$v_\pm(x) = A_\pm \cos(ax) + B_\pm \cos(bx),$$

where

$$A_\pm = \frac{b^2 \mp \sqrt{2}}{a^2 - b^2} \qquad \textit{and} \qquad B_\pm = -\frac{a^2 \mp \sqrt{2}}{a^2 - b^2}.$$

In the second lemma we concentrate on the critical points of the solutions $v_\pm$ of problem (6.36). In particular, it will be important for our shooting arguments in Section 6.6 that we know the location of these point with respect to the $v = 0$ axis (*above* or *below*) and the sign of the third derivative $v_\pm'''$ at these points.

Lemma 6.31 *Let ζ be a critical point of the solution v of problem* (6.36)*, i.e. $v'(\zeta) = 0$. Then*

$$\begin{aligned} &\sin(a\zeta) + \sin(b\zeta) = 0 && \textit{in Case I,} \\ &\sin(a\zeta) - \sin(b\zeta) = 0 && \textit{in Case II} \end{aligned}$$

and

$$\begin{aligned} v_\pm(\zeta) &= \frac{a^2 \mp \sqrt{2}}{a^2 - b^2}\left(\mp\frac{b}{a}\cos(a\zeta) - \cos(b\zeta)\right), \\ v_\pm'''(\zeta) &= b(a^2 \mp \sqrt{2})\sin(b\zeta). \end{aligned}$$

In both cases,

$$\operatorname{sign} v(\zeta) = -\operatorname{sign}\bigl(\cos(b\zeta)\bigr),$$

and

$$\operatorname{sign} v'''(\zeta) = \operatorname{sign}\bigl(\sin(b\zeta)\bigr).$$

The proof of Lemma 6.31 is elementary, and makes use of the observation that $A_\pm/B_\pm = \pm\, b/a$ and that $ab = \sqrt{2}$.

6.6 Even periodic solutions: $-\sqrt{8} \le q < q_3$

In this section, and in Section 6.8, we investigate the existence of even periodic solutions for which $\mathcal{E}[u] = 0$. As was explained in the introduction, we find it convenient to label the solutions according to the number of monotone segments, or *laps*, that go in a half-period. Thus, the solutions of Theorem 6.20 are called 1-lap solutions. In the subsequent theorems of Section 6.4 we have shown that for every $q \in (-\sqrt{8}, \sqrt{8})$ there exist n-lap solutions for any $n \geq 1$.

Numerical evidence suggest that the 1-lap solutions of Theorem 6.20 no longer exist for $q > \sqrt{8}$, but that 2-lap solutions still exist for some values of $q > \sqrt{8}$, and that n-lap solutions exist for n large enough. Specifically, we prove the following result. Let

$$q_n = \sqrt{2}\Big(n + \frac{1}{n}\Big), \qquad n \geq 1.$$

Theorem 6.32 *For each $n \geq 2$ there exist two families of even periodic n-lap solutions when $q \in (-\sqrt{8}, q_n)$. At the points of symmetry ζ_n we have $u(\zeta_n) > 1$ for all $n \geq 1$.*

Whereas in Section 6.4 we could use the properties of the functional $\mathcal{H}(u)$, when $q > \sqrt{8}$ this is no longer possible. The main ingredients used in the proof will now be the linear analysis given in Section 6.5, a powerful Comparison Lemma (Lemma 6.12) valid for $q \in (-\sqrt{8}, q_3)$ and, in Section 6.8, a counting argument which is reminiscent of a topological degree argument.

The Comparison Lemma enables us to obtain information about the location of all the critical values of the solutions with respect to the line $u = 1$. In the present section we assume that $-\sqrt{8} < q < q_3$, so that the Comparison Lemma holds. In Section 6.8 we allow q to be arbitrary large, and develop the counting argument.

In the first result of this section we show that when $-\sqrt{8} < q < q_2$, then there exist two families of n-lap solutions with $n \geq 2$. We begin with a pair of 2-lap solutions.

Theorem 6.33 *Let $-\sqrt{8} < q < q_2$. Then there exist two even 2-lap periodic solutions u_{2a} and u_{2b} of Equation* (6.1) *such that $\mathcal{E}[u_{2a}] = 0$ and $\mathcal{E}[u_{2b}] = 0$, with the following properties:*

(a) *The solution u_{2a} is even with respect to ξ_2, and $u''_{2a}(0) < 0$, and $u_{2a}(\xi_1) > 1$.*

(b) *The solution u_{2b} is even with respect to η_1, and $u''_{2b}(0) > 0$, and $u_{2b}(\eta_1) > 1$.*

We denote the branches of these solutions by, respectively, Γ_{2a} and Γ_{2b}. These branches, as well as graphs of two specific solutions u_{2a} and u_{2b}, are presented in Figure 6.28.

Proof. We first consider Part (b) and prove the existence of the solution u_{2b} for which $u''(0) > 0$ (Case I). For convenience we have dropped the subscript $2b$. We distinguish three cases:

$$(i)\ \sqrt{8} \leq q < q_{3,5} \qquad (ii)\ q = q_{3,5}, \qquad (iii)\ q_{3,5} < q < q_2.$$

According to (6.38) and (6.39a), these cases correspond to

$$(i)\ 1 \leq \frac{a}{b} < \frac{5}{3}, \qquad (ii)\ \frac{a}{b} = \frac{5}{3}, \qquad (iii)\ \frac{5}{3} < \frac{a}{b} < 2.$$

Case (i): Recall from Section 6.5 that v denotes the solution of Equation (6.36a), the linearisation of Equation (6.1) around $u = 1$. It follows from Lemma 6.31 that in this case

$$v(\xi_1) > 0 \qquad \text{and} \qquad v(\eta_1) < 0.$$

Hence, it follows from continuity that there exists a $\delta > 0$ such that

$$u(\xi_1) > 1 \quad \text{and} \quad u(\eta_1) < 1 \qquad \text{for } 1 - \delta < \alpha < 1.$$

Recall that $u(\xi_1) = M_- < 1$ when $\alpha = -M_-$, and define

$$a_1 = \inf\{\alpha < 1 \,|\, u(\xi_1) > 1 \text{ on } (\alpha, 1)\}.$$

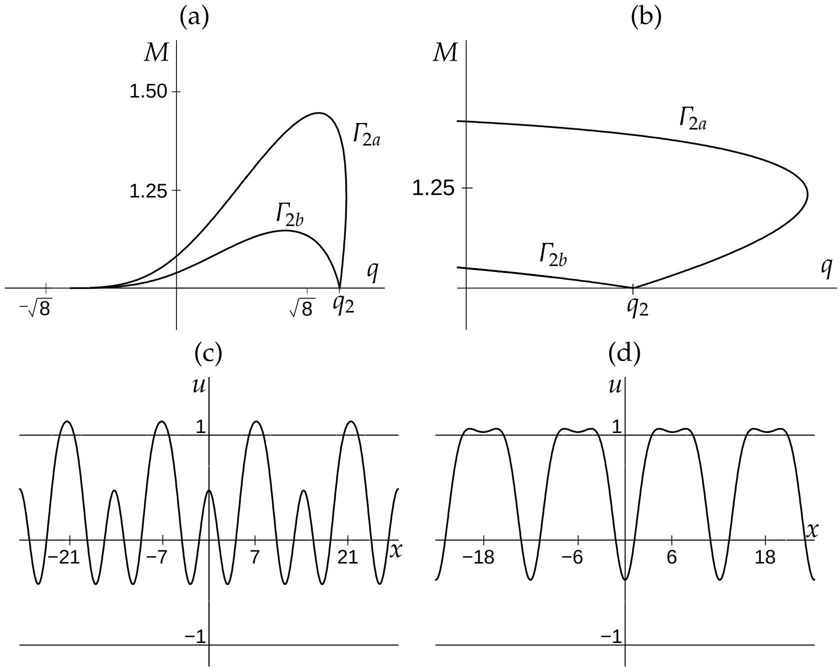

Figure 6.28: The solutions obtained in Theorem 6.33: (a) the branches Γ_{2a} and Γ_{2b} and (b) a blowup at q_2; (c) graph of u_{2a} and (d) of u_{2b}, both at $q = 0.5$.

Then by Lemma 6.9, $u(\eta_1) > 1$ for $a_1 < \alpha < a_1 + \varepsilon$ for some small $\varepsilon > 0$. However, $u(\eta_1) < 1$ for α close to 1. Therefore

$$b_1 = \sup\{\alpha > a_1 \,|\, u(\eta_1) > 1 \text{ on } (a_1, \alpha)\} \in (a_1, 1).$$

Plainly, at b_1 we have $u(\eta_1) = 1$ and by Lemma 6.4, $u'''(\eta_1) < 0$. Since $u'''(\eta_1) > 0$ at a_1 and $u'''(\eta_1) < 0$ at b_1, it follows that $u'''(\eta_1)$ has a zero for some $\alpha^* \in (a_1, b_1)$, and again we use symmetry to conclude that $u(x, \alpha^*)$ is a periodic solution with the desired properties.
Case (*iii*): From Lemma 6.31 we see that in this case

$$v(\xi_1) > v(\eta_1) > 0 \qquad \text{and} \qquad v'''(\eta_1) < 0.$$

Hence, there exists a $\delta > 0$ such that

$$u(\xi_1) > u(\eta_1) > 1 \quad \text{and} \quad u'''(\eta_1) < 0 \qquad \text{for } 1 - \delta < \alpha < 1. \tag{6.41}$$

As before, we define

$$a_1 = \inf\{\alpha < 1 \,|\, u(\xi_1) > 1 \text{ on } (\alpha, 1]\}.$$

and

$$b_1 = \sup\{\alpha > a_1 \,|\, u(\eta_1) > 1 \text{ on } (a_1, \alpha)\} \in (a_1, 1].$$

Recall that $u'''(\eta_1) > 0$ at a_1, so that $b_1 > a_1$ by Lemma 6.9. If $b_1 = 1$, then (6.41) implies that $u'''(\eta_1) < 0$ for α near b_1. On the other hand, if $b_1 < 1$, then $u'''(\eta_1) < 0$ at b_1. Thus, in both cases $u'''(\eta_1)$ changes sign on (a_1, b_1). Once again the existence of a periodic solution of the desired type follows.
Case (*ii*): From Lemma 6.31 we see that in this case,

$$v(\xi_1) > v(\eta_1) = 0 \qquad \text{and} \qquad v'''(\eta_1) < 0.$$

Thus, there exists a $\delta > 0$ such that

$$u(\xi_1) > 1 \quad \text{and} \quad u'''(\eta_1) < 0 \qquad \text{for } 1 - \delta < \alpha < 1.$$

Fix $\alpha \in (1-\delta, 1)$. If, for this value of α, one has $u(\eta_1) < 1$ then the proof is completed as in Case (*i*), and if $u(\eta_1) \geq 1$, then one can complete it as in Case (*iii*). This finishes the proof of Part (b).

Next, we prove Part (a). Recall that ξ_2 denotes the first positive local maximum of u since $u''(0) < 0$. From Lemma 6.31 we conclude that

$$v(\eta_1) < 0 \quad \text{and} \quad v(\xi_2) > 0, \quad v'''(\xi_2) < 0$$

for the entire interval $q_1 \leq q < q_2$. Hence, there exists a $\delta > 0$ such that

$$u(\eta_1) < 1 \quad \text{and} \quad u(\xi_2) > 1 \quad u'''(\xi_2) < 0 \qquad \text{for } 1-\delta < \alpha < 1. \tag{6.42}$$

Set

$$a_2 = \inf\{\alpha < 1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, 1)\}.$$

Then $M_- < a_2 < 1$. Plainly, $u'''(\xi_2) > 0$ at a_2. Since by (6.42), $u'''(\xi_2) < 0$ for α close to 1, it follows that there exists a point $\alpha_2^* \in (a_2, 1)$ such that $u'''(\xi_2) > 0$ at α_2^*, and hence $u(x, \alpha_2^*)$ is an even periodic solution with period $2\xi_2(\alpha_2^*)$. Finally, $u(\xi_2, \alpha_2^*) > 1$ since $a_2 < \alpha_2^* < 1$. This completes the proof of Theorem 6.33. □

Remark 6.34 We have now shown that the solutions u_{2a} and u_{2b} exist over the entire interval $(-\sqrt{8}, q_2)$. Our numerical investigation (Figure 6.28a) indicates that u_{2b} does not exist past q_2 although it appears that u_{2a} does exist on a small q-interval beyond q_2. Our experiments also indicate that the branches Γ_{2a} and Γ_{2b}, on which these solutions lie, disappear at q_2 as a result of a bifurcation from the constant solution $u = 1$. In Section 6.7 we shall give an explanation for the behaviour of the two branches Γ_{2a} and Γ_{2b} near the point $(q, M) = (q_2, 1)$. •

Theorem 6.33, together with Lemma 6.12, enables us to establish the existence of two families of even n-lap periodic solutions, for any $n \geq 2$.

Theorem 6.35 *Let $-\sqrt{8} < q < q_2$, and let $n \geq 2$. Then there exist two families of even n-lap periodic solutions u of Equation* (6.1) *such that $\mathcal{E}[u] = 0$, one for which $u''(0) > 0$ and one for which $u''(0) < 0$. They are symmetric with respect to the n^{th} critical point ζ_n. The critical values have the properties*

$$u(\zeta_k) < 1 \quad \textit{if } k \leq n-2 \ (n \geq 3) \qquad \textit{and} \qquad u(\zeta_n) > 1,$$

and

$$\begin{aligned} u(\zeta_{n-1}) &< 1 \qquad \textit{if } \zeta_n \textit{ is a maximum,} \\ u(\zeta_{n-1}) &> 1 \qquad \textit{if } \zeta_n \textit{ is a minimum.} \end{aligned}$$

Proof. The proof proceeds very much along the lines of the proofs of Theorems 6.22 and 6.24. To show how Lemma 6.12 is used, we give the proof for the 3-lap solution when $u''(0) > 0$. This solution is symmetric with respect to ξ_2. Otherwise, we leave the proof to the reader. Let

$$\alpha_3 = \sup\{\alpha > -M_- \,|\, u(\xi_1) < 1 \text{ on } (-M_-\alpha)\}.$$

It follows from Theorem 6.33 that $\alpha_3 \in (-M_-, 1)$. In addition,

$$u(\xi_1) = 1 \quad \text{and} \quad u'''(\xi_1) > 0 \qquad \text{at } \alpha_3.$$

Thus, in view of Lemma 6.12,

$$u(\xi_2) > 1 \quad \text{and} \quad u'''(\xi_2) < 0 \qquad \text{at } \alpha_3.$$

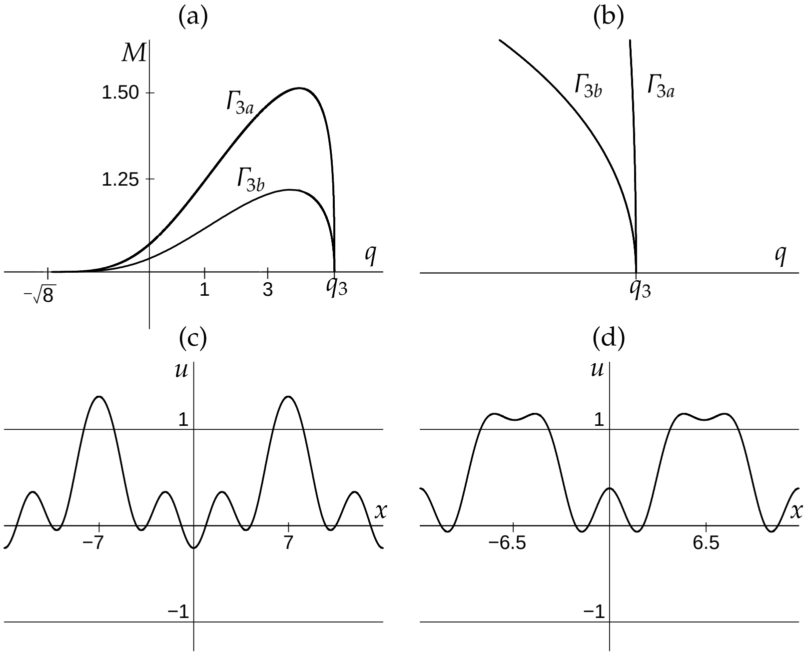

Figure 6.29: The solutions obtained in Theorem 6.35: (a) the branches Γ_{3a} and Γ_{3b} and (b) a blowup at q_3; (c) graph of u_{3a} and (d) of u_{3b}, both at $q = 2$.

Next, let

$$\tilde{\alpha}_3 = \inf\{\alpha < \alpha_3 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, \alpha_3)\}.$$

Then $\tilde{\alpha}_3 \in (-M_-, \tilde{\alpha}_3)$, and

$$u(\xi_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \qquad \text{at } \tilde{\alpha}_3.$$

Therefore, $u'''(\xi_2)$ changes sign, and thus has a zero α_3^*, on $(\tilde{\alpha}_3, \alpha_3)$. It is clear from the construction that the solution $u(x, \alpha_3^*)$ has the required properties. □

Next, we turn our attention to the interval $(-\sqrt{8}, q_3)$, and show that n-lap solutions continue to exist up to q_3 for $n \ge 3$. We first focus on 3-lap solutions.

Theorem 6.36 *Let $-\sqrt{8} < q < q_3$. Then there exist two even 3-lap periodic solutions u_{3a} and u_{3b} of Equation* (6.1)*, such that $\mathcal{E}[u_{3a}] = 0$ and $\mathcal{E}[u_{3b}] = 0$, with the following properties:*

(a) *The solution u_{3a} is even with respect to ξ_2,*

$$u_{3a}''(0) > 0 \qquad \textit{and} \qquad u_{3a}(\xi_1) < 1, \quad u_{3a}(\xi_2) > 1.$$

(b) *The solution u_{3b} is even with respect to η_2,*

$$u_{3b}''(0) < 0 \qquad \textit{and} \qquad u_{3b}(\eta_2) > 1.$$

The branches Γ_{3a} and Γ_{3b} of these solutions, as well as graphs of the solutions u_{3a} and u_{3b} at a specific value of q are presented in Figure 6.29.

Proof. For $-\sqrt{8} < q < q_2$, the existence of solutions such as u_{3a} and u_{3b} has been established in Theorems 6.22, 6.29 and 6.35. Thus, it suffices to prove Theorem 6.36 for $q_2 \le q < q_3$.

We begin with Part (a). Set

$$a_1 = \sup\{\alpha > -M_- \,|\, u(\xi_1) < 1 \text{ on } (-M_-, \alpha)\}.$$

From Lemma 6.31 we know that when $q_2 \leq q < q_3$, then

$$v(\xi_1) > 1 \quad \text{and} \quad v(\eta_2) < 1.$$

Hence, by continuity, there exists a constant $\delta > 0$ such that

$$u(\xi_1) > 1 \quad \text{and} \quad u(\eta_2) < 1 \qquad \text{for } 1 - \delta < \alpha < 1.$$

Therefore $a_1 \in (-M_-, 1)$, and

$$u(\xi_1) = 1, \quad u''(\xi_1) = 0 \quad \text{and} \quad u'''(\xi_1) > 0 \qquad \text{at } \alpha = a_1.$$

This implies that $u(\xi_2) > 1$ and, by Lemma 6.12, that $u'''(\xi_2) < 0$. Let

$$a_2 = \inf\{\alpha < a_1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, a_1)\}.$$

Then

$$u(\xi_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \qquad \text{at } \alpha = a_2,$$

because $u(\eta_1) < u(\xi_1) < 1$. Therefore, $u'''(\xi_2)$ changes sign on (a_2, a_1), so that there exists a point $a_2^* \in (a_2, a_1)$ such that $u'''(\xi_2) = 0$ at a_2^*, and $u(x, a_2^*)$ is a periodic solution which is symmetric with respect to ξ_2. By construction $u(\xi_1, a_2^*) < 1$ and $u(\xi_2, a_2^*) > 1$, as required. This ends the proof of Part (a).

Turning our attention to Part (b), we analyse the following three cases separately:

$$(i)\ q_2 \leq q < q_{3,7} \qquad (ii)\ q = q_{3,7}, \qquad (iii)\ q_{3,7} < q < q_3.$$

Again, according to (6.38) and (6.39a), these cases correspond to

$$(i)\ 2 \leq \frac{a}{b} < \frac{7}{3}, \qquad (ii)\ \frac{a}{b} = \frac{7}{3}, \qquad (iii)\ \frac{7}{3} < \frac{a}{b} < 3.$$

Case (i): It follows from Lemma 6.31 that in this case

$$v(\xi_2) > 0, \quad v(\eta_2) < 0 \quad \text{and} \quad v'''(\eta_2) < 0,$$

so that for some small $\delta > 0$,

$$u(\xi_2) > 1 \quad \text{and} \quad u(\eta_2) < 1 \qquad \text{for } 1 - \delta < \alpha < 1. \tag{6.43}$$

Define

$$a_1 = \inf\{\alpha < 1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, 1)\}.$$

Because $u(\xi_2) = M_- < 1$ when $\alpha = M_-$, it follows that $M_- < a_1 < 1$, and hence

$$u(\xi_2) = 1 \quad \text{and} \quad u'''(\xi_2) > 0 \qquad \text{at } a_1,$$

since $u(\eta_1) < M_- < 1$ at a_1. This means, according to Lemma 6.9, that $u(\eta_2) > 1$ for $a_1 < \alpha < a_1 + \varepsilon$ where $\varepsilon > 0$ is some small constant. Define

$$b_1 = \sup\{\alpha > a_1 \,|\, u(\eta_2) > 1 \text{ on } (a_1, \alpha)\}. \tag{6.44}$$

As we have seen in (6.43), $u(\eta_2) < 1$ for α close to 1. Therefore $b_1 \in (a_1, 1)$. Since $u(\xi_2) > 1$ at b_1 it follows that $u'''(\eta_2) < 0$ at b_1. Hence, in view of (6.44), $u'''(\eta_2)$ changes sign, and therefore has a zero at a point $\alpha_1^* \in (a_1, b_1)$. Thus, $u(x, \alpha_1^*)$ is an even periodic solution which, by construction, is symmetric with respect to η_2, and $u(\eta_2, \alpha_1^*) > 1$, as required.

Case (iii): In this case

$$v(\xi_2) > 0, \quad v(\eta_2) > 0 \quad \text{and} \quad v'''(\eta_2) < 0,$$

so that

$$u(\xi_2) > 1, \quad u(\eta_2) > 1 \quad \text{and} \quad u'''(\eta_2) < 0 \quad \text{for } 1 - \delta < \alpha < 1 \tag{6.45}$$

when $\delta > 0$ is sufficiently small. As before, we define

$$a_2 = \inf\{\alpha < 1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, 1)\},$$

and since $u(\xi_2) = M_-$ when $\alpha = M_-$, it follows that $a_2 \in (M_-, 1)$. Because in this case, $u(\eta_1) < \alpha < 1$ we deduce that

$$u(\xi_2) = u(\eta_2) = 1 \quad \text{and} \quad u'''(\eta_2) > 0 \qquad \text{if } \alpha = a_2. \tag{6.46}$$

By Lemma 6.9, $u(\eta_2) > 1$ in a right-neighbourhood of a_2. Let

$$b_2 = \sup\{\alpha > a_2 \,|\, u(\eta_2) > 1 \text{ on } (a_2, \alpha)\}.$$

We claim that $u'''(\eta_2)$ changes sign on (a_2, b_2). If $b_2 = 1$, then this assertion follows at once from (6.45), (6.46), and continuity. If $b_2 < 1$, then $u(\eta_2) = 1$ at b_2, and because $u(\xi_2) > 1$ at b_2, it follows that $u'''(\eta_2) < 0$, so that in view of (6.46), $u'''(\eta_2)$ also changes sign on (a_2, b_2).

Thus, there exists a point $\alpha_2^* \in (a_2, b_2)$ such that $u(x, a_2^*)$ is an even periodic solution which is symmetric with respect to η_2, such that

$$u(\eta_1) < 1, \quad u(\xi_2) > 1 \quad \text{and} \quad u(\eta_2) > 1.$$

Case (*ii*): By Lemma 6.31,

$$v(\xi_2) > 0, \quad v(\eta_2) = 0 \quad \text{and} \quad v'''(\eta_2) < 0,$$

so that for some small $\delta > 0$,

$$u(\xi_2) > 1 \quad \text{and} \quad u'''(\eta_2) < 0 \qquad \text{for } 1 - \delta < \alpha < 1. \tag{6.47}$$

Fix $\alpha \in (1 - \delta, 1)$. If $u(\eta_2) < 1$ we continue as in the proof of Case (*i*) and if $u(\eta_2) \ge 1$ we continue the proof as in Case (*iii*). This completes the proof of Theorem 6.36. □

About the families of n-lap solutions we can repeat the claim of Theorem 6.35 for the larger range of q-values: $q \in (-\sqrt{8}, q_3)$. However, the minimum number of laps is now raised from 2 to 3.

Until now we have studied branches of even multi-bump periodic solutions with $\mathcal{E} = 0$ on intervals of the form $(-\sqrt{8}, q_n)$ for $n \ge 1$. These branches appear to bifurcate from the points $(q_n, 1)$. In addition to these solutions there exist families of even periodic solutions which exist on intervals of the form $(-\sqrt{8}, q_{m,n})$, where $q_{m,n}$ is given in (6.6), and $n > m \ge 2$. The corresponding branches appear to bifurcate from the points $(q_{m,n}, 1)$. We shall not go into a general analysis of such solutions. Instead, we provide the details for one example, and choose $m = 2$ and $n = 3$. We denote the two branches of even periodic solutions by $\Gamma_{2,3a}$ and $\Gamma_{2,3b}$, and the solutions that lie on these branches by $u_{2,3a}$ and $u_{2,3b}$, respectively. Two such solutions are presented in Figure 6.30b,c at $q = 2$.

We note that $q_{2,3} = \frac{13}{6}\sqrt{2} \in (q_1, q_2)$. Therefore, we may again use Lemma 6.12 to prove the existence of these two new families of solutions.

Theorem 6.37 *Let* $-\sqrt{8} < q < q_{2,3}$. *Then there exist two even 3-lap periodic solutions* $u_{2,3a}$ *and* $u_{2,3b}$ *of Equation* (6.1) *such that* $\mathcal{E}[u_{2,3a}] = 0$ *and* $\mathcal{E}[u_{2,3b}] = 0$ *with the following properties:*

(a) *The solution* $u_{2,3a}$ *is symmetric with respect to* η_2, *and*

$$u_{2,3a}(0) < 1, \quad u''_{2,3a}(0) < 0, \quad u_{2,3a}(\xi_2) > 1, \quad u_{2,3a}(\eta_2) < 1.$$

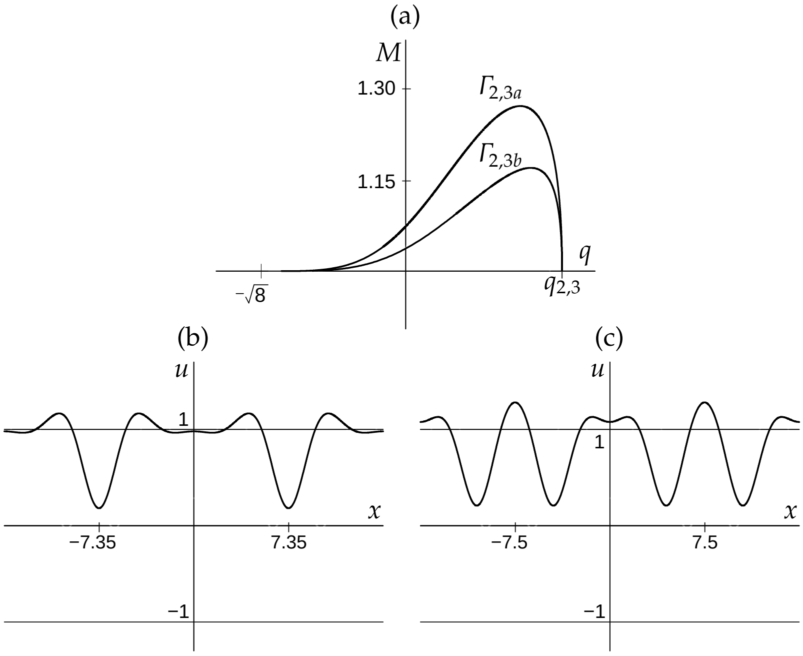

Figure 6.30: The solutions obtained in Theorem 6.37: (a) the branches $\Gamma_{2,3a}$ and $\Gamma_{2,3b}$; (b) graph of $u_{2,3a}$ and (c) of $u_{2,3b}$, both at $q = 2$.

(b) *The solution $u_{2,3b}$ is symmetric with respect to ξ_2, and*

$$u_{2,3b}(0) > 1, \quad u''_{2,3b}(0) > 0, \quad u_{2,3b}(\eta_1) < 1, \quad u_{2,3a}(\xi_2) > 1.$$

Proof. We start with Part (a). If $q_1 < q < q_{2,3}$, then, according to Lemma 6.31,

$$v(0) = -1, \quad v(\eta_1) < 0, \quad v(\xi_2) > 0, \quad v(\eta_2) < 0 \quad \text{and} \quad v'''(\eta_2) > 0.$$

Hence, for any $q \in (q_1, q_{2,3})$ there exists by continuity a $\delta > 0$ such that for $\alpha \in (1-\delta, 1)$

$$u(\eta_1) < 1, \quad u(\xi_2) > 1, \quad u(\eta_2) < 1, \quad \text{and} \quad u'''(\eta_2) > 0. \tag{6.48}$$

Let

$$a_1 = \inf\{\alpha < 1 \,|\, u(\xi_2) > 1 \text{ on } (\alpha, 1)\}.$$

Because $u(\xi_2) < 1$ when $\alpha = M_-$, it follows that $a_1 \in (M_-, 1)$, and

$$u(\xi_2) = 1 \quad \text{and} \quad u(\eta_2) = 1 \qquad \text{at } a_1.$$

Next, let

$$a_2 = \inf\{\alpha < 1 \,|\, u(\eta_2) < 1 \text{ on } (\alpha, 1)\}.$$

Then $a_2 > a_1$ by Lemma 6.9 and hence

$$u(\eta_2) = 1, \quad u(\xi_2) > 1 \quad \text{and} \quad u'''(\eta_2) < 0 \qquad \text{at } a_2.$$

Remembering (6.48), we conclude that $u'''(\eta_2)$ changes sign on $(a_2, 1)$. Thus, by continuity there exists an $a_2^* \in (a_2, 1)$ such that $u'''(\eta_2) = 0$ at a_2^*. Therefore, $u^*(x) = u(x, a_2^*)$ is a periodic solution which is symmetric with respect to η_2 and has the properties

$$u^*(0) < 1, \quad u^*(\eta_1) < 1, \quad u^*(\xi_2) > 1, \quad u^*(\eta_2) < 1,$$

as required.

Next we prove Part (b). We now take $u(0) > 1$ and $u''(0) < 0$, as opposed to the statement in the theorem (we will come back to this shortly). If $q_1 < q < q_{2,3}$, then we have to analyse three different cases separately, as in the proof of Part (b) of Theorem 6.36 (in the present case $q = q_{5,7}$ is the separating value). After some manipulations along the by now usual lines of which we omit the details, we find an $\alpha > 1$ such that

$$u(\eta_1) < 1, \quad u(\xi_2) > 1, \quad u(\eta_2) > 1, \quad \text{and} \quad u'''(\eta_2) = 0,$$

and thus u is symmetric with respect to $x = 0$ and $x = \eta_2$. The shift $\tilde{u}(x) = u(x - \eta_2)$ is now the solution desired in Part (b) of the theorem.

Proving that the solutions $u_{2,3a}$ and $u_{2,3b}$ exist for $-\sqrt{8} < q \leq q_1$ as well is left to the persevering reader. $\square$

6.7 Local behaviour near q_2

In Figure 6.10 (or 6.28) we saw that the numerically computed branches Γ_{2a} and Γ_{2b} of even 2-lap periodic solutions approach the bifurcation point $(q_2, 1)$ in the (q, M)-plane from different directions. In this section we present a local analysis at the point $(q_2, 1)$, and compute the angles θ_a and θ_b which these branches make with the positive q-axis at the bifurcation point. Specifically we obtain the following result.

Proposition 6.38 . *Let θ_a and θ_b be the angles with which the branches Γ_{2a} and Γ_{2b} approach $(q_2, 1)$. Then*

$$\tan\theta_a = \frac{2\sqrt{2}}{3} \qquad \textit{and} \qquad \tan\theta_b = -\frac{\sqrt{2}}{3}.$$

We shall discuss the two branches in succession.

The branch Γ_{2a}. The solutions that lie on Γ_{2a} have the properties

$$u''(0) < 0 \quad \text{and} \quad u'''(\xi_2) = 0,$$

and ξ_2 is the first point of symmetry. We use $\varepsilon = 1 - u(0) > 0$ as a small parameter, and we make the *Ansatz*:

$$\begin{aligned} u(\varepsilon) &= 1 + \varepsilon v + \varepsilon^2 w + O(\varepsilon^3), && (6.49a)\\ q(\varepsilon) &= q_0 + \varepsilon q_1 + O(\varepsilon^2), && (6.49b)\\ \xi(\varepsilon) &= \xi_0 + \varepsilon \xi_1 + O(\varepsilon^2). && (6.49c) \end{aligned}$$

To keep the notation simple, we have denoted the *zeroth* order terms in the expansion of q and ξ_2 by, respectively, q_0 and ξ_0. Thus, in this notation, $q_0 = q_2$ and $\xi_0 = \xi_2|_{\varepsilon=0}$. We recall that

$$q_0 = \frac{5}{\sqrt{2}} \quad \text{and hence} \quad a = 2^{3/4} \text{ and } b = 2^{-1/4} \quad (a = 2b),$$

and we obtain from Lemma 6.30 that

$$\xi_0 = \frac{\pi}{b} = 2^{1/4}\pi.$$

In this notation, we need to compute

$$\tan\theta_a = \frac{v(\xi_0)}{q_1}.$$

When we substitute the Ansatz (6.49) into Equation (6.1), use the initial conditions

$$u(0) = 1 - \varepsilon, \quad u'(0) = 0, \quad u''(0) = -\frac{1}{\sqrt{2}}(1 - u^2), \quad u'''(0) = 0,$$

and equate terms of equal order in ε, we obtain

$$\begin{cases} v^{(iv)} + q_0 v'' + 2v = 0 & (6.51a) \\ v(0) = -1, \quad v'(0) = 0, \quad v''(0) = -\sqrt{2}, \quad v'''(0) = 0, & (6.51b) \end{cases}$$

and

$$\begin{cases} w^{(iv)} + q_0 w'' + 2w = -q_1 v''(x) - 3v^2(x) & (6.52a) \\ w(0) = 0, \quad w'(0) = 0, \quad w''(0) = +\frac{1}{\sqrt{2}}, \quad w'''(0) = 0. & (6.52b) \end{cases}$$

The problem for v has been discussed in Section 6.5, where we found that

$$v(x) = \cos(ax) - 2\cos(bx).$$

Note that v does not depend on the value of q_1. For later use we note that

$$v(\xi_0) = 3, \quad v'(\xi_0) = 0, \quad v''(\xi_0) = -6b^2, \quad v'''(\xi_0) = 0, \quad v^{(iv)}(\xi_0) = 18b^4. \tag{6.53}$$

Thus, we find that

$$\begin{aligned} u(\xi, \varepsilon, q) &= 1 + \varepsilon v(\xi_0 + O(\varepsilon)) + O(\varepsilon^2) \\ &= 1 + \varepsilon v(\xi_0) + O(\varepsilon^2) \\ &= 1 + 3\varepsilon + O(\varepsilon^2). \end{aligned} \tag{6.54}$$

In order to compute ξ_1 we write

$$\begin{aligned} u'(\xi, \varepsilon, q) &= \varepsilon v'(\xi_0 + \varepsilon\xi_1 + O(\varepsilon^2)) + \varepsilon^2 w'(\xi_0 + O(\varepsilon), q_1) + O(\varepsilon^3) \\ &= \varepsilon v'(\xi_0) + \varepsilon^2 \xi_1 v''(\xi_0) + \varepsilon^2 w'(\xi_0, q_1) + O(\varepsilon^3) = 0. \end{aligned}$$

Hence

$$\xi_1 = -\frac{w'(\xi_0, q_1)}{v''(\xi_0)}. \tag{6.55}$$

To compute q_1 we observe that we can write

$$w(x, q_1) = w_0(x) + q_1 w_1(x),$$

and an easy computation shows that

$$w_0(x) = f(x) - 3g(x), \tag{6.56}$$

where f is given by

$$f(x) = \frac{1}{\sqrt{2}(a^2 - b^2)}\{-\cos(ax) + \cos(bx)\}.$$

and g is given by

$$g(x) = \frac{1}{a^2 - b^2}\int_0^x K(x-t) v^2(t)\, dt,$$

in which

$$K(x) = \frac{\sin(bx)}{b} - \frac{\sin(ax)}{a}.$$

For $w_1(x)$ we find

$$w_1(x) = -\frac{1}{a^2 - b^2}\int_0^x K(x-t) v''(t)\, dt. \tag{6.57}$$

We wish to choose q_1 in such a way that $u'''(\xi, \varepsilon, q) = 0$. Differentiating u we obtain

$$\begin{aligned} u'''(\xi, \varepsilon, q) &= `\varepsilon v'''(\xi_0) + \varepsilon^2 \xi_1 v^{(iv)}(\xi_0) + \varepsilon^2\left(w_0'''(\xi_0) + q_1 w_1'''(\xi_0)\right) + O(\varepsilon^3) \\ &= \varepsilon^2\left(\xi_1 v^{(iv)}(\xi_0) + w_0'''(\xi_0) + q_1 w_1'''(\xi_0)\right) + O(\varepsilon^3), \end{aligned} \tag{6.58}$$

because $v'''(\xi_0) = 0$ by (6.53). Remembering from (6.55) the expression for ξ_1 we write

$$\xi_1 v^{(iv)}(\xi_0) = -\frac{v^{(iv)}(\xi_0)}{v''(\xi_0)}\Big(w_0'(\xi_0) + q_1 w_1'(\xi_0)\Big). \tag{6.59}$$

We saw in (6.53) that $v^{(iv)}(\xi_0)/v''(\xi_0) = -\frac{3}{\sqrt{2}}$. When we use this in (6.59) and substitute the result into (6.58), we find that

$$u'''(\xi,\varepsilon,q) = \varepsilon^2 X(q_1) + O(\varepsilon^3),$$

where

$$X(q_1) = w_0'''(\xi_0) + \frac{3}{\sqrt{2}} w_0'(\xi_0) + q_1\Big(w_1'''(\xi_0) + \frac{3}{\sqrt{2}} w_1'(\xi_0)\Big).$$

Thus we need to choose q_1 so that $X(q_1) = 0$. Using the expression for w_0 given in (6.56) and for w_1 given in (6.57), we find that

$$w_0'(\xi_0) = 0 \qquad \text{and} \qquad w_0'''(\xi_0) = -\frac{3\pi}{b}$$
$$w_1'(\xi_0) = -\frac{\pi}{3b} \qquad \text{and} \qquad w_1'''(\xi_0) = \frac{7\pi}{3} b.$$

Therefore

$$X(q_1) = -\frac{3\pi}{b} + q_1\Big(\frac{7\pi}{3} b - \frac{3}{\sqrt{2}}\frac{\pi}{3b}\Big) = -\frac{3\pi}{b} + q_1 \frac{4\pi}{3} b,$$

so that $X(q_1) = 0$ if

$$q_1 = \frac{9}{4b^2} = \frac{9}{2\sqrt{2}}.$$

Remembering (6.54), we conclude that the branch Γ_{2a} leaves the point $(q, M) = (q_2, 1)$ under an angle θ_a given by

$$\tan\theta_a = \frac{3}{q_1} = \frac{2\sqrt{2}}{3}.$$

The branch Γ_{2b}. The solutions that lie on Γ_{2b} have the properties

$$u''(0) > 0 \quad \text{and} \quad u'''(\eta_1) = 0.$$

In the notation introduced in (6.49), we again need to compute

$$\tan\theta_b = \frac{v(\xi_0)}{q_1},$$

where now ξ_0 denotes the location of the first maximum of v when $q = q_0$.

For solutions on this branch, η_1 is the first point of symmetry. We expand u and q as in (6.49) drop the subscript 1 from η_1, and write

$$\eta(\varepsilon) = \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2).$$

From Lemma 6.30 we find that

$$v(x) = -\frac{1}{3}\big(\cos(ax) + 2\cos(bx)\big),$$

and

$$\xi_0 = \frac{\pi}{3b}, \quad \eta_0 = \frac{\pi}{b}, \quad v(\xi_0) = \frac{1}{2}, \quad v''(\eta_0) = \frac{\sqrt{2}}{3}, \quad v^{(iv)}(\eta_0) = -\frac{7}{3},$$

so that

$$\frac{v^{(iv)}(\eta_0)}{v''(\eta_0)} = -\frac{7}{\sqrt{2}}.$$

We now proceed exactly as in the previous case, and we find that

$$u'''(\xi,\varepsilon,q) = \varepsilon^2 Y(q_1) + O(\varepsilon^3),$$

where

$$Y(q_1) = w_0'''(\eta_0) + \frac{7}{\sqrt{2}} w_0'(\eta_0) + q_1 \Big(w_1'''(\eta_0) + \frac{7}{\sqrt{2}} w_1'(\eta_0) \Big).$$

For w_0 and w_1 and their derivatives we obtain at η_0:

$$\begin{aligned} w_0'(\eta_0) &= \frac{2\pi}{9b^3} \qquad \text{and} \qquad w_0'''(\eta_0) = -\frac{5\pi}{9b}, \\ w_1'(\eta_0) &= \frac{\pi}{3b} \qquad \text{and} \qquad w_1'''(\eta_0) = -\pi b. \end{aligned}$$

Therefore

$$Y(q_1) = \frac{\pi}{b} + q_1 \frac{4\pi}{3} b,$$

so that $Y(q_1) = 0$ if

$$q_1 = -\frac{3}{2\sqrt{2}}.$$

Since $v(\xi_0) = \frac{1}{2}$, it follows that

$$\tan\theta_b = \frac{1}{2}\frac{1}{q_1} = -\frac{\sqrt{2}}{3}.$$

Near $q = q_2$ and $\alpha = 1$ ($\varepsilon = 0$), we may use (6.49a) and the information about v and w obtained in this section to make the following observations.

1. There exists a $\delta > 0$ such that in the case that $u''(0) < 0$ we have for $q = q_2$

$$u(\xi_2) > 1 \quad \text{and} \quad u'''(\xi_2) < 0 \qquad \text{for } 1 - \delta < \alpha < 1.$$

   Therefore, the proof of the existence of u_{2a} in Theorem 6.32 also holds for $q = q_2 = \frac{5}{\sqrt{2}}$.
2. Again in the case that $u''(0) < 0$, for every $\delta > 0$ we have

$$\begin{aligned} u'''(\xi_2) &< 0 \qquad \text{for} \quad q = q_2 + \varepsilon(q_1 + \delta) \\ u'''(\xi_2) &> 0 \qquad \text{for} \quad q = q_2 + \varepsilon(q_1 - \delta) \end{aligned}$$

   for $\varepsilon > 0$ sufficiently small. Hence, fixing $\delta > 0$, there exists an $\varepsilon_0 > 0$ such that for $q_2 < q < q_2 + \varepsilon_0$ there are *two* periodic solutions u_{2a} and $\tilde{u}_{2a}$ which are symmetric with respect to ξ_2 and such that $u(\xi_2) > 1$. The first one has $1 - \frac{q-q_2}{q_1-\delta} < \alpha < 1 - \frac{q-q_2}{q_1+\delta}$, and it is found by varying α between these values and searching for an α such that $u'''(\xi_2) = 0$. The second one has $\alpha < 1 - \frac{q-q_2}{q_1-\delta}$ and can be found as in the proof of Theorem 6.32. A similar statement holds for the solutions of type u_{2b}, but there is only one solution of this type. Continuity of the branches near $q = q_2$ and $\alpha = 1$ can be proved using the Implicit Function Theorem, but we will not going into that here.

6.8 Even periodic solutions: $q \geq q_3$

In Section 6.4 we have exhibited the existence of an even single bump, or 1-lap periodic solution for $-\sqrt{8} < q < q_1 = \sqrt{8}$. In Section 6.6, we extended this result and showed that there exist *two* even 2-lap periodic solutions for $-\sqrt{8} < q < q_2$ and *two* even 3-lap periodic solutions $-\sqrt{8} < q < q_3$. One solution is *convex* at the origin ($u''(0) > 0$) and one solution is *concave* at the origin ($u''(0) < 0$). The goal of this section is to extend these results to the parameter regime $q \geq q_3$. It is important to note that the sharp qualitative results obtained in Sections 6.4 and 6.6 were proved by means of Lemma 6.12. However, this lemma no longer applies in the range $q \geq q_3$. Thus, to determine the existence and qualitative properties of periodic solutions, we shall develop further analytical techniques.

We recall that an even periodic solution u is called an n-lap periodic solution if it is even, and symmetric with respect to its n^{th} positive critical point ζ_n (and not symmetric with respect to any of the critical points in between 0 and ζ_n). In order to determine if an n-lap periodic solution u attains a relative maximum or minimum at the point of symmetry, one needs to know whether n is odd or even, as well as whether $u''(0)$ is positive or negative. For convenience we list the correspondence in the following table:

$$\left.\begin{array}{ll} \zeta_n = \xi_{(n+1)/2} & \text{when } n \text{ is odd} \\ \zeta_n = \eta_{n/2} & \text{when } n \text{ is even} \end{array}\right\} \quad \text{if } u''(0) > 0,$$

$$\left.\begin{array}{ll} \zeta_n = \xi_{(n+2)/2} & \text{when } n \text{ is even} \\ \zeta_n = \eta_{(n+1)/2} & \text{when } n \text{ is odd} \end{array}\right\} \quad \text{if } u''(0) < 0.$$

We will also always assume that $u(0) \in (-1, 1)$.

In the present section we extend the existence theorems to n-lap periodic solutions for arbitrary $n \geq 2$.

Theorem 6.39 *Let $n \geq 2$, and let*

$$-\sqrt{8} < q < q_n.$$

Then there exist two even n-lap periodic solutions, u_a and u_b, such that

$$u_{a,b}(\zeta_n) > 1 \quad \text{and} \quad u'''_{a,b}(\zeta_n) = 0, \tag{6.60}$$

while $u''_a(0) > 0$ and $u''_b(0) < 0$, and $u_{a,b}(0) \in (-1, 1)$.

We prove Theorem 6.39 in two steps: first, we establish the existence of n-lap solutions which satisfy (6.60) in intervals of the form $(q_{3,2n-1}, q_n)$, and then we show that these n-lap solutions exist on the whole interval $(-\sqrt{8}, q_n)$. Note that the following lemma does not state that ζ_n is the *first* point of symmetry (this fact is postponed until Lemma 6.46).

Lemma 6.40 *Let $n \geq 2$. Then for all $q_{3,2n-1} < q < q_n$ there exist two even periodic solution u_a and u_b which are symmetric with respect to their n^{th} positive critical point ζ_n, such that*

$$u_{a,b}(\zeta_n) > 1 \qquad \text{and} \qquad u''_a(0) > 0, \quad u''_b(0) < 0.$$

Proof. For the cases $n = 2$ and $n = 3$ we refer to the stronger results of Section 6.6. Thus, in the remainder of this proof we assume that $n \geq 4$. We only consider the case where $u''(0) > 0$. The case $u''(0) < 0$ is analogous. Let $u(x, \alpha_-)$ be the *small* amplitude, even, single bump periodic solution for which $u''(0) > 0$. Then $\alpha_- = -M_-$, where $M_- = \|u(\cdot, \alpha_-)\|_\infty < 1$. In particular, $u(\xi_k(\alpha_-), \alpha_-) < 1$ and $u(\eta_k(\alpha_-), \alpha_-) > -1$ for all $k \geq 1$ (see Figure 6.21).

Fix $n \geq 4$ and define $m = (n+1)/2$ if n is *odd*, and $m = n/2$ if n is even. We set

$$a = \sup\{\alpha > \alpha_- \mid u(\xi_i(\alpha), \alpha) \leq 1 \text{ for } i = 1, \ldots, m\}. \tag{6.61}$$

We stress that this definition is completely different from the ones used so far. For all $\alpha > a$ at least one of the maxima ξ_i with $i = 1, 2, \ldots m$ lies in the region $\{u > 1\}$.

We assert that $a < 1$. To see this we study the behaviour of $u(x, \alpha)$ when α is close to 1. Let v be the solution of the problem obtained by linearising around $u = 1$, which was introduced in Section 6.5. Then, according to Lemma 6.31,

$$v(\xi_m) > 0 \quad \text{and} \quad v'''(\xi_m) < 0 \qquad \text{if } q \in (q_{3,2n-1}, q_n).$$

Therefore, there exists a constant $\delta > 0$ such that

$$u(\xi_m(\alpha), \alpha) > 1 \quad \text{and} \quad u'''(\xi_m(\alpha), \alpha) < 0 \qquad \text{for } 1 - \delta < \alpha < 1. \tag{6.62}$$

From the first inequality in (6.62) we deduce that $a < 1$, as claimed.

It follows from the definition of a and Lemma 6.4 that

$$u(\xi_k(a), a) < 1 \text{ for } k = 1, \dots, m-1 \quad \text{and} \quad u(\xi_m(a), a) = 1. \tag{6.63}$$

From (6.63) and the energy identity (6.5) it follows that $u'''(\xi_m) \neq 0$ at a. Because $u(\eta_{m-1}) < u(\xi_{m-1}) < 1$ it follows that

$$u'''(\xi_m(a), a) = u'''(\eta_m(a), a) = u'''(\zeta_n(a), a) > 0, \tag{6.64}$$

where we remark that $\zeta_n = \xi_m$ if n is odd, and $\zeta_n = \eta_m$ if n is even. We now define

$$b = \sup\{\alpha \in (a, 1) \,|\, u(\zeta_n) > 1 \text{ on } (a, \alpha)\}, \tag{6.65}$$

which is well-defined because of the definition of a, Equation (6.63) and Lemma 6.9. Besides, we define (in view of (6.64))

$$c = \sup\{\alpha \in (a, 1) \,|\, u'''(\zeta_n) > 0 \text{ on } (a, \alpha)\}.$$

We now first consider the case that $\zeta_n = \xi_m$ (i.e. n odd). If $b = 1$, then it follows from (6.62) that $c < 1$, thus $u(x, c)$ is an even n-lap periodic solution which is symmetric with respect to ξ_m and $u(\xi_m) > 1$.

If $b < 1$ then we use the following result to obtain a solution.

Lemma 6.41 *Let a and b be defined as in* (6.61) *and* (6.65). *If $b < 1$, then $u'''(\zeta_n) < 0$ at b.*

We postpone the proof of Lemma 6.41 for a moment and first finish the proof of Lemma 6.40. We conclude from Lemma 6.41 that $c < b$, and as we saw before, this implies that there exists an even periodic solution which is symmetric with respect to ξ_m such that $u(\xi_m) > 1$.

The case that $\zeta_n = \eta_m$ (i.e. n even) is dealt with in a similar manner, but we have to distinguish three cases (the situation is similar to the proof of Part (b) of Theorem 6.33). According to Lemma 6.31 we have

$$v'''(\eta_m) < 0 \quad \text{for all } q \in (q_{3,2n-1}, q_n),$$

and

$$\begin{aligned} v(\eta_m) &> 0 \qquad \text{if } q_{3,2n+1} < q < q_n, \\ v(\eta_m) &= 0 \qquad \text{if } q = q_{3,2n+1}, \\ v(\eta_m) &< 0 \qquad \text{if } q_{3,2n-1} < q < q_{3,2n+1}. \end{aligned}$$

If $q_{3,2n+1} < q < q_n$ then the proof is finished in the same way as above. If $q_{3,2n-1} < q < q_{3,2n+1}$ then $b < 1$ and the proof is finished with the help of Lemma 6.41. Finally, if $q = q_{3,2n+1}$ then we have, for $\delta > 0$ small enough,

$$u(\xi_m) > 1 \text{ and } u'''(\eta_m) < 0 \quad \text{for } 1 - \delta < \alpha < 1.$$

We now choose an $\alpha \in (1 - \delta, 1)$. If $u(\eta_m) > 1$ then we finish the proof as in the case where $q_{3,2n+1} < q < q_n$, whereas if $u(\eta_m) \leq 1$ then we finish the proof as in the case where $q_{3,2n-1} < q < q_{3,2n+1}$. This completes the proof of Lemma 6.40. □

Before we give the proof of Lemma 6.41 we introduce some notation. We define the sets of maxima and minima in the region $\{u > 1\}$ by

$$\begin{aligned} \mathcal{C}_+ &= \{1 \leq k \leq m-1 \,|\, u(\xi_k) > 1\}, \\ \mathcal{C}_- &= \{1 \leq k \leq m-1 \,|\, u(\eta_k) > 1\}. \end{aligned}$$

The following proposition shows that for all $\alpha \in [a,b]$ we have $u(\xi_k) > 1$ if and only if $u(\eta_k) > 1$. Besides, if $u(\xi_\ell) > 1$ for some $1 \leq \ell \leq m-1$ then $u(\zeta) > 1$ for all critical points ζ in between ξ_ℓ and ξ_m.

Proposition 6.42 *For all $\alpha \in [a,b]$ we have*

$$\mathcal{C}_+ = \mathcal{C}_- = \varnothing \quad \textit{or} \quad \mathcal{C}_+ = \mathcal{C}_- = \{\ell, \ell+1, \ldots, m-1\} \textit{ for some } \ell \in \{1, 2, \ldots m-1\}. \tag{6.66}$$

Proof. We first notice that $u(0) < 1$ and $u(\xi_m) > 1$ for all $\alpha \in (a,b)$ (note that $u(\eta_m) > 1$ implies that $u(\xi_m) > 1$). For $\alpha = a$ Equation (6.66) holds since $\mathcal{C}_+ = \mathcal{C}_- = \varnothing$. It follows from the definition of a and Lemma 6.9 that $\mathcal{C}_+ = \mathcal{C}_- = \varnothing$ for $\alpha \in (a, a+\delta)$ with $\delta > 0$ sufficiently small. We now use a continuation argument in α to show that Equation (6.66) holds for all $\alpha \in [a,b)$. We define

$$\alpha_* = \sup\{\alpha \in (a,b) \,|\, \text{Equation (6.66) holds on } (a,\alpha)\},$$

and we suppose, by contradiction, that $\alpha_* < b$. Then at α_* we have $u(\xi_n) = 1$ for some $1 \leq n \leq m-1$. There are now two possibilities: either $u(\eta_{n-1}) = 1$ or $u(\eta_n) = 1$.

Concerning the first case, it follows from Lemma 6.4 that at $\alpha = \alpha_*$

$$u(\xi_{n-1}) > 1 \quad \text{and} \quad u(\eta_n) < 1, \tag{6.67}$$

and by continuity (6.67) holds for $\alpha \in (\alpha_* - \delta, \alpha_*)$ with $\delta > 0$ sufficiently small. However, using Lemma 6.9 one finds that this contradicts the fact that Equation (6.66) holds for all $\alpha \in (a, \alpha_*)$. Thus at α_* there is no $k \in \{1, 2, \ldots, m-1\}$ such that $u(\xi_k) = u(\eta_{k-1}) = 1$.

Therefore, we must have $u(\xi_n) = u(\eta_n) = 1$ at α_* for some $1 \leq n \leq m-1$. As before, it follows that

$$u(\xi_{n+1}) > 1 \quad \text{and} \quad u(\eta_{n-1}) < 1.$$

We assert that this implies that $u(\xi_{n-1}) < 1$. Namely, the possibility $u(\xi_{n-1}) > 1$ is excluded by the definition of α_* and the fact that $u(\eta_{n-1}) < 1$. Besides, $u(\xi_{n-1}) = 1$ would imply that $u(\eta_{n-2}) = 1$ which has already been excluded above. Hence $u(\xi_{n-1}) < 1$, and thus also $u(\eta_{n-2}) < 1$. A repeated argument shows that $u(\xi_k) < 1$ for all $1 \leq k \leq n-1$. Analogously it is proved that $u(\eta_k) > 1$ for all $n+1 \leq k \leq m-1$. By continuity this also holds for α close to α_*.

Finally, by the definition of α_* there must exist a sequence $\alpha_i \downarrow \alpha_*$ such that

$$u(\xi_n(\alpha_i), \alpha_i) > 1 \quad \text{and} \quad u(\eta_n(\alpha_i), \alpha_i) < 1.$$

The existence of such a sequence is excluded by the proof of Lemma 6.9, which can be found in [121, Lemma 6.13]. Hence, having obtained a contradiction, we have proved that Equation (6.66) holds on the entire interval $\alpha \in [a,b)$. The case $\alpha = b$ follows by continuity. □

Remark 6.43 It follows from the previous proposition that $u(\xi_k)$ and $u(\eta_k)$ can only enter and leave the region $\{u > 1\}$ together. •

Proof of Lemma 6.41. To prove Lemma 6.41 we argue by contradiction. Thus, suppose that $u'''(\zeta_n) > 0$ at b. Then $\zeta_n = \xi_m = \eta_m$, and it follows that $u(\eta_{m-1}) < 1$ at $\alpha = b$. Therefore, $m-1 \notin \mathcal{C}_-$. By Proposition 6.42, this implies that $u(\eta_k) < 1$ for all $1 \le k \le m-1$, so that $\mathcal{C}_- = \varnothing$. Besides, since $\mathcal{C}_- = \mathcal{C}_+$ by Proposition 6.42, this also implies that $\mathcal{C}_+ = \varnothing$. Thus, we conclude that $u(\xi_k) \le 1$ for all $1 \le k \le m-1$ and $u(\xi_m) = u(\eta_m) = 1$ at $\alpha = b$. Since a was defined as the largest value of α for which this situation occurs, this situation is excluded and we have reached a contradiction. □

We recall the definition

$$c = \sup\{\alpha > a \,|\, u'''(\zeta_n) > 0 \text{ on } (a, \alpha)\}.$$

Clearly $c < b \le 1$ by the proof of Lemma 6.40.

Proposition 6.44 *For all $\alpha \in [a, c]$ we have that*

$$u'''(\xi_k) > 0 \quad \textit{and} \quad u'''(\eta_k) > 0 \qquad \textit{for all } k \in \mathcal{C}_+ = \mathcal{C}_-.$$

Proof. We first notice that since $u(\xi_k)$ and $u(\eta_k)$ only enter $\{u > 1\}$ together, we must have $u'''(\xi_k) > 0$ and $u'''(\eta_k) > 0$ at the point of entry. Suppose now, by contradiction, that there exists a smallest $\alpha \in [a, c]$, for which Proposition 6.44 does not hold: let

$$d = \sup\{\alpha > a \,|\, u'''(\xi_k) > 0,\, u'''(\eta_k) > 0 \text{ for all } k \in \mathcal{C}_+\},$$

and suppose that $d \le c$. Then at $\alpha = d$ there is a critical point $\zeta \in (0, \zeta_n)$ such that $u(\zeta) > 1$ and $u'''(\zeta) = 0$, i.e., ζ is a point of symmetry.

Since by Proposition 6.42 all the critical values between ζ and ζ_n lie above $u = 1$, it follows that $u(x) > 1$ for $x \in [\zeta, \zeta_n]$. In fact, since ζ is a point of symmetry, we have that $u(x) > 1$ for all $x \in [2\zeta - \zeta_n, \zeta_n]$. Since $u(0) < 1$, this means that $2\zeta - \zeta_n > 0$. By symmetry, $2\zeta - \zeta_n$ is a critical point, and from the definition of d we see that $u'''(2\zeta - \zeta_n) \ge 0$. Therefore, again by symmetry, $u'''(\zeta_n) \le 0$, so that from definition of c it follows that $c \le d$. Since by assumption, $d \le c$, we conclude that $d = c$, and $u'''(\zeta_n) = 0$. But then u is symmetric with respect to both ζ and ζ_n. This means that $u(x) > 1$ for all $x \in \mathbb{R}$, which contradicts the fact that $u(0) < 1$. □

Remark 6.45 Another way of obtaining the final contradiction above, is via the observation that

$$(u''' + qu')' = u(1 - u^2) < 0 \qquad \text{on } (\zeta, \zeta_n).$$

Upon integrating over (ζ, ζ_n) we obtain that $u'''(\zeta_n) < 0$ at $\alpha = d$, contradicting the assumption that $d \le c$. •

Lemma 6.46 *Let $n \ge 4$. For every $q \in [q_{n-1}, q_n)$ there exist two even n-lap solutions u_a and u_b such that $u_{a,b}(\zeta_n) > 1$ and $u'''_{a,b}(\zeta_n) = 0$, while $u''_a(0) > 0$ and $u''_b(0) < 0$.*

Proof. Since $q_{3,2n-1} < q_{n-1}$ for $n \ge 4$, it follows immediately from Lemma 6.40 that there exist two periodic solutions which are symmetric with respect to ζ_n, and such that $u_{a,b}(\zeta_n) > 1$. We assert that ζ_n is the *first* point of symmetry. Proposition 6.44 ensures that ζ_n is the first point of symmetry in the region $\{u > 1\}$. Besides, there is no critical point $0 < \zeta < \zeta_n$ with $u'''(\zeta) = 0$ in the region $\{u \le 1\}$, since that would imply (by Proposition 6.42 and the symmetry with respect to 0) that $u(x) \le 1$ for all $x \in \mathbb{R}$, contradicting the fact that $u(\zeta_n) > 1$. □

The following Lemma is a reformulation of Theorem 6.39. It shows that the n-lap solutions obtained in Lemma 6.46 exists for all $q \in (-\sqrt{8}, q_n)$.

Lemma 6.47 *Let $n \geq 2$ and let $-\sqrt{8} \leq q < q_n$. For any $N \geq n$ there exist two even N-lap solutions u_a and u_b such that $u_{a,b}(\zeta_N) > 1$ and $u'''_{a,b}(\zeta_N) = 0$, while $u''_a(0) > 0$ and $u''_b(0) < 0$.*

Proof. The range $q \in (-\sqrt{8}, q_3)$ was already covered in Sections 6.4 and 6.6. For $q \in [q_{n-1}, q_n)$ with $n \geq 4$ we see from Lemma 6.46 that there exists two n-lap solutions. For $N > n$ we again restrict our attention to the case $u''(0) > 0$, the other case being completely analogous. We can define a, b and c as before for both n and N. We have $a_n < c_n < b_n \leq 1$ and $a_N < c_N \leq b_N \leq 1$. If $c_N < 1$ then clearly we have an N-lap solution. Arguing by contradiction we assume that $c_N = 1$. It follows directly from the definition of a and (6.63) that $a_N < a_n$. Proposition 6.44 shows that $u'''(\zeta_n) > 0$ if $u(\zeta_n) > 1$ for all $\alpha \in (a_N, 1]$. However, $u'''(\zeta_n) = 0$ for $\alpha = c_n > a_n > a_N$, a contradiction. □

6.9 Proof of Lemma 6.12

In this section we prove Lemma 6.12. We recall the setting: we assume that u is a solution of Equation (6.1), and that $a \in \mathbb{R}$ is a critical point where u has the following properties:

$$u(a) = 1, \quad u'(a) = 0, \quad u''(a) = 0 \quad \text{and} \quad u'''(a) > 0. \tag{6.68}$$

Then $u' > 0$ in a right-neighbourhood of a so that the point

$$b = \sup\{x > a \,|\, u' > 0 \text{ on } (0, x)\} \tag{6.69}$$

is well defined. By Lemma 6.6, it is also finite.

Let us now recall Lemma 6.12

Lemma 6.48 *Suppose that*

$$-\sqrt{8} < q < q_3 = \sqrt{2}\Big(3 + \frac{1}{3}\Big).$$

Let u be a solution of Equation (6.1) *which at a point $a \in \mathbb{R}$ has the properties listed in* (6.68)*. Then at its next critical point b defined by* (6.69)*, we have*

$$u(b) > 1, \quad u'(b) = 0, \quad u''(b) < 0 \quad \textit{and} \quad u'''(b) < 0.$$

Proof. If $-\sqrt{8} < q \leq \sqrt{8}$ we use the function H introduced in Section 6.4. Since $u > 1$ on (a, b), it follows that $H'' > 0$ on (a, b). At critical points of u, we have by (6.24)

$$H' = u''u'''.$$

Hence $H' = 0$ at a, and therefore $H' > 0$ at b. Since $u''(b) \leq 0$, and even $u''(b) < 0$ by the first integral, it follows that $u'''(b) < 0$, as asserted.

If $q > \sqrt{8}$ we can no longer prove that $H'' > 0$ on (a, b) and we have to proceed differently. Without loss of generality we may assume that $a = 0$ and consider the initial value problem

$$\begin{cases} u^{(iv)} + qu'' + u^3 - u = 0, & \text{(6.70a)} \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = 0 \quad u'''(0) = \lambda, & \text{(6.70b)} \end{cases}$$

where the initial values are derived from (6.68) and λ is a positive number. We denote the solution of problem (6.70) by $u(x, \lambda)$, and its first critical point, corresponding to b, by

$\zeta(\lambda)$:

$$\zeta(\lambda) = \sup\{x > 0 \,|\, u'(\cdot, \lambda) > 0 \text{ on } (0, x)\}.$$

We need to show that

$$u'''(\zeta(\lambda), \lambda) < 0 \quad \text{for every } \lambda > 0. \tag{6.71}$$

As a first step we show that (6.71) is satisfied for λ large enough. This can be proved by means of a simple scaling argument. With the variables

$$x = \lambda^{-1/5} t \quad \text{and} \quad u(x, \lambda) = \lambda^{2/5} w(t, \lambda), \qquad \lambda > 0,$$

the initial value problem (6.70) can be written as

$$\begin{cases} w^{(iv)} + q\lambda^{-2/5} w'' + w^3 - \lambda^{-4/5} w = 0, \\ w(0) = \lambda^{-2/5}, \quad w'(0) = 0, \quad w''(0) = 0, \quad w'''(0) = 1. \end{cases}$$

By standard ODE arguments, $w(t, \lambda) \to W(t)$ on compact intervals, as $\lambda \to \infty$, where W is the solution of the limit problem

$$\begin{cases} W^{(iv)} + W^3 = 0, \\ W(0) = 0, \quad W'(0) = 0, \quad W''(0) = 0, \quad W'''(0) = 1. \end{cases}$$

Plainly,

$$T = \sup\{t > 0 \,|\, W' > 0 \text{ on } (0, t)\} < \infty$$

and $W'''(T) < 0$. Hence, by continuity, for λ large enough, $w'''(\cdot, \lambda) < 0$ at the first zero of $w'(\cdot, \lambda)$. Thus, $u'''(\zeta(\lambda), \lambda) < 0$ for λ large enough.

Proceeding with the proof of Lemma 6.48, we suppose that (6.71) is false and that $u'''(\zeta(\lambda_0), \lambda_0) \geq 0$ for some $\lambda_0 > 0$. Then there exist a constant $\lambda^* \in [\lambda_0, \infty)$ such that $u'''(\zeta(\lambda^*), \lambda^*) = 0$. At $\zeta^* = \zeta(\lambda^*)$ we then have

$$u' > 0 \text{ on } (0, \zeta^*), \qquad u'(\zeta^*) = 0 \quad \text{and} \quad u'''(\zeta^*) = 0. \tag{6.72a}$$

This means that $u(\cdot, \lambda^*)$ is symmetric with respect to ζ^* and that

$$u' < 0 \text{ on } (\zeta^*, 2\zeta^*), \qquad u(2\zeta^*) = 1 \quad \text{and} \quad u'(2\zeta^*) = 0. \tag{6.72b}$$

For further reference, we introduce the notation

$$u(\zeta^*) = \alpha^* > 1 \qquad \text{and hence} \qquad u''(\zeta^*) = \frac{1}{\sqrt{2}}\big(1 - (\alpha^*)^2\big). \tag{6.72c}$$

In the remainder of the proof we show that if $1 < a/b < 3$, where a and b have been defined in (6.37b), i.e. if $\sqrt{8} < q < q_3$, then a solution $u(x, \lambda^*)$ with the properties listed in (6.72) cannot exist. We shift the origin to $x = \zeta^*$ and $u = 1$ and write

$$x = \zeta^* + \tilde{x} \quad \text{and} \quad u(x) = 1 + w(\tilde{x}).$$

Then w is the solution of the problem

$$\begin{cases} w^{(iv)} + qw'' + 2w = -w^2(3 + w), & \text{(6.73a)} \\ w(0) = \alpha^* - 1, \quad w'(0) = 0, \quad w''(0) = \dfrac{1}{\sqrt{2}}\big(1 - (\alpha^*)^2\big), \quad w'''(0) = 0, & \text{(6.73b)} \end{cases}$$

where we used (6.72) and have dropped the tilde. We shall compare the solution $w = w(x, \alpha^*)$ of problem (6.73) with the solution v of the linear problem

$$\begin{cases} v^{(iv)} + qv'' + 2v = 0, & \text{(6.74a)} \\ v(0) = \alpha - 1, \quad v'(0) = 0, \quad v''(0) = \dfrac{1}{\sqrt{2}}(1 - \alpha^2), \quad v'''(0) = 0, & \text{(6.74b)} \end{cases}$$

where $\alpha > 1$ is an arbitrary number, which eventually will be chosen equal to α^*. We denote the solution of problem (6.74) by $v(x)$ or $v(x, \alpha)$. An elementary computation shows that $v(x)$ can be written as

$$v(x) = A\cos(ax) + B\cos(bx), \tag{6.75}$$

where we recall that a and b are the positive roots of the equation

$$k^4 - qk^2 + 2 = 0,$$

given by

$$a^2 = \frac{1}{2}(q + \sqrt{q^2 - 8}) \qquad \text{and} \qquad b^2 = \frac{1}{2}(q - \sqrt{q^2 - 8}). \tag{6.76}$$

The coefficients A and B are given by

$$A = -\frac{b^2(\alpha - 1) + \beta}{a^2 - b^2} \quad \text{and} \quad B = \frac{a^2(\alpha - 1) + \beta}{a^2 - b^2}, \qquad \beta = -\frac{1}{\sqrt{2}}(\alpha^2 - 1). \tag{6.77}$$

Let

$$x_0(\alpha) = \sup\{x > 0 \,|\, v(\cdot, \alpha) > 0 \text{ on } [0, x)\}.$$

In order to proceed with our comparison argument we need the following result.

Lemma 6.49 *Suppose that $\sqrt{8} < q < q_3$ or, equivalently, $1 < \frac{a}{b} < 3$. Then for any $\alpha > 1$ we have $x_0(\alpha) < \infty$ and $v'(\cdot, \alpha) < 0$ on $(0, x_0(\alpha)]$.*

We postpone the proof of this lemma until after the proof of Lemma 6.48 has been completed.

We now continue with the proof of Lemma 6.48. Let

$$y_0(\alpha) = \sup\{x > 0 \,|\, w(\cdot, \alpha) > 0 \text{ on } [0, x)\}.$$

Our goal is to prove that $w'(y_0(\alpha), \alpha) < 0$ for any $\alpha > 0$. Translating this result back to the variables x and u, we find in particular that $u'(2\zeta^*) < 0$, which contradicts (6.72b) and completes the proof.

By the variation of constants formula, we find that

$$w(x) = v(x) - \frac{1}{a^2 - b^2}\int_0^x \Big\{\frac{1}{b}\sin(bt) - \frac{1}{a}\sin(at)\Big\} h(x - t)\, dt,$$

where $h(s) = w^2(s)\{3 + w(s)\} \geq 0$ as long as $w \geq 0$, i.e. on $[0, y_0]$. Note that

$$w'(x) = v'(x) - \frac{1}{a^2 - b^2}\int_0^x \{\cos(bt) - \cos(at)\} h(x - t)\, dt. \tag{6.78}$$

At the first zero x_0 of v (which exists by Lemma 6.49) we have

$$w(x_0) = -\frac{1}{a^2 - b^2}\int_0^x \Big\{\frac{1}{b}\sin(bt) - \frac{1}{a}\sin(at)\Big\} h(x_0 - t)\, dt.$$

Hence, if

$$K(t) \stackrel{\text{def}}{=} \frac{1}{b}\sin(bt) - \frac{1}{a}\sin(at) > 0 \quad \text{for } 0 < t < x_0, \tag{6.79}$$

then $w < v$ on $(0, x_0]$, and therefore $y_0 < x_0$. In addition, if

$$K'(t) = \cos(bt) - \cos(at) > 0 \quad \text{for } 0 < t < x_0, \tag{6.80}$$

then $w' < v' < 0$ on $(0, y_0]$ by (6.78). In particular, $w'(y_0) < 0$, as asserted.

The conclusion of the proof consists of a detailed analysis of the function $K(t)$ to show that (6.79) and (6.80) hold on $(0, y_0]$. An elementary computation shows that

$$K(0) = 0, \quad K'(0) = 0, \quad K''(0) = 0, \quad K'''(0) = a^2 - b^2 > 0.$$

Hence $K > 0$ and $K' > 0$ in a right-neighbourhood of the origin. We set

$$t_1 = \sup\{t > 0 \,|\, K > 0 \text{ on } (0, t)\},$$

and

$$t_0 = \sup\{t > 0 \,|\, K' > 0 \text{ on } (0, t)\}.$$

Plainly $0 < t_0 < t_1$ and $K'(t_1) \leq 0$. We recall the assumption that

$$1 < \frac{a}{b} < 3 \quad \Leftrightarrow \quad \frac{\pi}{2a} < \frac{\pi}{2b} < \frac{3\pi}{2a} < \frac{3\pi}{2b}. \tag{6.81}$$

Because $a/b > 1$, it follows from (6.79) and (6.81) that $t_1 \in (\frac{\pi}{2b}, \frac{3\pi}{2b})$, and hence, that $t_0 < \frac{3\pi}{2b}$. On the other hand, since $\frac{\pi}{2a} < \frac{\pi}{2b}$ it is clear that $K' > 0$ on $(0, \frac{\pi}{2a}]$ so that $t_0 > \frac{\pi}{2a}$. Observe that

$$\cos(at) < 0 \quad \text{on } \left(\tfrac{\pi}{2a}, \tfrac{3\pi}{2a}\right) \qquad \text{and} \qquad \cos(bt) > 0 \quad \text{on } \left(0, \tfrac{\pi}{2b}\right).$$

Because $a/b < 3$ and hence $\frac{3\pi}{2a} > \frac{\pi}{2b}$, it follows that $K' > 0$ on $[\frac{\pi}{2a}, \frac{\pi}{2b}]$, and we conclude that

$$\frac{\pi}{2b} < t_0 < \frac{3\pi}{2b}. \tag{6.82}$$

The value of v at t_0 can easily be computed by using the formula (6.80) for K' in the expression (6.75) for v. We obtain

$$v(t_0) = \frac{\alpha - 1}{a^2 - b^2} \cos(bt_0),$$

which, in view of (6.82), shows that $v(t_0) < 0$. We conclude that $x_0 < t_0 < t_1$, and hence that the properties (6.79) and (6.80) of respectively K and K' are true. This completes the proof of Lemma 6.48. □

Proof of Lemma 6.49. We first show that the assertion is true for

$$\alpha > q\sqrt{2} - 1. \tag{6.83}$$

Note that

$$(v''' + qv')' = -2v < 0 \quad \text{as long as} \quad v > 0, \tag{6.84}$$

and hence, still as long as $v > 0$,

$$v''(x) + qv(x) \leq v''(0) + qv(0) = (1 - \alpha)\Big(\frac{1 + \alpha}{\sqrt{2}} - q\Big).$$

Therefore, if (6.83) holds then

$$v'' < -qv < 0 \quad \text{as long as} \quad v > 0,$$

so that $x_0(\alpha) < \infty$ and $v'(\cdot, \alpha) < 0$ for $0 < x \leq x_0(\alpha)$.

Let

$$\alpha^* = \inf\{\hat{\alpha} > 1 \,|\, x_0(\alpha) < \infty \text{ and } v'(\cdot, \alpha) < 0 \text{ for } \hat{\alpha} < \alpha < \infty\}.$$

If $\alpha^* = 1$, then the assertion is proved. Therefore, we suppose that $\alpha^* > 1$. We distinguish two cases:

(*i*) $x_0(\alpha^*) = \infty$ and $v'(\cdot, \alpha^*) \leq 0$ for $0 \leq x < \infty$, or

(*ii*) $x_0(\alpha^*) < \infty$ and $v'(x_1, \alpha^*) = 0$, for some $x_1 \leq x_0(\alpha^*)$.

To keep the notation as simple as possible, we shall henceforth omit the asterisk when referring to α^*.
Case (i): Because $v' = 0$ and $v''' = 0$ at the origin, integration of (6.84) over $(0, x)$ shows that

$$v'''(x) + qv'(x) < 0 \quad \text{for } x > 0.$$

Hence, integrating (6.84) again, but now over $(1, x)$ we conclude that

$$(v''(x) + qv(x))' < v'''(1) + qv(1) = -\delta \quad \text{for } x > 1,$$

where δ is a positive constant. Hence

$$v''(x) + qv(x) < C - \delta(x-1) \quad \text{and therefore} \quad v''(x) < C - \delta(x-1) \quad \text{for } x > 1,$$

where $C = v''(1) + qv(1)$ is a constant. Thus $v''(x) \to -\infty$ as $x \to \infty$, which implies that $x_0(\alpha) < \infty$, a contradiction.
Case (ii): Suppose that $x_1 < x_0(\alpha)$. Then

$$v(x_1) > 0, \quad v'(x_1) = 0 \quad \text{and} \quad v''(x_1) = 0. \tag{6.85}$$

When we multiply Equation (6.74a) by v', integrate over $(0, x)$ and use the initial conditions, we obtain

$$v'v''' - \frac{1}{2}(v'')^2 + \frac{q}{2}(v')^2 + v^2 = (\alpha - 1)^2\Big\{1 - \frac{1}{4}(1+\alpha)^2\Big\} \quad \text{for } x \in [0, x_0],$$

Evaluating the left-hand side at x_1, using the properties of v at x_1 listed in (6.85), we obtain

$$v^2(x_1) = (\alpha - 1)^2\Big\{1 - \frac{1}{4}(1+\alpha)^2\Big\}. \tag{6.86}$$

Since $\alpha > 1$, the right-hand side of (6.86) is negative, while the left-hand side is nonnegative, a contradiction.

It remains to consider the case that $x_1 = x_0$, so that

$$v(x_0) = 0 \quad \text{and} \quad v'(x_0) = 0. \tag{6.87}$$

We begin with a preliminary result.

Lemma 6.50 *Suppose that* $1 < a/b < 3$. *Then, if the solution* v *of problem* (6.74) *has the properties* (6.87), *the point* x_0 *must lie in the interval* $(0, \frac{\pi}{2b})$.

Proof. We recall the formula for v:

$$v(x) = A\cos(ax) + B\cos(bx), \tag{6.88}$$

where a and b are given in (6.76) and A and B are given in (6.77). We discuss in succession the three cases:

$$(i)\ A = 0, \quad (ii)\ A < 0 \quad \text{and} \quad (iii)\ A > 0.$$

Case (i): $A = 0$. Since $A + B = \alpha - 1$, it follows that in this case $B = \alpha - 1$ and hence

$$v(x) = (\alpha - 1)\cos(bx).$$

Thus, $x_0 = \frac{\pi}{2b}$. However, $v'(x_0) < 0$, which contradicts (6.87), so that A cannot be zero.
Case (ii): $A < 0$. Observe that in this case

$$B\cos(bx_0) = |A|\cos(ax_0),$$

and $B = \alpha - 1 - A = \alpha - 1 + |A| > |A|$. Remembering that $a > b$, we conclude that

$$B\cos(bx) > |A|\cos(ax) \ge 0 \qquad \text{if } 0 \le x \le \frac{\pi}{2a}.$$

By assumption

$$\frac{\pi}{2} < \frac{a}{b}\frac{\pi}{2} < \frac{3\pi}{2}. \tag{6.89}$$

and hence

$$B\cos(bx) \geq 0 > |A|\cos(ax) \qquad \text{if } \frac{\pi}{2a} < x \leq \frac{\pi}{2b}.$$

Therefore $\frac{\pi}{2b} < x_0 < \frac{3\pi}{2b}$. This implies that

$$v'' + a^2 v = (a^2 - b^2)\cos(bx) < 0 \qquad \text{on } \Big(\frac{\pi}{2b}, x_0\Big),$$

and hence, because $v' < 0$ on $(0, x_0)$,

$$\{(v')^2 + a^2 v^2\}' = 2(v'' + a^2 v)v' > 0 \qquad \text{on } \Big(\frac{\pi}{2b}, x_0\Big).$$

When we integrate this inequality over $(\frac{\pi}{2b}, x_0)$, and use (6.87), we find that

$$\{(v')^2 + a^2 v^2\}\big|_{\pi/2b} < 0,$$

a contradiction, hence $A < 0$ cannot occur either.
Case (iii): $A > 0$. We have

$$v\Big(\frac{\pi}{2b}\Big) = A\cos\Big(\frac{a}{b}\frac{\pi}{2}\Big).$$

But, in view of (6.89), $\cos\big(\frac{a}{b}\frac{\pi}{2}\big) < 0$. This means that $v\big(\frac{\pi}{2b}\big) < 0$, so that $x_0 < \frac{\pi}{2b}$. Summarising, we have found that the constant A in (6.88) must be positive, and hence, that $x_0 < \frac{\pi}{2b}$. □

We continue the proof of Lemma 6.49. Using the explicit expression (6.88) for v, we deduce from (6.87) that the constants A and B must satisfy the equations

$$A\cos(ax_0) + B\cos(bx_0) = 0 \qquad \text{and} \qquad aA\sin(ax_0) + bB\sin(bx_0) = 0. \tag{6.90}$$

Since $A + B = \alpha - 1$, they cannot both be equal to zero. Therefore, the determinant of the system (6.90) of equations must be zero, and hence

$$a\cos(bx_0)\sin(ax_0) = b\cos(ax_0)\sin(bx_0). \tag{6.91}$$

By Lemma 6.50, $\cos(bx_0) > 0$. Hence, if $\cos(ax_0) = 0$, then $\sin(ax_0) = 0$ as well, and this is impossible. Thus, $\cos(ax_0) \neq 0$ and we may divide (6.91) by $\cos(ax_0)\cos(bx_0)$. We thus find that x_0 must be a solution of the equation

$$\tan(bx) = \frac{a}{b}\tan(ax) \qquad \text{in} \quad \Big(0, \frac{\pi}{2a}\Big) \cup \Big(\frac{\pi}{2a}, \frac{\pi}{2b}\Big), \tag{6.92}$$

because $\frac{3\pi}{2a} > \frac{\pi}{2b}$ by assumption. With $\lambda = a/b$ and $bx = t$, we can write (6.92) as

$$\phi(t) \stackrel{\text{def}}{=} \lambda\tan(\lambda t) - \tan(t) = 0 \qquad \text{in} \quad I \stackrel{\text{def}}{=} \Big(0, \frac{\pi}{2\lambda}\Big) \cup \Big(\frac{\pi}{2\lambda}, \frac{\pi}{2}\Big). \tag{6.93}$$

Thus, we seek a root $\tau = bx_0$ of Equation (6.93). However, the following Lemma shows that such roots do not exist, which completes the proof of Lemma 6.49 □

Lemma 6.51 *If $1 < \lambda < 3$, then equation $\phi(t) = 0$ has no roots in the set I.*

Proof. Observe that

$$\lambda\tan(\lambda t) > \tan(t) \qquad \text{if } 0 < t < \frac{\pi}{2\lambda} \tag{6.94a}$$

and

$$\lambda\tan(\lambda t) < 0 < \tan(t) \qquad \text{if } \frac{\pi}{2\lambda} < t < \frac{\pi}{\lambda}. \tag{6.94b}$$

If $\lambda \leq 2$, then $\pi/2\lambda \geq \pi/2$, so that it is immediately clear from (6.94) that ϕ cannot have a zero in I.

If $2 < \lambda < 3$, then $\pi/\lambda < \pi/2 < 3\pi/2\lambda$ and it follows that if $\phi(t)$ has a zero in I, then it must lie in the interval $(\frac{\pi}{\lambda}, \frac{\pi}{2})$, where both terms in $\phi(t)$ are positive. Plainly,

$$\phi(\frac{\pi}{\lambda}) = -\tan\Big(\frac{\pi}{\lambda}\Big) < 0 \quad \text{and} \quad \phi(t) \to -\infty \qquad \text{as } t \to \frac{\pi}{2}^{-}. \tag{6.95}$$

Suppose that $\phi(t)$ has a zero, and that t_0 is the largest zero on $(\frac{\pi}{\lambda}, \frac{\pi}{2})$. Then (6.95) implies that $\phi'(t_0) \leq 0$. However, an easy computation shows that

$$\phi'(t) = \lambda^2 - 1 > 0 \quad \text{when } \phi(t) = 0.$$

Thus, we have a contradiction, and $\phi(t)$ cannot have a zero in I. This completes the proof of Lemma 6.51. □

6.10 Linearisation

We linearise around the constant solution $u = 1$ of the equation

$$u^{(iv)} + qu'' + u^3 - u = 0.$$

Looking at the initial value problem (6.22), we write $u = 1 + \varepsilon v$ and $\alpha = 1 - \varepsilon$. Omitting higher order terms, we obtain

$$\begin{cases} v^{(iv)} + qv'' + 2v = 0. & (6.96a) \\ v(0) = -1, \quad v'(0) = 0, \quad v''(0) = \pm\sqrt{2}, \quad v'''(0) = 0. & (6.96b) \end{cases}$$

We denote the corresponding solutions by $v_{\pm}(x)$. Substitution of $v(x) = e^{\lambda x}$ yields the characteristic equation

$$\lambda^4 + q\lambda^2 + 2 = 0. \tag{6.97}$$

We distinguish two cases:

$$(i)\ \ -\sqrt{8} < q < \sqrt{8} \qquad \text{and} \qquad (ii)\ \sqrt{8} \leq q < \infty.$$

Case (i): Equation (6.97) has roots

$$\lambda = \pm a \pm ib \tag{6.98a}$$

in which $a > 0$ and $b > 0$ are given by

$$a = \frac{1}{2}\sqrt{\sqrt{8} - q} \qquad \text{and} \qquad b = \frac{1}{2}\sqrt{\sqrt{8} + q}. \tag{6.98b}$$

An elementary computation shows that the solutions $v_+(x)$ and $v_-(x)$ of problem (6.96) are given by

$$v_{\pm}(x) = -\cosh(ax)\cos(bx) + K_{\pm}\sinh(ax)\sin(bx),$$

where

$$K_+ = \frac{a}{b} = \sqrt{\frac{\sqrt{8} - q}{\sqrt{8} + q}} \qquad \text{and} \qquad K_- = -\frac{b}{a}.$$

Thus,

$$v'_+(x) = (K_+ a + b)\cosh(ax)\sin(bx),$$
$$v'_-(x) = (K_- b - a)\sinh(ax)\cos(bx).$$

We see that for $u(x, \alpha)$ with $u''(0) < 0$ (i.e. corresponding to v_-) we obtain

$$\eta_1(\alpha) \to \frac{\pi}{\sqrt{\sqrt{8}+q}} \quad \text{and} \quad \xi_2(\alpha) \to \frac{3\pi}{\sqrt{\sqrt{8}+q}} \qquad \text{as } \alpha \to 1.$$

and, since $\varepsilon = 1 - \alpha$,

$$u_-(\eta_1(\alpha), \alpha) \sim 1 - (1-\alpha)\frac{b}{a}\sinh\Big(\frac{\pi a}{2b}\Big) \qquad \text{as } \alpha \to 1.$$

Case (*ii*): Equation (6.97) has roots

$$\lambda = \pm ia \quad \text{and} \quad \lambda = \pm ib,$$

where

$$a^2 = \frac{1}{2}(q + \sqrt{q^2 - 8}) \qquad \text{and} \qquad b^2 = \frac{1}{2}(q - \sqrt{q^2 - 8}).$$

For the solutions $v_\pm(x)$ of problem (6.96) we find

$$v_\pm(x) = A_\pm \cos(ax) + B_\pm \cos(bx),$$

in which

$$A_\pm = \frac{b^2 \mp \sqrt{2}}{a^2 - b^2} \qquad \text{and} \qquad B_\pm = -\frac{a^2 \mp \sqrt{2}}{a^2 - b^2}.$$

Properties of the solutions $v_\pm(x)$ and their critical points and critical values are given in Lemma 6.31.

Chapter 7

Second order Lagrangians and Twist maps

7.1 Introduction

Various mathematical models for problems in nonlinear elasticity, nonlinear optics, solid mechanics, etc. are derived from *second order* Lagrangian principles, i.e., the differential equations are obtained as the Euler-Lagrange equations of a Lagrangian L that depends on a state variable u, and its first and second order derivatives. The Euler-Lagrange differential equations are fourth order and are of conservative nature.

In scalar models the Lagrangian action is defined by $J[u] = \int L(u, u', u'')\, dt$. A second order Lagrangian system is, under suitable assumptions on the u''-dependence of L, equivalent to a Hamiltonian system on $\mathbb{R}^4$. Trajectories of the Lagrangian system, and thus Hamiltonian system, lie on three dimensional sets $M_E \stackrel{\text{def}}{=} \{H = E\}$, where H is the Hamiltonian (conserved quantity). The sets M_E are smooth manifolds for all regular E values of H (i.e. $\nabla H|_{M_E} \neq 0$), and are non-compact for all $E \in \mathbb{R}$. It turns out that for Hamiltonian systems that come from second order Lagrangians, one can find a natural two dimensional section $\{u' = 0\} \cap M_E$ which bounded trajectories have to intersect finitely or infinitely many times (possibly only in the limit) [96]. This section will be denoted by Σ_E and $\Sigma_E = N_E \times \mathbb{R}$, where N_E is a one dimensional set defined by:

$$N_E = \left\{(u, u'') \;\middle|\; \frac{\partial L}{\partial u''}u'' - L(u, 0, u'') = E\right\} \tag{7.1}$$

(see Section 7.1.1 for more details). The Hamiltonian flow induces a return map to the section Σ_E, and closed trajectories (closed characteristics) correspond to fixed points of iterates of this map. In many situations the return map is an analogue of a monotone area-preserving Twist map (see e.g. [13, 102, 98]). The theory developed in this chapter will be centred around this property. Lagrangian systems that allow such Twist maps will be referred to as *Twist systems*. Definitions and a precise analysis will be given in the forthcoming sections. This chapter will be concerned with the basic properties of Twist systems and the study of simple closed characteristics. These are periodic trajectories that, when represented in the (u, u')-plane (configuration plane of the Lagrangian system), are simple closed curves. In Chapter 8 we will investigate more elaborate types of closed characteristics via a Morse type theory. One of the main results of this chapter is the following.

Theorem 7.1 *Consider a Twist system with Lagrangian L, and let E be a regular value. If N_E has a compact connected component $\tilde{N}_E$, then there exists at least one simple closed characteristic at energy level E with $u(t) \in \pi^u \tilde{N}_E$ for all $t \in \mathbb{R}$ (where $\pi^u \tilde{N}_E$ is the projection of $\tilde{N}_E$ onto the u-coordinate).*

A precise statement of this result will be presented in Section 7.3.1 together with information about the location and the Morse index of the trajectory (Theorem 7.11). The results in this chapter are proved for Twist systems. We can safely conjecture that Theorem 7.1 remains true even without the Twist property. This can for example be achieved via continuation to a Twist system within the class of simple closed curves in the (u, u')-plane. This exploits the idea that no simple closed characteristics exist on the boundary of the class of simple closed curves. Certain mild growth conditions on L are needed in this case (also for the continuation). This idea will be subject of future study.

For singular energy levels a similar theorem can be proved (Theorem 7.13). The bottom line is that under the same compactness assumptions there exists a simple closed characteristic in the broader sense of the word, i.e., depending on possible singularities a closed characteristics is either a regular simple closed trajectory, a simple homoclinic loop, or a simple heteroclinic loop. We also explain how singularities can lead to multiplicity of closed characteristics (this issue is addressed in full in Chapter 8).

In Section 7.4 we give some more background information on Twist maps including a few observations deduced from numerical calculations. We also briefly discuss the analogues of KAM-tori/circles for second order Lagrangian systems, and the issue of integrability versus non-integrability.

Throughout the chapter specific examples of physical systems will be given such as the Extended Fisher-Kolmogorov (EFK) and Swift-Hohenberg equations ($u'''' - \alpha u'' + F'(u) = 0$ with $\alpha \in \mathbb{R}$). The theory developed in this chapter also applies to systems on $M = S^1$ by simply assuming L to be periodic in u.

The organisation of the chapter is as follows. We introduce the concepts that play a major role in our analysis in Sections 7.1.1–7.1.3. The definition of the Twist property is stated in Section 7.2.1. Some examples of Twist systems are given in Section 7.2.2 and in Section 7.2.3 we explain to what extent the assumptions can be weakened. Subsequently, we apply the theory to Twist systems on energy levels with (Sections 7.3.2 and 7.3.3) and without singular points (Sections 7.3.1). We deal with non-compact interval components in Section 7.3.4. In Section 7.4 we list some concluding remarks. Finally, Sections 7.5 and 7.6 are devoted to the classification of equilibrium points and the proof of the Twist property for a specific class of second order Lagrangians.

7.1.1 Second order Lagrangians

Let $L : \mathbb{R}^3 \to \mathbb{R}$ be a C^2-function of the variables u, v, w. For any smooth function $u : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, define the functional $J[u] = \int_I L(u, u', u'')\, dt$, which is called is the *(Lagrangian) action* of u. The function L may be regarded as a function on 2-jets on $\mathbb{R}$, and is generally referred to as the *Lagrangian*[1]. The pair (L, dt) is called a second order Lagrangian system on $\mathbb{R}$. The action J of the Lagrangian system is said to be stationary at a function u if $\delta J[u] = 0$

[1]In the case of a general smooth 1-dimensional manifold M one defines L as a smooth function on 2-jet space of M. The action is then defined by considering functions $(u, u', u'') : I \to J^2M$.

with respect to variations $\delta u \in C_c^\infty(I,\mathbb{R})$, i.e.

$$\begin{aligned}\delta J[u] &= \delta \int_I L(u,u',u'')dt = \int_I \left[\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial v}\delta u' + \frac{\partial L}{\partial w}\delta u''\right] dt \\ &= \int_I \left[\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial v} + \frac{d^2}{dt^2}\frac{\partial L}{\partial w}\right]\delta u dt = 0.\end{aligned}$$

A stationary function u thus satisfies the differential equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u''} = 0,$$

which is called the *Euler-Lagrange equation* of the Lagrangian system (L, dt). The Lagrangian action J is invariant under the $\mathbb{R}$-action $t \mapsto t + c$, which by Noether's Theorem yields the conservation law

$$\Big(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\Big)u' + \frac{\partial L}{\partial u''}u'' - L(u,u',u'') = \text{ constant} \tag{7.2}$$

(see for instance [100]). This conservation law is called the *Hamiltonian*.

If L is strictly convex in the w-variable then the Lagrangian system (L, dt) is equivalent to a Hamiltonian system on $\mathbb{R}^4$ with the standard symplectic structure. Therefore we assume:

(H) $\partial_w^2 L(u,v,w) \geq \delta > 0$ *for all* (u,v,w).

The correspondence between a Lagrangian system (L, dt) on $\mathbb{R}$ and a Hamiltonian system (H,ω) on $\mathbb{R}^4$ can be explained as follows. Let $x = (p_u, p_v, u, v)$ be symplectic coordinates for $\mathbb{R}^4$ with the symplectic form given by $\omega = dp_u \wedge du + dp_v \wedge dv$. Define the Hamiltonian $H(x) = p_u v + L^*(u,v,p_v)$, where $L^*(u,v,p_v) = \max_{w\in\mathbb{R}}\{p_v w - L(u,v,w)\}$ is the Legendre transform of L. Since L is strictly convex in w we have that L^* is strictly convex in p_v. Moreover, $\partial_{p_v}L^* = \big(\partial_w L\big)^{-1}(p_v) = w$, hence $H(x) = p_u v + p_v\big(\partial_w L\big)^{-1}(p_v) - L(u,v,\big(\partial_w L\big)^{-1}(p_v))$. For any function $x : I \to \mathbb{R}$ the Hamiltonian action is defined by $\mathcal{A}[x] = \int_I \big[p_u u' + p_v v' - H(x)\big]dt$. A function x is stationary for $\mathcal{A}$ if and only if the u-coordinate is stationary for J. In particular, the Euler-Lagrange equations for $\mathcal{A}$ are of the form $x' = X_H(x)$, where $X_H = \mathcal{J}\nabla H$ and $\mathcal{J}$ is defined by $\omega(x, \mathcal{J}y) = \langle x, y\rangle$ (where $\langle x, y\rangle$ is the standard inner product in $\mathbb{R}^4$). X_H is called the Hamiltonian vector field associated to H. The correspondence between u and its derivatives and x is given by: $v = u'$, $p_u = \partial_{u'}L - p_v'$, and $p_v = \partial_{u''}L$. See for example [9] for more details on this correspondence. The state space $\mathbb{R}^4$ of the Hamiltonian system (H,ω) is often referred to as the *phase space* and $J^1\mathbb{R} = \mathbb{R}^2$ is called the *configuration space*[2].

If the Hamiltonian is sufficiently smooth then the Hamiltonian system $x' = X_H(x)$ generates a local flow on $\mathbb{R}^4$. If we assume strict convexity of L in the w-variable then H is of class C^1. Under hypothesis (H) the Hamiltonian $H(x)$ is a C^2-function[3], which in return generates a local C^1-flow ϕ_H^t on $\mathbb{R}^4$ via the equation $x' = X_H(x)$.

Stationary functions of J satisfy Equation (7.2), which is equivalent to $H(x) = E \in \mathbb{R}$. For the associated Hamiltonian system (H,ω) this means that the stationary motions lie

[2]In the general case the configuration space is J^1M and the phase space is TJ^1M.

[3]In order to study stationary points of $\mathcal{A}$ additional regularity for H is not required. One does usually need proper growth conditions on H.

on the 3-dimensional sets $M_E = \{x \in \mathbb{R}^4 \mid H(x) = E\}$. If $\nabla H \neq 0$ on M_E then E is called a *regular value* and M_E is a smooth non-compact manifold without boundary. The vector field X_H restricted to M_E is non-singular when E is a regular value. Indeed, the singular points of the vector field X_H, i.e. points x_* such that $X_H(x_*) = 0$, are exactly the critical points of the Hamiltonian, and thus only occur at singular energy levels. Singular points are of the form $x_* = (p_u, p_v, u, 0)$ and are given by: $\partial_u L(u,0,0) = 0$, $p_u = \partial_v L(u,0,0)$ and $p_v = \partial_w L(u,0,0)$. Equivalently, for a Lagrangian system an energy level E is said to be regular if and only if $\frac{\partial L}{\partial u}(u,0,0) \neq 0$ for all points $u \in \mathbb{R}$ that satisfy the relation $-L(u,0,0) = E$.

A bounded characteristic of a Lagrangian system (L, dt) is a function $u \in C^2_b(\mathbb{R},\mathbb{R})$ for which $\delta \int_I L(u,u',u'')\,dt = 0$ with respect to variations $\delta u \in C^2_c(I,\mathbb{R})$ for any compact interval $I \subset \mathbb{R}$. Since the Lagrangian is a C^2-function of the variables (u,v,w) it follows from the Euler-Lagrange equations that $u \in C^3_b(\mathbb{R},\mathbb{R})$, $\frac{\partial L}{\partial w}(\cdot) \in C^2_b(\mathbb{R},\mathbb{R})$, and $\big(\frac{d}{dt}\frac{\partial L}{\partial w} - \frac{\partial L}{\partial v}\big)(\cdot) \in C^2_b(\mathbb{R},\mathbb{R})$ (regularity of critical points of L). This is equivalent to having a function $x \in C^2_b(\mathbb{R},\mathbb{R}^4)$ which is stationary for $\mathcal{A}[x]$: a bounded characteristic for the associated Hamiltonian system (H,ω).

The question now arises, given an energy value E, do there exist bounded and/or closed characteristics (see Section 7.1.3 for a definition) on M_E, and how many, and how are these questions related to geometric and topological properties of M_E.

7.1.2 Cross-sections and area-preserving maps

From (H) it follows that bounded solutions of the Euler-Lagrange equations only have isolated extrema (well-posedness of the initial value problem for $x' = X_H(x)$). Therefore a bounded characteristic has either finitely, or infinitely many isolated local extrema. For the associated Hamiltonian system this means that a bounded trajectory always intersects the section $\Sigma_E = \{v = 0\} \cap M_E = \{(p_u, p_v, u, 0) \mid p_u \in \mathbb{R},\ p_v = \partial_w L(u,0,w),\ (u,w) \in N_E\}$, where N_E is defined by (7.1)[4]. In the case that there are only finitely many (or zero) intersections, $x(t)$ must be asymptotic as $t \to \pm\infty$ to singular points of X_H, and thus critical points of H. If E is a regular value this possibility is excluded. A bounded solution u is therefore a concatenation of monotone laps between extrema (an increasing lap followed by a decreasing lap and vice versa), at least if we assume that u does not have critical inflection points, i.e. Σ_E is not intersected in a point where $w = 0$. In this context it is important to note that if E is a regular value then critical inflection points can only occur at the boundary of

$$\pi^u N_E \stackrel{\text{def}}{=} \{u \mid (u,w) \in N_E \text{ for some } w \in \mathbb{R}\} = \{u \mid L(u,0,0) + E \geq 0\}.$$

The last equality follows from the definition of N_E and the fact that $\partial_w(w\partial_w L - L) = w\partial^2_w L$ in combination with hypothesis (H). We will be interested in bounded characteristics that avoid critical inflection points. It will follow later on that at regular energy values critical inflection points cannot occur (Lemma 7.7).

Recalling that $w = \partial_{p_v} L^*$ we define $N^+_E = \{(u,p_v) \in N_E \mid \partial_{p_v} L^*(u,0,p_v) > 0\}$, $N^-_E = \{(u,p_v) \in N_E \mid \partial_{p_v} L^*(u,0,p_v) < 0\}$, and $N^0_E = \{(u,p_v) \in N_E \mid \partial_{p_v} L^*(u,0,p_v) = 0\}$. It follows from hypothesis (H) that N^+_E and N^-_E are smooth graphs over the u-axis and $\pi^u N^+_E =$

[4] It is sometimes convenient to define N_E in terms of coordinates (u, p_v) by using the formula $p_v = \partial_w L$.

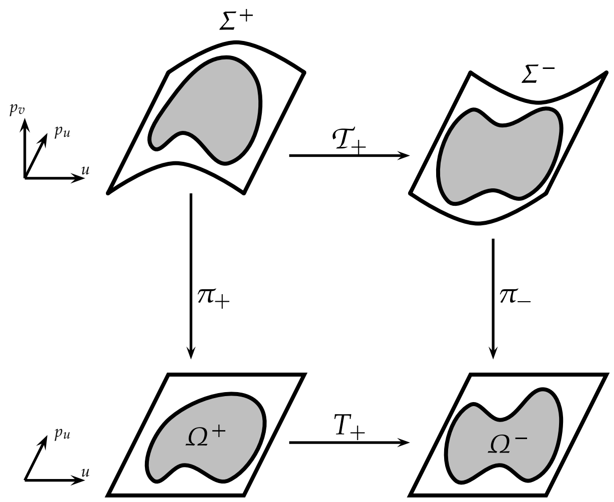

Figure 7.1: The map $\mathcal{T}_+$, which is induced by the flow, and its projection T_+.

$\pi^u N_E^-$. The sets $\Sigma_E^\pm = N_E^\pm \times \mathbb{R}$ are smooth surfaces over the (p_u, u)-plane. Thus, the projections $\pi_\pm : \Sigma_E^\pm \to \pi^u N_E^\pm \times \mathbb{R}$ are invertible. For a given bounded trajectory $x(t)$ we therefore only need to know the (p_u, u)-coordinates of the intersections of $x(t)$ with $\Sigma_E^\pm$. Consequently, bounded characteristics can be identified with sequences of points (p_{u_i}, u_i) in the (p_u, u)-plane.

In the following we fix the energy level E and drop the subscript in the notation. The vector field X_H is transverse to the section $\Sigma^+ \cup \Sigma^-$ (non-transverse at Σ^0). It therefore makes sense to consider the Poincaré return maps, i.e., maps from Σ^+ to Σ^- and from Σ^- to Σ^+, by following the flow φ_H^t starting at Σ^+ until it intersects Σ^-. It may happen that φ_H^t does not intersect Σ^- at all. For the points in Σ^+ for which the flow does intersect Σ^- we have defined a map $\mathcal{T}_+$ from Σ^+ to Σ^-[5]. The same can be done for the map $\mathcal{T}_-$ mapping from Σ^- to Σ^+. Since $\Sigma^\pm$ are graphs over the (p_u, u)-plane the above defined maps induce maps $T_\pm = \pi_\mp \mathcal{T}_\pm \pi_\pm^{-1}$ between open regions $\Omega^\pm \subset \pi_\pm \Sigma^\pm$, i.e. $T_\pm : \Omega^\pm \to \Omega^\mp$ (see also Figure 7.1). For any point $(p_u, u) \in \Omega^\pm$, $T_\pm$ is a local C^1-diffeomorphism (since there are no critical inflection points in $N^\pm$).

Since bounded characteristics consist of increasing laps followed by decreasing laps we seek fixed points of iterates of the composition map $T = T_- \circ T_+$ (or $T = T_+ \circ T_-$). Fixed points are contained in the set

$$\Omega^* = \bigcap_{n \in \mathbb{Z}} \big(T_- \circ T_+\big)^n (\Omega^+) \subset \mathbb{R}^2.$$

The maps $T_\pm$ are area-preserving maps with respect to the area form $\alpha = dp_u \wedge du$. This means that for any region $U \subset \Omega^\pm$ it holds that $\int_U \alpha = \int_{T_\pm U} T_\pm^* \alpha$ (locally area-preserving). This was proved in [96] for the EFK-equation. We will give a different proof of this fact here. Let $(p_u, u) \in U \subset \Omega^+$, and recall that $\omega = dp_u \wedge du + dp_v \wedge dv$. Now $\mathcal{T}_+$ maps $\pi_+^{-1} U \subset \Sigma^+$ to $\mathcal{T}_+ \pi_+^{-1} U \subset \Sigma^-$. Since $\mathcal{T}_+$ preserves ω, and because $\Sigma^\pm \subset \{v = 0\}$ it follows that the 2-form $\alpha = dp_u \wedge du$ is preserved, and thus T_+, as a map from Ω^+ to Ω^-, is area-preserving. This implies that

$$p_{u_2} du_2 - p_{u_1} du_1 = dS_*(p_{u_1}, u_1), \tag{7.3}$$

[5] In ODE theory the study of this map is often called a *shooting* method.

where $(p_{u_1}, u_1) \in U$ and $(p_{u_2}, u_2) = T_+(p_{u_1}, u_1) \in T_+U$, and S_* is a C^1-function of (p_{u_1}, u_1).

The map T_+ is a (local) *Twist map* if $u_2 = u_2(p_{u_1}, u_1)$ is strictly increasing in $-p_{u_1}$. It then follows from (7.3) that there exists a C^1-function $S_E(u_1, u_2) = S_*(p_{u_1}(u_1, u_2), u_1)$ such that $\partial_1 S_E = -p_{u_1}$ and $\partial_2 S_E = p_{u_2}$. This function is called the *generating function* of the Twist map. A similar construction can be carried out for T_-. We refer to [9] for more details.

The function S_E can be used to formulate a variational principle in terms of the u_i-variables. In the next chapter we will make a connection with the variational principle for the Lagrangian action

$$J_E[u] = \int_0^\tau \big(L(u, u', u'') + E\big) dt,$$

where the integration over $[0, \tau]$ is between two consecutive extrema of $u(t)$. In relation to this connection we note the following (which does not depend on T_+ being a Twist map or not).

Lemma 7.2 *Let $S_*(p_{u_1}, u_1) = J_E[u]$, where $u(t)$ is the trajectory starting at $\pi_+^{-1}(p_{u_1}, u_1) \in \Sigma^+$, and $\tau = \tau(p_{u_1}, u_1)$ is the first intersection time at Σ^-. Then S_* satisfies Equation* (7.3).

Proof. Define the Hamiltonian action $\mathcal{A}_E[x] = \int_0^T \{p_u u' + p_v v' - H(x) + E\} dt$, and let $(p_{u_1}, u_1) \in \Omega^+$. Consider the trajectory $\{\varphi_H^t(\pi_+^{-1}(p_{u_1}, u_1))\}_{t=0}^{t=\tau(p_{u_1}, u_1)}$, where $\tau(p_{u_1}, u_1)$ is the first intersection time at Σ^-. These trajectories vary smoothly with $(p_{u_1}, u_1) \in \Omega^+$. We now consider variations with respect to $(p_{u_1}, u_1) \in \Omega^+$. Using the fact that (p_u, p_v, u, v) obeys the Hamilton equations and $v(\tau(p_{u_1}, u_1)) = 0$, we obtain

$$\begin{aligned} \delta \mathcal{A}_E[x] &= p_u \delta u|_0^\tau + p_v \delta v|_0^\tau + \big[p_u u' + p_v v' - H(x) + E\big]_\tau \delta\tau \\ &= p_u \delta u(\tau) - p_u \delta u(0) + p_v \big[\delta v(\tau) + v'(\tau)\delta\tau\big] \\ &= p_{u_2} \delta u_2 - p_{u_1} \delta u_1, \end{aligned}$$

where $(p_{u_2}, u_2) = T_+(p_{u_1}, u_1)$. It may be clear that $\mathcal{A}_E[x] = J_E[u]$, which proves the lemma. □

If T_+ is a Twist map then for J_E this implies that there exists a local continuous family $u(t; u_1, u_2)$ of critical points (and $\tau(u_1, u_2)$ varies continuously). Conversely, we will show in the next chapter that the continuity conditions on the family of critical points $u(t; u_1, u_2)$ imply the Twist property.

We remark that instead of studying the maps $T_\pm$ one can study a related area-preserving map which is well defined when $T_\pm$ are Twist maps. From $T_\pm$ we construct the map $\tilde{T}$

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \tilde{T} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad u_{n-1}, u_n, u_{n+1} \in \pi^u N_E.$$

For this map we can use the generating function $S_E(u_1, u_2)$ to retrieve the maps $T_\pm$. We refer to [6, 13] for more details.

7.1.3 Closed characteristics

A special class of bounded characteristics are closed characteristics. These are functions u that are stationary for $J[u]$ and are τ-periodic for some period τ. If we seek closed characteristics at a given energy level E we can invoke the following variational principle:

$$\text{Extremise}\{J_E[u] \,|\, u \in \Omega_{\text{per}}\}, \tag{7.4}$$

where $J_E[u] = \int_0^\tau (L(u,u',u'') + E)\,dt$ and $\Omega_{\text{per}} = \cup_{\tau>0} C^2(S^1,\tau)$. It may be clear that τ is also a parameter in this problem. Problem (7.4) is equivalent to

$$\text{Extremise}\{J_E[v,\tau] \,|\, (v,\tau) \in C^2(S^1,1) \times \mathbb{R}^+\}, \tag{7.5}$$

where $J_E[v,\tau] = \int_0^1 (L(v, \frac{v'}{\tau}, \frac{v''}{\tau^2}) + E)\tau ds$. This equivalent variational characterisation is convenient for technical purposes. Notice that the variations in τ guarantee that any critical point of (7.4) has energy $H(x) = E$. The variational problem of finding closed characteristics for a given energy value E can also be formulated in terms of unparametrised closed curves in the configuration plane.

The Morse index of a closed characteristic u is defined as the number of negative eigenvalues of the linear operator $d^2 J_E[u]$ on $T_u\Omega_{\text{per}} \simeq C^2(S^1,1) \times \mathbb{R}$. The nullity is the dimension of the kernel of $d^2 J_E[u]$. The large Morse index is defined as the sum of the Morse index and the nullity.

7.2 Twist systems

7.2.1 Generating functions

In this section we will introduce a class of Lagrangian systems which satisfy a variant of the Twist property. Such systems can be studied via generating functions. We start with systems for which the generating function is of class C^2. In Section 7.2.2 we will give a number of examples of such systems. In Section 7.2.3 we explain how the theory also works with C^1-generating functions which allows a weaker version of the Twist property (see hypothesis (T′) in Section 7.2.3).

For a regular energy value E the set $\pi^u N_E$ is a union of closed intervals. Connected components of $\pi^u N_E$ are denoted by I_E and will be referred to as *interval components*. Since E is regular it holds that $L(u,0,0) + E > 0$ for $u \in \text{int}(I_E)$, and $L(u,0,0) + E = 0$ for $u \in \partial I_E$. In terms of N_E this means that connected components of N_E topologically are copies of $\mathbb{R}$ and/or S^1. Let $\Delta = \{(u_1,u_2) \in I_E \times I_E \,|\, u_1 = u_2\}$ be the diagonal, then for any pair $(u_1,u_2) \in I_E \times I_E \setminus \Delta$ we define

$$S_E(u_1,u_2) = \inf_{\substack{u \in X_\tau \\ \tau \in \mathbb{R}^+}} \int_0^\tau \Big(L(u,u',u'') + E\Big)dt, \tag{7.6}$$

where $X_\tau = X_\tau(u_1,u_2) = \{u \in C^2([0,\tau]) \,|\, u(0) = u_1,\, u(\tau) = u_2,\, u'(0) = u'(\tau) = 0,\, u'|_{(0,\tau)} > 0 \text{ if } u_1 < u_2 \text{ and } u'|_{(0,\tau)} < 0 \text{ if } u_1 > u_2\}$. We remark that the notation S_E is slightly suggestive since it is not a priori clear that this definition of S_E is equivalent to the one in Section 7.1.2 (however, compare Lemma 7.2). If there is no ambiguity about the choice of E we simply write $S(u_1,u_2)$. At this point it is not clear whether S is defined on all of $I_E \times I_E \setminus \Delta$.

Collections of monotone pieces, or *laps*, of u from u_1 to u_2 that minimise $\int(L+E)$, are the analogues of broken geodesics. Our goal now is to formulate a variational problem in terms of the u_i-coordinates of bounded characteristics replacing the 'full' variational problem for $J_E[u]$. This will be a direct analogue of the method of broken geodesics.

As in (7.5) there is an equivalent formulation of the variational problem above. In view of this we consider the pair (v,τ), with $v(s) = u(t)$ and $s = t/\tau$. For the special points

$(u_1, u_2) \in \Delta$ we define $v(s) = u_1$ for all $s \in [0,1]$ and $\tau = 0$ (and $S(u_1, u_1) = 0$). A Lagrangian system (L, dt) is said to satisfy the *Twist property* on an interval component I_E if (with E a regular energy value):

(T) $\inf\{J_E[u] \mid u \in X_\tau(u_1, u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimiser* $u(t; u_1, u_2)$ *for all* $(u_1, u_2) \in I_E \times I_E \setminus \Delta$, *and* u *and* τ *are* C^1*-smooth functions of* (u_1, u_2).

To be precise, by C^1-smoothness we mean that $(u_1, u_2) \to (v, \tau)$ is a C^1-function from $\mathrm{int}(I_E \times I_E \setminus \Delta)$ to $C^2([0,1]) \times \mathbb{R}^+$ and a C^0-function on $I_E \times I_E$. The results presented in this chapter will apply whenever the Twist property is satisfied on an interval component I_E[6].

If E is a singular energy level with non-degenerate critical points then we have the same formulation of the Twist property with the following exceptions. First, C^1-smoothness is only required for all $(u_1, u_2) \in \mathrm{int}(I_E \times I_E \setminus \Delta)$ such that u_1 nor u_2 is a critical point[7]. Second, when an equilibrium point $u_* \in I_E$ is a saddle-focus or a center then $\tau(u_1, u_2)$ is not continuous at (u_*, u_*)[8]. In the case that u_1 and/or u_2 is an equilibrium point of real saddle type then τ can be ∞[9]. We refer to Section 7.3.2 and Section 7.5 for more information on singular energy levels and equilibrium points.

Definition 7.3 *A Lagrangian system* (L, dt) *is called a Twist system on an interval component* I_E *if both hypotheses* (H) *and* (T) *are satisfied.*

Using hypothesis (T) we can derive the following regularity properties for S.

Lemma 7.4 *Let* E *be a regular value. If* (L, dt) *is a Twist system on an interval component* I_E*, then the function* $S_E(u_1, u_2)$ *is of class* $C^2(\mathrm{int}(I_E \times I_E \setminus \Delta)) \cap C^1(I_E \times I_E \setminus \Delta) \cap C^0(I_E \times I_E)$.

Proof. Due to the smoothness assumption in (T) and the regularity of solutions of the Euler-Lagrange equations (see Section 7.1.1), we have that $u(t; u_1, u_2)$ varies smoothly with (u_1, u_2) with values in C^2. It is easily seen that $S_E(u_1, u_2) = J_E[u(t; u_1, u_2)]$ is a C^1-function on $I_E \times I_E \setminus \Delta$. Lemma 7.2 and Equation (7.3) show that $\partial_1 S(u_1, u_2) = -p_{u_1}$ and $\partial_2 S(u_1, u_2) = p_{u_2}$. It follows from the smoothness assumption in (T) and the fact that all solutions obey (7.2) that p_{u_1} and p_{u_2} are C^1-functions of (u_1, u_2)[10], hence S_E is a C^2-function on $\mathrm{int}(I_E \times I_E \setminus \Delta)$. Continuity of S_E at the diagonal follows either via a simple estimate in the variational problem, or by analysing the shooting map[11]. □

If S is considered on $I_E^1 \times I_E^2$, where I_E^i, $i = 1, 2$ are different connected components of $\pi^u N_E$, then one does not expect S_E to be defined on all of $I_E^1 \times I_E^2$. The next lemma

[6] Most of the results in this chapter also hold for slightly weaker conditions. For example, when we do not require the family of solutions/extrema to be minimisers of $J_E(u_1, u_2)$ then we obtain the same results, the information on the index excluded. For the case where the family is continuous but not C^1 we refer to Section 7.2.3.

[7] Singular energy levels connected components of $\pi^u N_E$ can have internal critical points. This will be discussed in Section 7.3.3.

[8] It still holds that $J_E[u(t; u_1, u_2)] \to 0$ as $(u_1, u_2) \to (u_*, u_*)$.

[9] We then consider u on either $[0, \infty)$, $(-\infty, 0]$ or $\mathbb{R}$ (whichever is appropriate) and require that $u(t; u_1, u_2)$ converges on compact sets as u_1 and/or u_2 tends to the critical point.

[10] At points $t \in (0, \tau)$ we have $p_u = (-\frac{\partial L}{\partial u''} u'' + L + E)/u'$ which depends smoothly on (u_1, u_2) since $u' \neq 0$ for $t \in (0, \tau)$. The smooth dependence of the initial value problem for the Hamiltonian flow now ensures that $p_u(0)$ and $p_u(\tau)$ depend smoothly on (u_1, u_2) as well.

[11] In both cases the corner points $\partial\Delta$ have to be dealt with separately.

reveals some important properties of the generating function S. For the remainder of this section we assume that E is a regular value and we consider interval components I_E on which (L, dt) is a Twist system.

Lemma 7.5 *Let E be a regular value. Then*

(a) $\partial_1 S(u_1, u_2) = -p_{u_1}$ *and* $\partial_2 S(u_1, u_2) = p_{u_2}$ *for all* $(u_1, u_2) \in I_E \times I_E \setminus \Delta$,

(b) $\partial_1\partial_2 S(u_1, u_2) > 0$ *for all* $(u_1, u_2) \in \mathrm{int}(I_E \times I_E \setminus \Delta)$*, and*

(c) $\partial_{n_\pm} S|_{\mathrm{int}(\Delta)} = +\infty$*, where* $n_\pm = (\mp 1, \pm 1)^T$.[12]

Proof. Part (a) has been dealt with in the proof of Lemma 7.4. For Part (b) of this lemma we argue as follows: $\partial_1\partial_2 S(u_1, u_2) = \frac{\partial p_{u_2}}{\partial u_1} = -\frac{\partial p_{u_1}}{\partial u_2}$. Because of the uniqueness of the initial value problem for $x' = X_H(x)$ the variable $-p_{u_1}$ is a strictly increasing function of u_2 (u_1 fixed)[13]. In exactly the same way p_{u_2} is a strictly increasing function of u_1 (u_2 fixed). Therefore $\partial_1\partial_2 S(u_1, u_2) \geq 0$. On the other hand using the smooth dependence on initial data for $x' = X_H(x)$ and the smoothness of $\tau(u_1, u_2)$, it follows that both $u_2 = u_2(u_1, p_{u_1})$ and $u_1 = u_1(u_2, p_{u_2})$ are smooth functions. This implies that $\frac{\partial p_{u_1}}{\partial u_2} \neq 0$ and $\frac{\partial p_{u_2}}{\partial u_1} \neq 0$, and thus $\partial_1\partial_2 S(u_1, u_2) > 0$.

As for Part (c) we only consider the derivative in the direction n_+ (the other case is similar). We have that $u'''(0), u'''(\tau) \to -\infty$ as $u_1 \to u_2$ since $u''(0), u''(\tau) \neq 0$ on $\mathrm{int}(\Delta)$. For p_u it holds that $p_u = \partial_v L(u, 0, w) - \partial^2_{vw} L(u, 0, w)u'' - \partial^2_w L(u, 0, w)u'''$ and thus $p_{u_i} \to \infty$, $i = 1, 2$. □

The question of finding bounded characteristics for (L, dt) can now best be formulated in terms of S. Extremising the action J_E over a space of 'broken geodesics' now corresponds to finding critical points of the formal sum $\sum_{n\in\mathbb{Z}} S(u_n, u_{n+1})$. Formally we seek critical points (bounded sequences) of the infinite sum

$$W(\cdots, u_{-1}, u_0, u_1, \cdots) = \sum_{i\in\mathbb{Z}} S(u_i, u_{i+1}).$$

Since this sum is usually not well-defined for bounded sequences $(u_i)_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, we say that a sequence is a *critical sequence*, or critical point of W, if:

$$\partial_2 S(u_{i-1}, u_i) + \partial_1 S(u_i, u_{i+1}) = 0, \quad \text{for all } i \in \mathbb{Z}. \tag{7.7}$$

Such equations are called *second order recurrence relations* (see e.g. [13, 102] for related recurrence relations in the context of Twist diffeomorphisms). If (7.7) is satisfied for all $i \in \mathbb{Z}$ then u-laps can be glued to a C^3-function for which all derivatives up to order three match. Indeed, Equation (7.7) means that the third derivatives match[14]. Since every u-lap satisfies the Euler-Lagrange equations we then get a C^3_b-function u that is stationary for $J[u]$. Of course, if we seek periodic sequences, i.e., sequences $(u_i)_{i\in Z}$ with $u_{i+2n} = u_i$, where $2n$ is called the period, we may look for critical points of the restricted action $W_{2n} = \sum_{i=1}^{2n} S(u_i, u_{i+1})$ defined on I_E^{2n}.[15] This corresponds to finding closed characteris-

[12]This should be read as follows: when we approach a point $(\tilde{u}, \tilde{u}) \in \mathrm{int}(\Delta)$ from within the region $\{u_2 > u_1\}$ then $\partial_{n_+} S \to \infty$ as $(u_1, u_2) \to (\tilde{u}, \tilde{u})$.

[13]It could also be strictly decreasing but this is excluded by Part (c) of the lemma

[14]It holds that $\partial_2 S(u_{i-1}, u_i) + \partial_1 S(u_i, u_{i+1}) = \partial^2_w L(u_i, 0, u_i'')(-u_i''' + \tilde{u}_i''')$, where u_i''' is the third derivative on the left and $\tilde{u}_i'''$ is the third derivative on the right.

[15]The function W_{2n} is continuous on I_E^{2n} and is of class C^2 on the set $\{(u_1, ..., u_{2n}) \in \mathrm{int}(I_E^{2n}) \,|\, u_i \neq u_{i+1}$ for all $i = 1, .., 2n\}$, with $u_{2n+1} \equiv u_1$.

tics for (L, dt). The period can be linked to various topological properties of u and x (in the Hamiltonian system (H, ω)) such as knotting and linking of closed characteristics. Moreover, periodic sequences as critical points of W_{2n} have a Morse index, which is exactly the Morse index of a closed characteristic u as critical point of $J|_{\Omega_{\mathrm{per}}}$.

Lemma 7.6 *Let E be a regular value. Let $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ with $u_i \in \mathrm{int}(I_E)$ be a periodic sequence with period $2n$, which is a stationary point of W_{2n} with index $\mu(\boldsymbol{u}) \le 2n$. Then the associated closed characteristic u for (L, dt) is stationary for $J[u]$ and the Morse index of u is also $\mu(\boldsymbol{u})$, and vice versa.*

Proof. Let $\boldsymbol{u}$ be stationary for W_{2n}, i.e. $dW_{2n}(\boldsymbol{u}) = 0$. Concatenating the u-laps between the consecutive extrema u_i yields a τ-periodic C^3-function u that satisfies the Euler-Lagrange equations of (L, dt). It may be clear that the function u is a critical point of (7.4). The statement concerning the Morse index $\mu(\boldsymbol{u}) = \mu(u)$ can be proved as follows. The assumption that $u_i \in \mathrm{int}(I_E)$ implies that $u'' \ne 0$ at extrema of $u(t)$. This implies that the number of monotone laps is conserved under small perturbations in $\Omega_{\mathrm{per}} = \cup_{\tau>0} C^2(S^1, \tau)$.

A function w in a small neighbourhood of u can be characterized by the heights of the extrema $u_1, \dots, u_{2n}$ (cylic), the distances between the extrema $\tau_1, \dots, \tau_{2n}$, and the deviations $v_i(t) \in C_0^2([0,1])$ of the minimizing laps, namely

$$w(t) = u\big(\tfrac{\tau(u_i,u_{i+1})}{\tau_i}(t - T_i); u_i, u_{i+1}\big) + v_i\big(\tfrac{1}{\tau_i}(t - T_i)\big) \qquad \text{for all } t \in [T_i, T_{i+1}],\ i = 1, \dots, 2n,$$

where $u(t; u_i, u_{i+1})$ and $\tau(u_i, u_{i+1})$ is the minimizing pair defined in hypothesis (T), and $T_i = \sum_{k=1}^{i-1} \tau_k$. Consequently, $T_u\Omega_{\mathrm{per}}$ can be identified with $(C_0^2([0,1]) \times \mathbb{R})^{2n} \oplus \mathbb{R}^{2n}$, seperating the dependence on the heights of the extrema from the other contributions. The linear operator $d^2 J_E[u]$ induces a linear operator on $(C_0^2([0,1]) \times \mathbb{R})^{2n}$, which is non-negative. Consequently the Morse index of $d^2 J_E[u]$ is equal to the Morse index of the induced operator on $\mathbb{R}^{2n}$. This induced operator is in fact $d^2 W_{2n}[\boldsymbol{u}]$ (for more details see e.g. [103]: case of broken geodesics). □

For points on the boundary ∂I_E additional information about S can be obtained. Denote the left boundary point of I_E by u^- and right boundary point by u^+.

Lemma 7.7 *Let E be a regular value. Let $u^- \in \partial I_E$ (assuming that there exists a left boundary point) then $\partial_1 S(u^-, \tilde{u}) > -\partial_v L(u^-, 0, 0)$ and $\partial_2 S(\tilde{u}, u^-) > \partial_v L(u^-, 0, 0)$ for $\tilde{u} > u^-$. Similarly, if $u^+ \in \partial I_E$ then $\partial_1 S(u^+, \tilde{u}) < -\partial_v L(u^+, 0, 0)$ and $\partial_2 S(\tilde{u}, u^+) < \partial_v L(u^+, 0, 0)$ for all $\tilde{u} < u^+$.*

Proof. Let us prove the above inequalities for $\partial_1 S$ as the case for $\partial_2 S$ leads to an analogous argument. We start with the left boundary point u^-. We seek an increasing lap from u^- to $\tilde{u}$. At $u_1 = u^-$ it holds that $-L(u^-, 0, 0) = E$, $u_1'' = 0$ and $\partial_u L(u^-, 0, 0) > 0$, which implies that $u_1'''(0) > 0$. By contradiction, suppose that $u'(0) = u''(0) = u'''(0) = 0$. On one hand we have $p_v' = \partial_v L - p_u$ and on the other hand $p_v' = \partial_{uw}^2 L u' + \partial_{vw}^2 L u'' + \partial_w^2 L u'''$. From the former and the Euler-Lagrange equation we see that $p_v''(0) = -\partial_u L(0) < 0$, so that

$$\lim_{\varepsilon\to 0} \frac{(\partial_{uw}^2 L u' + \partial_{vw}^2 L u'' + \partial_w^2 L u''')(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\partial_w^2 L u'''(\varepsilon)}{\varepsilon} = -\partial_u L(0) < 0.$$

We conclude (using condition (H)) that $u'''(t) < 0$ in a right neighbourhood of 0, which contradicts the fact that we are dealing with an increasing lap.

It now follows that $p_{u_1} = \partial_v L(u^-,0,0) - \partial_w^2 L(u^-,0,0)u_1'''(0) < \partial_v L(u^-,0,0)$. Therefore $\partial_1 S(u^-,\tilde{u}) = -p_{u^-} > -\partial_v L(u^-,0,0)$. For the right boundary point u^+ we get $\partial_1 S(u^+,\tilde{u}) = -p_{u^+} < -\partial_v L(u^+,0,0)$, since u is a decreasing lap. □

7.2.2 Examples of Twist systems

An example of a class of Lagrangians for which we can verify the Twist property in various cases is given by $L(u,u',u'') = \frac{1}{2}u''^2 + K(u,u')$. Most of the fourth order equations coming from physical models are derived from Lagrangians of this form. We could tag such systems as *fourth order mechanical systems* based on the analogy with second order mechanical systems given by Lagrangians of the form $L(u,u') = \frac{1}{2}u'^2 + K(u)$ (integrable systems). The Lagrangian L clearly satisfies hypothesis (H) and (L,dt) is thus equivalent to the Hamiltonian system (H,ω) with ω the standard symplectic form on $\mathbb{R}^4$ (see Section 7.1.1) and $H(x) = p_u v + \frac{1}{2}p_v^2 - K(u,v)$. For a regular energy value E the set $\pi^u N_E$ is given by $\pi^u N_E = \{u \,|\, K(u,0) + E \geq 0\}$. If E is regular it holds that $K(u,0)+E > 0$ for $u \in \mathrm{int}(I_E)$, and $K(u,0)+E = 0$ for $u \in \partial I_E$.

Lemma 7.8 *Let I_E a connected component of $\pi^u N_E$ (E not necessarily regular*[16]*). Assume that*

(a) $\frac{\partial K}{\partial v}v - K(u,v) - E \leq 0$ *for all $u \in I_E$ and $v \in \mathbb{R}$,*

(b) $\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\{\frac{\partial K}{\partial v}v - K(u,v) - E\} \geq 0$ *for all $u \in I_E$ and $v \in \mathbb{R}$.*

Then for any pair $(u_1,u_2) \in I_E \times I_E \setminus \Delta$ Problem (7.6) has a unique minimiser $(u,\tau) \in X_\tau \times \mathbb{R}^+$ (in fact the only critical point), and the minimiser $u(t;u_1,u_2)$ depends C^1-smoothly on (u_1,u_2) for $(u_1,u_2) \in \mathrm{int}(I_E \times I_E \setminus \Delta)$[17].

For the proof of this lemma we refer to Section 7.6.

At this point we are not able to prove that the Twist property holds for more general systems under some mild growth conditions on K without assuming (a) and (b). However, numerical experiments (see Section 7.4.1) for various Lagrangians suggest that Lemma 7.8 is still valid, although we do not have a proof of this fact. Milder conditions on K sometimes only allow the existence of a continuous family $u(t;u_1,u_2)$. We come back to this case in Section 7.2.3. The conditions given in Lemma 7.8 already allow for a large variety of Lagrangians that occur in various physical models. We will give a few examples of such systems now.

7.2.2.1 The EFK/Swift-Hohenberg system

The EFK/Swift-Hohenberg Lagrangian is given by $L(u,u',u'') = \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u)$, where $\alpha \in \mathbb{R}$ and F is a smooth potential function[18]. The Hamiltonian in this case is $H(x) = p_u v + \frac{1}{2}p_v^2 - \frac{\alpha}{2}v^2 - F(u)$. Connected components of $\pi^u N_E$ are sets of the from $\{u \,|\, F(u) + E \geq 0\}$.

In the case that $\alpha > 0$ this L is referred to as the EFK-Lagrangian (see e.g. [90, 89, 88]), and in the case $\alpha \leq 0$ it is usually referred to as the Swift-Hohenberg Lagrangian [106, 23,

[16] If E is a singular energy level then we require the critical points to be non-degenerate.

[17] If E is a singular energy level then C^1-regularity holds for all $(u_1,u_2) \in \mathrm{int}(I_E \times I_E \setminus \Delta)$ for which u_1 nor u_2 is a critical point.

[18] Note that in this chapter the potential $F(u)$ is defined with the opposite sign compared to Chapter 1.

137]. For example, $F(u) = \frac{1}{4}(u^2-1)^2$ is the classical EFK/Swift-Hohenberg potential [117, 120], $F(u) = \frac{1}{3}u^3 - \frac{1}{2}u^2$ gives the water-wave model [35], $F(u) = -\frac{1}{4}(u^2-1)^2$ is the potential of a nonlinear optics model [1].

If $\alpha \leq 0$ then the conditions (a) and (b) are satisfied for any interval component I_E. The Swift-Hohenberg systems is therefore a Twist system for all interval components. For $\alpha > 0$ this is not immediately clear (conditions (a) and (b) are not satisfied)[19]. More details on EFK/Swift-Hohenberg systems are given in Section 7.3.4.

7.2.2.2 The suspension-bridge model

The suspension bridge model is a special case of the Swift-Hohenberg equation, namely $L(u,u',u'') = \frac{1}{2}u''^2 - \frac{c^2}{2}u'^2 + F(u)$, with $F(u) = e^u - u - 1$ (see [121]). Clearly, the suspension bridge model is a Twist system for all $c \in \mathbb{R}$. For more details see Section 7.3.4. This model is particularly intriguing due to the specific form of the potential function F. The growth of F for $u \to \infty$ is essentially different from the growth for $u \to -\infty$ which has far reaching consequences for the set of closed characteristics.

7.2.2.3 The fifth order KdV equation

Consider $L(u,u',u'') = \frac{1}{2}u''^2 + K(u,u')$, where $K(u,u') = \frac{1}{2}(\alpha + 2\mu u)u'^2 + F(u)$, with $F(u) = \frac{\kappa}{3}u^3 + \frac{\sigma}{2}u^2$, which describes travelling waves in a fifth order Korteweg-de Vries equation (see e.g. [40, 115]). In order for the theory to be applicable the conditions in Lemma 7.8 on K imply that $\alpha + 2\mu u \leq 0$ for $u \in I_E$. The case $\mu = 0$ is the Swift-Hohenberg equation again. Let us assume for example that $\kappa, \sigma > 0$, then one finds compact intervals I_E for values $-\frac{\sigma^3}{6\kappa^2} < E \leq 0$. These intervals are contained in $[-\frac{3\sigma}{2\kappa}, 0]$. For $\mu > 0$ the condition becomes $u < -\frac{\alpha}{2\mu}$, which is for instance satisfied for all $u \in I_E$ if $\alpha < 0$. For $\mu < 0$ the condition becomes $u > -\frac{\alpha}{2\mu}$, which is satisfied for all $u \in I_E$ if $\alpha < 2\frac{\sigma\mu}{\kappa}$. Many more combination can be found by also varying the signs of κ and σ.

7.2.3 The C^0-Twist property

As we already remarked before, the theory developed in this chapter can be adjusted for C^1-generating functions. We will point out the difficulties and how the theory has to be adjusted at the end of this section. First we start with a weaker version of the Twist property that ensures the existence of C^1-generating functions.

(T′) $\inf\{J_E[u] \,|\, u \in X_\tau(u_1,u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimiser* $u(t;u_1,u_2)$ *for all* $(u_1,u_2) \in I_E \times I_E \setminus \Delta$, *and* u *and* τ *are continuous functions of* (u_1,u_2).

Hypothesis (T′) is often easier to verify than the stronger hypothesis (T). Let I_E be an interval component and (L, dt) is a Twist system on I_E with respect to hypothesis (T′). Then $p_{u_2}(u_1,u_2)$ is strictly increasing in u_1 and $-p_{u_2}(u_1,u_2)$ is strictly increasing in u_2, and both are continuous in (u_1,u_2). The maps $T_\pm$ as described in Section 7.1.2 are therefore monotone (C^1-) Twist maps, which have a C^1-generating function $S_E(u_1,u_2) = J_E[u(t;u_1,u_2)]$.

[19] J. Kwapisz [95] proves that the C^0-Twist property (T′) (see Section 7.2.3) is satisfied for the EFK-Lagrangian ($\alpha > 0$) on interval components I_E for which $F(u) + E$ has at most one internal extremum (a maximum).

Lemma 7.9 *Let I_E be an interval component. If (L, dt) is a Twist system with respect to hypothesis* (T′)*, then S_E is a C^1-generating function on $I_E \times I_E \setminus \Delta$.*

Part (b) of Lemma 7.5 is now replaced by the property that $\partial_1 S$ and $\partial_2 S$ are increasing functions of u_1 and u_2 respectively. The difficulties in working with C^1-generating functions are the definition of the Morse index and the gradient flow of $W = \sum_i S(u_i, u_{i+1})$. In Section 7.3 we use the gradient flow of W to find other critical points besides minima and maxima. One way to deal with this problem is to approximate S by C^2-functions. A C^1-Morse/Conley index can then be defined (see for instance [17, 18]). An analogue of Lemma 7.6 can also be proved now. Other properties of S that we use in this chapter, such as construction of isolating neighbourhoods, do not need the C^2-regularity. For this reason we will continue with C^2-function keeping in mind that all result carry over to the C^1-case.

7.3 Existence

7.3.1 Simple closed characteristics for compact sections N_E

The properties of S listed in the Section 7.2.1 can be used to derive an existence result for simple closed characteristics. Before stating the theorem we need to introduce some additional notation: $I_E \times I_E \setminus \Delta = D_E^+ \cup D_E^-$, where $D_E^+ = \{(u_1, u_2) \in I_E \times I_E \setminus \Delta \,|\, u_2 > u_1\}$, and D_E^- is defined analogously. The function $W_2(u_1, u_2) = S(u_1, u_2) + S(u_2, u_1)$ is a C^2-function on $\mathrm{int}(I_E \times I_E \setminus \Delta)$. Since $W_2(u_1, u_2) = W_2(u_2, u_1)$ we can restrict our analysis to D_E^+.

Throughout this section we again assume that E is regular and (L, dt) is a Twist system on I_E.

Lemma 7.10 *Assume that $\pi^u N_E$ contains a compact interval component I_E. Then W_2 has at least one maximum on D_E^+*[20].

Proof. We have that $W_2|_\Delta = 0$ and W_2 is strictly positive near $\mathrm{int}(\Delta)$ by Lemma 7.5c. Since the set $\overline{D_E^+}$ is compact, W_2 must attain a maximum on set $\overline{D_E^+}$. It follows that $\max_{(u_1,u_2)\in\overline{D_E^+}} W_2(u_1, u_2) > 0$.

Writing $I_E = [u^-, u^+]$ we denote by $n_1 = (1, 0)^T$ the inward pointing normal on the left boundary $B_1 = \{(u^-, u_2) \,|\, u_2 \in I_E\}$ and by $n_2 = (0, -1)^T$ the inward pointing normal on $B_2 = \{(u_1, u^+) \,|\, u_1 \in I_E\}$. Using Lemma 7.7 we can now compute $\frac{\partial W_2}{\partial n_1}$ and $\frac{\partial W_2}{\partial n_2}$. For example let $u_1 = u^-$, then

$$\frac{\partial W_2}{\partial n_1} = \partial_1 S(u^-, u_2) + \partial_2 S(u_2, u^-) > -\partial_v L(u^-, 0, 0) + \partial_v L(u^-, 0, 0) = 0.$$

Similarly, using Lemma 7.7, we derive that $\frac{\partial W_2}{\partial n_2}|_{B_2} > 0$. Since both $\frac{\partial W_2}{\partial n_1}|_{B_1} > 0$, $\frac{\partial W_2}{\partial n_2}|_{B_2} > 0$, and $W_2|_\Delta = 0$, the maximum is attained in $\mathrm{int}(D_E^+)$ (see also Figure 7.2). $\square$

If we study W_{2n}, $n > 1$ we do not necessarily find new closed characteristics for (L, dt). i.e. critical points of W_{2n} of higher index may be the same closed characteristic traversed more than once. In the next sections we will describe some mechanisms that yield more geometrically distinct closed characteristics.

[20]From straightforward Morse theory for W_2 on D_E^+ we obtain in addition that $b_0 \geq 0$, $b_1 - b_0 \geq 0$ and $b_2 - b_1 + b_0 = 1$, where b_i is the number of critical points of index i (in the case that W_2 is a Morse function).

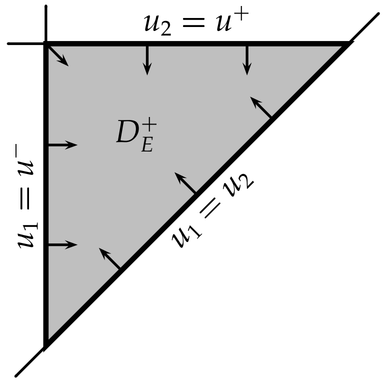

Figure 7.2: A picture of D_E^+. The arrows denote the direction of the gradient ∇W_2 schematically (of course the gradient is not perpendicular to the boundary everywhere). Clearly the maximum of W_2 is attained in the interior of D_E^+.

The above lemma can be slightly rephrased for Lagrangian systems (see Lemma 7.6). We do not have information about the nullity of $d^2 J_E(u_1, u_2)$, so that the large Morse index[21] of the solutions may be greater than 2, but the Morse index is certainly smaller than or equal to 2.

Theorem 7.11 *Assume that $\pi^u N_E$ contains a compact interval component I_E. Then (L, dt) contains at least one simple closed characteristic $u(t) \in \mathrm{int}(I_E)$ with large Morse index greater than or equal to 2 and Morse index less than or equal to 2.*

Theorem 7.11 states that the associated Hamiltonian system (H, ω) has at least one closed characteristic on M_E. The above theorem is reminiscent of first order Lagrangian systems: $L(u, u')$ with Euler-Lagrange equation $\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} = 0$. Such systems may be labeled as mechanical systems if $\partial_{u'} L > 0$. On the compact components of $\{(u, u') \mid \frac{\partial L}{\partial u'} u' - L(u, u') = E\}$ closed characteristics exist (integrable system).

If L is invariant with respect to $t \mapsto -t$, then it holds that $L(u, v, w) = L(u, -v, w)$ for all $(u, v, w) \in \mathbb{R}^3$. A consequence of this symmetry is that $S(u_1, u_2) = S(u_2, u_1)$ which implies that we can study just S (instead of W_2) to find simple closed characteristics in this case. Moreover, this symmetry of L carries over to the simple closed characteristic: $u(t)$ is symmetric with respect to its extrema. Some Lagrangian systems are also invariant under the symmetry $u \mapsto -u$ which yields the relation $L(u, v, w) = L(-u, -v, -w)$. If $0 \in \pi^u N_E$ then there is at least one closed characteristic on the anti-diagonal $u_1 = -u_2$. If the global maximum of W_2 is not on the anti-diagonal $u_1 = -u_2$ then there are at least 2 more closed characteristics (by symmetry).

If we consider non-compact interval components I_E there is no topological restriction that forces the existence of closed characteristics, and there need not exist any. In order to deal with this case (in forthcoming sections) more information about L is needed: asymptotic behaviour (see Section 7.3.4).

7.3.2 Singular energy levels

If E is a singular energy level then there exist points $u \in \pi^u N_E$ for which $\partial_u L(u, 0, 0) = 0$ and $L(u, 0, 0) + E = 0$. For a singular value E the connected components of N_E are

[21] The large Morse index is defined as the sum of the Morse index and the nullity.

either smooth manifolds ($\mathbb{R}$ or S^1), or they are characterised as: $N_E^c \simeq (\mathbb{R}\vee)S^1 \vee \cdots \vee S^1(\vee\mathbb{R})$. The points in $\mathbb{R}^2$ on which N_E fails to be a manifold lie on the u-axis, and are exactly the points u for which $\partial_u L(u,0,0) = 0$ and $L(u,0,0) + E = 0$. The set of such critical points is denoted by $\mathcal{C}(I_E)$. As before, $\pi^u N_E$ is a union of closed intervals. An *interval component* I_E is defined as a subset of $\pi^u N_E$ such that $L(u,0,0) + E > 0$ for all $u \in \mathrm{int}(I_E)$ and $L(u,0,0) + E = 0$ for $u \in \partial I_E$. Since E is singular two interval components I_E^1 and I_E^2 may have non-empty intersection, i.e. $I_E^1 \cap I_E^2 = \{\text{one point}\} \subset \mathcal{C}(I_E)$. Concatenations of interval components are discussed in Section 7.3.3. If we consider interval components with critical points geometric properties come into play. We assume that (L, dt) is a Twist system for the interval components that we consider.

Let I_E be an interval component for which $u^- \in \partial I_E$ is a critical point. In order to prove the analogue of Lemma 7.7 we need to know whether $u'''(0)$ is zero or not. This is determined by $\tau(u^-, u_2)$, i.e. if $\tau(u^-, u_2) < \infty$ then $u'''(0) > 0$ (assuming $u_2 > u^-$), and if $\tau(u^-, u_2) = \infty$ then $u'''(-\infty) = 0$ (in the case that $\tau = \infty$ we consider u on $[-\tau, 0]$ using translation invariance). These two cases can be distinguished by studying the singularity at u^-. We can compute the spectrum of u^- which we will denote by $\sigma(u^-)$. We assume that we are dealing with *non-degenerate* singular points, i.e. $0 \notin \sigma(u^-)$. Critical points on the boundary of interval components obey $\partial_u^2 L(u,0,0) > 0$. It is shown in Section 7.5 that there are three possible behaviours for $\sigma(u^-)$: $\sigma(u^-) \subset \mathbb{R}$, $\sigma(u^-) \subset i\mathbb{R}$, or $\sigma(u^-) \subset \mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$. In the latter case there is one eigenvalue in each quadrant. The three possible behaviours are categorised as *real saddle*, *center* and *saddle-focus* respectively. If $\sigma(u^-) \subset \mathbb{C} \setminus \mathbb{R}$ (center or saddle-focus), then $\tau(u^-, u_2) < \infty$ for all $u_2 \in I_E$. It is immediately clear that Theorem 7.11 is still valid in that case. Also, if both u^+ and u^- are critical points and have their spectrum in $\mathbb{C} \setminus \mathbb{R}$, Theorem 7.11 remains true.

Theorem 7.12 *Let E be a singular value and assume that $\sigma(\mathcal{C}(I_E)) \subset \mathbb{C} \setminus \mathbb{R}$. Then (L, dt) contains at least one simple closed characteristic $u(t) \in \mathrm{int}(I_E)$ with large Morse index greater than or equal to 2 and Morse index less than or equal to 2.*

The way to attack the problem of finding closed characteristics at singular energy levels in general is to again consider the function $W_2(u_1, u_2) = S(u_1, u_2) + S(u_2, u_1)$. Since $W_2|_\Delta \equiv 0$ and strictly positive near Δ, W_2 attains its global maximum in $I_E \times I_E \setminus \Delta$. As was already pointed out before, the maximum is attained in the interior of $I_E \times I_E \setminus \Delta$ if there are no critical points of $L(u,0,0)$ in ∂I_E, or if critical points of $L(u,0,0)$ have complex spectrum (Theorem 7.12). Thus in order for W_2 to attain its global maximum on the boundary, the interval component ∂I_E needs to contain at least one critical point of $L(u,0,0)$ with real spectrum.

The next question is: suppose W_2 attains its maximum at $\partial(I_E \times I_E \setminus \Delta)$, does this maximum correspond to a simple closed trajectory for (L, dt)? Again from the previous we know that at a point $(u_1, u_2) \in \partial(I_E \times I_E \setminus \Delta)$ it holds that $\partial_1 W_2 \geq 0$ if $u_1 = u^-$ and $\partial_2 W_2 \leq 0$ if $u_2 = u^+$. A boundary maximum for which $u_1 = u^-$ and $u_2 = u^+$ is called a co-dimension 2 point, and the remaining boundary points are called co-dimension 1 points. It is clear that at a co-dimension 1 point, for example at $u_1 = u^-$, it holds that $\partial_2 W_2 = 0$. Since we are assuming that this point is a maximum, and because $\partial_1 W_2 \geq 0$ it follows that the maximum is in fact a critical point. The same holds for a co-dimension 1 point at $u_2 = u^+$. Such points correspond to solutions $u(t)$ for which $u'''(-\infty) = u'''(\infty) = 0$, and $u(-\infty) = u(\infty) = u^-$, and $u(t)$ is thus a homoclinic orbit. By the same reasoning codimension 2 points are also critical points. Such a point corresponds to a heteroclinic loop (two heteroclinic connections that form a loop).

Summarising, we can introduce the notion of closed characteristic in the broad sense of the word: a simple closed periodic orbit, a simple homoclinic loop, or a simple heteroclinic loop (they all form a simple closed loop in the configuration plane). If we use this definition we obtain the following theorem.

Theorem 7.13 *Assume that $\pi^u N_E$ has a compact interval component I_E then (L, dt) has at least one simple closed characteristic in the broad sense.*

It is clear from the previous that a necessary condition for (L, dt) to have a simple homoclinic loop to u^- is that u^- is a critical point of $L(u, 0, 0)$ that has real spectrum (real saddle). The same holds for u^+. A necessary condition to find a simple heteroclinic loop between u^- and u^+ is that both u^- and u^+ are real saddles. Unfortunately, these conditions need not be sufficient[22].

One way to guarantee the existence of a simple homoclinic loop to $u^- \in \mathcal{C}(I_E)$ is that $\tau(u^-, u_2) = \tau(u_2, u^-) = \infty$ for all $u_2 \in I_E$[23], and either $u^+ \notin \mathcal{C}(I_E)$ or u^+ has complex spectrum. In that case $\partial_1 S(u^-, u_2) = -\partial_v L(u^-, 0, 0)$ for all $u_2 \in I_E$. In terms of W_2 this yields that $\partial_1 W_2(u^-, u_2) = 0$ for all $u_2 \in I_E$. We can now restrict W_2 to the line-segment $\{u_1 = u^-\} \times I_E$. Define $W_1(u) = W_2|_{\{u_1=u^-\}\times I_E} = S(u^-, u) + S(u, u^-)$. It easily follows that (compare Lemma 7.7) $W_1(u^-) = 0$, $W_1(u^- + \varepsilon) > 0$ for $\epsilon > 0$ sufficiently small[24] and $W_1'(u^+) < 0$, and thus W_1 has at least one global maximum u_* on (u^-, u^+). The point u_* corresponds to a homoclinic orbit to $u = u^-$.

Regarding the Morse index of this point/orbit we note the following. If u_* is a (local) maximum of W_2 on D_E^+ then the large Morse index is again equal to 2. The corresponding homoclinic orbit has large Morse index greater than or equal to 2 and Morse index less than or equal to 2. However, restricted to the class of functions that are homoclinic to u^- it has large Morse greater than or equal to 1 and Morse index less than or equal to 1 (mountain-pass critical point)[25].

7.3.3 Concatenation of interval components

Up to this point we have only considered single interval components I_E. When E is a singular value then two interval components I_E^1 and I_E^2 may have a common boundary point. This boundary point is then necessarily a critical point. The concatenation of the interval components I_E^i, $i = 1, 2$, will be denoted by $I_E^\#$, and the critical point in $I_E^1 \cap I_E^2$ is denoted by u_*. If (L, dt) is a Twist system on both interval components I_E^1 and I_E^2 it does

[22]For the EFK Lagrangian with $F(u) = \frac{1}{4}(u^2 - 1)^2$ it has been shown that the simple closed characteristic found in Theorem 7.13 corresponds to a heteroclinic loop if and only if the equilibrium points are real saddles (see Chapter 2 and [117, 96]).

[23]It follows from Lemma 7.5b that it in fact suffices that $\tau(u^-, u^+) = \tau(u^+, u^-) = \infty$.

[24]It follows from the linearisation around u^- that $p_{u_2} < 0$ for $u^- < u_2 < u^- + \varepsilon$ when ε is small enough.

[25]For the EFK Lagrangian with $F_a(u) = \int_1^u (s^2 - 1)(s - a)ds$, $0 \le a < 1$ and $\alpha \ge 2\sqrt{2(1 - a)}$ the Twist property is satisfied on the interval component $I_0 = [u^-, 1]$ and $\tau(u^-, 1) = \infty$. Therefore there exists a homoclinic loop is this case. The existence of such solutions for this problem was first proved in [116] by means of a different method. If the case $a = 0$ is considered one obtains a heteroclinic loop (see e.g. [89]).

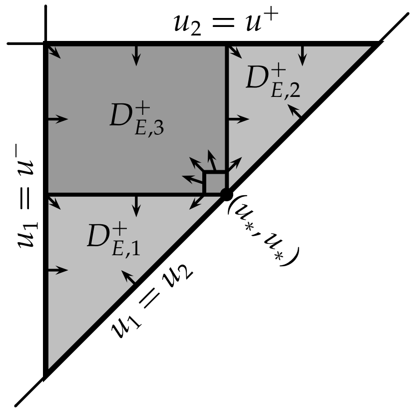

Figure 7.3: The triangle D^+_E when a connected component of $\pi^u N_E$ consists of two compact interval components. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has maximum in $D^+_{E,1}$ and $D^+_{E,2}$ and a saddle point in $D^+_{E,3}$.

not necessarily mean that (L, dt) is a Twist system on the concatenated interval $I^\#_E$. One can easily give examples where (L, dt) fails to satisfy the Twist property on $I^\#_E$.[26] However, if (L, dt) is Twist system on $I^\#_E$, and this is indeed true in many cases, then more solutions can be found. In order to study this case we will use the gradient flow of W_2:

$$\frac{du_1}{dt} = \partial_2 S(u_2, u_1) + \partial_1 S(u_1, u_2), \tag{7.8a}$$

$$\frac{du_2}{dt} = \partial_2 S(u_1, u_2) + \partial_1 S(u_2, u_1), \tag{7.8b}$$

with $u_i \in \operatorname{int}(I^\#_E)$ for $i = 1, 2$.

As before we can restrict our analysis to D^+_E. Define $D^+_{E,1} = \{(u_1, u_2) \in I^1_E \times I^1_E \,|\, u_2 > u_1\}$, $D^+_{E,2} = \{(u_1, u_2) \in I^2_E \times I^2_E \,|\, u_2 > u_1\}$, and $D^+_{E,3} = I^1_E \times I^2_E \setminus (u_*, u_*)$. On the domains $D^+_{E,1}$ and $D^+_{E,2}$ one can again apply Theorem 7.13 which yields the existence of maxima on each of these components. Note that this is independent of the type of u_* (spectrum $\sigma(u_*)$). The following theorem will crucially use the fact that u_* is a critical point for which $\sigma(u_*) \subset \mathbb{C} \setminus \{\mathbb{R} \cup i\mathbb{R}\}$, i.e. a saddle-focus.

Lemma 7.14 *Let $I^\#_E$ be a concatenation of two compact interval components I^1_E and I^2_E and assume that the critical point $u_* \in I^1_E \cap I^2_E$ is a saddle-focus. Then W_2 has at least one maximum on each of the components $D^+_{E,i}$, $i = 1, 2$ and W_2 has a saddle point (critical point with large Morse index equal to 1) on the component $D^+_{E,3}$.*

Proof. The existence of at least one maximum on each of the components $D^+_{E,i}$, $i = 1, 2$, follows directly from Theorem 7.14. As for the existence of saddle points we argue as follows (see also Figure 7.3). Applying Lemma 7.7 we obtain that $\partial_1 W_2|_{\partial I^1_E \times I^2_E} > 0$ and $\partial_2 W_2|_{I^1_E \times \partial I^2_E} > 0$. In order to successfully apply Conley's Morse theory we need to choose an appropriate subset of $D_{E,3}$ which will serve as an isolating neighbourhood. Near $(u_1, u_2) = (u_*, u_*)$ we can find a small solution of the Euler-Lagrangian equation by perturbing from a linear solution. Consider the unique monotone lap $u(t)$ for which $u(0) = u_1 = u_* - \delta$ and $u(\tau) = u_* + \delta$. Since u_* is critical point of saddle-focus type,

[26] For example consider the EFK Lagrangian with $F(u) = \frac{1}{4}(u^2 - 1)^2$. Take $E = 0$, then $\pi^u N_0 = \mathbb{R}$ is the concatenation of three intervals. If $\alpha \geq 2\sqrt{2}$ then (L, dt) is not a Twist system on $I^\#_0 = \mathbb{R}$. However for $\alpha \leq 0$ the Twist property is satisfied on $\mathbb{R}$, and numerical experiments indicate the same for $2\sqrt{2} > \alpha > 0$. This is related to the behaviour of the singularities $u = \pm 1$ (see Section 7.4.1).

it follows that $u'''(0) < 0$ and $u'''(\tau) < 0$ for δ sufficiently small[27]. Straightforward calculation shows that $\partial_1\partial_2 W_2 = \partial_1\partial_2 S(u_1,u_2) + \partial_1\partial_2 S(u_2,u_1) > 0$. These two facts combined show that $\partial_1 W_2(u_* - \delta, u_2) < 0$ for all $u_2 \le u_* + \delta$, and $\partial_2 W_2(u_1, u_* + \delta) > 0$ for all $u_1 \ge u_* - \delta$. Define $N_\delta = D^+_{E,3} \setminus \{(u_1,u_2)\,|\,u_* - \delta < u_1 < u_*, u_* + \delta > u_2 > u_*\}$. The set N_δ is a closed subset of $D^+_{E,3}$ and is isolating with respect to the gradient flow of W_2.[28] The next step is to compute the Conley index of the maximal invariant set $\mathrm{Inv}(N_\delta) \subset N_\delta$. It suffices here to compute the homological index (see [48]) of $\mathrm{Inv}(N_\delta)$. In order to do so we need to find an index pair for $\mathrm{Inv}(N_\delta)$. Let $\partial I^1_E = \{a_1^-, a_1^+\}$, $\partial I^2_E = \{a_2^-, a_2^+\}$. Let $N_\delta^- = \{u_1 = u_*, u_* + \delta \le u_2 \le a_2^+\} \cup \{a_1^- \le u_1 \le u_* - \delta, u_2 = u_*\}$, then (N_δ, N_δ^-) is an index pair for $\mathrm{Inv}(N_\delta)$, and $CH_*(\mathrm{Inv}(N_\delta)) = H_*(N_\delta, N_\delta^-)$. Consequently $CH_1(\mathrm{Inv}(N_\delta)) \simeq \mathbb{Z}$ and $CH_k(\mathrm{Inv}(N_\delta)) = 0$ for $k \ne 1$. The fact that the homological Conley index is non trivial for $k = 1$ and because (7.8) is a gradient flow we conclude that there exists at least one critical point of W_2 in N_δ with large Morse index equal to 1. □

With regard to the relative position of the extrema of W_2 we note the following. Let (b_i, c_i) be the maximum in $D^+_{E,i}$ for $i = 1, 2$. Since $\nabla W_2(b_i, c_i) = 0$ it follows from Lemma 7.5b that $\partial_1 W_2(b_1, u_2) > 0$ for all $u_2 > c_1$ and $\partial_2 W_2(u_1, c_2) < 0$ for all $u_1 < b_2$. Therefore, we may as well use $\tilde{D}^+_{E,3} = \{b_1 \le u_1 \le u_*,\ u_* \le u_2 \le c_2\} \setminus (u_*, u_*)$ instead of $D^+_{E,3}$. We then obtain a saddle point $(b_3, c_3) \in \tilde{D}^+_{E,3}$ with $b_1 < b_3 < b_2$ and $c_1 < c_3 < c_2$.

In terms of closed characteristics for a Lagrangian systems the above lemma yields

Theorem 7.15 *Let $\pi^u N_E$ contain a concatenation $I^\#_E$ of two compact intervals I^1_E and I^2_E, and assume that (L, dt) is a Twist system on $I^\#_E$. If $u_* \in I^1_E \cap I^2_E$ is of saddle-focus type, then there exist at least 3 geometrically distinct closed characteristics.*

An analogue of the above theorem can also be proved for concatenations of more than two interval components. We leave this to the interested reader.

7.3.4 Non-compact interval components

As already indicated in the previous sections the theory developed in this chapter is applicable to various model equations that we know from physics, such as the EFK/Swift-Hohenberg type equations, fifth order KdV equations, suspension bridge model, etc. (see Section 7.2.2). In this section we will take a closer look at the class of EFK/Swift-Hohenberg type equations. This family of equations is given by a Lagrangian of the form: $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u)$, where F is the potential, which is an arbitrary C^2-function of u. We have already proved that such Lagrangian systems are always Twist systems if $\alpha \le 0$ (and we believe the same to be true also for $\alpha > 0$ (Twist property on interval components)). The results obtained in this chapter prove that for any energy level E for which the set $\{u\,|\,F(u) + E \ge 0\}$ contains a compact interval component I_E, there exist a simple closed characteristic $u(t) \in \mathrm{int}(I_E)$. Let us by means of example consider a double equal-well potential F (like $\frac{1}{4}(u^2-1)^2$) with $\min_u F(u) = 0$. In this case the set $\{u\,|\,F(u) + E \ge 0\}$ always contains non-compact interval components. Without further geometric know-

[27]This follows for example from an explicit calculation of the solution for the linearised problem.
[28]The flow is not well-defined on the boundary of D_{E_3}, but we can choose a slightly smaller isolating neighbourhood inside $D_{E,3}$ with the same Conley index (alternatively we can use the Morse index for C^1-functions (see also Section 7.2.3)).

ledge of the energy manifold M_E a general topological result proving existence of closed characteristics does not seem likely. Therefore we will consider a specific example here. Consider the energy level $E = 0$, then $I_0 \stackrel{\text{def}}{=} \pi^u N_0 = \mathbb{R}$, and I_0 is a concatenation of three interval components. The Lagrangian system with $\alpha \le 0$ is a Twist system on I_0 and therefore S is well-defined on $\mathbb{R}^2$. One way to deal with this non-compact case is to compactify the system (see Chapter 5). This however requires detailed information about the asymptotic behaviour of F. There is a weaker assumption that one can use in order to restrict the analysis of W_2 to a compact subset of D_E^+. This boils down to the following geometric property:

(D) *There exists a pair* $(u_1^*, u_2^*) \in D_E^+$ *(with* $|u_1^*|$ *and* $|u_2^*|$ *large) such that* $u_{a,b}'''(0) < 0$ *and* $u_{a,b}'''(\tau) < 0$ *for the unique minimisers* $u_a = u(t; u_1^*, u_2^*)$ *and* $u_b = u(t; u_2^*, u_1^*)$ *of* (7.6)[29].

If (L, dt) satisfies hypothesis (D) on a (non-compact) interval component I_E, then the system is said to be *dissipative* on $I_E^* = [u_1^*, u_2^*]$.

Lemma 7.16 *If a Lagrangian system is dissipative on* I_E^**, then it holds that* $\partial_1 W_2(u_1^*, u_2) < 0$ *for all* $u_2 \in (u_1^*, u_2^*]$ *and* $\partial_2 W_2(u_1, u_2^*) > 0$ *for all* $u_1 \in [u_1^*, u_2^*)$.

Proof. It follows from (D) that $\partial_1 W_2(u_1^*, u_2^*) < 0$. Lemma 7.5b implies that $\partial_1 W_2(u_1^*, u_2)$ is increasing as a function of u_2. It easily follows that $\partial_1 W_2(u_1^*, u_2) < 0$ for all $u_2 \le u_2^*$. The other assertion is proved in exactly the same way. □

For many nonlinearities $F(u)$ it can be proved that the EFK/Swift-Hohenberg system is dissipative on some interval $I_E^* = [u_1^*, u_2^*]$ with $u_1^* < -1$ and $u_2^* > +1$ [30]. Notice that S need not have any critical points, for example for $E \gg 0$ (see Chapter 5). For $E = 0$ there are two equilibrium points which will force S to have critical points.

Lemma 7.17 *If the Swift-Hohenberg Lagrangian is dissipative on* I_0^* *(with* $\{\pm 1\} = \mathcal{C}(I_0^*)$*) then it has at least two geometrically distinct simple closed characteristics (large and small amplitude). Moreover, if* $u = \pm 1$ *are both saddle-foci then there exist two more geometrically distinct simple closed characteristics.*

Proof. We consider the function W_2 on $I_E^* \times I_E^*$ and as before we define $D_E^+ = I_E^* \times I_E^* \cap \{u_2 > u_1\}$ (see also Figure 7.4). Define $A_1 = \{-1 < u_1 < u_2 < 1\}$ and $A_2 = D_E^+ \cap \{u_1 < -1, u_2 > 1\}$. As in the proof of Lemma 7.10 we have that $\partial_1 W_2(\pm 1, u_2) > 0$ and $\partial_2 W_2(u_1, \pm 1) < 0$. We now see from Lemma 7.16 that the gradient of W_2 points outwards on ∂A_2 and inwards on ∂A_1. Hence, on A_1 the function W_2 attains a maximum and on A_2 the function W_2 attains a minimum (index 2 and index 0 points respectively), which proves the first part of the lemma.

As for the second part we argue as in the proof of Lemma 7.12. Since $u = \pm 1$ are saddle-foci one finds index 1 saddle points in both $A_3 = D_E^+ \cap \{-1 < u_1 < 1, u_2 > 1\}$ and $A_4 = D_E^+ \cap \{u_1 < -1, -1 < u_2 < 1\}$. □

Concerning the relative position of the extrema of W_2, the same reasoning as at the end of Section 7.3.3 can be followed. Denoting by (b_i, c_i) the extremum in A_i (for $i = 1, 2, 3, 4$)

[29] Notice that $u_a(t) = u_2^* - u_b(\tau - t)$ if $L(u, v, w)$ is symmetric in v.

[30] For example, when $F(u) \sim |u|^n$ as $|u| \to \infty$ for some $n > 2$ then this follows from a scaling argument. After scaling the Euler-Lagrange equation tends to $u'''' = -|u|u^{n-2}$. For this equation it is easy to see that $u(0) = u_1 < 0$, $u'''(0) = 0$ implies that $u(\tau) = u_2 > 0$ and $u'''(\tau) < 0$. A perturbation argument then shows that (D) is satisfied for the original equation for some (u_1^*, u_2^*) with $-u_1^*$ and u_2^* large.

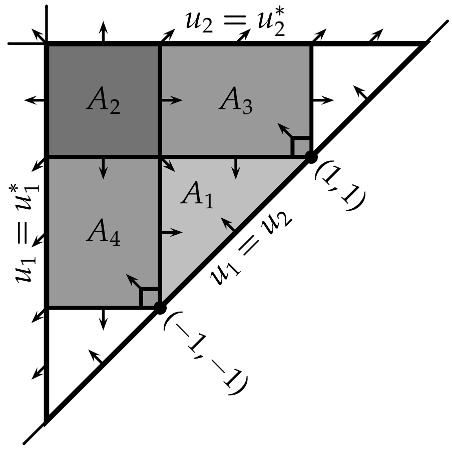

Figure 7.4: The triangle $D_E^+ = I_E^* \times I_E^* \cap \{u_2 > u_1\}$ for the case of a double-well potential. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has at least one maximum and one minimum. Additionally, when the equilibrium points are saddle-foci then W_2 has two saddle points.

we find that $b_2 < b_4 < b_1 < b_3$ and $c_4 < c_1 < c_3 < c_2$.

The result proved above have already been found in Chapter 6 and [106] for the special case $F(u) = \frac{1}{4}(u^2 - 1)^2$ without information about the index of the solutions. Many more examples can be considered with non-compact interval components. A rather tricky system is the suspension bridge model (see Section 7.2.2.2). The Lagrangian is given by $L(u, u', u'') = \frac{1}{2}u''^2 - \frac{c^2}{2}u'^2 + F(u)$, where $F(u) = e^u - u - 1$. This nonlinearity is especially hard to deal with when trying to compactify D_E^+. In this context it is interesting to note that there is no a priori L^∞ bound on the set of bounded solutions (see [121]) as opposed to nonlinearities with super-quadratic growth. From the analysis in [121] it follows that there exists a point $(u_1^*, u_2^*) \in D_E^+$ such that $\partial_1 S(u_1^*, u_2^*) > 0$, $\partial_2 S(u_1^*, u_2^*) > 0$, and $\partial_1 S(u_1^*, u_2) > 0$ for all $u_2 > 0$. This is a different dissipativity condition. Upon examining W_2 (for $E = 0$) on $I_E^* \times I_E^*$ we find at least one index 1 simple closed characteristic for the suspension bridge problem (this was already proved in [121], without information on the Morse index). In order for the argument to work the equilibrium point 0 has to be a saddle-focus. Moreover, for the dissipativity condition to be satisfied the coefficient in front of the second term in the Lagrangian has to be strictly positive. In [121] more complicated closed characteristics are also found. This will be subjected to a further study.

7.4 Concluding remarks

7.4.1 Numerical evidence for the Twist property

In Lemma 7.8 we prove the Twist property for a class of Lagrangians including the well-known Swift-Hohenberg Lagrangian. Numerical evidence suggests that the Twist property holds for a large class of other Lagrangians as well. As an example we depict in Figure 7.5 solutions of the EFK equation (i.e., the EFK Lagrangian with $F(u) = \frac{1}{4}(u^2 - 1)^2$). For $\alpha \leq 0$ the Twist property is always satisfied by Lemma 7.8. Numerical evidence suggests that the Twist property is satisfied for all $E > 0$ and all $\alpha \in \mathbb{R}$ (with $I_E = \mathbb{R}$). At the singular energy level $E = 0$ there are (for $\alpha < 0$) two different cases, namely where the

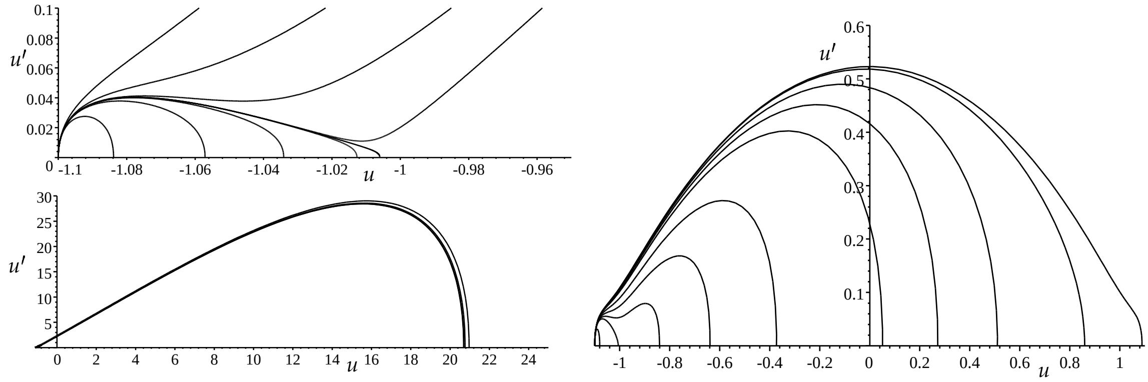

Figure 7.5: For fixed $u_1 = -1.1$ characteristics in the energy level $E = 0$ of the EFK Lagrangian are shown (in the (u, u')-plane). On the left the equilibrium points $u = \pm 1$ are real saddles ($\alpha = 5$). Notice the different scales needed to obtain an overall picture of the situation. The Twist property is only satisfied for $u_2 \in (u_1, -1)$. On the right the equilibrium points are saddle-foci ($\alpha = 1$). In this case the Twist property seemingly holds for all $u_2 > u_1$.

equilibrium points are real saddles and saddle-foci. While the Twist property certainly is not satisfied on the whole of $\mathbb{R}$ (it *is* satisfied on the interval component $[-1, 1]$) for the real saddle case, we conjecture that the Twist property holds on $\mathbb{R}$ as long as the equilibrium points are saddle-foci.

We also performed numerical calculations on the fifth order KdV equation (see Section 7.2.2.3) and it seems that the same is true for this system. It is of course impossible to make statements about the rich class of second order Lagrangians as a whole, but the Twist property appears to hold for a large subclass.

7.4.2 Local behaviour near equilibrium points

In Section 7.3.2 we indicated that the critical points u_* with $\partial_u^2 L(u_*, 0, 0) > 0$ can be categorised into three classes: $\sigma(u_*) = \{\pm\lambda_1, \pm\lambda_2\}$ (real saddle), $\sigma(u_*) = \{\pm a \pm bi\}$ (saddle-focus), and $\sigma(u_*) = \{\pm ai, \pm bi\}$ (center). The fourth possibility, which occurs for equilibrium points with $\partial_u^2 L(u_*, 0, 0) < 0$, is $\sigma(u_*) = \{\pm\lambda, \pm ai\}$ (saddle-center). Such points do not occur as boundary points of interval components and one may ask how they fit in.

Consider a compact interval component I_E, then $L(u, 0, 0) + E > 0$ for all $u \in \mathrm{int}(I_E)$ and $\partial_u L|_{\partial I_E} \geq 0$ (if $\partial_u L = 0$ at a boundary point then necessarily $\partial_u^2 L > 0$). There exists a point $u_* \in \mathrm{int}(I_E)$ such that $\partial_u L(u_*, 0, 0) = 0$ and $\partial_u^2 L(u_*, 0, 0) < 0$. As a matter of fact there may be many minima and maxima. Now let E decrease until the next singular level is reached. If the extremum in this level is a minimum then I_E splits into two components, and if this extremum is a maximum then I_E simply shrinks to the point u_*. Conversely, if u_* is a saddle-center equilibrium point at energy level E_*, then there exists an $\epsilon > 0$ such that $\pi^u N_{E_*+\epsilon}$ contains a compact interval component $I_{E_*+\epsilon}$ which shrinks to u_* as $\epsilon \to 0$.

The local theory for saddle-centers reveals the existence of a family of closed characteristics on $I_{E_*+\epsilon}$ parametrised by ϵ (Lyapunov Center Theorem). Our theory not only provides the existence of closed characteristics for $E_* < E < E_* + \epsilon$ but also guarantees the existence of closed characteristics *for all $E > E_*$ as long as the interval component I_E containing u_* remains compact*. We should emphasise again the resemblance with the classical mechanical system $\frac{\partial L(u,u')}{\partial u} - \frac{d}{dt}\frac{\partial L(u,u')}{\partial u'} = 0$.

7.4.3 KAM theory

For the Lagrangian systems that we study in this chapter one may wonder whether such systems can be completely integrable. A Lagrangian system (L, dt) is said to completely integrable if the associated Hamiltonian system (H, ω) is completely integrable[31]. Many of the examples that we consider such as the EFK/Swift-Hohenberg system with $\alpha \le 0$ are far from being integrable. An example of an integrable system is given by the Lagrangian $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{1}{4}u^4$ (see Chapter 5 for a proof). Integrability can also be addressed at the level of the Twist maps in the Lagrangian systems. Without going into too much detail let us look at a specific example. Consider again the EFK/Swift-Hohenberg family defined by the $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + \frac{1}{4}(u^2 - 1)^2$, $\alpha \le 0$. Now let $E < 0$ and consider the area-preserving map T on $\mathbb{R}^2$ as discussed in Section 7.3.3. It follows from the compactification results in Chapter 5 that $\mathbb{R}^2 \setminus B_r(0)$ contains only invariant curves for the map T for $r > 0$ sufficiently large. Inside the ball $B_r(0)$ the map T can be chaotic (depending on the character of the equilibrium points). The invariant curves in $\mathbb{R}^2 \setminus B_r(0)$ can be interpreted as the invariant tori/circles of an integrable system, comparable to the conserved invariant tori in KAM theory. To get a feel for integrability of the map T on compact interval components we can look at the quadratic Lagrangian $L(u, u', u'') = \frac{1}{2}u''^2 - \frac{1}{2}u^2$. We will leave this to the interested reader.

The question of integrability versus non-integrability for second order Lagrangian systems may be fairly complex. The results in [89, 88] and those proved in Section 7.3.3 seem to suggest that equilibrium points of saddle-focus and center type in combination with geometric and topological conditions on the system create regions of non-integrability. With the techniques presented in this chapter and the methods in Chapter 8 we are trying to understand some of the dynamics of the system in this case. These questions will be subject of future study.

7.5 Classification of equilibrium points

The equilibrium solutions of the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u''} = 0,$$

are given by the relation $\frac{\partial L}{\partial u}(u_*, 0, 0) = 0$. The sign of $\frac{\partial^2 L}{\partial u^2}(u_*, 0, 0)$ divides the behaviours of the equilibrium points onto two groups. We will not consider the case $\frac{\partial^2 L}{\partial u^2}(u_*, 0, 0) = 0$ which requires information on higher order derivatives. Equilibrium points for which $\frac{\partial^2 L}{\partial u^2}(u_*, 0, 0) \neq 0$ are usually called *non-degenerate*. In order to study the local structure of singular points we need to consider the second variation of $J[u]$ around an equilibrium solution $u(t) \equiv u_*$. This yields the following linear differential equation for the variations φ:

$$\frac{\partial^2 L}{\partial u^2}\varphi + \Big(2\frac{\partial^2 L}{\partial u \partial u''} - \frac{\partial^2 L}{\partial u'^2}\Big)\varphi'' + \frac{\partial^2 L}{\partial u''^2}\varphi'''' = 0,$$

where all partial derivatives of L are evaluated at $(u, u', u'') = (u_*, 0, 0)$. The characteristic equation is given by $\partial_u^2 L + \big(2\partial_{uu''}^2 L - \partial_{u'}^2 L\big)\lambda^2 + \big(\partial_{u''}^2 L\big)\lambda^4 = 0$. For non-degenerate equilib-

[31] Note that for a system to be completely integrable it is not necessary that one is able to write down all the conserved quantities explicitly.

rium solutions the following classification holds:

Lemma 7.18 *Let $u(t) \equiv u_*$ be an equilibrium solution.*

(a) *If $\partial_u^2 L < 0$, then $\sigma(u_*) = \{\pm\lambda, \pm ai\}$ (saddle-center).*

(b) *If $\partial_u^2 L > 0$, then $\sigma(u_*) = \{\pm\lambda_1, \pm\lambda_2\}$, $\sigma(u_*) = \{\pm ai, \pm bi\}$, or $\sigma(u_*) = \{\pm a \pm bi\}$ (real saddle, center, and saddle-focus respectively) depending on $\partial_{uu''}^2 L$ and $\partial_{u'}^2 L$.*

Here $a, b, \lambda, \lambda_1, \lambda_2 > 0$.

Proof. From the characteristic equation we derive

$$\lambda_\pm^2 = \frac{-\left(2\partial_{uu''}^2 L - \partial_{u'}^2 L\right) \pm \sqrt{D}}{2\partial_{u''}^2 L}, \qquad \text{where} \quad D = \left(2\partial_{uu''}^2 L - \partial_{u'}^2 L\right)^2 - 4\left(\partial_{u''}^2 L\right)\left(\partial_u^2 L\right)$$

Clearly if $\partial_u^2 L < 0$, then $\sqrt{D} > |2\partial_{uu''}^2 L - \partial_{u'}^2 L|$ and thus $\lambda_-^2 < 0$ and $\lambda_+^2 > 0$. This forces the spectrum to be $\{\pm\lambda, \pm ai\}$. If $\partial_u^2 L > 0$, then $\sqrt{|D|} < |2\partial_{uu''}^2 L - \partial_{u'}^2 L|$ and there are three possibilities:

1. $D > 0$, then $\sqrt{D} < |2\partial_{uu''}^2 L - \partial_{u'}^2 L|$ and $\lambda_\pm^2$ are both positive or negative. This depends on $\partial_{uu''}^2 L$ and $\partial_{u'}^2 L$. If both eigenvalues are negative the spectrum is given by $\{\pm ai, \pm bi\}$, and if both eigenvalues are positive the spectrum is $\{\pm\lambda_1, \pm\lambda_2\}$.
2. $D = 0$, then the same possibilities as in the previous case hold, with the additional property that the eigenvalues all have multiplicity two.
3. $D < 0$, then $\lambda_\pm^2 \in \mathbb{C} \setminus \mathbb{R}$ and there for the spectrum is $\{\pm a \pm bi\}$.

This proves the lemma. □

As indicated before we do not study the case $\partial_u^2 L = 0$. In order to analyse degenerate equilibrium solutions a normal form analysis is required. An example of such type of analysis for a nonlinear saddle-focus can be found in Chapter 5. The results proved in Chapter 5 for nonlinear saddle-foci would suffice for the purposes of the present chapter.

7.6 The proof of Lemma 7.8

Stationary functions of the action functional $J_E[u]$, with $L(u, u', u'') = \frac{1}{2}u''^2 + K(u, u')$, satisfy the equation

$$u'''' - \frac{d}{dt}\frac{\partial K}{\partial u'} + \frac{\partial K}{\partial u} = 0. \tag{7.9}$$

Solutions of (7.9) satisfy the Hamiltonian relation $-u'u''' + \frac{1}{2}u''^2 + \frac{\partial K}{\partial u'}u' - K(u, u') - E = 0$. For an increasing lap from u_1 to u_2 the derivative u' can be represented as a function of u. Set $z(u) = u'\sqrt{u'}$ (see for example [10, 117] were similar transformations are used). Using the Hamiltonian relation we find that z satisfies the equation

$$\begin{cases} \frac{d^2 z}{du^2} = g(u, z) \\ z(u) > 0 \quad \text{for } u \in (u_1, u_2) \\ z(u_1) = z(u_2) = 0, \end{cases} \qquad \text{where } g(u, z) = \frac{3}{2}\,\frac{\frac{\partial K}{\partial u'}u' - K(u, u') - E}{z^{5/3}}.$$

The same holds for decreasing laps ($z < 0$). If

$$\frac{\partial K}{\partial u'}u' - K(u, u') - E \leq 0 \qquad \text{and} \qquad \frac{\partial^2 K}{\partial u'^2}u'^2 - \frac{5}{2}\left(\frac{\partial K}{\partial u'}u' - K(u, u') - E\right) \geq 0,$$

for all $u \in I_E$, and $z \geq 0$ (condition (a) and (b) in Lemma 7.8), then $g(u,z) \leq 0$ and $\frac{\partial g}{\partial z}(u,z) \geq 0$ respectively.

It follows from results in [51] that the boundary value problem for the z-equation has a unique strictly concave positive solution. Consequently the u-laps from u_1 to u_2 are unique, and we thus obtain a family $u(t;u_1,u_2)$. These functions are global minimisers of J_E[32]. From the smooth dependence of the initial value problem of (7.9) we deduce that these functions depend continuously on $\lambda = (u_1,u_2) \in \Lambda \stackrel{\text{def}}{=} I_E \times I_E \setminus \Delta$, and that the time $\tau(u_1,u_2)$ it takes for u to (monotonically) go from u_1 to u_2 depends continuously on u_1 and u_2 as well[33] and $\tau(u_1,u_2) < \infty$ for all $(u_1,u_2) \in \Lambda$[34].

The remainder of this proof will be concerned with showing that $u(t;\lambda)$ varies smoothly with respect to λ for all $\lambda \in \mathrm{int}(\Lambda)$ that are away from possible equilibrium points. Rescale the u-variable as $s = \frac{u-u_1}{u_2-u_1}$ and set $y(s) = z(u)$. From the z-equation we obtain the following equation for y:

$$y'' = \tilde{g}(s,y;\lambda), \quad y(0) = y(1) = 0, \quad y > 0 \text{ on } (0,1).$$

Moreover $\tilde{g} \leq 0$ and $\frac{\partial \tilde{g}}{\partial y} \geq 0$, and we can write $\tilde{g}(s,u;\lambda) = \frac{h(s,y;\lambda)}{y^{5/3}}$ with $h(s,y;\lambda)$ a continuous function.

In order to obtain smooth dependence on the parameter λ we first consider the following equation: $y_\epsilon'' = \tilde{g}(s,y_\epsilon;\lambda)$, $y_\epsilon(0) = y_\epsilon(1) = \epsilon$ and $y_\epsilon > \epsilon$ on $(0,1)$. It follows from the maximum principle that $0 < y_\epsilon - y_0 \leq \epsilon$. For the y_ϵ-problem it is not difficult to show that $y_\epsilon(\cdot;\lambda)$ depends smoothly on λ. To prove this we consider the map $F(y_\epsilon,\lambda) = y_\epsilon'' - g(s,y_\epsilon;\lambda)$, where F maps from $X_\epsilon \times \Lambda$ (with $X_\epsilon = \epsilon + H_0^1(0,1)$) to $H^{-1}(0,1)$, and $F \in C^1(X_\epsilon \times \Lambda, H^{-1})$. From the Implicit Function Theorem we derive that

$$\frac{d}{d\lambda} y_\epsilon(\cdot;\lambda) = -\big(F_y(y_\epsilon,\lambda)\big)^{-1} F_\lambda(y_\epsilon,\lambda) \in C(\Lambda, X_\epsilon).$$

Our goal now is to derive a similar expression for $\frac{d}{d\lambda} y_0(\cdot;\lambda)$. We cannot apply the Implicit Function Theorem to y_0 directly because of the singularity of $\tilde{g}$ at $y = 0$.

We define $\Phi_\epsilon(\lambda) \stackrel{\text{def}}{=} F_y(y_\epsilon(\cdot;\lambda),\lambda) = \frac{d^2}{ds^2} - \frac{\partial g}{\partial y}(s,y_\epsilon;\lambda) = \frac{d^2}{ds^2} - \frac{k(s,y_\epsilon;\lambda)}{y_\epsilon^{8/3}}$, where k is a continuous function. For $\lambda \in \Lambda$ away from the equilibrium points the asymptotic behaviour of y_0 at $s = 0,1$ is $y_0(s) = O(s^{3/4})$ as $s \downarrow 0$ and $y_0(s) = O((1-s)^{3/4})$ as $s \uparrow 1$. We now conclude from Hardy's inequality that $\Phi_0(\lambda) \in B(H_0^1, H^{-1})$ for all $\lambda \in \Lambda$.

It holds that $\Phi_\epsilon(\lambda) \to \Phi_0(\lambda)$ in $B(H_0^1, H^{-1})$ as $\epsilon \to 0$, and the same holds for the inverses in $B(H^{-1}, H_0^1)$ since $\Phi_\epsilon(\lambda)$ is uniformly bounded in ϵ. We obtain that (writing $k_\epsilon = k(\cdot, y_\epsilon;\lambda)$)

$$\|\Phi_\epsilon(\lambda) - \Phi_0(\lambda)\| \leq C \|k_\epsilon \Big(\frac{y_0}{y_\epsilon}\Big)^{\frac{8}{3}} - k_0\|_{L^2}.$$

From the L^∞-convergence of y_ϵ to y_0 we then conclude that $\Phi_\epsilon(\lambda) \to \Phi_0(\lambda)$ as $\epsilon \to 0$. In order to obtain the above inequality we again used Hardy's inequality in combination

[32] In z-variables we have $J_E = \int_{u_1}^{u_2} \big(\frac{2}{9} z'^2 + \frac{K(u,z^{2/3})+E}{z^{2/3}}\big)\, du$. The condition $\frac{\partial g}{\partial z} \geq 0$ implies that this functional is convex.

[33] Away from equilibrium points this is obvious. At equilibrium points this follows either by taking limits and using the uniqueness, or from the local analysis performed in [120, Lemma 5.8].

[34] It follows from $g \leq 0$ and the analysis in Section 7.5 that equilibrium points (which are non-degenerate by assumption) can only be of saddle-focus or center type.

with the asymptotic behaviour of y_0 at $s = 0, 1$[35].

We now assert that $F_\lambda(y_\epsilon, \lambda) \to F_\lambda(y_0, \lambda)$ in H^{-1} as $\epsilon \to 0$. We find that

$$\|F_\lambda(y_\epsilon, \lambda) - F_\lambda(y_0, \lambda)\|_{H^{-1}} \leq C\|\frac{\partial h_\epsilon}{\partial \lambda}\left(\frac{y_0}{y_\epsilon}\right)^{\frac{5}{3}} - \frac{\partial h_0}{\partial \lambda}\|_{L^2}.$$

As before, due to the L^∞-convergence of y_ϵ to y_0 the assertion follows.

We conclude that $\frac{d}{d\lambda}y_\epsilon(\cdot;\lambda)$ converges to $\left(F_y(y_0,\lambda)\right)^{-1}F_\lambda(y_0,\lambda) \stackrel{\text{def}}{=} \zeta_\lambda$. The next step is to consider the difference quotient $D_h y(\cdot;\lambda) = \frac{y(\cdot;\lambda+h)-y(\cdot;\lambda)}{h}$. We have that $D_h y_\epsilon \to D_h y_0$ in L^∞ as $\epsilon \to 0$, and $D_h y_\epsilon \to \frac{d}{d\lambda}y_\epsilon(\cdot;\lambda)$ as $h \to 0$ for $\epsilon > 0$. Combining these facts we obtain $\|\zeta_\lambda - D_h y_0\|_{L^\infty} \leq \|\zeta_\lambda - \frac{d}{d\lambda}y_\epsilon(\cdot;\lambda)\|_{L^\infty} + \|\frac{d}{d\lambda}y_\epsilon(\cdot;\lambda) - D_h y_\epsilon\|_{L^\infty} + \|D_h y_\epsilon - D_h y_0\|_{L^\infty} \to 0$ as $\epsilon, h \to 0$. This gives

$$\frac{d}{d\lambda}y_0(\cdot;\lambda) = -\left(F_y(y_0,\lambda)\right)^{-1}F_\lambda(y_0,\lambda) \in H_0^1(0,1), \quad \text{for all } \lambda \in \operatorname{int}(\Lambda).$$

Finally, an estimate similar to the ones above shows that $\frac{d}{d\lambda}y_0(\cdot;\lambda)$ depends continuously on λ for all $\lambda \in \operatorname{int}(\Lambda)$ that are away from equilibrium points. It then follows from the differential equation that $y_0'(s;\lambda)$ and $y_0''(s;\lambda)$ are C^1-functions of λ for all $s \in (0,1)$, i.e., $y_0(\cdot;\lambda)$ is continuously differentiable as a C^2-function on any compact subset of $(0,1)$. This implies that $u(\cdot;\lambda)$ is continuously differentiable as a C^3-function (at least away from its extrema). Finally, a simple application of the Implicit Function Theorem shows that $\tau(\lambda)$ is continuously differentiable for all $\lambda \in \operatorname{int}(\Lambda)$ that are away from equilibrium points.

[35]If k has a zero at $s = 0$ or $s = 1$ the asymptotic behaviour of y_0 will be different (i.e. $y_0 = O(s)$ near $s = 0$). In this case a slightly different inequality holds which proves the same statement.

Chapter 8

Braided closed characteristics

8.1 Introduction

This chapter extends the investigation of periodic solutions of second order Lagrangians, which was started in Chapter 7. There *simple* closed characteristics were studied, whereas in this chapter the focus is on so-called *braided* closed characteristics. The main idea is to reduce the variational problem to a finite dimensional setting where only the extrema of a profile are varied. A *Twist property* ensures the uniqueness of the monotone laps between the extrema. For simple closed characteristics the new setting is two dimensional, which greatly simplifies the analysis. The reduced problem for non-simple closed characteristics is set in a higher dimensional phase space. Since the analysis of the (equilibria of the) associated gradient flow is quite complex, one is forced to introduce a new perspective: braid diagrams (we come back to this shortly). In this chapter we give an overview of the method; for a complete treatment we refer to [74].

We summarise the machinery introduced in Chapter 7. Consider a second order Lagrangian system (L, dt), where $L = L(u, u', u'')$ is the Lagrangian. Assume that $L \in C^2(\mathbb{R}^3; \mathbb{R})$ satisfies the non-degeneracy hypothesis $\partial_w^2 L(u, v, w) \geq \delta > 0$. Our aim is to find bounded functions, or *bounded characteristics*, $u : \mathbb{R} \to \mathbb{R}$, which are stationary for the action integral $J[u] = \int L(u, u', u'')\, dt$. Such stationary points u are bounded solutions of the Euler-Lagrange equation

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial u''} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0. \tag{8.1}$$

Solutions of (8.1) satisfy the energy constraint

$$\Big(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\Big)u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u'') = E = \text{ constant}.$$

By transforming to a Hamiltonian context, one finds that characteristics reside on non-compact three-dimensional energy surfaces in $\mathbb{R}^4$.

An energy value E is called regular if $\frac{\partial L}{\partial u}(u, 0, 0) \neq 0$ for all u that satisfy $L(u, 0, 0) + E = 0$. For a fixed regular energy value E the extrema of a characteristic are contained in the closed set $\{u \,|\, L(u, 0, 0) + E \geq 0\}$. The connected components I_E of this set are called *interval components*. In order to set up a variational principle for bounded characteristics in terms of the extrema of u, the following *Twist hypothesis* was introduced in Chapter 7:

(T) $\inf\{J_E[u] \equiv \int_0^\tau \big(L(u, u', u'') + E\big)dt \,|\, u \in X_\tau(u_1, u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimiser* $u(t; u_1, u_2)$ *for all* $\{(u_1, u_2) \in I_E \times I_E \,|\, u_1 \neq u_2\}$, *and* u *and* τ *are* C^1*-smooth functions of* (u_1, u_2).

Here $X_\tau = X_\tau(u_1, u_2) = \{u \in C^2([0, \tau]) \,|\, u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = u'(\tau) = 0$, and $u'|_{(0,\tau)} > 0$ if $u_1 < u_2$, and $u'|_{(0,\tau)} < 0$ if $u_1 > u_2\}$. It may be convenient to think of Hypothesis (T) as an assumption on the uniqueness of the monotone laps between minima and maxima, although the uniqueness assumption is not completely equivalent to hypothesis (T). In

Chapter 7 it has been proved that hypothesis (T) holds for a large class of Lagrangians L, including the Swift-Hohenberg Lagrangian $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{\beta}{2}u'^2 + F(u)$ with $\beta \leq 0$, and numerics suggest that (T) is more generally satisfied on interval components of regular energy surfaces.

We recast the problem of finding periodic orbits for a given energy level E into solving second order recurrence relations. This is accomplished via a method comparable to that of *broken geodesics*, which in the present context are concatenations of the monotone laps given by (T) (see Chapter 7). In this introduction we want to give a concise survey of the method, and therefore we do not go into full detail, but in subsequent sections precise definitions are given.

A closed characteristic u at energy level E is a (C^2-smooth) function $u : [0, \tau] \to \mathbb{R}$, $0 < \tau < \infty$, which is stationary for the action $J_E[u]$ with respect to variations $\delta u \in C^2_{\mathrm{per}}([0, \tau])$, and $\delta\tau \in \mathbb{R}^+$. The Twist hypothesis (T) allows one to encode a characteristic by its extrema $\{u_i\}$. A broken geodesic $u : [0, \tau] \to I_E$ is a closed characteristic at a regular energy level E if and only if the sequence of its extrema (u_i) satisfies $\nabla W_{2p}(u_i, \ldots, u_{i+2p}) = 0$, where $2p$ is the number of extrema in one period, $W_{2p} = \sum_{i=0}^{2p-1} S(u_i, u_{i+1})$, and $S(u_i, u_{i+1})$ is the action of the lap connecting u_i and u_{i+1}. This function S is a generating function and the functional W_{2p} is a discrete action defined on the space of $2p$-periodic sequences.

The problem of finding critical points of W_{2p} can be rephrased via the recurrence relation

$$\mathcal{R}(u_{i-1}, u_i, u_{i+1}) \stackrel{\mathrm{def}}{=} \partial_2 S(u_{i-1}, u_i) + \partial_1 S(u_i, u_{i+1}) = 0. \tag{8.2}$$

The analysis of (8.2) is facilitated by the study of the gradient flow $u_i' = \mathcal{R}(u_{i-1}, u_i, u_{i+1}) = \nabla W$ on a space of sequences. We may assume, without loss of generality, that $(-1)^i u_i < (-1)^i u_{i+1}$, with $u_i, u_{i+1} \in I_E$. In this context, the Twist hypothesis (T) translates into the *Twist property* for $\mathcal{R}$:

$$\partial_1 \mathcal{R} > 0 \quad \text{and} \quad \partial_3 \mathcal{R} > 0. \tag{8.3}$$

In order to have a smooth flow on a compact space we consider two natural boundary conditions for the generating function S, which are derived from the behaviour of S near $\partial(I_E \times I_E)$. In the compact case we can find a compact interval $I \subset I_E$ such that $\partial(I \times I)$ is repelling, and in case $I_E = \mathbb{R}$ we assume (the natural condition) that there exists a compact interval $I \subset \mathbb{R}$ such that $\partial(I \times I)$ is attracting (dissipativity assumption):

(C) *compact*: large amplitudes are repelling;
(D) *dissipative*: large amplitudes are attracting.

For the third possibility, that of mixed boundary conditions, we refer to [74].

As mentioned before, the Twist hypothesis (T) allows one to encode a characteristic by its extrema $\{u_i\}$, and without loss of generality we take u_0 to be a local minimum. We can construct a piecewise linear graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ by straight line segments (see Figure 8.1a,b). If u is a closed characteristics then its critical points are encoded by a finite sequence $\{u_i\}_{i=0}^{2p-1}$, where $2p$ is the discrete period. The piecewise linear graph, called a *strand*, is really cyclic: one restricts to $0 \leq i \leq 2p$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n strands. We place on these diagrams a *braid structure* by assigning a crossing type (positive) to every transverse intersection of the graphs:

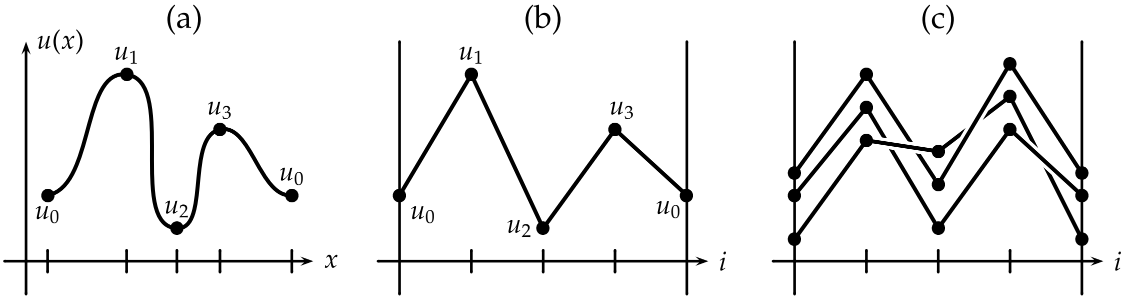

Figure 8.1: (a) A periodic function and (b) its piecewise linear graph; (c) a braid consisting of 3 strands.

larger slope crosses over smaller slope (see Figure 8.1c). We thus represent periodic sequences of extrema in the space of closed, positive, piecewise linear braid diagrams. Since for bounded characteristics local minima and maxima occur alternately, we require that $(-1)^i(u_{i\pm1} - u_i) > 0$: the (natural) *up-down* restriction. This space of up-down piecewise linear braids is denoted by $\mathcal{E}^n_{2p}$, where $2p$ is the period and n is the number of strands. The completion $\overline{\mathcal{E}}^n_{2p}$ includes singular braid diagrams. Definitions are provided in Section 8.2. Notice that the problem is invariant under even shifts of the index i.

The gradient flow of $W_{2p}(u_0, \ldots, u_{2p-1})$ on $2p$-periodic sequences immediately translates to a flow on $\overline{\mathcal{E}}^n_{2p}$. The Twist property (T) or, equivalently, property (8.3), appends additional structure to the gradient flow on the piecewise linear braid diagrams; the gradient flow associated to Equation (8.2) is tightly linked with the braid structure. Namely, the complexity of braid diagrams or, more precisely, the number of intersections in the braid diagram, decreases along the flow. This property is the discrete analogue of the lap number theorem for second order parabolic equations. The strategy is to construct isolating neighbourhoods for the gradient flow of W_{2p} on $\overline{\mathcal{E}}^n_{2p}$ and compute its Conley homology. Nontrivial Conley homology implies the existence of closed characteristics.

Consider the special situation of $(n+1)$-strand braid diagrams where n designated strands, the *skeleton*, corresponds to a collection of closed characteristics. Since these closed characteristics are stationary for the gradient flow of W_{2p}, it induces a flow on a ($2p$-dimensional) invariant subset of $\overline{\mathcal{E}}^{n+1}_{2p}$, the *relative braid diagrams*: only one of the strands exhibits dynamics under the gradient flow of W_{2p}. The space $\overline{\mathcal{E}}^{n+1}_{2p}$ is partitioned into braid classes by co-dimension 1 'walls' of singular braids. This also induces a partitioning of the relative braid diagrams. These equivalence classes of braid types are candidates for isolating neighbourhoods.

Under either of the boundary conditions (C) or (D), consider a braid class for which the $(n+1)^{\text{st}}$ strand is non-isotopic to any of the strands of the skeleton (i.e., none of the strands of the skeleton is contained in the boundary). The fact that the number of intersections decreases along the flow implies that the closure of such a braid class is a proper isolating neighbourhood for the induced flow. Consequently the Conley homology is well-defined.

We carry out the above construction for two special braid classes depicted in Figure 8.2. In the compact case we consider a skeleton of two linked strands with period $2p$ and non-zero linking number r (i.e. crossing number $2r$), where $0 < r \leq p$. The third strand (dashed) has linking number $q < r$ with the skeleton. We denote this braid class by $X^q_{p,r}$. In the dissipative case we consider a skeleton of two strands of period $2p$ with non-

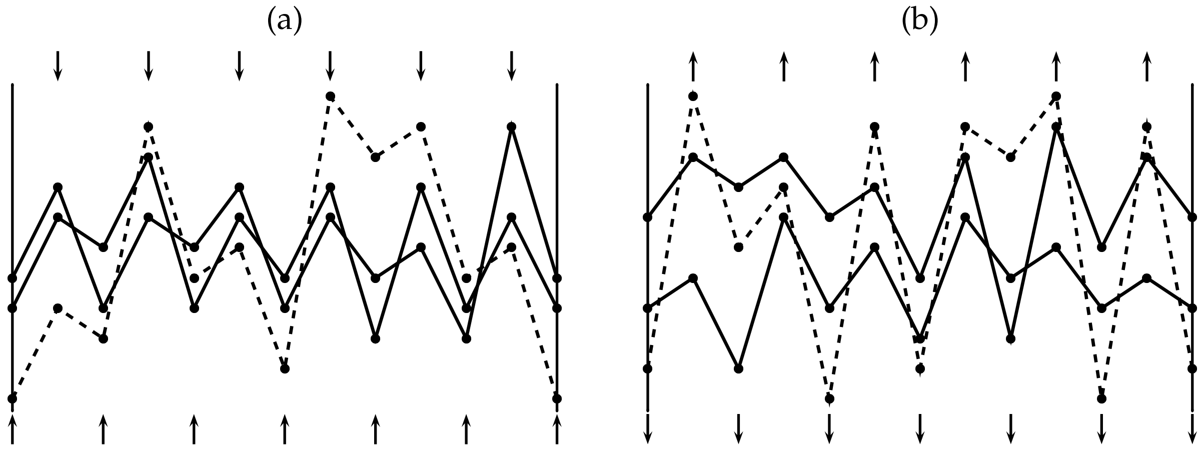

Figure 8.2: Two examples of relative braid classes (dashed) whose Conley homology with respect to the fixed strands (solid) is nontrivial. (a) Compact boundary conditions: $X^r_{p,q}$ with $p = 6, r = 3, q = 2$; (b) dissipative boundary conditions: $Y^r_{p,q}$ with $p = 6, r = 1, q = 4$.

maximal linking number $0 \leq r < p$. The third strand (dashed) has linking number $q > r$ with the skeleton. We denote this braid class by $Y^q_{p,r}$.

Theorem 8.1 *Consider the braid classes $X^q_{p,r}$ (with $0 < q < r \leq p$) and $Y^q_{p,r}$ (with $0 \leq r < q < p$) given in Figure 8.2. The Conley homology of the gradient flow of W_{2p} on these braid classes is well-defined and given by*

$$CH_k(X^q_{p,r}) = \begin{cases} \mathbb{Z} & k = 2q-1 \text{ or } 2q, \\ 0 & \text{else.} \end{cases} \qquad CH_k(Y^q_{p,r}) = \begin{cases} \mathbb{Z} & k = 2q \text{ or } 2q+1, \\ 0 & \text{else.} \end{cases}$$

Nontrivial Conley homology of a braid class, together with the gradient nature of the flow, implies the existence of a critical point in that class. One easily constructs an infinite family of closed characteristics with distinct braid types forced by the pair of non-maximally linked (including unlinked) orbits for dissipative boundary conditions or linked orbits for compact boundary conditions, by taking higher covers of the base orbits (i.e., taking multiples of p and r) and applying Theorem 8.1 iteratively (see Section 8.5).

Theorem 8.2 *Consider Equation* (8.1) *for a regular energy level E under the Twist hypothesis* (T). *The following are sufficient conditions for the existence of infinitely many distinct (in particular having distinct braid types) closed characteristics:*

(a) *a compact interval component I_E and the existence of a pair of closed orbits whose braid representations are linked.*

(b) *an interval component $I_E = \mathbb{R}$ with dissipative asymptotic behaviour and the existence of a pair of closed orbits whose braid representations are unlinked or non-maximally linked.*

Note that in both cases the existence of a single non-simple closed characteristic u is a sufficient condition. Indeed, two even shifts of the braid representation of u yield a 2-strand braid that is necessarily linked but not maximally linked.

The outline of this chapter is as follows. In Section 8.2 we introduce the necessary definitions of braid classes. Next, in Section 8.3 we introduce a class of flows that respect the braid structure. The Conley index of braid classes with respect to such flows is defined

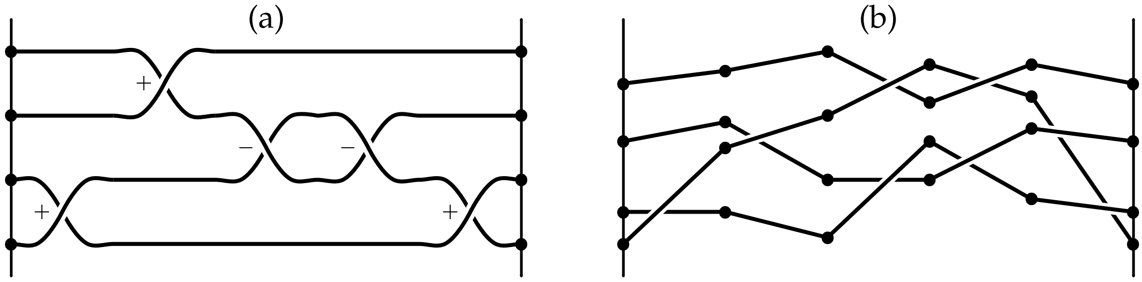

Figure 8.3: (a) A braid with representation $\sigma_1\sigma_3\sigma_2^{-2}\sigma_1$, where the positive and negative crossings are marked by $+$ and $-$ respectively. (b) A piecewise linear braid in $\mathcal{D}_5^4$.

in Section 8.4. Finally, in Section 8.5 we apply the theory to second order Lagrangian systems and, after a calculation of the Conley index (Theorem 8.1) we obtain the existence results of Theorem 8.2. We only give an outline of the techniques here; a complete description can be found in [74], as well as generalisations in various directions.

8.2 Spaces of closed braid diagrams

Recall the definition of a braid (see [27] for a detailed introduction). A (geometric) braid β on n strands is a collection of embeddings $\{\beta^k : [0,1] \to \mathbb{R}^3\}_{k=1}^n$ with disjoint images such that

1. $\beta^k(0) = (0,0,k)$;
2. $\beta^k(1) = (0,0,\tau(k))$ for some permutation τ;
3. $\frac{\partial}{\partial s}\beta^k(s) > 0$ for all $s \in [0,1]$.

The last condition implies that the braid is to be 'read' from left to right. Two such braids are said to be equivalent, or of the same *topological braid type*, if they are homotopic in the space of braids. In particular, no intersections are permitted; the strands must remain disjoint. There is a natural group structure on the space of braids with n strands, B_n, given by concatenation. Using generators σ_k which interchange the k^{th} and $(k+1)^{st}$ strands (with a positive crossing) yields the representation:

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i\sigma_j & = & \sigma_j\sigma_i & ; \quad |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & ; \quad i < n-1 \end{array} \right\rangle.$$

Any braid can be written in terms of generators σ_i (see Figure 8.3a for an example).

Braids find their greatest applications in knot theory via taking their closures. Algebraically, the closed braids on n strands can be defined as the group of conjugacy classes in B_n. Geometrically, in closing the braids one quotients out the range of the braid embeddings via the equivalence relation $(0,y,z) \sim (1,y,z)$ (i.e., identifying begin- and endpoints), and alters the restriction on the position of the endpoints to be $\beta^k(0) \sim \beta^{\tau(k)}(1)$. Thus, a closed braid is a collection of disjoint embedded loops in $S^1 \times \mathbb{R}^2$ which are everywhere transverse to the $\mathbb{R}^2$-planes.

The specification of a braid type may be accomplished unambiguously by a labelled projection to the (x,y)-plane: a *braid diagram*. Any braid may be perturbed slightly so that all strand crossings in the projection are pairwise transversal; in this case, a marking of $(+)$ or $(-)$ serves to indicate whether the crossing is 'right over left' or 'left over right' respectively (see also Figure 8.3a).

In the sequel we will restrict to a special type of braid diagrams: *piecewise linear* (or *discretised*) braid diagrams, which we will now define (see also Figure 8.3b).

Definition 8.3 *Denote by $\mathcal{D}_d^n$ the space of all closed piecewise linear braid diagrams (PL-braid diagrams) on n strands with period d. That is, the space of all (unordered) collections $\beta = \{\beta^k\}_{k=1}^n$ of continuous maps $\beta^k : [0,1] \to \mathbb{R}$ such that*

(a) *β^k is affine linear on $[\frac{i}{d}, \frac{i+1}{d}]$ for all k and for all $i = 0, \ldots, d-1$;*

(b) *$\beta^k(0) = \beta^{\tau(k)}(1)$ for some permutation τ;*

(c) *for any s such that $\beta^k(s) = \beta^l(s)$ with $k \neq l$, the crossing is transversal: for ϵ sufficiently small*
$$\big(\beta^k(s-\epsilon) - \beta^l(s-\epsilon)\big)\big(\beta^k(s+\epsilon) - \beta^l(s+\epsilon)\big) < 0;$$

(d) *any such crossing is marked with a $(+)$ crossing sign.*

In our applications the permutation τ in property (b) will generally be the identity, so that each strand forms a closed loop. Notice that any PL-braid is of course completely determined by the points $\beta^k(\frac{i}{d})$, which we denote by u_i^k, and we will alternate between the notation β and $\boldsymbol{u} = (u_i^k)$ for a PL-braid diagram throughout.

By definition all crossings of the strands occur at isolated points, and that path components of $\mathcal{D}_d^n$ comprise *closed braid types* $[\beta]$; one cannot change the braid type by a continuous deformation through diagrams in $\mathcal{D}_d^n$. In order to define proper topological invariants for the path components of $\mathcal{D}_d^n$ we need to know how these components fit together. This can be achieved by considering 'singular' braid diagrams. The singular diagrams act as gates between the path components of $\mathcal{D}_d^n$.

Definition 8.4 *Denote by $\overline{\mathcal{D}}_d^n$ the space of all PL-braid diagrams β which satisfy properties* (a) *and* (b) *of Definition 8.3 (strong closure). On the subset $\mathcal{D}_d^n \subset \overline{\mathcal{D}}_d^n$ retain the* $(+)$ *crossing convention. Denote by $\Sigma = \overline{\mathcal{D}}_d^n \setminus \mathcal{D}_d^n$ the set of* singular *braid diagrams.*

The set Σ is a variety in $\overline{\mathcal{D}}_d^n$ consisting of numerous co-dimension one walls which mutually intersect along higher co-dimension faces.

Definition 8.4 implies that singular braid diagrams do not satisfy condition (3) of Definition 8.3. To be more precise, for any singular braid $\beta \in \Sigma$ there exist times $t \in \{\frac{k}{d}\}_{k=0}^d$ and indices $i \neq j$ such that $\beta^i(s) = \beta^j(s)$, and
$$\big(\beta^i(s-\epsilon) - \beta^j(s-\epsilon)\big)\big(\beta^i(s+\epsilon) - \beta^j(s+\epsilon)\big) \geq 0,$$
for sufficiently small $\epsilon > 0$. The number of such distinct occurrences is the co-dimension of the singular braid diagram $\beta \in \Sigma$.

For singular braids of sufficiently high co-dimension entire components of the braid diagram can coalesce. We define these *collapsed singularities* Σ^- as follows:
$$\Sigma^- = \{\beta \in \Sigma \mid \beta^k(s) = \beta^l(s) \text{ for all } s \text{ and some } k \neq l\}.$$
Clearly the co-dimension of singularities in Σ^- is at least d. Since for braid diagrams in Σ^- the number of strands reduces, such singularities consist of the spaces $\overline{\mathcal{D}}_d^{n'}$, $n' < n$, i.e. $\Sigma^- = \bigcup_{n'<n} \overline{\mathcal{D}}_d^{n'}$. If $n = 1$ then $\Sigma^- = \varnothing$.

Given $\beta_0 \in \overline{\mathcal{D}}^n_d$ and $\beta_1 \in \overline{\mathcal{D}}^m_d$ the union $\beta_0 \cup \beta_1 \in \overline{\mathcal{D}}^{n+m}_d$ is naturally defined. We can now introduce the notion of relative braid type in $\mathcal{D}^{n+m}_d$. Given $\beta_1 \in \mathcal{D}^m_d$ define

$$\mathcal{D}^n_d \text{ rel } \beta_1 = \{\beta_0 \cup \beta_1 \mid \beta_0 \in \mathcal{D}^n_d\} \cap \mathcal{D}^{n+m}_d$$

The path components of $\mathcal{D}^n_d$ rel β_1 comprise the *relative braid types* $[\beta_0 \text{ rel } \beta_1]$, which gives a partitioning of $\mathcal{D}^n_d$ (β_1-dependent). The braid β_1 is called the *skeleton* in this setting. The set of singular braids Σ rel β_1 are those singular braids in $\Sigma \subset \overline{\mathcal{D}}^{n+m}_d$ of the form $\beta_0 \cup \beta_1$, β_1 fixed. The associated collapsed singular braids are denoted by Σ^- rel β_1. As before, the set $\mathcal{D}^n_d$ rel $\beta_1 \cup \Sigma$ rel β_1 is the closure of $\mathcal{D}^n_d$ rel β_1, and is denoted by $\overline{\mathcal{D}}^n_d$ rel β_1. Two relative braid types $[\beta_0 \text{ rel } \beta_1]$ and $[\beta'_0 \text{ rel } \beta'_1]$ in $\mathcal{D}^n_d$ rel β_1 and $\mathcal{D}^n_d$ rel β'_1 respectively, are called *equivalent*, notation $[\beta_0 \text{ rel } \beta_1] \sim [\beta'_0 \text{ rel } \beta'_1]$, if $[\beta_1] = [\beta'_1]$ and $[\beta_0 \cup \beta_1] = [\beta'_0 \cup \beta'_1]$. Later on we will assign a topological invariant to the type $[\beta_0 \text{ rel } [\beta_1]]$.

Let us now define the subclass of up-down braid diagrams. The reason for this construction is that closed characteristics (i.e., periodic solutions of Equation (8.1)) consist of an alternation of decreasing and decreasing laps.

Definition 8.5 *The space* $\mathcal{E}^n_{2p}$ *of* up-down *PL-braid diagrams on* n *strands with period* $2p$ *is the subset of* $\mathcal{D}^n_{2p}$ *determined by the relation* $(-1)^i(u^k_{i+1} - u^k_i) > 0$*, for* $k = 1, \ldots, n$ *and* $i = 0, \ldots, 2p-1$*, where* $u^k_i = \beta^k(\frac{i}{2p})$*.*

In this definition we choose the first lap to be increasing. Let $\overline{\mathcal{E}}^n_{2p}$ be the subset of all braid diagrams in $\overline{\mathcal{D}}^n_{2p}$ satisfying $(-1)^i(u^k_{i+1} - u^k_i) > 0$. As before the singular braid diagrams are defined as $\Sigma = \overline{\mathcal{E}}^n_{2p} - \mathcal{E}^n_{2p}$. The path components in $\mathcal{E}^n_{2p}$ comprise the up-down braid types $[\boldsymbol{u}]_{\mathcal{E}}$, where $\boldsymbol{u} = (u^k_i)$. The path components in $\mathcal{E}^n_{2p}$ rel $\boldsymbol{v}$ make up the relative up-down braid types $[\boldsymbol{u} \text{ rel } \boldsymbol{v}]_{\mathcal{E}}$.

In contrast to $\overline{\mathcal{D}}^n_d$, the set $\overline{\mathcal{E}}^n_{2p}$ is a subset of $\mathbb{R}^{2pn}$ *with* boundary (which is given by $\partial\overline{\mathcal{E}}^n_{2p} = \mathrm{cl}\big(\mathcal{E}^n_{2p}\big)\backslash\mathcal{E}^n_{2p}$). The boundary $\partial\overline{\mathcal{E}}^n_{2p}$ can be characterised as follows:

$$\partial\overline{\mathcal{E}}^n_{2p} = \{\boldsymbol{u} \in \overline{\mathcal{E}}^n_{2p} \mid u^k_i = u^k_{i+1} \text{ for at least one } i \text{ and } k\}.$$

Such boundary braids are called *horizontal singularities*.

8.3 Parabolic flows and recurrence relations

In this section we will introduce a class of recurrence relations which we will use to define particular types of flows on the spaces $\overline{\mathcal{D}}^n_d$. The recurrence relations in this section are defined for general domains and we will specify particular choices of domains where needed.

Let Ω be the sequence space $\Omega = \mathbb{R}^{\mathbb{Z}}$; an element in Ω is denoted by $u = (u_i)_{i\in\mathbb{Z}}$. Define the recurrence relation $\mathcal{R} = (\mathcal{R}_i)_{i\in\mathbb{Z}}$, with the sequence of functions $\mathcal{R}_i$ satisfying the following axioms:

(A1) *smoothness*: $\mathcal{R}_i = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) \in C^1(\mathbb{R}^3)$ for all $i \in \mathbb{Z}$.

(A2) *monotonicity*: $\partial_1\mathcal{R}_i > 0$ and $\partial_3\mathcal{R}_i \geq 0$ for all $u \in \Omega$ and all $i \in \mathbb{Z}$; or
$\partial_1\mathcal{R}_i \geq 0$ and $\partial_3\mathcal{R}_i > 0$ for all $u \in \Omega$ and all $i \in \mathbb{Z}$.

(A3) *periodicity*: for some $d \in \mathbb{N}$, $\mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$.

The choice $d = \infty$ in Axiom (A3) means no periodicity requirements. Now consider the recurrence relation

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = 0, \quad (u_i)_{i\in\mathbb{Z}} \in \Omega,\ i \in \mathbb{Z}. \tag{8.4}$$

If Axioms (A1)-(A3) are met then (8.4) is called a *parabolic recurrence relation*.

For applications to Lagrangian dynamics a variational structure needs to be present. At the level of recurrence relations this implies that $\mathcal{R}$ is a gradient. This property is captured by the following axiom:

(A4) *exactness*: there exist functions $S_i \in C^2(\mathbb{R}^2)$ such that

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_{i-1}(u_{i-1}, u_i) + \partial_1 S_i(u_i, u_{i+1}) \qquad \text{for all } u \in \Omega \text{ and } i \in \mathbb{Z}.$$

The functions S_i are called the generating functions. In Lagrangian problems the action functional naturally defines the functions S_i. We can also define the formal action in this case: $W(u) = \sum_i S_i(u_i, u_{i+1})$, and $\mathcal{R} = \nabla W$. Recurrence relations that satisfy (A1)-(A4) are called *exact parabolic recurrence relations*.

In order to define parabolic flows we regard $\mathcal{R}$ as a vector field on Ω, and consider the differential equations

$$\frac{du_i}{dt} = \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}), \qquad u \in \Omega. \tag{8.5}$$

If Axioms (A1)-(A2) are satisfied one can show with some effort that Equation (8.5) defines a C^1-flow ψ^t on $X(\Omega)$ (see [5] for details). For our purpose we can restrict to the flow on the spaces of periodic sequences

$$\Omega_{kd} = \{u \in \Omega \mid u_{i+kd} = u_i \text{ for all } i \in \mathbb{Z}\}.$$

If Axiom (A3) is satisfied then (8.5) defines a flow on Ω_{kd} for any $k \in \mathbb{N}$. We will use the notations $\psi^t(u(0)) = u(t) = (u_i(t))$ interchangeably.

For a pair of sequences u and v in X_{kd} one can define the intersection number $I(u, v)$ as follows. Consider u and v as piecewise linear graphs β_u and β_v. If the two piecewise linear graphs β_u and β_v intersect transversely the intersection number is defined as the number of intersections over one period kd. Clearly the intersection number is even.

Axiom (A2) implies that a parabolic flow ψ^t acts in a natural way with respect to the intersection number. The following result is a direct consequence of a result by J. Smillie [135] for specific tridiagonal systems of ordinary differential equations. We will use extensions of this result as given in [8, 70].

Lemma 8.6 *Let ψ^t be the parabolic flow on X_{kd} defined by* (8.5). *Then for any $u \not\equiv v \in X_{kd}$, the set of t-values for which $\psi^t(u)$ and $\psi^t(v)$ do not intersect transversely is discrete. If $\psi^t(u)$ and $\psi^t(u)$ are non-transverse at $t = t_0$, then $I(\psi^t(u), \psi^t(v))|_{t=t_0^-} > I(\psi^t(u), \psi^t(v))|_{t=t_0^+}$, i.e., $I(\psi^t(u), \psi^t(v))$ is a non-increasing function of t for any pair $u \not\equiv v$.*

It is this property (which is analogous to the lap number theorem for second order parabolic partial differential equations) that inspires us to attach the name 'parabolic' to these flows.

Our objective now is to define flows on the finite dimensional spaces of PL-braids $\overline{\mathcal{D}}^n_d$. Recall that elements β of $\overline{\mathcal{D}}^n_d$ are also denoted by $\boldsymbol{u} = (u^k_i)$, where $u^k_i = \beta^k(\frac{i}{d})$. One considers the same equations as (8.5):

$$\frac{du_i^k}{dt} = \mathcal{R}_i(u_{i-1}^k, u_i^k, u_{i+1}^k), \qquad \boldsymbol{u} \in \overline{\mathcal{D}}_d^n. \tag{8.6}$$

The flow on $\overline{\mathcal{D}}_d^n$ generated by (8.6) is denoted by Ψ^t, and it is called a *parabolic flow* on PL-braid diagrams.

The behaviour of ψ^t with respect to the intersection number (see Lemma 8.6) now transfers to Ψ^t. This can be described in terms of the word metric for braids in $\mathcal{D}_d^n$. A PL-braid diagram can be expressed in terms of the positive generators $\{\sigma_j\}_{j=1}^n$. While this *word* is not necessarily unique (since only positive generators are considered), the length of the word is unique, and this length is the *word metric*. This word metric now acts as a Lyapunov function on $\overline{\mathcal{D}}_d^n$: it is non-increasing along the flow.

Lemma 8.7 *Let Ψ^t be a parabolic flow on $\overline{\mathcal{D}}_d^n$.*

(a) *For each point $\boldsymbol{u} \in \Sigma \setminus \Sigma^-$ the local orbit $\{\Psi^t(\boldsymbol{u}) \,|\, t \in [-\varepsilon, \varepsilon]\}$ intersects Σ uniquely at $\boldsymbol{u}$ for all ε sufficiently small.*

(b) *For any $\boldsymbol{u} \in \Sigma \setminus \Sigma^-$ the word metric of $\Psi^t(\boldsymbol{u})$ for $t > 0$ is strictly less than that of $\Psi^t(\boldsymbol{u})$ for $t < 0$.*

The above construction carries over to the class of up-down braid diagrams $\overline{\mathcal{E}}_{2p}^n$. The recurrence relations defined in (8.4) define a parabolic flow Ψ^t via (8.5) on the space $\overline{\mathcal{E}}_{2p}^n$. As before, Ψ^t is topologically transverse to $\Sigma \setminus \Sigma^-$ and Ψ^t acts on $\Sigma \setminus \Sigma^-$ as to strictly decrease the word metric. Besides, we will assume (since this holds in our applications) that the horizontal singularities are repelling, i.e., $\mathcal{R}_i$ satisfies

$$\lim_{u_i \downarrow u_{i\pm1}} \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = +\infty \qquad \text{if } i \text{ is odd,} \tag{8.7a}$$

$$\lim_{u_i \uparrow u_{i\pm1}} \mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = -\infty \qquad \text{if } i \text{ is even.} \tag{8.7b}$$

This guarantees that Ψ^t behaves is a favourable way near horizontal singularities, i.e., Ψ^t respects the up-down restriction in forward time.

8.4 The Conley index for braids

Consider a relative braid class $[\beta_0 \text{ rel } \beta_1]$. Let Ψ^t be a parabolic flow such that $\Psi^t(\beta_1) = \beta_1$ (one can prove that such a flow exists, see [74]). We now want to define the Conley index of the braid types $[\beta_0 \text{ rel } \beta_1]$ with respect to Ψ^t.

Let us recall the notion of isolating neighbourhood as was introduced by Conley [48]. A compact set N is an isolating neighbourhood for a flow Ψ^t if the maximal invariant set $M(N) = \{x \in N \,|\, \mathrm{cl}\{\Psi^t(x)\}_{t\in\mathbb{R}} \subset N\}$ is contained in the interior of N. The invariant set $M(N)$ is then called a *compact isolated invariant set* for Ψ^t. In [48] Conley proves that every compact isolated invariant set M admits a pair (N, N^-), such that (following the definitions given in [104])

1. $M = M(\mathrm{cl}(N \setminus N^-))$, and $N \setminus N^-$ is neighbourhood of M;
2. N^- is positively invariant in N;
3. N^- is an exit set for N.[1]

[1]The set N^- is an exit set for N if every orbit that leaves N in forward time, leaves N through/via N^-.

Such a pair is called an *index pair* for M. The Conley index h(M) is then defined as the homotopy type of the pointed space $(N/N^-, [N^-])$, which is denoted by $[N/N^-]$. This definition is independent of the choice of the index pair. The Conley index is stable under perturbations, and invariant under continuation. Since the homotopy type of a space is notoriously difficult to compute, one often passes to homology. The homological Conley index $CH_*(N)$ (or $CH_*(M)$) is defined as the relative homology $H_*(N, N^-)$. Most important for our purposes is the following property of the homological Conley index, which is in direct analogy with the Morse index (and the degree): if $CH_*(N) \neq 0$, then there exists a nontrivial invariant set of Ψ^t within the interior of N. Besides, in the case that Ψ^t is a gradient flow, i.e., when $\mathcal{R}$ satisfies Axiom (A4), then if $CH_*(N) \neq 0$ there must be a stationary point in N.

Let $\beta_1 \in \mathcal{D}_d^m$ (skeleton), and consider the relative braid class $[\beta_0 \text{ rel } \beta_1]$ in $\mathcal{D}_d^n$ rel β_1. Not all braid classes are isolating neighbourhoods for the flow Ψ^t. The situation to avoid is when a component of the braid diagram collapses. In other words, the set of collapsed braid diagrams $\Sigma^- \subset \Sigma$ is an invariant set under the flow and thus forms an obstruction to obtaining isolation. We therefore restrict our attention to braid types of the following form.

Definition 8.8 *A relative braid type* $[\beta_0 \text{ rel } \beta_1]$ *is called* proper *if:*

(a) $\text{cl}[\beta_0 \text{ rel } \beta_1] \cap \Sigma^- \text{ rel } \beta_1 = \varnothing$*; and*

(b) $[\beta_0 \text{ rel } \beta_1] \subset \mathcal{D}_d^n \text{ rel } \beta_1$ *is bounded.*

The first condition precisely excludes the possibility of two strands collapsing as discussed above. The second condition is a compactness condition. We now have the following theorem.

Theorem 8.9 *Let* $[\beta_0 \text{ rel } \beta_1]$ *be a proper relative braid type and let* Ψ^t *be a parabolic flow for which* β_1 *is stationary. Then*

(a) $N = \text{cl}[\beta_0 \text{ rel } \beta_1]$ *is an isolating neighbourhood for the flow* Ψ^t*, which thus yields a well-defined Conley index* $h(\beta_0 \text{ rel } \beta_1)$.

(b) *The index* $h(\beta_0 \text{ rel } \beta_1)$ *is independent of the choice of the parabolic flow* Ψ^t *(as long as* $\Psi^t(\beta_1) = \beta_1$*).*

(c) *The index* $h(\beta_0 \text{ rel } \beta_1)$ *is independent of the choice of* β_1 *within its PL-braid class* $[\beta_1]$.

The Conley index of any proper relative braid type $[\beta_0 \text{ rel } \beta_1]$ can in fact be defined intrinsically, independent of any notion of parabolic flows, see [74].

Since these concepts may be somewhat hard to grasp, let us give two low-dimensional examples. Consider the proper 2-periodic braid illustrated in Figure 8.4a. There is exactly one free strand, so that the configuration space $\mathcal{D}_2^1$ rel β_1 is two-dimensional. The point in the middle, u_1, is free to move vertically between the fixed points on the skeleton. When u_1 meets a point of the skeleton, one has a singular braid in Σ which is on the exit set, since a slight perturbation sends this singular braid to a different braid class with fewer crossings. The end critical point, u_0 $(= u_2)$ can freely move vertically in between the two fixed points on the skeleton. The singular boundaries are in this case not on the exit set, since pushing u_0 across the skeleton increases the number of crossings. Since the points u_0 and u_1 can be moved independently, the configuration space N in this case is the product

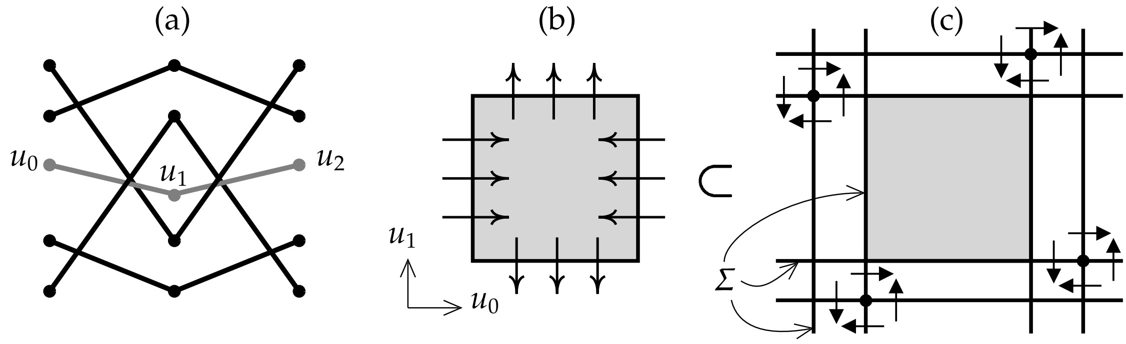

Figure 8.4: (a) A proper braid class with a skeleton consisting of four strands and one free strand (grey); (b) the associated configuration space with parabolic flow; (c) an expanded view of $\mathcal{D}_2^1$ rel β_1.

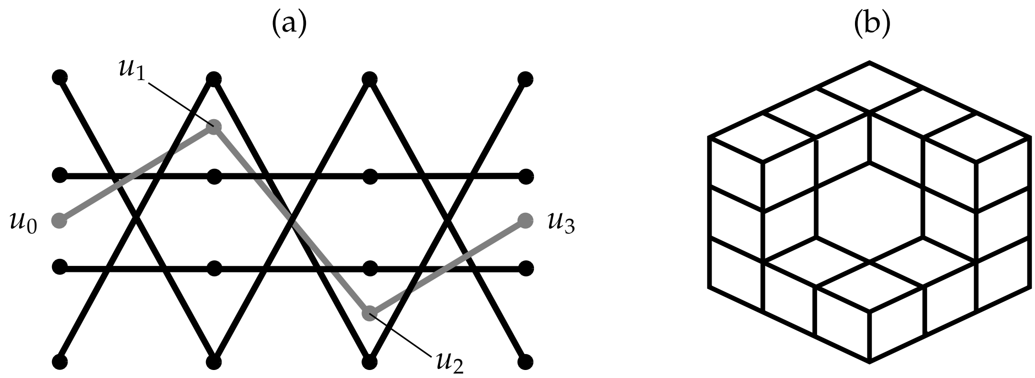

Figure 8.5: (a) A proper 3-periodic braid type with in (b) the associated configuration space N.

of two compact intervals. The exit set N^- consists of those points on ∂N for which u_1 is a boundary point, see Figure 8.4b. Thus, the homotopy index of this relative braid is $[N/N^-] \simeq S^1$. In Figure 8.4c an expanded view of $\mathcal{D}_2^1$ rel β_1 is depicted, where the fixed points of the flow correspond to the four fixed strands in the skeleton β_1. The braid classes adjacent to these fixed points are not proper.

A second example is the proper relative braid presented in Figure 8.5a. Since there is one free strand of period three, the configuration space N is determined by the position vector (u_0, u_1, u_2). This example differs greatly from the previous example. For instance, the point $u_0 = u_3$ (as represented in the figure) may pass through the nearest strand of the skeleton above and below without changing the braid type. The points u_1 and u_2 may not pass through any strands of the skeleton without changing the braid type *unless* u_0 has already passed through. In this case, either u_1 or u_2 (depending on whether the upper or lower strand is crossed) becomes free to move. To simplify the analysis, consider (u_0, u_1, u_2) as all of $\mathbb{R}^3$ (allowing for the moment singular braids and other braid classes as well). The position of the skeleton induces a cubical partition of $\mathbb{R}^3$ by planes, the equations being $u_i = v_i^k$ for the various strands v^k of the skeleton $\boldsymbol{v}$ (corresponding to β_1). The braid class N is thus some collection of cubes in $\mathbb{R}^3$. In Figure 8.5b we illustrate this cube complex. It is homeomorphic to $D^2 \times S^1$. In this case, the exit set N^- happens to be the entire boundary ∂N.

Via the results of the previous section, the homotopy index is an invariant of the

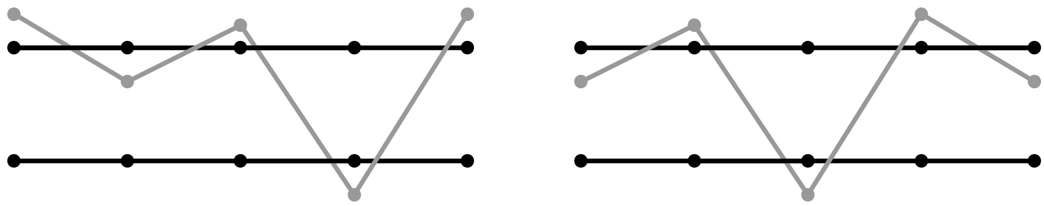

Figure 8.6: An example of two non-free PL-braids which are of the same topological braid type but define disjoint PL-braid classes.

PL-braid type: keeping the period fixed and moving within a connected component of the space of relative PL-braids leaves the index invariant. The *topological* braid type, as defined in Section 8.2, does not have an implicit notion of period. The effect of changing the discretisation of a topological closed braid is not obvious: not only does the dimension of the index pair change, the homotopy types of the isolating neighbourhood and the exit set often change as well under changing the discretisation. It is thus perhaps remarkable that any changes are correlated under the quotient operation $[N/N^-]$: the homotopy index is in many cases an invariant of the *topological* closed braid type, as will become clear in the following. On the other hand, given a complicated braid, it is intuitively clear that a certain number of discretisation points are necessary to capture the topology correctly. If the period d is too small, then $\mathcal{D}_d^n$ rel β_1 may contain more than one path component with the same topological braid type.

Definition 8.10 *A relative braid type* $[\beta_0 \text{ rel } \beta_1]$ *in* $\mathcal{D}_d^n$ rel β_1 *is called* free *if any other PL-braid in* $\mathcal{D}_d^n$ rel β_1 *which has the same topological braid type as* β_0 rel β_1*, is in* $[\beta_0 \text{ rel } \beta_1]$.

In our applications we will generally only encounter free braid types. However, not all PL-braid types are free, see Figure 8.6 for an example of a non-free braid type.

Define the *extension map* $\Phi : \overline{\mathcal{D}}_d^n \to \overline{\mathcal{D}}_{d+1}^n$ via concatenation with the trivial braid of period one:

$$(\Phi\beta)_i^k \stackrel{\text{def}}{=} \begin{cases} u_i^k & \text{for } i = 0, \ldots, d, \\ u_d^k & \text{for } i = d+1. \end{cases} \tag{8.8}$$

The reader may note (though not without some effort) that the non-free braids of Figure 8.6 become free under the image of Φ.

It is a pleasant surprise that Φ preserves the homotopy index of a free, proper braid.

Theorem 8.11 *If* $[\beta_0 \text{ rel } \beta_1]$ *and* $[\Phi\beta_0 \text{ rel } \Phi\beta_1]$ *define free proper braid types, then the Conley homotopy indices are equivalent:*

$$h(\Phi\beta_0 \text{ rel } \Phi\beta_1) = h(\beta_0 \text{ rel } \beta_1).$$

Theorem 8.11 is an important tool in the calculation of the (homological) Conley index of braid types. The proof comes from a recasting of the situation as a singular perturbation problem.

Finally, we turn our attention to the class of up-down braids. The definition of a proper relative braid type in this context is the same as in Definition 8.8, but now with relative braids in $\mathcal{E}_{2p}^n$ instead of $\mathcal{D}_{2p}^n$. From the considerations in Section 8.3 (in particular the behaviour near horizontal singularities) it follows, as in Theorem 8.9a, that for proper braid types the set $N = \mathrm{cl}_{\overline{\mathcal{E}}}[\boldsymbol{u} \text{ rel } \boldsymbol{v}]_{\mathcal{E}}$ is an isolating neighbourhood. Thus, for any parabolic flow Ψ^t induced by a parabolic recurrence relations which satisfies (8.7), the Conley index $h(\boldsymbol{u} \text{ rel } \boldsymbol{v}, \mathcal{E})$ of the maximal invariant set $M = M(N)$ is well-defined.

We now make a connection between the Conley index of the up-down braid type $h(\boldsymbol{u} \text{ rel } \boldsymbol{v}, \mathcal{E})$ and the Conley index of a (regular) PL-braid type. For $\boldsymbol{v} \in \mathcal{E}^m_{2p}$ define an augmented braid $\boldsymbol{v}^*$ as follows. Choose two strands $\boldsymbol{v}^-$ and $\boldsymbol{v}^+$, $v_i^\pm = v_\pm + (-1)^{i+1}\delta$ with $v_\pm$ large and $\delta > 0$ small enough, so that $v_i^- < v_i^k < v_i^+$, for all k and i. Let $\boldsymbol{v}^* = \boldsymbol{v}^- \cup \boldsymbol{v}^+ \cup \boldsymbol{v}$, i.e. $\boldsymbol{v}^* \in \mathcal{E}^{m+2}_{2p}$. When $[\boldsymbol{u} \text{ rel } \boldsymbol{v}]_\mathcal{E}$ is a proper braid type in $\mathcal{E}^n_{2p}$ rel $\boldsymbol{v}$, then $[\boldsymbol{u} \text{ rel } \boldsymbol{v}^*]_\mathcal{E}$ corresponds precisely to the subset of $\mathcal{E}^n_{2p}$ defined by $[\boldsymbol{u} \text{ rel } \boldsymbol{v}]_\mathcal{E}$, and clearly $[\boldsymbol{u} \text{ rel } \boldsymbol{v}^*]_\mathcal{E}$ is a proper braid type in $\mathcal{E}^n_{2p}$ rel $\boldsymbol{v}^*$.

Let $[\boldsymbol{u} \text{ rel } \boldsymbol{v}^*] = [\boldsymbol{u} \text{ rel } \boldsymbol{v}^*]_\mathcal{D}$ be the path component in $\mathcal{D}^n_{2p}$ rel $\boldsymbol{v}^*$ containing $[\boldsymbol{u} \text{ rel } \boldsymbol{v}^*]_\mathcal{E}$. It now follows that $[\boldsymbol{u} \text{ rel } \boldsymbol{v}^*]$ is a proper (regular) PL-braid type. The Conley index of $[\boldsymbol{u} \text{ rel } \boldsymbol{v}]_\mathcal{E}$ can be related to that of $[\boldsymbol{u} \text{ rel } \boldsymbol{v}^*]$:

Theorem 8.12 *For any proper up-down braid type* $[\boldsymbol{u} \text{ rel } \boldsymbol{v}]_\mathcal{E}$ *we have*

$$h(\boldsymbol{u} \text{ rel } \boldsymbol{v}, \mathcal{E}) = h(\boldsymbol{u} \text{ rel } \boldsymbol{v}^*).$$

Since the strands $\boldsymbol{v}^\pm$ only serve to retain compactness when the up-down restriction is lifted (i.e., when extending from $\mathcal{E}$ to $\mathcal{D}$), no strands need to be added to $\boldsymbol{v}$ if there are already strands in $\boldsymbol{v}$ which lie completely above and below all other strands. In that case one may take $\boldsymbol{v}^* = \boldsymbol{v}$.

8.5 Second order Lagrangian systems

As was already indicated in Section 8.1, our main application of the theory in the preceding sections is derived from the problem of finding periodic solutions of *second order Lagrangian systems*. An important motivation for studying such systems comes from the stationary *Swift-Hohenberg model* [137, 52], which is described by the fourth order equation

$$\left(1 + \frac{d^2}{dt^2}\right)^2 u - \alpha u + u^3 = 0, \qquad \alpha \in \mathbb{R}. \tag{8.9}$$

More generally, our results apply to a broad class of second order Lagrangian systems $L = L(u, u', u'')$. As explained in Section 8.1 and Chapter 7, the variational principle for finding closed characteristics in a fixed energy level can be discretised when the Lagrangian L satisfies the Twist hypothesis (T). The variational principle, i.e., finding critical points of $J = \int L$ in some function space, is then reduced to a finite dimensional setting. Namely, one searches for critical points of $W_{2p}(u_0, \ldots, u_{2p-1}) = \sum_{i=0}^{2p-1} S(u_i, u_{i+1})$, with $u_{2p} = u_0$. Here the u_i correspond to the extrema of the function u, and $S(u_i, u_{i+1}) = \int L(u(t), u'(t), u''(t))\, dt$ is the action of the (unique) monotone lap $u(t)$ from u_i to u_{i+1}.

This *generating function* S satisfies a number of properties as was proved in Chapter 7. Let $I = (u_-, u_+) \subset I_E$ be any open sub-interval of I_E[2]. Define the diagonal $\Delta = \{(u_1, u_2) \in I \times I \,|\, u_1 = u_2\}$. The generating function S has the following properties:

1. *smoothness*: $S \in C^2(I \times I \backslash \Delta)$.
2. *monotonicity*: $\partial_1 \partial_2 S(u_1, u_2) > 0$ for all $u_1 \neq u_2 \in I$.

[2] Recall from Section 8.1 that I_E is a connected component of $\{u \,|\, L(u, 0, 0) + E \geq 0\}$.

3. *diagonal singularity*:

$$\lim_{u_1 \uparrow u_2} -\partial_1 S(u_1, u_2) = \lim_{u_2 \downarrow u_1} \partial_2 S(u_1, u_2) = \lim_{u_1 \downarrow u_2} \partial_1 S(u_1, u_2) = \lim_{u_2 \uparrow u_1} -\partial_2 S(u_1, u_2) = \infty.$$

Critical points of W_{2p} satisfy the recurrence relation

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) \stackrel{\text{def}}{=} \partial_2 S(u_{i-1}, u_i) + \partial_1 S(u_i, u_{i+1}) = 0,$$

where $\mathcal{R}_i(u_{i-1}, u_i, u_{i+1})$ is defined on

$$\Omega_i = \{(u_{i-1}, u_i, u_{i+1}) \in I^3 \,|\, (-1)^i (u_{i\pm1} - u_i) > 0\}.$$

Here we have chosen u_0 to corresponds to a local minimum. The properties of S imply that $\mathcal{R}$ satisfies Axioms (A1)-(A4) with $d = 2$, i.e., $\mathcal{R}$ is an exact parabolic recurrence relation. The fact that $\mathcal{R}_i$ is defined on the domains Ω_i, subjects sequences (u_i) to the constraint $(-1)^i (u_{i\pm1} - u_i) > 0$. This (natural) up-down restriction is the reason that we need the concept of up-down braids. The properties of the generating function S listed above yield that the gradient flow Ψ^t associated to $\mathcal{R}$ (see (8.6)) is a parabolic flow which respects the up-down restriction in forward time.

Theorem 8.12 will be used to find a family of up-down braid types with non-trivial Conley homology. These braid classes then necessarily contain critical points, i.e. closed characteristics. We consider two cases: compact interval components, and non-compact interval components $I_E = \mathbb{R}$ with a certain asymptotic behaviour[3].

Before stating our main theorems, we introduce some notions of complexity for braids and closed characteristics, see Figure 8.7 for examples. Two closed characteristics u^1 and u^2 (and the braid diagram formed by the pair) are said to be *unlinked* if the associated strands in the PL-braid diagram are strictly ordered. We say that two closed characteristics are *linked*, if the stands form a non-trivial braid diagram (i.e., when they are not unlinked). Note that u^2, seen as a strand, may be an even shift of u^1 (and one could refer to this case as self-linking). In either case their braid diagram is said to contain a (non-trivial) link. Closed characteristics can also be represented as closed curves in the (u, u')-plane. A closed characteristic is *simple* if its representation in the (u, u')-plane is a simple closed curve. A non-simple closed characteristic also yields a braid diagram which is (non-trivially) linked by considering the braid which has all its even translates as strands. Finally, two closed characteristics of period d form a *maximal* link if the associated braid diagram is described by the braid-word σ_1^d.

Let L satisfy the Twist property (T), and let E be a regular energy level for which there exists a compact interval component I_E.

Theorem 8.13 *Suppose I_E contains a link. Then there exists an infinity of non-simple, geometrically distinct, closed characteristics in I_E.*

In the proof of this theorem we will see what the nature of the infinite family of links is, how they relate, what their Morse index are, etcetera.

The idea of the proof is the following (see [74] for full details). Let $\{v^1, v^2\}$ be a non-trivial braid in $\mathcal{E}_{2p}^2$ for some $p \in \mathbb{N}$, i.e., v^1 and v^2 form a link. In order to avoid problems

[3]For analogous results on the third possibility, a semi-infinite interval, we refer to [74].

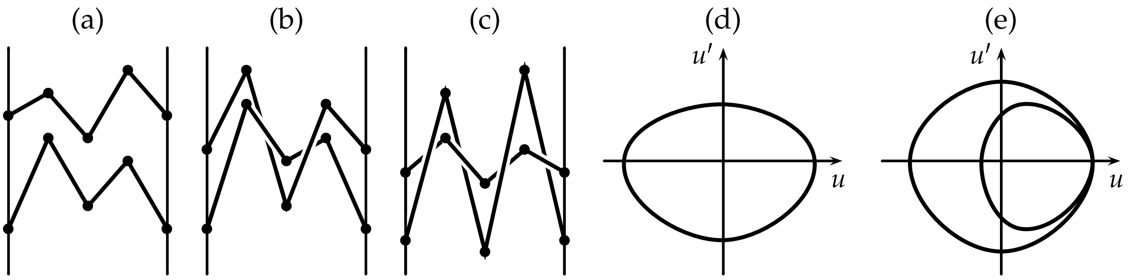

Figure 8.7: On the left examples of braid diagrams which are (a) unlinked, (b) linked, and (c) maximally linked. On the right representations in the (u, u')-plane of closed characteristics which are (d) simple, and (e) non-simple.

near the boundary of $I_E = [u_-, u_+]$ and near the diagonal $u_i = u_{i+1}$, we choose

$$\begin{aligned} \Omega_i^\delta &= \{(u_{i-1}, u_i, u_{i+1}) \in I_E^3 \,|\, u_- + \delta < u_{i\pm 1} < u_i - \delta < u_+ - 2\delta\} \qquad \text{for } i \text{ odd}, \\ \Omega_i^\delta &= \{(u_{i-1}, u_i, u_{i+1}) \in I_E^3 \,|\, u_- + 2\delta < u_i + \delta < u_{i\pm 1} < u_+ - \delta\} \qquad \text{for } i \text{ even}. \end{aligned} \tag{8.10}$$

The set Ω_{2p} is now the set of $2p$ periodic sequences (u_i) such that $(u_{i-1}, u_i, u_{i+1}) \in \Omega_i^\delta$. Here we choose $\delta > 0$ sufficiently small, so that the vector field $\mathcal{R} = (\mathcal{R}_i)$ is everywhere transverse to $\partial\Omega_{2p}$, and moreover points inwards. Hence Ω_{2p} is positively invariant for the induced parabolic flow.

Define

$$\begin{aligned} C_+ &= \{(u_i) \in \Omega_{2p} \,|\, u_i > v_i^k \text{ for } k = 1,2 \text{ and all } i = 0, \ldots, 2p\}, \\ C_- &= \{(u_i) \in \Omega_{2p} \,|\, u_i < v_i^k \text{ for } k = 1,2 \text{ and all } i = 0, \ldots, 2p\}. \end{aligned}$$

The two sets $C_\pm$ can be interpreted as subsets of the set of relative up-down braid diagrams $\mathcal{E}_{2p}^1$ rel β_1, where β_1 is the braid formed by v^1 and v^2. The fact that the braid diagram β_1 contains a (non-trivial) link and the properties of parabolic flows, imply that on the boundaries of the C_- and C_+, the vector field $\mathcal{R}_i$ is everywhere transverse and pointing inwards. Thus, C_- and C_+ are positively invariant with respect to the parabolic flow Ψ^t. Consequently, W_{2p} has global maxima v^- and v^+ on int(C_-) and int(C_+) respectively. The maxima v^- and v^+ have the property that $v_i^- < v_i^{1,2} < v_i^+$. Seen as a braid diagram, $\boldsymbol{v} = \{v^1, v^2, v^-, v^+\}$ is a stationary skeleton for the induced parabolic flow Ψ^t.

Consider the relative braid classes $[u \text{ rel } \boldsymbol{v}]_{\mathcal{E}}$ which are presented in Figure 8.8a, and which are indicated by $X_{p,r}^q \subset \mathcal{E}_{2p}^1$ rel $\boldsymbol{v}$. Here r is the linking number of v^1 and v^2, i.e., $2r$ is the number of crossings of v^1 and v^2. Since v^1 and v^2 are linked one has $0 < 2r \leq 2p$. The strand u is chosen to cross the strands $v^{1,2}$ $2q$ times, whereas it does not cross $v^\pm$. While this description does not completely characterise the braid class, Figure 8.8a indeed fixes the braid type under consideration. An important restriction is that $0 < q < r$.

The relative up-down braid type $X_{p,r}^q$ is a proper free braid type, provided $0 < q < r$. It follows from Theorem 8.12 that $h(X_{p,r}^q, \mathcal{E}) = h(X_{p,r}^q)$, i.e., the up-down restriction may be disregarded. The homology of $h(X_{p,r}^q)$ can now be calculated.

Lemma 8.14 *The Conley homology of $h = h(X_{p,r}^q)$ is given by:*

$$CH_k(X_{p,r}^q) = \begin{cases} \mathbb{Z} & k = 2q - 1 \text{ or } 2q, \\ 0 & \text{else.} \end{cases}$$

The proof of Lemma 8.14 relies on Lemma 8.11 and the continuation (which leaves the

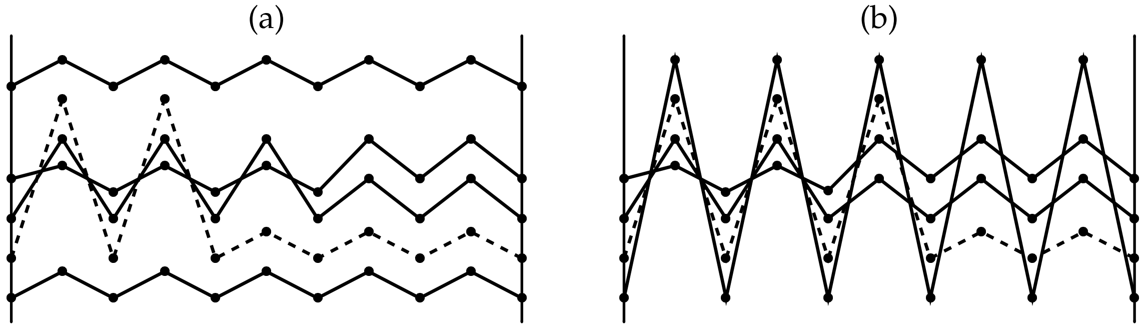

Figure 8.8: Two up-down braid types (a) $X^q_{p,r}$ with $p = 5$, $r = 3$, $q = 2$, and (b) $Y^q_{p,r}$ with $p = 5$, $r = 2$, $q = 3$. All crossings are positive, i.e., the larger slope crosses on top of the smaller slope.

Conley index invariant) to a system for which one can calculate the invariant set completely (an 'integrable' system).

Since the Conley homology is non-trivial, we obtain the existence of a critical point in $X^q_{p,r}$. In fact (see for example [6, Section 6]), we derive from Morse theory that there exist at least two distinct critical points (generically of index $2q$ and $2q-1$), for each q satisfying $0 < q < r \leq p$. This way the number of solutions depends on r and p. In order to find infinitely many closed characteristics, we consider all multiples of $2p$, i.e., let the skeleton be contained in $\mathcal{E}^4_{2pm}$, $m \geq 1$. Now q must satisfy $0 < q < rm \leq pm$. By choosing triples (q, p, m) such that q and pm are relative prime, we obtain the same Conley homology as above, and therefore an infinity of pairs of geometrically distinct critical points of W, corresponding to stationary point of the action J. Let q' be the linking number around the $v^{1,2}$, and let p' be the period, then admissible ratios $\frac{q'}{p'}$ for closed characteristics are determined by the relation

$$0 < \frac{q'}{p'} < \frac{r}{p}.$$

Thus if v^1 and v^2 are maximally linked, i.e. $r = p$, then closed characteristics exist for all ratios in $\mathbb{Q} \cap (0, 1)$. Finally, note that $v^{\pm}$ correspond to the case $q' = 0$.

On non-compact interval components W_{2p} need not have any critical points. In order to obtain more insight in non-compact interval components, some knowledge about asymptotic behaviour of the system seems to be necessary. In Chapter 7 this issue was addressed, and the *dissipativity* condition introduced there reads as follows: there exists a (large) pair (u_1^*, u_2^*) such that

$$\partial_1 S(u_1^*, u_2^*) < 0 \quad \text{and} \quad \partial_2 S(u_1^*, u_2^*) > 0. \tag{8.11}$$

An example of a sufficient condition on L so that the above dissipativity hypothesis holds, is the following asymptotic behaviour (for all (u, v, w)):

$$\lim_{\lambda \to \infty} \lambda^{-s} L\big(\lambda u, \lambda^{(s+2)/4} v, \lambda^{s/2} w\big) = c_1 w^2 + c_2 |u|^s \qquad \text{for some } s > 2 \text{ and } c_1, c_2 > 0.$$

For dissipative Lagrangians we can prove the following general result. Let L satisfy the Twist property (T) and the dissipativity condition (8.11), and let E be a regular energy value such that $I_E = \mathbb{R}$.

Theorem 8.15 *Suppose that $I_E = \mathbb{R}$ contains a non-maximal link. Then there exists an infinity of non-simple, geometrically distinct, closed characteristics in I_E.*

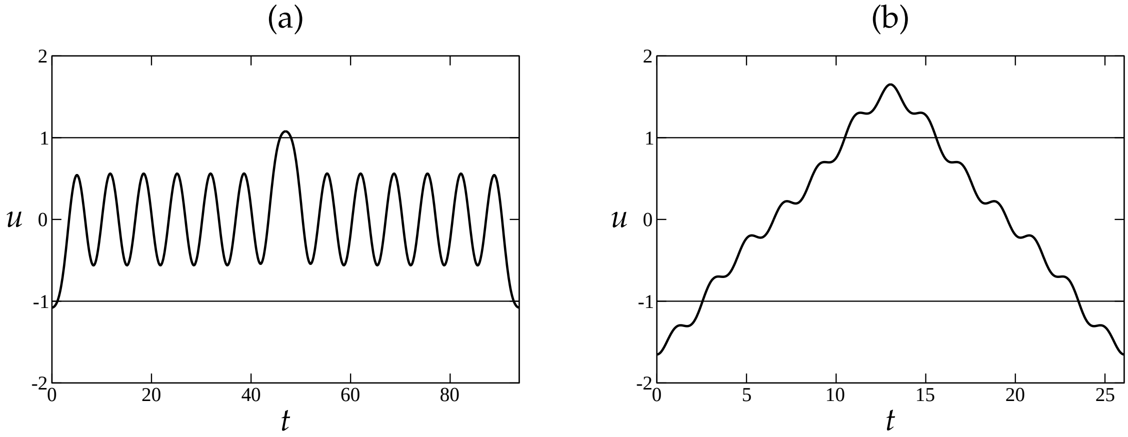

Figure 8.9: Solutions of (8.13) with energy $E = 0$ corresponding to the braid type $Y^1_{13,0}$ for (a) $Q = 0$, and (b) $Q = 10$.

The proof is very similar to that of Theorem 8.13. Let $\{v^1, v^2\}$ form a non-maximally linked braid in $\mathcal{E}^2_{2p}$ for some $p \in \mathbb{N}$, i.e., having linking number $0 \le r < p$. Choose Ω_i^δ as in (8.10) with I_E replaced by $[u_1^*, u_2^*]$ (see (8.11)). Define

$$C = \{u \in \Omega_{2p} \,|\, u \text{ is maximally linked with } v^{1,2}\}.$$

As before, the set C can be interpreted as a subset of the relative up-down braid diagrams $\mathcal{E}^1_{2p}$ rel β_1, where β_1 is the braid formed by v^1 and v^2. Since v^1 and v^2 are not maximally linked, the vector field $\mathcal{R}_i$ is transverse to the boundary ∂C, and is pointing outwards on ∂C. Therefore C is negatively invariant for the induced parabolic flow Ψ^t, and consequently there exists a global minimum $v^3 \in \text{int}(C)$. Define the skeleton $\boldsymbol{v}$ to be $\boldsymbol{v} = \{v^1, v^2, v^3\}$. Now consider the relative up-down braid type $[u \text{ rel } \boldsymbol{v}]_{\mathcal{E}}$ depicted in Figure 8.8b, denoted by $Y^q_{p,r}$. It can be described as follows. The linking number of v^1 and v^2 is r, where $0 \le r < p$. The strand $u \in Y^q_{p,r}$ satisfies $(-1)^i u_i \ge (-1)^i v^3_i$ for all i, and u has linking number q with the strands $v^{1,2}$. For $r < q < p$, $Y^q_{p,r}$ is a proper free braid type. By Theorem 8.12 we have that $h(Y^q_{p,r}) = h(u \text{ rel } \boldsymbol{v}, \mathcal{E}) = h(u \text{ rel } \boldsymbol{v}^*)$.

Lemma 8.16 *The Conley homology of $h = h(Y^q_{p,r})$ is given by:*

$$CH_k(Y^q_{p,r}) = \begin{cases} \mathbb{Z} & k = 2q \text{ or } 2q+1, \\ 0 & \textit{else.} \end{cases}$$

In the same manner as in the proof of Theorem 8.13 infinitely many solutions are found. The relation for the admissible ratios here reads

$$\frac{r}{p} < \frac{q'}{p'} < 1. \tag{8.12}$$

Hence if v^1 and v^2 are unlinked, i.e. $r = 0$, then closed characteristics exist for all ratios in $\mathbb{Q} \cap (0, 1)$. Finally, note that v^3 corresponds to the case $q' = p'$.

As an example, we apply Theorem 8.15 to Equation (8.9) for $\alpha > 1$. After rescaling we obtain

$$u'''' + Qu'' - u + u^3 = 0, \qquad \text{with } Q = \frac{2}{\sqrt{\alpha - 1}}. \tag{8.13}$$

The corresponding Lagrangian is given by $L(u, u', u'') = \frac{1}{2}u''^2 - \frac{Q}{2}u'^2 + \frac{1}{4}(u^2 - 1)^2$. It is shown in Chapter 7 that the Swift-Hohenberg model satisfies the Twist property (T) for all $Q \geq 0$ (i.e. all $\alpha > 1$). The potential[4] $F(u) = \frac{1}{4}(u^2 - 1)^2$ has two non-degenerate global minima, at (singular) energy level $E = 0$. For each regular $E \in (0, \epsilon)$ with ε sufficiently small, there exist at least two unlinked simple closed characteristics[5]. For all $E > 0$ one has that $I_E = \mathbb{R}$, and the dissipative boundary conditions are met for any sufficiently large subinterval $I \in \mathbb{R}$. Theorem 8.15 now yields an infinity of non-simple closed characteristics for all $E \in (0, \epsilon)$. A periodic solution exists for any ratio $\frac{q'}{p'} \in \mathbb{Q} \cap (0, 1)$, see (8.12). These characteristics still exist in the limit $E = 0$. In the limit $E \to 0$ the two unlinked simple periodic solutions may collapse onto the two equilibrium points. Nevertheless, because all the solutions produced by Theorem 8.15 link around both simple closed characteristics, the infinite family of solutions still exists in the limit $E = 0$. Since the singular level $E = 0$ contains singular points, oscillations of closed characteristics may coalesce. However, it is not difficult to show the extrema of these solutions may coalesce only in pairs at the equilibrium points. Notice that the nature of the equilibrium points is not important for this result: it holds both when the equilibria $u = \pm 1$ are saddle-foci ($0 \leq Q < \sqrt{8}$), and when they are centers ($Q \geq \sqrt{8}$). A solution corresponding to $q = 1$ and $p = 13$ is shown in Figure 8.9 for both $Q = 0$ and $Q = 10$. Observe that although there is a big difference in shape, the braid type is the same.

[4]Note that the potential $F(u)$ is defined with the opposite sign compared to Chapter 1.

[5]With some effort this can be deduced this from the analysis in Section 7.3.4.

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Samenvatting

Dit proefschrift gaat over de dynamica en evenwichtstoestanden die beschreven worden door vierde orde differentiaalvergelijkingen. Het uitgangspunt hierbij is wiskundig, maar dit soort vergelijkingen is van belang bij de beschrijving van vele fysische verschijnselen.

De taal waarin natuurwetten worden geformuleerd is wiskundig. Omdat deze wetten de verandering van bepaalde grootheden beschrijven, is het te verwachten dat differentiaalvergelijkingen hierbij een grote rol spelen. Een bekend voorbeeld is de wet van Newton in de klassieke mechanica, die de versnelling van een voorwerp of deeltje relateert aan de kracht die erop wordt uitgeoefend. De versnelling is de tweede afgeleide van de plaats (de eerste afgeleide is de snelheid), zodat dit een *tweede orde* differentiaalvergelijking is.

Net als zulke fundamentele natuurwetten, worden vele natuurkundige verschijnselen gemodelleerd met behulp van differentiaalvergelijkingen. Een voorbeeld hiervan is de warmte-vergelijking, die de verandering van temperatuur als functie van plaats en tijd beschrijft. Dit is ook een tweede orde differentiaalvergelijking, maar hij is van een heel andere aard dan de wet van Newton. Die laatste is namelijk een *gewone* differentiaalvergelijking (de positie hangt van één variabele af: de tijd), terwijl de warmte-vergelijking een *partiële* differentiaalvergelijking is (de temperatuur varieert zowel in de tijd als in de ruimte).

Ondanks de verschillen hebben deze twee voorbeelden van differentiaalvergelijkingen een gemeenschappelijk kenmerk: wanneer de begintoestand bekend is, wordt de toekomstige evolutie, of *dynamica*, van het systeem volledig bepaald door de differentiaalvergelijking. Dit is karakteristiek voor wat in de wiskunde een *dynamisch systeem* genoemd wordt. Het feit dat de gebeurtenissen in zo'n systeem volledig vastliggen wanneer de beginsituatie bekend is, impliceert dat we te maken hebben met een deterministisch proces. Dat wil echter niet zeggen dat we altijd precies kunnen voorspellen wat er gaat gebeuren, want de beginsituatie is nooit exact bekend. Een zeer kleine afwijking in de beginsituatie kan na verloop van tijd tot geheel andere gevolgen leiden. Deterministische processen kunnen daarom toch chaotische verschijnselen veroorzaken. Een goed voorbeeld daarvan zijn weersvoorspellingen, maar ook bijvoorbeeld de trekking van de lottoballetjes, die volledig beschreven wordt door de wet van Newton, maar die toch een 'willekeurige' uitkomst heeft.

De natuurkunde is niet de enige wetenschap waarin differentiaalvergelijkingen een prominente rol vervullen. Zij worden op vele gebieden toegepast als model voor het te bestuderen probleem. Om een paar voorbeelden te noemen: de reacties en verspreiding van chemicaliën, de groei en afname van populaties in de biologie, de ontwikkeling en behandeling van ziektes in de geneeskunde, en de stroming van een gas of vloeistof, waarbij de toepassingen variëren van fundamentele sterrenkunde tot weersvoorspellingen tot industriële processen.

Naast deze toepassingsmogelijkheden is er nog een andere reden om differentiaalvergelijkingen te bestuderen: de wiskundige uitdaging. De theorie van differentiaalvergelijkingen heeft connecties met vele takken binnen de wiskunde. Een verdere ontwikkeling van de theorie maakt daarom niet alleen de toepasbaarheid groter, maar vergroot ook het inzicht in de structuur van de wiskunde die eraan ten grondslag ligt. En uiteindelijk is dat toch de grootste motivatie voor het onderzoek dat in dit proefschrift beschreven wordt: proberen de onderliggende structuur van een wiskundig probleem te begrijpen.

We gaan nu dieper op het onderwerp van dit proefschrift in; het is uiteraard onvermijdelijk dat daarbij enige wiskundige notatie om de hoek komt kijken. De *vierde orde* partiële differentiaalvergelijking waar het in dit proefschrift om draait is

$$\frac{\partial f}{\partial t} = -\gamma \frac{\partial^4 f}{\partial x^4} + \beta \frac{\partial^2 f}{\partial x^2} + f - f^3 \qquad \gamma > 0,\ \beta \in \mathbb{R}. \tag{A}$$

De functie f hangt van de tijd t en de plaats x af, en de betekenis van f is afhankelijk van de toepassing waarin men geïnteresseerd is (we komen daar zo op terug). In de differentiaalvergelijking staan twee getallen β en γ, de *parameters* (ook daar komen we later op terug). Overigens is vergelijking (A) slechts een voorbeeld van een brede klasse van vergelijkingen die in dit proefschrift wordt bestudeerd.

Wanneer een oplossing f van deze differentiaalvergelijking niet verandert in de tijd, dan noemen we dat een stationaire oplossing of ook wel een evenwichtsoplossing. Zo'n stationaire oplossing varieert dus alleen in de ruimte en niet in de tijd, oftewel de functie f hangt alleen van x af. Zulke stationaire oplossingen voldoen aan de *gewone* differentiaal vergelijking

$$-\gamma \frac{d^4 f}{dx^4} + \beta \frac{d^2 f}{dx^2} + f - f^3 = 0, \qquad \gamma > 0,\ \beta \in \mathbb{R}. \tag{B}$$

Het feit dat er in de vergelijkingen (A) en (B) een derde macht van f voorkomt, impliceert dat zij *niet-lineair* zijn. Dit is in feite de belangrijkste reden dat deze vergelijkingen vanuit wiskundig oogpunt interessant zijn. Het zorgt ervoor dat we de oplossingen van deze vergelijkingen in het algemeen niet expliciet kunnen bepalen. We kunnen ze wel (met de computer) benaderen, maar we kunnen ze bijna nooit als formule opschrijven. Het is echter vaak wel mogelijk om uitspraken te doen over kwalitatieve eigenschappen van de oplossingen van zulke niet-lineaire vergelijkingen.

Vierde orde differentiaalvergelijkingen van bovenstaand type komen naar voren in zeer uiteenlopende toepassingen, zoals de vibraties in het wegdek van een hangbrug, de vorming van geologische patronen in aardlagen, het zenden van een lichtpuls door een glasfiber, of de voortbeweging van een golf in een ondiep kanaal. Maar er zijn ook minder tastbare toepassingen, zoals in de beschrijving van fase-overgangen (het smelten van ijs is een voorbeeld van een fase-overgang) en de stroming van een vloeistof tussen twee parallel geplaatste platen met verschillend temperatuur. De betekenis van de grootheid f is bijvoorbeeld de temperatuur van een gas, de hoogte van een oppervlak of de intensiteit van een lichtstraal.

We zien verder dat er in bovenstaande vergelijkingen twee *parameters* voorkomen: β en γ. Wanneer deze parameters worden veranderd kunnen de oplossingen van de vergelijkingen ander gedrag vertonen. Het doel van het onderzoek is om te bestuderen hoe de oplossingen zich gedragen voor alle mogelijke parameterwaarden. In toepassingen komt het variëren van de parameters overeen met het kiezen van een andere temperatuur, andere afmetingen of een ander materiaal, kortom het instellen van alle factoren die het proces beïnvloeden.

De differentiaalvergelijkingen (A) en (B) kunnen worden beschouwd als vierde orde uitbreidingen van respectievelijk de eerder genoemde warmte-vergelijking en de wet van Newton. Wanneer we $\gamma = 0$ nemen dan reduceren de vergelijkingen tot tweede orde differentiaalvergelijkingen. Het is dus te begrijpen dat een van de resultaten in dit proefschrift is dat voor kleine waarden van γ de oplossingen van de vierde orde vergelijking erg op die van de tweede orde vergelijking lijken.

We gaan nu iets verder in op de verschillen tussen tweede en vierde orde vergelijkingen, en daarbij beperken we ons voor de eenvoud tot de gewone differentiaalvergelijking (B), hoewel er ook voor de partiële differentiaalvergelijking essentiële verschillen zijn. Het grote verschil tussen een tweede en een vierde orde differentiaalvergelijking is dat een tweede orde differentiaalvergelijking geen chaotisch gedrag kan vertonen en een vierde orde differentiaalvergelijking wel. Dat wil zeggen dat de oplossingen van tweede orde differentiaalvergelijkingen zich altijd heel gestructureerd en geordend gedragen, terwijl voor vierde orde differentiaalvergelijkingen er de meest wilde verschijnselen kunnen optreden.

Oplossingen kunnen worden voorgesteld als banen in de ruimte; voor tweede orde vergelijkingen zijn dit banen in een twee-dimensionale ruimte en voor vierde orde vergelijkingen in een vier-dimensionale ruimte. In twee dimensies (het platte vlak) is er voor de oplossingen niet genoeg bewegingsvrijheid om al te gecompliceerd gedrag te vertonen, terwijl daar in vier dimensies wel voldoende ruimte voor is. Men zou dit kunnen vergelijken met het feit dat zoveel meer auto's op elkaar botsen dan vliegtuigen, die immers een dimensie extra hebben om zich in te bewegen. Of, om het probleem nog wat verder te simplificeren, op een smalle weg kunnen auto's elkaar niet inhalen en moeten zij dus allemaal ordelijk achter elkaar blijven rijden, terwijl zij elkaar op een snelweg links (en rechts) kunnen passeren zodat het er veel chaotischer aan toe kan gaan.

Bij het bestuderen van differentiaalvergelijkingen zijn er veel verschillende soorten oplossingen waarin men geïnteresseerd is; dit hangt vaak ook van de toepassing af. Men kan zich bijvoorbeeld beperken tot tijdonafhankelijke oplossingen (ook al omdat het probleem dan minder moeilijk is). Ook kan men bijvoorbeeld op zoek gaan naar periodieke (oscillerende) oplossingen, of juist naar oplossingen die alleen maar stijgen. Overigens beperkt men zich bij een dergelijk wiskundig probleem vaak niet tot één van deze mogelijkheden. Meestal is het als het volgen van een weg waarvan je niet precies weet of hij wel ergens naar toe gaat, laat staan waar hij precies naar toe gaat.

In dit proefschrift worden dan ook verschillende soorten oplossingen bestudeerd. De manier waarop dat gebeurt varieert sterk en hangt niet alleen af van het soort oplossingen, maar ook van de waarden van de parameters β en γ. Laat ons kort een beschrijving geven van de twee belangrijkste technieken: schietmethoden en variationele methoden.

Bij een schietmethode kiezen we eerst een beginsituatie. Zoals eerder beschreven is, wordt de dynamica dan volledig bepaald door de differentiaalvergelijking. We proberen vervolgens te bepalen hoe de oplossing er uitziet. Omdat de vergelijkingen niet-lineair zijn kunnen we dat nooit precies berekenen, maar we kunnen vaak wel het kwalitatieve gedrag begrijpen. Wanneer dit gedrag niet het gewenste gedrag is dan veranderen we de beginsituatie een klein beetje en kijken of het er beter op wordt. Door dit proces op een systematische manier te herhalen vinden we uiteindelijk een oplossing met de gewenste karakteristieken. Of niet natuurlijk, want de oplossingen van een differentiaalvergelijking doen wat de differentiaalvergelijking hen oplegt, dus we kunnen niet verwachten dat er oplossingen zijn die voldoen aan alle mogelijke voorwaarden die we kunnen verzinnen. Soms bestaan er bijvoorbeeld geen periodieke oplossingen, of gaan alle oplossingen naar oneindig, zodat het zoeken naar een oplossing die eindig blijft hopeloos is. Niettemin verschaft de bevinding dat een bepaald soort oplossing niet kan bestaan natuurlijk ook veel inzicht.

Een variationele methode is op een totaal ander leest geschoeid. We bekijken nu functies die al het gewenste gedrag hebben, maar die niet noodzakelijkerwijs oplossingen van de differentiaalvergelijking zijn. Vervolgens gebruiken we het feit dat veel fysische systemen zich zo gedragen dat zij een toestand proberen te bereiken met een zo laag mogelijke energie. Het wonderbaarlijke is nu dat die functie waarvoor de bijbehorende energie de laagst mogelijke is, een oplossing van de differentiaalvergelijking is. Dit principe heet het *variationele principe*, en het wordt veelvuldig gebruikt om oplossingen met een bepaald gezocht gedrag te vinden. In tegenstelling tot een schietmethode speelt een variationele methode zich af in een oneindig dimensionale ruimte. Hoewel dat enige moeilijkheden met zich meebrengt, is men er vaak toch op aangewezen om bepaalde extra informatie over oplossingen te verkrijgen, of omdat een schietmethode eenvoudigweg niet werkt.

We behandelen nu zeer kort de inhoud van de verschillende hoofdstukken. We benadrukken dat dit proefschrift voor het overgrote deel bestaat uit bewijzen van wiskundige stellingen, en dat toepassingen een minder belangrijke rol spelen.

Na een inleidende hoofdstuk wordt in hoofdstuk 2 aangetoond dat, voor positieve waarden van β en niet al te grote waarden van γ (om precies te zijn $\gamma \leq \frac{\beta^2}{8}$), de begrensde oplossingen van de vierde orde gewone differentiaalvergelijking (B) precies corresponderen met die van de tweede orde vergelijking. De parameterwaarden $\gamma > \frac{\beta^2}{8}$ komen in hoofdstuk 3 aan de orde, waar met behulp van variationele methoden periodieke en chaotische oplossingen worden bestudeerd. In hoofdstuk 4 ligt de nadruk op de partiële differentiaalvergelijking (A), met name op stabiliteit van stationaire oplossingen. We onderzoeken dus wat er gebeurt wanneer een evenwichtsoplossing een klein beetje verstoord wordt. Er wordt bewezen dat, als we een eindig ruimtelijk gebied bekijken met zogenaamde 'vrije' randvoorwaarden, er voor $\gamma \leq \frac{\beta^2}{8}$ altijd precies twee stabiele evenwichtstoestanden zijn, terwijl voor $\gamma > \frac{\beta^2}{8}$ het aantal stabiele toestanden exponentieel groeit wanneer het gebied dat we beschouwen groter wordt. Uit deze drie hoofdstukken blijkt dat er een scherpe overgang is tussen de parameterwaarden waarvoor de vergelijking zich tam gedraagt, en de parameterwaarden waarvoor het gedrag chaotisch is. Het is uitzonderlijk dat deze overgang zo nauwkeurig in kaart kan worden gebracht.

Hoofdstuk 5 gaat over een heel ander soort oplossingen van vergelijking (A), namelijk over zogenaamde *lopende golven*. Deze oplossingen hebben een onveranderlijke vorm, maar dit vaste profiel beweegt zich met een constante snelheid van links naar rechts (of andersom). In hoofdstuk 6 komen negatieve waarden van de parameter β aan bod. Met behulp van een schietmethode worden verschillende families van periodieke oplossingen bestudeerd. De schietmethode en de variationele methode worden in hoofdstuk 7 tot een geheel gesmeed. In dit hoofdstuk worden voornamelijk eenvoudig periodieke oplossingen bestudeerd, terwijl in hoofdstuk 8 meer gecompliceerde vormen aan bod komen. In dit afsluitende hoofdstuk worden periodieke oplossingen geïnterpreteerd als knopen in een drie-dimensionale ruimte.

Nawoord

Dit proefschrift was niet geworden wat het is zonder de inbreng en steun van vele vrienden, collega's en bekenden. Allereerst wil ik de personen noemen die direct bij de totstandkoming van dit proefschrift betrokken zijn geweest. Zij vinden zichzelf terug in het voorwoord. Verschillende leden van de promotiecommissie hebben door hun opmerkingen en vragen de presentatie van de resultaten overzichtelijker gemaakt en het aantal fouten en onnauwkeurigheden verminderd. Daarnaast ben ik ook enkele (anonieme) referees erkentelijk.

Financiële ondersteuning voor congres- en werkbezoek heb ik gekregen van NWO en via het TMR programma 'Nonlinear Parabolic PDEs: Methods and Applications'. Het Mathematisch Instituut heeft mij alle vrijheid gegeven om ongestoord te doen wat ik wilde; het was misschien soms zelfs iets te rustig. Gelukkig liepen er dan altijd wel een paar collega's rond met wie ik tijdens de lunch de discussie aan kon gaan.

Velen, wiskundigen en niet-wiskundigen, hebben mij van inspiratie voorzien en daardoor aan dit proefschrift bijgedragen, vaak ook zonder dat zij zich ervan bewust zijn. Eveline, Roderick en vooral Heleen hebben, als huisgenoten en vrienden, mij vele plezierige jaren bezorgd. De leden van Christiaan Huygens hebben zowel mijn blikveld verruimd, als voor vrolijke avonden vol samenhorigheid gezorgd. Van Berend heb ik bijvoorbeeld niet alleen voor het eerst over de kwadratuur van de cirkel gehoord, maar ook hebben onze gesprekken veel aan mijn zelfkennis bijgedragen. De ontspanning die ik bij tijd en wijle vond in het huis van de familie Flinterman, zelfs al was het voor een dag, heeft mij altijd zeer veel goed gedaan.

In de anderhalf jaar die ik bij de vakgroep 'Moleculen in Aangeslagen Toestand' heb doorgebracht, heb ik ontzettend veel geleerd, maar ben ik vooral ook onder de indruk geraakt van de manier waarop aan die onderzoeksgroep vorm wordt gegeven. Het jaar dat Barbera en Fernando als post-docs in Leiden doorbrachten heeft een blijvende invloed op mij gehad. Hun relativeringsvermogen en sympathie heb ik als bijzonder prettig ervaren. Hoewel Didier geen Nederlands wenst te spreken of verstaan, heeft onze samenwerking in Parijs mij erg veel plezier en energie gebracht.

Willem is waarschijnlijk mijn enige vriend buiten de wiskunde die iets van de inhoud van dit proefschrift begrijpt. Maar meer nog waardeer ik de avonden waarop we discussiëren over het leven, en ik hoop dat dat nog lang zo mag blijven. De etentjes met Frouke, Ella en Michelle zijn altijd een hoogtepunt; hun vriendschap is hartverwarmend.

De onvoorwaardelijk steun van Els, Emiel, Annette en Pieter is voor mij van onschatbare waarde. En tenslotte, Heleen laat mij elke dag weer zien dat het leven de moeite meer dan waard is.

Curriculum vitae

De auteur werd op 11 maart 1973 in Haarlem geboren. Hij deed in 1991 eindexamen op het Stedelijk Gymnasium Haarlem. Daarna begon hij met de studies wiskunde en natuurkunde aan de Universiteit Leiden, en in 1992 behaalde hij beide propedeuses. Zijn afstudeeronderzoek bij natuurkunde deed hij in de vakgroep 'Moleculen in Aangeslagen Toestand' van prof. dr. J. Schmidt en dr. E.J.J. Groenen. Daar deed hij spectroscopische experimenten aan C_{60}-moleculen. Het afstudeerwerk bij wiskunde vond plaats onder leiding van prof. dr. ir. L.A. Peletier, en bestond uit numerieke berekeningen aan vierde orde differentiaalvergelijkingen. Begin 1996 studeerde hij zes maanden aan de Ecole Polytechnique in Parijs. In november 1996 legde hij het doctoraal examen af in zowel natuurkunde als wiskunde.

In december 1996 begon hij zijn promotie-onderzoek in de groep van prof. Peletier. Gedurende vier jaar heeft hij, eerst als beurspromovendus en later als AIO, onderzoek verricht aan vierde orde differentiaalvergelijkingen, wat geresulteerd heeft in dit proefschrift. Tijdens deze periode bezocht hij conferenties en zomerscholen in Snowbird (Utah), Bath, Leiden, Cortona en Twente. Naast deze conferenties gaf hij voordrachten in Pittsburgh, Atlanta en Parijs. Hij bracht drie werkbezoeken aan het Center for Dynamical Systems and Nonlinear Studies van het Georgia Institute of Technology en verbleef twee maanden aan de universiteit Paris VI. Bovendien nam hij deel aan vier studiegroepen 'Wiskunde met de Industrie'.

Als vertegenwoordiger van de promovendi zat hij in de faculteitsraad en nam hij deel aan de stafvergaderingen van de vakgroep wiskunde. Als student-assistent en als promovendus gaf hij werkcolleges analyse 1, 2 en 4, functionaalanalyse en gewone en partiële differentiaal vergelijkingen. Hij was daarnaast drie jaar lang verantwoordelijk voor het college 'Wiskundige procesbeschrijving 2' voor biologiestudenten. Tevens was hij een van de docenten van het interdisciplinaire college 'Dynamical systems and nonequilibrium pattern formation'. Zijn leven buiten de universiteit werd onder meer gevuld met lezen, koken, sporten, wandelen en televisie kijken, maar ook was hij lid van het disputgezelschap Christiaan Huygens, waarvan hij enkele jaren abactis en een jaar praeses was.

Vanaf januari 2001 zal Jan Bouwe gedurende twee jaar werkzaam zijn aan de universiteit van Nottingham als EPSRC research fellow.