

RIGOROUS COMPUTATION OF A RADIALY SYMMETRIC LOCALISED SOLUTION IN A GINZBURG-LANDAU PROBLEM

J.B. VAN DEN BERG*, C.M. GROOTHEDDE†, AND J.F. WILLIAMS‡

Abstract. We present a rigorous numerical method for proving the existence of a localised radially symmetric solution for a Ginzburg-Landau type equation. This has a direct application to the problem of finding spots in the Swift-Hohenberg equation. The method is more generally applicable to finding radially symmetric solutions of stationary PDEs on the entire space. One can rewrite such a problem in the form of a singular ODE. We transform this ODE to a finite domain and use a Green's function approach to formulate an appropriate integral equation. We then construct a mapping whose fixed points coincide with solutions to the ODE, and show via computer-aided analytic estimates that the mapping is contracting on a small neighbourhood of a numerically determined approximate solution.

Key words. computer-assisted proof, radial symmetry, localised solution, non-autonomous ODE, Ginzburg-Landau equation, Swift-Hohenberg equation

AMS subject classifications. 34B40, 35B36, 35Q56, 37M99

1. Introduction. Computer-assisted proofs in dynamical systems have a long tradition, going back to the proof of universality of the Feigenbaum constant [10]. Confirming the existence of the chaotic attractor in the Lorenz system [18] is another, more recent, pinnacle of this approach. The past decade has seen structural advances in the techniques for the rigorous computer-aided study of systems of ordinary differential equations (ODEs), see e.g. [3, 1, 9] (and the references therein), as well as the survey paper [2]. Nevertheless, relatively little attention has gone to the study of solutions on unbounded domains for *non-autonomous* ODEs. Such problems, for example, appear naturally when considering radially symmetric solutions of partial differential equations (PDEs). In this paper we develop a technique that combines analytical estimates and numerical computations to provide a rigorous existence result for the non-autonomous problem

$$(1.1) \quad A''(s) + \frac{1}{s}A'(s) - \frac{1}{4s^2}A(s) - A(s) + A(s)^3 = 0, \quad \text{for } s \in (0, \infty),$$

with $A(0) = 0$ and $\lim_{s \rightarrow \infty} A(s) = 0$. The present work answers the open question raised in [12]. The equation is of Ginzburg-Landau type, a family of problems that is ubiquitous in the study of pattern formation, see e.g. [16] and the references therein.

Before discussing the origin of equation (1.1) in detail, we remark that it fits into a larger class of problems stemming from PDEs posed on the entire space. The problem of proving the existence of stationary radially symmetric solutions of PDEs on the entire space remains a challenge, despite numerous advances both in analytic and numerical techniques. In particular, a gap remains between what is feasible in the realm of rigorous proofs using purely analytic methods on the one hand, and what is observed in numerical simulations of complex pattern forming systems on the other. The novel computer-assisted method developed in this paper contributes to filling this

*VU University Amsterdam, Department of Mathematics, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands, janbouwe@few.vu.nl; partially supported by NWO-VICI grant 639033109.

†VU University Amsterdam, Department of Mathematics, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands, c.m.groothedde@vu.nl.

‡Simon Fraser University, Department of Mathematics, 8888 University Drive Burnaby, BC, V5A 1S6, Canada, jfwillia@sfu.ca.

gap. While we apply the method to (1.1), it can be used more generally to obtain existence results for radially symmetric solutions to PDEs on the entire space. As an example, consider the 3-dimensional stationary Schrödinger equation $\Delta\psi - \psi + \psi^3 = 0$. When one looks for radially symmetric solutions, the problem reduces to

$$(1.2) \quad u'' + \frac{2}{r}u' - u + u^3 = 0,$$

with $r \in (0, \infty)$ and $\psi(x) = u(|x|)$. As we shall see in Remark 2.1, equation (1.2) can be rewritten in a way that closely resembles (1.1). As a consequence, our methods are applicable to equation (1.2) as well. We note that (1.2) has already been studied intensively (see e.g. [8] for an overview). In particular, a positive “ground state” solution was proven to exist [13], as well as a countable family of sign changing solutions [15]. Instead of using our method to (re)prove the existence of the ground state solution of (1.2), we shall apply our techniques to the yet unsolved, but similar, problem given by (1.1).

The main motivation for studying equation (1.1) comes from another stationary radially symmetric PDE problem, presented in [12]. In this paper, McCalla and Sandstede consider the planar Swift-Hohenberg equation

$$(1.3) \quad u_t = -(1 + \Delta)^2 u - \mu u + \nu u^2 - u^3.$$

This equation was first considered by Swift and Hohenberg in the context of thermal convection [17]. The dependent variable u , which depends on time and space, is an order parameter for the fluid velocity, while the parameter μ encodes how far the temperature of the system is from the threshold temperature at which convection occurs. The parameter ν breaks the symmetry of the system, which allows the appearance of an even larger variety of patterns. Since its derivation in the 1970’s, this equation has served as a model for pattern forming processes.

In [12], McCalla and Sandstede specifically study the existence of *spots* of the Swift-Hohenberg equations: radially symmetric localised stationary solutions which “concentrate” near the origin. It was already shown in [11] that one such spot exists for every $\nu > 0$ and every sufficiently small $0 < \mu \ll 1$. The amplitude of this spot scales as $\mu^{1/2}$ as μ tends to zero. In [12], the authors provide a construction for a second spot whenever $\nu > \sqrt{27/38}$ and $0 < \mu \ll 1$, of which the amplitude scales as $\mu^{3/8}$.

We will now explain how (1.1) comes up in the analysis of the existence of the second spot for the Swift-Hohenberg equation (1.3). Restricting to radially symmetric stationary solutions, equation (1.3) becomes

$$0 = - \left(1 + \partial_r^2 + \frac{1}{r} \partial_r \right)^2 u - \mu u + \nu u^2 - u^3.$$

The next step is rewriting this ODE as a five-dimensional first order autonomous system:

$$(1.4) \quad \frac{d}{dr} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \alpha \end{pmatrix} = \begin{pmatrix} u_3 \\ u_4 \\ -u_1 - \alpha u_3 + u_2 \\ -u_2 - \alpha u_4 - \mu u_1 + \nu u_1^2 - u_1^3 \\ -\alpha^2 \end{pmatrix}.$$

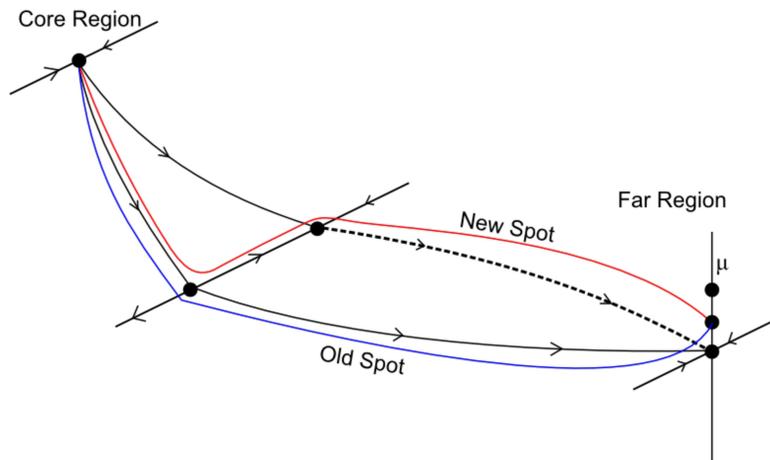


FIG. 1. A simplified version of Figure 2 from [12], representing schematically the dynamics of the five-dimensional system (1.4). The red line represents the desired (new) spot, the blue line represents the original spot and the dotted line represents the solution to (1.1). The equilibria in the far field depend on the parameter μ .

Spots are described by a connecting orbit between the two-dimensional manifold of small solutions that stay bounded as $r \rightarrow 0$ (the “core” solutions) and the two-dimensional manifold of solutions that decay as $r \rightarrow \infty$ (the “far” solutions). In the (singular) limit $\mu \rightarrow 0$, the construction of the second spot requires a connecting orbit that is described by a reduced equation, namely (1.1). In particular, equation (1.1) is derived by changing to complex coordinates, considering the singular limit $\mu \rightarrow 0$ and one further change of coordinates. The details of these transformations can be found in [11]. The resulting (complex) second-order ODE is then given by

$$(1.5) \quad A''(s) + \frac{1}{s}A'(s) - \frac{1}{4s^2}A(s) - A(s) = c|A(s)|^2A(s)^3,$$

where $c = \frac{3}{4} - \frac{19}{18}\nu^2$ and $c < 0$ exactly when $\nu > \sqrt{27/38}$. A localised solution to this equation provides us, through the asymptotic analysis in [11], with a corresponding connecting orbit of the five-dimensional system (1.4) for small μ , as illustrated in Figure 1. Equation (1.1) is obtained from (1.5) by scaling out the constant c and by restricting to real solutions.

Since the new (second) spot is basically obtained by perturbing localised solutions of (1.1), two facts need to be checked to complete a rigorous proof of its existence. First, it needs to be shown that a localised solution \hat{A} to (1.1) indeed exists. Second, this solution needs to be non-degenerate (hence “robust” under perturbations in μ) in the sense that the only bounded solution of the linearisation of (1.1) around \hat{A} is the trivial solution.

These two requirements are not trivial and the authors of [12] point out the fallacy in an earlier proof. Because of this, McCalla and Sandstede present the existence, as well as the assumption on the linearisation, as a hypothesis. This hypothesis, which

we will state in the form of a theorem that we prove in this paper, states the following.

THEOREM 1.1. *The ODE*

$$(1.6) \quad A''(s) + \frac{1}{s}A'(s) - \frac{1}{4s^2}A(s) - A(s) = A(s)^3$$

has a bounded non-trivial localised solution on $[0, \infty)$ with the property that the linearisation of (1.6) around this solution does not have a non-trivial uniformly bounded solution. In addition, the constructed solution is positive on $(0, \infty)$ and decays as e^{-s} as $s \rightarrow \infty$.

Our proof is computer-aided and based on the functional analytic setting and associated techniques developed in [22, 23]. There is by now a substantial literature on this approach, see [9, 19] and the references therein. We point out that there is a complementary computer-assisted method based on careful rigorous integration of the ODE (e.g. using the CAPD software package [1]), which may also be employed to construct a proof of the theorem. It is largely a matter of taste which of these computer-assisted techniques is preferable. The proof presented in this paper is “elementary” in the sense that we analytically derive a set of inequalities that need to be checked by a computer-aided computation. This latter step requires no special software except the interval arithmetic package *Intlab* [14] for Matlab. The files containing the numerical approximation of the solutions, as well as the code for checking the final step of the proof of Theorem 1.1 can be found in [20].

We shall start, in Section 2, by rewriting the problem (1.6), which is posed on an unbounded domain, in terms of an equivalent integral formulation on the bounded interval $[0, 1]$. This is obviously advantageous from a computational point of view, and in Section 3 we discuss the discretization that leads to a numerically obtained approximate solution. In Section 4 we construct the fixed point map T that critically involves information obtained from our numerical approximation. By the Banach fixed-point theorem, it then suffices to show that this map is a contraction, which requires analytic estimates on the derivative of T . In Section 5 we prove that our technique “automatically” guarantees that the linearised equation mentioned in Theorem 1.1 has no non-trivial bounded solutions. In Section 6 we show how to evaluate certain logarithmic integrals that come up in our formulation using interval arithmetic. This evaluation is then used in Sections 7 and 8, where we derive all the necessary analytic estimates needed to show that T is a contraction. To evaluate these estimates explicitly we need the aid of a computer, and in Section 9 it is outlined how this then indeed leads to a proof of Theorem 1.1.

2. Construction of the integral equation. Our main goal in this section is to rewrite (1.6), which is posed on the half-line, as an integral equation on a finite interval, to which we can then apply a fixed point argument in order to prove the existence of a solution. We shall first rescale A to arrive at a simpler equation. By then making the substitution $t = e^{-s}$, we arrive at an ODE on the interval $(0, 1)$. Finally, we use one additional rescaling and a Green’s function method to obtain an integral equation on the interval $[0, 1]$ with “free” boundary conditions, i.e., posed on $C[0, 1]$ without any additional boundary conditions.

The first transformation of (1.6) was already observed in [12] and is obtained by writing $u(s) = s^{1/2}A(s)$, leading to

$$(2.1) \quad u''(s) = u(s) - \frac{1}{s}u(s)^3.$$

REMARK 2.1. *It is worth noting that this equation is very similar to (1.2) in the sense that by substituting $u(s) = f(s)/s$ in (1.2) we arrive at $f''(s) = f(s) - \frac{1}{s^2}f(s)^3$, which differs from (2.1) by a power of s in the nonlinearity only. Because of this similarity, our proof of the existence of solutions of (1.6) can, with minor alterations to the code, be adapted to also show the existence of a radially symmetric localised solution to (1.2).*

Turning our attention back to (2.1), we define v on $(0, 1]$ by $v(e^{-s}) = u(s)$, which satisfies

$$(2.2) \quad t^2 v''(t) + tv'(t) - v(t) = \frac{v(t)^3}{\log t},$$

where $t \in (0, 1)$. Since we want u (and A) to be localised, we need to supplement equation (2.2) with the boundary conditions $v(0) = 0$ and $v(1) = 0$ (the latter follows directly from the singularity in the equation at $t = 1$). But we can say a little more. We expect that as $s \rightarrow \infty$, equation (2.1) reduces to $u''(s) = u(s)$, hence we anticipate u to scale as e^{-s} . Furthermore, asymptotically balancing terms in equation (2.1) suggests that bounded solutions behave linearly as $s \rightarrow 0$. In terms of v we thus expect the solution to be linear at both boundaries $t = 0$ and $t = 1$. Because of this, we write $v(t) = t(1-t)w(t)$, for some bounded function w on $[0, 1]$, which satisfies

$$(2.3) \quad \frac{d}{dt} \left((1-t)^2 t^3 \frac{dw}{dt}(t) \right) - 3(1-t)t^2 w(t) = \frac{(1-t)^4 t^3 w(t)^3}{\log t}.$$

The main advantage of this formulation is that we no longer need to specify any boundary conditions.

The homogeneous equation, obtained by setting the right hand side of (2.3) to zero, has solutions $\frac{1}{1-t}$ and $\frac{1+t}{2t^2}$ (these are just the solutions e^{-s} and $e^s - e^{-s}$ of the linear part of (2.1) transformed to the (t, w) variables). This allows us to calculate the corresponding Green's function and we arrive at the integral equation

$$(2.4) \quad w(t) = -\frac{1}{2} \frac{1+t}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} w(s)^3 ds - \frac{1}{2} \frac{1}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} w(s)^3 ds.$$

Note that it follows from a direct estimate that, for every $w \in C[0, 1]$,

$$(2.5a) \quad \lim_{t \rightarrow 0} \frac{1+t}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} w(s)^3 ds = 0,$$

$$(2.5b) \quad \lim_{t \rightarrow 1} \frac{1}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} w(s)^3 ds = 0.$$

Hence, the right-hand side of (2.4) indeed defines an element of $C[0, 1]$.

By going back through the transformations, we conclude that if $w(t)$ is a bounded solution of (2.4), then

$$(2.6) \quad A(s) = s^{-1/2}(1 - e^{-s})e^{-s}w(e^{-s})$$

is a bounded localised solution of (1.6). In conclusion, we see that in order to show that there exists a non-trivial bounded solution of (1.6), it suffices to find a non-trivial bounded solution of (2.4). This is formalized in the following lemma.

LEMMA 2.2. Define the map $G : C[0, 1] \rightarrow C[0, 1]$ by

$$(2.7) \quad G(w) : t \mapsto w(t) + \frac{1}{2} \frac{1+t}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} w(s)^3 ds \\ + \frac{1}{2} \frac{1}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} w(s)^3 ds.$$

If there exists a $\hat{w} \in C[0, 1]$ such that $G(\hat{w}) = 0$, then this corresponds via (2.6) to a localised solution \hat{A} of (1.6). If \hat{w} is positive, then so is \hat{A} (with $\hat{A}(0) = 0$).

3. Finite dimensional approximation. The problem of finding bounded solutions to (1.6) is now reduced to finding zeroes of the map G defined in (2.7). In the remainder of this paper, we prove the existence of this zero by first using numerical methods to construct an approximation of this solution, and then showing the existence of a continuous (hence smooth) solution close to this approximation by means of a fixed point argument.

In order to obtain a numerical approximation, we split the map G into a finite dimensional part, which we can use to find a numerical approximate solution, and the corresponding infinite dimensional remainder, which we need to control analytically. We start by defining the mesh

$$\Delta_m \stackrel{\text{def}}{=} \{0 = t_0 < t_1 < \dots < t_m = 1\},$$

where $0 = t_0 < \dots < t_m = 1$. As we shall remark upon in Section 9, we choose a uniform mesh in the proof of Theorem 1.1. Nevertheless, we consider a general mesh throughout, since the estimates do not simplify significantly for the special case of a uniform mesh. Using this mesh, we define $S_m \subset C[0, 1]$ as the space of linear splines (continuous piecewise linear functions) with base-points in Δ_m . Note that we will trivially identify S_m and \mathbf{R}^{m+1} . We define a projection $\Pi_m : C[0, 1] \rightarrow S_m$, by

$$\Pi_m : w \mapsto \Pi_m w = (w(t_0), \dots, w(t_m)) \in \mathbf{R}^{m+1} \simeq S_m,$$

and its complementary projection $\Pi_\infty \stackrel{\text{def}}{=} I - \Pi_m$.

For $w \in C[0, 1]$ we denote $\Pi_m w$ by w_m . Since Π_m is a projection, we can decompose $C[0, 1]$ as

$$C[0, 1] = \Pi_m C[0, 1] \oplus \Pi_\infty C[0, 1] = S_m \oplus S_\infty,$$

where $S_\infty \stackrel{\text{def}}{=} (1 - \Pi_m)C[0, 1]$. It can easily be seen that the spaces $S_m, S_\infty \subseteq C[0, 1]$ are closed with respect to the usual supremum norm on $C[0, 1]$ and are therefore Banach spaces. Using this notation, we define a family of closed neighbourhoods of 0 by setting

$$B_\omega(r) \stackrel{\text{def}}{=} \{w \in C[0, 1] : \|\Pi_m w\|_\infty \leq r \text{ and } \|\Pi_\infty w\|_\infty \leq \omega r\}.$$

In this definition, $\omega > 0$ is a control parameter that we can alter to adapt the radius of the ‘‘tail’’. In the following, we will assume that $\omega > 0$ is a fixed constant (to be chosen later) and treat r as a variable parameter.

Using the above finite-dimensional reduction, we define a map $G^m : S_m \rightarrow S_m \simeq \mathbf{R}^{m+1}$ by $G^m : w \mapsto (\Pi_m G)(w_m)$. Explicitly, the k -th component of $G^m(w)$ is given

by

$$G^m(w)_k = w(t_k) + \frac{1}{2} \frac{1+t_k}{t_k^2} \int_0^{t_k} \frac{s^3(1-s)^3}{\log s} w(s)^3 ds \\ + \frac{1}{2} \frac{1}{1-t_k} \int_{t_k}^1 \frac{s(1+s)(1-s)^4}{\log s} w(s)^3 ds.$$

Note that by the limits in (2.5), the terms for $k = 0, m$ are also well defined.

By first using standard ODE shooting methods, we find a numerical approximate solution to (2.2). We then change variables and apply Newton iterations to G^m in order to refine this result, giving us an approximate zero \hat{w}_m . It is close to this approximate solution that we will find our rigorous solution to $G(w) = 0$.

REMARK 3.1. *As it turns out, \hat{w}_m is positive. This leads to the positivity result in Theorem 1.1, see Lemma 4.3. Moreover, we will use this fact to slightly simplify the estimates in what follows, but this is not at all a restriction on the method.*

4. Radii polynomials. Using our approximate zero \hat{w}_m , we shall construct a new map T whose fixed points coincide with zeros of G . We then prove the existence of a fixed point of T , and therefore a zero of G , by means of contraction argument.

Since we obtained \hat{w}_m by means of a Newton method, we expect the corresponding Newton map to be contracting near \hat{w}_m , hence we shall base the construction of T on this map. Since the Newton map requires an inverse of the derivative, we calculate numerically an approximate inverse A_m^\dagger of the Jacobian $DG^m(\hat{w}_m)$ of G^m at \hat{w}_m .

Using the approximate zero \hat{w}_m and the approximate inverse A_m^\dagger , we define the map $T : C[0, 1] \rightarrow C[0, 1]$ by (see also [22, 23])

$$(4.1) \quad T(w) \stackrel{\text{def}}{=} (\Pi_m - A_m^\dagger \Pi_m G)(w) + \Pi_\infty(w - G(w)).$$

Note that the first term of T is indeed similar to a Newton map for G^m , while in the second term we have basically approximated the linearization in the “tail” of G by the identity (since it turns out the tail parts of integral terms in (2.7) are relatively small).

Our aim is to find a closed neighbourhood $\hat{w}_m + B_\omega(r)$ of \hat{w}_m on which the map T is a contraction, implying that T has a unique fixed point in $\hat{w}_m + B_\omega(r)$. Furthermore, if T is a contraction, then we must have $\|DT(w)\| < 1$ on $\hat{w}_m + B_\omega(r)$, hence $I - DT(w)$ is invertible for each w in this neighbourhood. Since

$$I - DT(w) = A_m^\dagger \Pi_m DG(w) + \Pi_\infty DG(w),$$

the operator $I - DT(w)$ can only be injective if A_m^\dagger is injective, implying that $T(w) = w$ if and only if $G(w) = 0$ (this is easily derived from (4.1)). This means that if T is a contraction, then T has a unique fixed point in $\hat{w}_m + B_\omega(r)$ corresponding to a zero of G .

The most important step in constructing estimates on the balls $\hat{w}_m + B_\omega(r)$, $r > 0$, is defining a finite set of polynomials, called the *radii polynomials* [21, 9]. In order to construct these polynomials, we need bounds Y_0, \dots, Y_m and Y_∞ , and polynomials

Z_0, \dots, Z_m and Z_∞ such that the following hold:

$$(4.2a) \quad |(\Pi_m(T(\hat{w}_m) - \hat{w}_m))_k| \leq Y_k, \quad \text{for } k = 0, \dots, m$$

$$(4.2b) \quad \|\Pi_\infty(T(\hat{w}_m) - \hat{w}_m)\|_\infty \leq Y_\infty$$

$$(4.2c) \quad \sup_{w_1, w_2 \in B_\omega(r)} |(\Pi_m DT(\hat{w}_m + w_1)w_2)_k| \leq Z_k(r), \quad \text{for } k = 0, \dots, m$$

$$(4.2d) \quad \sup_{w_1, w_2 \in B_\omega(r)} \|\Pi_\infty DT(\hat{w}_m + w_1)w_2\|_\infty \leq Z_\infty(r).$$

Expressions for these bounds will be derived explicitly in Sections 7 and 8. Based on the bounds above we define the radii polynomials as follows.

DEFINITION 4.1. *Let Y_k , Y_∞ , $Z_k(r)$, and $Z_\infty(r)$ be as in (4.2), then we define the radii polynomials by*

$$p_k(r) \stackrel{\text{def}}{=} Y_k + Z_k(r) - r, \quad k = 0, \dots, m$$

$$p_\infty(r) \stackrel{\text{def}}{=} Y_\infty + Z_\infty(r) - \omega r.$$

Note that the ω used in this definition is the same ‘‘fixed’’ ω that we use in $B_\omega(r)$. The radii polynomials are a crucial tool in our contraction argument. Indeed, the above definitions and considerations lead to the following existence and uniqueness theorem (details of the proof can be found in [7, 6]), which is based on [23].

THEOREM 4.2. *Let $r > 0$ be such that $p_k(r) < 0$ for all $0 \leq k \leq m$ and $p_\infty(r) < 0$. Then the map T is a contraction on $\hat{w}_m + B_\omega(r)$. Consequently, G has a unique zero inside $\hat{w}_m + B_\omega(r)$.*

For the radii polynomials to all be negative for some $r > 0$, we need our Y -bounds and Z -bounds to be suitably small. The parameter ω allows us to shift the intervals on which our polynomials are negative, with the purpose of finding an interval on which *all* of them are negative. In order for our computations to be rigorous, we use interval arithmetic both to evaluate Y and Z , and to check that $p(r) < 0$.

To prove the final assertion in Theorem 1.1 we need a little bit more than the fact that $\hat{w}_m > 0$ (see Remark 3.1).

LEMMA 4.3. *Assume that the assumptions of Theorem 4.2 are satisfied. Suppose that $\min \hat{w}_m > r(1 + \omega)$. Then the zero \hat{w} of G found in Theorem 4.2 is strictly positive on $[0, 1]$, and the corresponding solution \hat{A} of (1.6) is strictly positive on $(0, \infty)$.*

Proof. Since $\hat{w} - \hat{w}_m \in B_\omega(r)$, we obtain, for all $t \in [0, 1]$,

$$\hat{w}(t) \geq \hat{w}_m(t) - |\hat{w}(t) - \hat{w}_m(t)| \geq \min \hat{w}_m - \|\hat{w} - \hat{w}_m\|_\infty \geq \min \hat{w}_m - r(1 + \omega) > 0.$$

The positivity of \hat{A} then follows from Lemma 2.2. \square

5. The linearised equation. The previous sections outline the framework in which we will prove the existence of a solution \hat{A} to (1.6). We also need to prove the second part of Theorem 1.1, namely that the linearisation of (1.6) around \hat{A} does not have any non-trivial bounded solutions. Fortunately, this follows almost directly from our proof of the existence of \hat{A} . Roughly, the reason is that a contraction method can only have a successful outcome for non-degenerate (‘‘transversal’’) problems. We give the details of the argument below.

Our starting point is a solution $\hat{w} \in C[0, 1]$, from which we obtain \hat{A} through (2.6). The linearised version of (1.6) is

$$(5.1) \quad \tilde{A}''(s) = -\frac{1}{s}\tilde{A}'(s) + \left(\frac{1}{4s^2} + 1 - 3\hat{A}(s)^2\right)\tilde{A}(s),$$

and the linearisation of (2.4) is given by

$$(5.2) \quad \begin{aligned} \tilde{w}(t) = & -\frac{1}{2} \frac{1+t}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} 3\hat{w}(s)^2 \tilde{w}(s) ds \\ & - \frac{1}{2} \frac{1}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} 3\hat{w}(s)^2 \tilde{w}(s) ds. \end{aligned}$$

Since the transformation described in (2.6) is linear, \tilde{A} and \tilde{w} also satisfy

$$(5.3) \quad \tilde{A}(s) = s^{-1/2}(1 - e^{-s})e^{-s}\tilde{w}(e^{-s}).$$

We now wish to prove that (5.1) does not have any non-trivial bounded solutions. We argue by contradiction. Hence we assume that a non-trivial bounded solution to (5.1) does exist. Furthermore, suppose that we have proven the first part of Theorem 1.1 using Theorem 4.2, i.e., by showing the existence of an $r > 0$ such that all radii polynomials are negative.

We proceed by showing that the assumption that (5.1) has a non-trivial bounded solution implies that (5.2) also has a non-trivial bounded solution. Since the factor $s^{1/2}e^s/(1 - e^{-s})$ in (5.3) is not bounded on $(0, \infty)$, this requires some work, in particular in the limits $s \rightarrow 0$ and $s \rightarrow \infty$.

LEMMA 5.1. *Suppose that there exists a bounded non-trivial solution to (5.1), then there exists a bounded non-trivial solution to (5.2).*

Proof. Let $\tilde{A}(s)$ be a bounded solution to (5.1), then the function $\tilde{u}(s) = s^{1/2}\tilde{A}(s)$ satisfies $|\tilde{u}(s)| \leq Cs^{1/2}$ for some $C > 0$ and all $s \in [0, \infty)$. Furthermore, \tilde{u} solves the linearisation of (2.1), given by

$$(5.4) \quad \tilde{u}''(s) = \tilde{u}(s) \left(1 - \frac{3\hat{u}^2}{s} \right),$$

where $\hat{u}(s) = s^{1/2}\hat{A}(s) = (1 - e^{-s})e^{-s}\hat{w}(e^{-s})$. Since \hat{w} is bounded, we see that $|\hat{u}(s)| \leq \hat{c}e^{-s}$ for some $\hat{c} > 0$ and all s , and $|\hat{u}(s)| \leq c_0s$ for some $c_0 > 0$ and all sufficiently small s .

Using the function \tilde{u} , we define \tilde{w} by $\tilde{w}(e^{-s}) = \tilde{u}(s)e^s/(1 - e^{-s})$, which solves (5.2). It remains to show that \tilde{w} is bounded, i.e., we need to show that, first, \tilde{u} decays as e^{-s} , and second, \tilde{u} is linear near $s = 0$. The former follows from an exponential dichotomy. Indeed, since $|\tilde{u}(s)| \leq Cs^{1/2}$, it follows from Lemma A.2 in the Appendix that $e^s\tilde{u}(s)$ is bounded as $s \rightarrow \infty$.

Finally, we need to show that $|\tilde{u}(s)| \leq c_1s$ for some $c_1 > 0$ and all sufficiently small s . From (5.4) we observe that $\tilde{u}''(s)$ is uniformly bounded for sufficiently small s , since \tilde{u} is bounded and $|\hat{u}(s)| \leq c_0s$. Hence, $\tilde{u}'(0)$ is well defined (bounded). Since $\tilde{u}(0) = 0$, it follows that $\tilde{u}(s)/s$ is uniformly bounded for sufficiently small s . \square

Using the previous lemma, we now give the proof of the second assertion of Theorem 1.1.

LEMMA 5.2. *Let $r > 0$ be such that all radii polynomials are negative, and let \hat{w} be the unique zero of G in the ball $\hat{w}_m + B_\omega(r)$ obtained in Theorem 4.2. Let $\hat{A}(s) = s^{-1/2}(1 - e^{-s})e^{-s}\hat{w}(e^{-s})$. Then any bounded solution to (5.1) is trivial.*

Proof. We argue by contradiction. Suppose that a non-trivial bounded solution to (5.1) exists, then by the Lemma 5.1 there exists a non-trivial bounded solution \tilde{w} to (5.2). Any bounded solution to (5.2) satisfies $DG(\hat{w})\tilde{w} = 0$, where DG denotes the Fréchet derivative of the map G defined in (2.7). Since all radii polynomials

are negative, the map T is a contraction on $\hat{w}_m + B_\omega(r)$. In particular, since $\hat{w} \in \hat{w}_m + B_\omega(r)$, we have that $\|DT(\hat{w})\| < 1$, which implies that $I - DT(\hat{w})$ is invertible. Now note that

$$I - DT(\hat{w}) = A_m^\dagger \Pi_m DG(\hat{w}) + \Pi_\infty DG(\hat{w}).$$

This means that if $DG(\hat{w})\tilde{w} = 0$, then also $(I - DT(\hat{w}))\tilde{w} = 0$. Since $I - DT(\hat{w})$ is invertible, its kernel is trivial, hence $\tilde{w} = 0$. This contradicts that \tilde{w} is non-trivial. \square

6. Interval arithmetic evaluation of logarithmic integrals. This section deals with the central technical aspect of our estimates. In particular, we introduce the notation for “evaluating” certain integrals in an interval arithmetic sense. Indeed, a crucial part of our estimates consists of estimating terms of the form $\int_a^b \frac{p(s)}{\log(s)} ds$ for polynomials p and $0 \leq a < b \leq 1$. Since \hat{w}_m is piecewise linear, it is immediate that evaluating G^m can be reduced to calculating such integrals. Furthermore, most of our estimates involve G^m and its derivatives in some way and hence also contain these types of integrals, see e.g. Lemmas 7.2, 8.1 and 8.2. In order to make our estimates sufficiently sharp, it will be crucial that these terms can be evaluated with high precision, especially in the context of interval arithmetic.

These “logarithmic integrals” can be easily decomposed as linear combinations of $\int_a^b \frac{s^n}{\log s} ds$, for $n \geq 0$. It suffice to obtain a good estimate for such “monomial” terms, unless $b = 1$, in which case we will have to be more careful, see (6.5).

First we note that these integrals can be written in two distinct ways, namely

$$(6.1a) \quad \int_0^t \frac{s^n}{\log s} ds = \text{Ei}(\log t^{n+1}),$$

$$(6.1b) \quad \int_0^t \frac{s^n}{\log s} ds = -\Gamma(0, -\log t^{n+1}).$$

Here Ei denotes the exponential integral function, while Γ denotes the incomplete gamma function. We need to evaluate the integrals both for t near 0, where (6.1b) is convenient, and for t near 1, where (6.1a) is convenient.

One well known expansion of the exponential integral is given by

$$(6.2) \quad \text{Ei}(x) = \gamma + \log|x| + \sum_{k=1}^{\infty} \frac{x^k}{kk!},$$

where $\gamma \approx 0.577216$ denotes the Euler-Mascheroni constant. This expansion converges rapidly for small $|x| \neq 0$. For $t = e^{x/(n+1)}$ close to 1, truncating the series thus provides us with a good approximation. Furthermore, since $x = \log t^{n+1}$ is negative for $t < 1$, this series is alternating, hence the error bounds are simply given by two consecutive partial sums of the series.

For t close to 0, the value of $\log t^{n+1}$ is very large, and (6.2) is not suitable. Instead we use (6.1b). We express the incomplete gamma function by means of a

continued fraction due to Legendre, given by

$$(6.3) \quad \Gamma(0, z) = \frac{e^{-z}}{z + \frac{1}{1 + \frac{1}{z + \frac{2}{1 + \frac{2}{z + \frac{3}{1 + \ddots}}}}}}$$

It can be shown that this continued fraction indeed converges rapidly for large values of $|z|$ and that the truncated fractions alternate around the limit, hence the true value of $\Gamma(0, z)$ must always lie between consecutive values in the sequence of truncated expansions [5].

We write $f_1(z) \stackrel{\text{def}}{=} e^{-z}/z$, $f_2(z) \stackrel{\text{def}}{=} e^{-z}/(z + (1/1))$ and so on for the partial continued fractions of (6.3). Then we define $L_{m_1, m_2} : [0, 1) \rightarrow \mathbf{R}$ by

$$L_{m_1, m_2}(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{for } t = 0, \\ -f_{m_2}(-\log t) & \text{for } 0 < t < e^{-1}, \\ \gamma + \log |\log t| + \sum_{k=1}^{m_1} \frac{(\log t)^k}{kk!} & \text{for } e^{-1} \leq t < 1, \end{cases}$$

which, for large m_1, m_2 approximates the logarithmic integral. For simplicity, we have chosen $t = e^{-1}$ as the boundary between “small” and “large” values of t in this definition.

By our previous remarks, the given expansions for Ei and Γ alternate around their limit, hence we can use this to define bounds $L_{m_1, m_2}^{\pm} : [0, 1) \rightarrow \mathbf{R}$ that estimate the logarithmic integrals:

$$\begin{aligned} L_{m_1, m_2}^{-}(t) &\stackrel{\text{def}}{=} \min\{L_{m_1, m_2}(t), L_{m_1+1, m_2+1}(t)\}, \\ L_{m_1, m_2}^{+}(t) &\stackrel{\text{def}}{=} \max\{L_{m_1, m_2}(t), L_{m_1+1, m_2+1}(t)\}. \end{aligned}$$

These allow us to make the following estimate.

LEMMA 6.1. *Let $t \in [0, 1)$, then for all $n, m_1, m_2 \geq 0$ we have*

$$L_{m_1, m_2}^{-}(t^{n+1}) \leq \int_0^t \frac{s^n}{\log s} ds \leq L_{m_1, m_2}^{+}(t^{n+1}).$$

While Lemma 6.1 provides us with an estimate for logarithmic integrals containing monomials, we shall mainly need estimates of logarithmic integrals that contain polynomial terms. For this reason we introduce for all $m_1, m_2 > 0$ and all polynomial functions $p(t) = a_0 + \dots + a_N t^N$ maps $\mathcal{L}_{m_1, m_2}^{\pm}[p] : [0, 1) \rightarrow \mathbf{R}$ defined by

$$(6.4) \quad \mathcal{L}_{m_1, m_2}^{\pm}[p](t) \stackrel{\text{def}}{=} \sum_{\substack{0 \leq j \leq N \\ a_j \neq 0}} a_j L_{m_1, m_2}^{\pm \text{sign}(a_j)}(t^{j+1}).$$

This map is not well-defined if $t = 1$. However, for the logarithmic integrals that we encounter, $p(s)$ always contains a factor $(1 - s)$, and it can therefore be written as $p(s) = (1 - s)p_0(s)$ for some other polynomial $p_0(s) = b_0 + \dots + b_{N-1}s^{N-1}$, with

the relation $a_j = b_j - b_{j-1}$. Since $(1-s)/\log s$ is bounded on $(0, 1)$, the logarithmic integrals containing these polynomials will in fact be well-defined for $t = 1$. This allows us to extend our definition of $\mathcal{L}_{m_1, m_2}^\pm[p]$ by setting

$$(6.5) \quad \mathcal{L}_{m_1, m_2}^\pm[p](1) \stackrel{\text{def}}{=} \int_0^1 \frac{(1-s)p_0(s)}{\log s} ds = \sum_{j=0}^{N-1} b_j \log \frac{1+j}{2+j},$$

which thus complements the expression (6.4), which is defined for $0 \leq t < 1$. These definitions allow us to formulate the following general result.

COROLLARY 6.2. *Let p be a polynomial containing a factor $(1-s)$. Then for all $m_1, m_2 \geq 1$ we have*

$$\mathcal{L}_{m_1, m_2}^-[p](t_2) - \mathcal{L}_{m_1, m_2}^+[p](t_1) \leq \int_{t_1}^{t_2} \frac{p(s)}{\log s} ds \leq \mathcal{L}_{m_1, m_2}^+[p](t_2) - \mathcal{L}_{m_1, m_2}^-[p](t_1)$$

for all $0 \leq t_1 < t_2 \leq 1$.

These estimates give high accuracy rigorous enclosures of the logarithmic integrals, which can be established through the use of interval arithmetic. It is worth noting that in practice we use $\mathcal{L}_{30, 260}^\pm$, which, for the mesh that we choose, yields an accuracy of the same order of magnitude as machine precision ($\epsilon \simeq 10^{-16}$).

In our constructions of the Y -bounds and Z -bounds, we shall repeatedly make use of the estimate provided in Corollary 6.2. As the notation for this evaluation quickly becomes cumbersome, we shall only present the details for the function

$$(6.6) \quad f(t) = \frac{1}{t^l} \int_0^t \frac{s^3(1-s)^3}{\log s} \hat{w}_m(s)^3 ds,$$

where $\hat{w}_m \in S_m$ and $t \in [0, 1]$ and $l \leq 4$. This function appears for example in the second term in the definition of G in (2.7).

First note that on each interval $[t_k, t_{k+1}]$ the function $\hat{w}_m \in S_m$ is linear, hence we can define p_k as the polynomial defined on each interval $[t_k, t_{k+1}]$ by

$$p_k(s) \stackrel{\text{def}}{=} s^3(1-s)^3(\hat{w}_m|_{[t_k, t_{k+1}]}(s))^3.$$

In general we need to make two different types of estimates. First, we want to estimate $f(t_k)$, and second, we want to estimate $f(t)$ uniformly for $t \in [t_k, t_{k+1}]$.

The first estimate follows almost directly from Corollary 6.2, although the estimate at $t_0 = 0$ requires a small observation. Namely, since \hat{w}_m is bounded, and $l \leq 4$, we have that $\lim_{t \rightarrow 0} f(t) = 0$. Hence if we set $\mathcal{E}_0^\pm(f) = 0$ and

$$(6.7) \quad \mathcal{E}_k^\pm(f) \stackrel{\text{def}}{=} \frac{1}{t_k^l} \sum_{j=0}^{k-1} (\mathcal{L}_{m_1, m_2}^\pm[p_j](t_{j+1}) - \mathcal{L}_{m_1, m_2}^\mp[p_j](t_j)),$$

for $1 \leq k \leq m$, then

$$\mathcal{E}_k^-(f) \leq f(t_k) \leq \mathcal{E}_k^+(f),$$

for all $k = 0, \dots, m$.

The second estimate is constructed primarily from the first one and the mean value theorem. In particular, we have that for $t \in [t_k, t_{k+1}]$,

$$(6.8) \quad f(t) = \frac{1}{t^l} \int_0^{t_k} \frac{p_{k+1}(s)}{\log s} ds + \frac{1}{t^l} \int_{t_k}^t \frac{p_{k+1}(s)}{\log s} ds.$$

The first term can be accurately estimated by

$$\min \left\{ \frac{\mathcal{E}_k^-(t^l f(t))}{t_k^l}, \frac{\mathcal{E}_k^-(t^l f(t))}{t_{k+1}^l} \right\} \leq \frac{1}{t^l} \int_0^{t_k} \frac{p_{k+1}(s)}{\log s} ds \leq \max \left\{ \frac{\mathcal{E}_k^+(t^l f(t))}{t_k^l}, \frac{\mathcal{E}_k^+(t^l f(t))}{t_{k+1}^l} \right\},$$

where for $k = 0$ one should interpret these bounds as being 0, since $\mathcal{E}_0^\pm = 0$.

For the second term in (6.8) we obtain from the mean value theorem that for $k = 1, \dots, m-2$ (i.e., excluding the first and last interval)

$$\frac{(t - t_k)}{t^l} \inf_{s \in [t_k, t]} \frac{p_{k+1}(s)}{\log s} \leq \frac{1}{t^l} \int_{t_k}^t \frac{p_{k+1}(s)}{\log s} ds \leq \frac{(t - t_k)}{t^l} \sup_{s \in [t_k, t]} \frac{p_{k+1}(s)}{\log s},$$

from which we infer the uniform bound (for $t \in [t_k, t_{k+1}]$)

$$\begin{aligned} \frac{(t_{k+1} - t_k)}{t_k^l} \min \left\{ 0, \inf_{s \in [t_k, t_{k+1}]} \frac{p_{k+1}(s)}{\log s} \right\} &\leq \frac{1}{t^l} \int_{t_k}^t \frac{p_{k+1}(s)}{\log s} ds \\ &\leq \frac{1}{t^l} \int_{t_k}^t \frac{p_{k+1}(s)}{\log s} ds \leq \frac{(t_{k+1} - t_k)}{t_k^l} \max \left\{ 0, \sup_{s \in [t_k, t_{k+1}]} \frac{p_{k+1}(s)}{\log s} \right\}. \end{aligned}$$

In our specific setting \hat{w}_m is positive, see Remark 3.1, hence all p_k are positive. The above estimate then reduces to

$$\frac{(t_{k+1} - t_k)}{t_k^l} \inf_{s \in [t_k, t_{k+1}]} \frac{p_{k+1}(s)}{\log s} \leq \frac{1}{t^l} \int_{t_k}^t \frac{p_{k+1}(s)}{\log s} ds \leq 0.$$

In practice we compute the infimum in the lower bound by evaluating $\frac{p_{k+1}(t)}{\log t}$ on $[t_k, t_{k+1}]$ using interval arithmetic.

REMARK 6.3. *The estimate above cannot be evaluated on the intervals $[0, t_1]$ and $[t_{m-1}, 1]$. We consider these intervals separately, where, to reduce the size of the expressions, we use that \hat{w}_m is positive (see Remark 3.1). For $t \in [0, t_1]$, we have that*

$$\begin{aligned} 0 &\geq \frac{1}{t^l} \int_0^t \frac{p_{k+1}(s)}{\log s} ds \geq \frac{1}{t^l} \int_0^t \frac{s^3}{\log t_1} \max\{\hat{w}_m(0)^3, \hat{w}_m(t_1)^3\} ds \\ &\geq \frac{1}{4} \frac{t_1^{4-l}}{\log t_1} \max\{\hat{w}_m(0)^3, \hat{w}_m(t_1)^3\}. \end{aligned}$$

Similarly, since $-1 \leq (1-s)/\log s \leq 0$ for $s \in (0, 1)$, we have that for $t \in [t_{m-1}, 1]$,

$$\begin{aligned} 0 &\geq \frac{1}{t^l} \int_{t_{m-1}}^t \frac{p_{k+1}(s)}{\log s} ds \geq -\frac{1}{t^l} \int_{t_{m-1}}^t (1-s)^2 \max\{\hat{w}_m(t_{m-1})^3, \hat{w}_m(1)^3\} ds \\ &\geq -\frac{1}{3} \frac{(1-t_{m-1})^3}{t_{m-1}^l} \max\{\hat{w}_m(t_{m-1})^3, \hat{w}_m(1)^3\}. \end{aligned}$$

From the above considerations we conclude that if we set

$$\begin{aligned} \mathcal{E}_{[t_k, t_{k+1}]}^-(f) &\stackrel{\text{def}}{=} \min \left\{ \frac{\mathcal{E}_k^-(t^l f(t))}{t_k^l}, \frac{\mathcal{E}_k^-(t^l f(t))}{t_{k+1}^l} \right\} + \frac{t_{k+1} - t_k}{t_k^l} \inf_{t \in [t_k, t_{k+1}]} \frac{p_{k+1}(t)}{\log t} \\ \mathcal{E}_{[t_k, t_{k+1}]}^+(f) &\stackrel{\text{def}}{=} \max \left\{ \frac{\mathcal{E}_k^+(t^l f(t))}{t_k^l}, \frac{\mathcal{E}_k^+(t^l f(t))}{t_{k+1}^l} \right\}, \end{aligned}$$

for $k = 1, \dots, m - 2$, and

$$\begin{aligned}\mathcal{E}_{[0,t_1]}^-(f) &\stackrel{\text{def}}{=} \frac{1}{4} \frac{t_1^{4-l}}{\log t_1} \max\{\hat{w}_m(0)^3, \hat{w}_m(t_1)^3\} \\ \mathcal{E}_{[0,t_1]}^+(f) &\stackrel{\text{def}}{=} 0 \\ \mathcal{E}_{[t_{m-1},1]}^-(f) &\stackrel{\text{def}}{=} \min \left\{ \frac{\mathcal{E}_m^-(t^l f(t))}{t_{m-1}^l}, \mathcal{E}_m^-(t^l f(t)) \right\} - \frac{1}{3} \frac{(1-t_{m-1})^3}{t_{m-1}^l} \max\{\hat{w}_m(t_{m-1})^3, \hat{w}_m(1)^3\} \\ \mathcal{E}_{[t_{m-1},1]}^+(f) &\stackrel{\text{def}}{=} \max \left\{ \frac{\mathcal{E}_m^+(t^l f(t))}{t_{m-1}^l}, \mathcal{E}_m^+(t^l f(t)) \right\},\end{aligned}$$

then

$$\mathcal{E}_{[t_k, t_{k+1}]}^-(f) \leq f(t) \leq \mathcal{E}_{[t_k, t_{k+1}]}^+(f),$$

for all $t \in [t_k, t_{k+1}]$, with $k = 0, \dots, m - 1$.

The $\mathcal{E}_{[t_k, t_{k+1}]}^\pm$ notation can also be used for more general functions than those defined by f in (6.6). In particular, we have computed and implemented completely analogous estimates for integrals of the form

$$\frac{1}{(1-t)^l} \int_t^1 \frac{(1-s)p_0(s)}{\log s} ds.$$

The details are repetitive and therefore we skip them here. We will use the notation \mathcal{E}_k^\pm and $\mathcal{E}_{[t_k, t_{k+1}]}^\pm$ for these type of integrals as well.

Finally, the estimates generalize in an obvious way to linear combinations of logarithmic integrals, namely by simply taking the appropriate linear combinations of the upper and lower bounds, depending on whether the coefficients are positive or negative, see (6.4). Henceforth, we will use the interval notation

$$\mathcal{E}_k(F) \stackrel{\text{def}}{=} [\mathcal{E}_k^-, \mathcal{E}_k^+] \quad \text{and} \quad \mathcal{E}_{[t_k, t_{k+1}]}(F) \stackrel{\text{def}}{=} [\mathcal{E}_{[t_k, t_{k+1}]}^-, \mathcal{E}_{[t_k, t_{k+1}]}^+],$$

where $F(t)$ is a logarithmic integral (or a linear combination of such integrals). In particular, for any $k = 0, \dots, m - 1$

$$\begin{aligned}F(t_k) &\in \mathcal{E}_k(F), \\ F(t) &\in \mathcal{E}_{[t_k, t_{k+1}]}(F) \quad \text{for all } t \in [t_k, t_{k+1}].\end{aligned}$$

We call these enclosures the *interval arithmetic evaluation* of a logarithmic integral.

7. Explicit construction of the Y-bounds. We begin our estimates by showing how to compute the Y-bounds (4.2a) and (4.2b). As mentioned above, all computations in this section will have to be made using interval arithmetic. In order to simplify many of the expressions that we shall encounter in this section, we introduce the twice continuously differentiable function $g : [0, 1] \rightarrow \mathbf{R}$ defined by

$$(7.1) \quad \begin{aligned}g(t) &\stackrel{\text{def}}{=} \frac{1}{2} \frac{1+t}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} \hat{w}_m(s)^3 ds \\ &\quad + \frac{1}{2} \frac{1}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} \hat{w}_m(s)^3 ds,\end{aligned}$$

so that $G(\hat{w}_m) = \hat{w}_m + g$.

As it turns out, the Y_k bounds are relatively easy to obtain using the interval arithmetic evaluation introduced in Section 6. Let $\mathcal{E}(g)$ be the vector of intervals given by $(\mathcal{E}(g))_k \stackrel{\text{def}}{=} \mathcal{E}_k(g)$, $k = 0, \dots, m$. We set

$$(7.2) \quad Y_k \stackrel{\text{def}}{=} \sup |(A_m^\dagger[\hat{w}_m + \mathcal{E}(g)])_k|,$$

where matrix multiplication is performed using interval arithmetic. The sup in (7.2) is just the right endpoint of the interval. Since $G(\hat{w}_m) = \hat{w}_m + g$, the bounds Y_k will be small when \hat{w}_m is a good approximation of a zero of G^m .

LEMMA 7.1. *With Y_k as defined in (7.2) we have*

$$|(\Pi_m(T(\hat{w}_m) - \hat{w}_m))_k| = |(A_m^\dagger[\hat{w}_m + \Pi_m g])_k| \leq Y_k.$$

The Y_∞ -bound requires a little more work. We observe that

$$\|\Pi_\infty(T(\hat{w}_m) - \hat{w}_m)\|_\infty = \|(I - \Pi_m)g\|_\infty,$$

hence we need a way to estimate $\|(I - \Pi_m)g\|_\infty = \|\Pi_\infty g\|_\infty$ for $g \in C^2[0, 1]$. A standard estimate can be used to show that, for all $t \in [t_k, t_{k+1}]$,

$$(7.3) \quad |g(t) - (\Pi_m g)(t)| \leq \frac{1}{8} |t_{k+1} - t_k|^2 \sup_{s \in [t_k, t_{k+1}]} |g''(s)|.$$

Here g'' can be written as $g''(t) = h_1(t) + h_2(t) + h_3(t)$, with

$$(7.4a) \quad h_1(t) \stackrel{\text{def}}{=} \frac{3+t}{t^4} \int_0^t \frac{s^3(1-s)^3}{\log s} \hat{w}_m(s)^3 ds$$

$$(7.4b) \quad h_2(t) \stackrel{\text{def}}{=} \frac{1}{(1-t)^3} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} \hat{w}_m(s)^3 ds$$

$$(7.4c) \quad h_3(t) \stackrel{\text{def}}{=} -\frac{(1-t)^2}{\log t} \hat{w}_m(t)^3.$$

The first two terms are estimated using the interval arithmetic evaluation from Section 6. The final term is simply evaluated using interval arithmetic on each interval $[t_k, t_{k+1}]$, $k = 1, \dots, m-2$, whereas the intervals $[t_0, t_1]$ and $[t_{m-1}, 1]$ are dealt with in an analogous (but simpler) manner to the arguments used in Remark 6.3. For the estimates thus obtained we use the notation $h_3(t) \in \mathcal{F}_{[t_k, t_{k+1}]}(h_3)$ for $t \in [t_k, t_{k+1}]$. This then gives us the following result.

LEMMA 7.2. *Define Y_∞ as*

$$Y_\infty \stackrel{\text{def}}{=} \max_{0 \leq k \leq m-1} \left\{ \frac{1}{8} |t_{k+1} - t_k|^2 \cdot \sup \left| \mathcal{E}_{[t_k, t_{k+1}]}(h_1 + h_2) + \mathcal{F}_{[t_k, t_{k+1}]}(h_3) \right| \right\},$$

then

$$\|\Pi_\infty(T(\hat{w}_m) - \hat{w}_m)\|_\infty \leq Y_\infty.$$

8. Explicit construction of the Z-bounds. In this section we make the estimates (4.2c) and (4.2d) explicit. For notational convenience we introduce, for $w_1, w_2 \in B_\omega(r)$,

$$z(w_1, w_2) \stackrel{\text{def}}{=} DT(\hat{w}_m + w_1)w_2.$$

We wish to estimate $|(\Pi_m z(w_1, w_2))_k|$ for $w_1, w_2 \in B_\omega(r)$ as a function of r . Writing $w_i = r\tilde{w}_i$, where $\tilde{w}_i \in B_\omega(1)$, we have

$$\begin{aligned} \Pi_m z(w_1, w_2) &= \Pi_m(DT(\hat{w}_m + w_1)w_2) = r\Pi_m\tilde{w}_2 - rA_m^\dagger\Pi_m DG(\hat{w}_m + r\tilde{w}_1)\tilde{w}_2 \\ &= r(I_m - A_m^\dagger D\Pi_m G(\hat{w}_m))\Pi_m\tilde{w}_2 \\ &\quad + rA_m^\dagger(D\Pi_m G(\hat{w}_m)\Pi_m\tilde{w}_2 - D\Pi_m G(\hat{w}_m + r\tilde{w}_1)\tilde{w}_2) \\ &= r(I_m - A_m^\dagger DG^m(\hat{w}_m))\Pi_m\tilde{w}_2 - rA_m^\dagger\eta'(0), \end{aligned}$$

where the auxiliary term η is given by

$$\eta(\tau) := \Pi_m G(\hat{w}_m + r\tilde{w}_1 + \tau\tilde{w}_2) - \Pi_m G(\hat{w}_m + \tau\Pi_m\tilde{w}_2).$$

We start by showing how to estimate the $(I_m - A_m^\dagger DG^m(\hat{w}_m))\Pi_m\tilde{w}_2$ term above. Viewing $DG^m(\hat{w}_m)$ as an $(m+1) \times (m+1)$ -matrix, we distinguish three cases. If $k_1 = k_2$, then

$$\begin{aligned} DG^m(\hat{w}_m)_{k_1, k_2} &= \phi_0(t_{k_1}, t_{k_2}) \\ &\stackrel{\text{def}}{=} 1 + \frac{1}{2} \frac{1+t_{k_1}}{t_{k_1}^2} \int_{t_{k_2-1}}^{t_{k_2}} \frac{s^3(1-s)^3}{\log s} 3\hat{w}_m(s)^2 \frac{s-t_{k_2-1}}{t_{k_2}-t_{k_2-1}} ds \\ &\quad + \frac{1}{2} \frac{1}{1-t_{k_1}} \int_{t_{k_2}}^{t_{k_2+1}} \frac{s(1+s)(1-s)^4}{\log s} 3\hat{w}_m(s)^2 \frac{t_{k_2+1}-s}{t_{k_2+1}-t_{k_2}} ds; \end{aligned}$$

if $k_1 < k_2$, then

$$\begin{aligned} DG^m(\hat{w}_m)_{k_1, k_2} &= \phi_{-1}(t_{k_1}, t_{k_2}) \\ &\stackrel{\text{def}}{=} \frac{1}{2} \frac{1}{1-t_{k_1}} \int_{t_{k_2-1}}^{t_{k_2}} \frac{s(1+s)(1-s)^4}{\log s} 3\hat{w}_m(s)^2 \frac{s-t_{k_2-1}}{t_{k_2}-t_{k_2-1}} ds \\ &\quad + \frac{1}{2} \frac{1}{1-t_{k_1}} \int_{t_{k_2}}^{t_{k_2+1}} \frac{s(1+s)(1-s)^4}{\log s} 3\hat{w}_m(s)^2 \frac{t_{k_2+1}-s}{t_{k_2+1}-t_{k_2}} ds; \end{aligned}$$

if $k_1 > k_2$, then

$$\begin{aligned} DG^m(\hat{w}_m)_{k_1, k_2} &= \phi_{+1}(t_{k_1}, t_{k_2}) \\ &\stackrel{\text{def}}{=} \frac{1}{2} \frac{1+t_{k_1}}{t_{k_1}^2} \int_{t_{k_2-1}}^{t_{k_2}} \frac{s^3(1-s)^3}{\log s} 3\hat{w}_m(s)^2 \frac{s-t_{k_2-1}}{t_{k_2}-t_{k_2-1}} ds \\ &\quad + \frac{1}{2} \frac{1+t_{k_1}}{t_{k_1}^2} \int_{t_{k_2}}^{t_{k_2+1}} \frac{s^3(1-s)^3}{\log s} 3\hat{w}_m(s)^2 \frac{t_{k_2+1}-s}{t_{k_2+1}-t_{k_2}} ds. \end{aligned}$$

Here, any integral with domain of integration outside $[0, 1]$ should be read as 0. Let $\mathcal{M}(\phi)$ be the matrix of intervals given by

$$\mathcal{M}(\phi)_{k_1, k_2} \stackrel{\text{def}}{=} \tilde{\mathcal{E}}(\phi_{\text{sign}(k_1-k_2)}(t_{k_1}, t_{k_2})),$$

where $\tilde{\mathcal{E}}$ denotes interval arithmetic evaluation following the methodology from Section 6, which is somewhat simpler here since the sum in (6.7) reduces to a single term per integral in the expressions for ϕ . Combining this with the fact that $|(\Pi_m \tilde{w}_2)_k| \leq 1$, and using the notation $\mathbf{1} = (1, 1, \dots, 1) \in \mathbf{R}^{m+1}$, the term $(I_m - A_m^\dagger DG^m(\hat{w}_m)) \Pi_m \tilde{w}_2$ can then be estimated by

$$(8.1) \quad \left| \left((I_m - A_m^\dagger DG^m(\hat{w}_m)) \Pi_m \tilde{w}_2 \right)_k \right| \leq \sup \left(|I_m - A_m^\dagger \mathcal{M}(\phi)| \mathbf{1} \right)_k,$$

where the absolute value on the right-hand side is taken element-wise for the matrix of intervals $I_m - A_m^\dagger \mathcal{M}(\phi)$.

Next, we can express $\eta(\tau) \in S_m$ as

$$\begin{aligned} \eta(\tau)_k &= r \tilde{w}_1(t_k) + \frac{1}{2} \frac{1+t_k}{t_k^2} \int_0^{t_k} \frac{s^3(1-s)^3}{\log s} \zeta_s(\tau) ds \\ &\quad + \frac{1}{2} \frac{1}{1-t_k} \int_{t_k}^1 \frac{s(1+s)(1-s)^4}{\log s} \zeta_s(\tau) ds, \end{aligned}$$

where $\zeta_s(\tau)$ is given by

$$\zeta_s(\tau) \stackrel{\text{def}}{=} (\hat{w}_m(s) + r \tilde{w}_1(s) + \tau \tilde{w}_2(s))^3 - (\hat{w}_m(s) + \tau(\Pi_m \tilde{w}_2)(s))^3.$$

Furthermore, note that for $\tilde{w}_1, \tilde{w}_2 \in B_\omega(1)$ we have that

$$(8.2) \quad \begin{aligned} |\zeta'_s(0)| &= |3(\hat{w}_m(s) + r \tilde{w}_1(s))^2 \tilde{w}_2(s) - 3\hat{w}_m(s)^2 (\Pi_m \tilde{w}_2)(s)| \\ &\leq 3\omega |\hat{w}_m(s)|^2 + 6(1+\omega)^2 |\hat{w}_m(s)| r + 3(1+\omega)^3 r^2. \end{aligned}$$

We again use the fact that \hat{w}_m is positive (see Remark 3.1) to slightly simplify the estimates and notation (the estimates can be adapted to the general case in a straightforward manner). Namely, we infer from (8.2) that

$$|\zeta'_s(0)| \leq 3\omega \hat{w}_m(s)^2 + 6(1+\omega)^2 \hat{w}_m(s) r + 3(1+\omega)^3 r^2.$$

Using the interval arithmetic evaluation of logarithmic integrals from Section 6 we thus find that

$$|\eta'(0)_k| \leq V_k^1 + V_k^2 r + V_k^3 r^2,$$

where the coefficients $V_k^i > 0$ are given by

$$\begin{aligned} V_k^1 &\stackrel{\text{def}}{=} \sup \mathcal{E}_k \left(-\frac{3\omega(1+t)}{2t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} \hat{w}_m(s)^2 ds - \frac{3\omega}{2(1-t)} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} \hat{w}_m(s)^2 ds \right) \\ V_k^2 &\stackrel{\text{def}}{=} \sup \mathcal{E}_k \left(-\frac{6(1+\omega)^2(1+t)}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} \hat{w}_m(s) ds - \frac{6(1+\omega)^2}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} \hat{w}_m(s) ds \right) \\ V_k^3 &\stackrel{\text{def}}{=} \sup \mathcal{E}_k \left(-\frac{3(1+\omega)^3(1+t)}{2t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} ds - \frac{3(1+\omega)^3}{2(1-t)} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} ds \right). \end{aligned}$$

By combining the above estimates, we obtain the following result.

LEMMA 8.1. *Let V_k^i be as above, and define the vector-valued $Z(r) \in \mathbf{R}^{m+1}$ by*

$$Z(r) \stackrel{\text{def}}{=} \left(\sup |I_m - A_m^\dagger \mathcal{M}(\phi)| \mathbf{1} + |A_m^\dagger| V^1 \right) r + |A_m^\dagger| V^2 r^2 + |A_m^\dagger| V^3 r^3,$$

then for all $w_1, w_2 \in B_\omega(r)$

$$|(\Pi_m z(w_1, w_2))_k| = |DT(\hat{w}_m + w_1)w_2| \leq Z_k(r) \quad \text{for } k = 0, \dots, m.$$

Finally, it remains to derive a bound Z_∞ on $\|\Pi_\infty(T(\hat{w}_m) - \hat{w}_m)\|_\infty$. Similar to (7.3), we observe that

$$\begin{aligned} \|\Pi_\infty z(w_1, w_2)\|_\infty &= r \|\Pi_\infty D(I - G)(\hat{w}_m + r\tilde{w}_1)\tilde{w}_2\|_\infty \\ &= r \sup_{t \in [0,1]} |\Pi_\infty \partial_1 \hat{\eta}(0, t)| \\ &\leq r \max_{0 \leq k \leq m-1} \left\{ \frac{1}{8} (t_{k+1} - t_k) \sup_{t \in [t_k, t_{k+1}]} |\partial_1 \partial_2^2 \hat{\eta}(0, t)| \right\}, \end{aligned}$$

where $\hat{\eta}(\tau, t)$ is given by

$$\begin{aligned} \hat{\eta}(\tau, t) &= (I - G)(\hat{w}_m(t) + r\tilde{w}_1(t) + \tau\tilde{w}_2(t)) \\ &= -\frac{1}{2} \frac{1+t}{t^2} \int_0^t \frac{s^3(1-s)^3}{\log s} \xi_s(\tau) ds \\ &\quad - \frac{1}{2} \frac{1}{1-t} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} \xi_s(\tau) ds, \end{aligned}$$

with $\xi_s(\tau) = (\hat{w}_m(s) + r\tilde{w}_1(s) + \tau\tilde{w}_2(s))^3$. Now note that

$$|\xi'_s(0)| = |3(\hat{w}_m(s) + r\tilde{w}_1(s))^2 \tilde{w}_2(s)| \leq \chi(r, s),$$

with

$$\begin{aligned} \chi(r, s) &\stackrel{\text{def}}{=} 3(1+\omega)\hat{w}_m(s)^2 + 6(1+\omega)^2\hat{w}_m(s)r + 3(1+\omega)^3r^2 \\ &\stackrel{\text{def}}{=} \chi^0(s) + \chi^1(s)r + \chi^2(s)r^2. \end{aligned}$$

Here we have used once again the fact that \hat{w}_m is positive to slightly simplify the estimates. It is straightforward to arrive at the estimate

$$|\partial_1 \partial_2^2 \eta(0, t)| \leq \sum_{i=0}^2 r^i [\hat{h}_1^i(t) + \hat{h}_2^i(t) + \hat{h}_3^i(t)]$$

with (cf. (7.4))

$$\begin{aligned} \hat{h}_1^i(t) &\stackrel{\text{def}}{=} -\frac{3+t}{t^4} \int_0^t \frac{s^3(1-s)^3}{\log s} \chi^i(s) ds \\ \hat{h}_2^i(t) &\stackrel{\text{def}}{=} -\frac{1}{(1-t)^3} \int_t^1 \frac{s(1+s)(1-s)^4}{\log s} \chi^i(s) ds \\ \hat{h}_3^i(t) &\stackrel{\text{def}}{=} -\frac{(1-t)^2}{\log t} \chi^i(t). \end{aligned}$$

For \hat{h}_1^i and \hat{h}_2^i we use the interval arithmetic evaluation of logarithmic integrals, whereas to estimate \hat{h}_3^i we use the same arguments as those used to estimate h_3 in Section 7. In particular, we use the notation $\hat{h}_3^i(t) \in \mathcal{F}_{[t_k, t_{k+1}]}(\hat{h}_3^i)$ for $t \in [t_k, t_{k+1}]$. The resulting estimate is summarized in the following lemma.

LEMMA 8.2. *Let $Z_\infty(r)$ be defined by*

$$Z_\infty(r) \stackrel{\text{def}}{=} \sum_{i=0}^2 r^{i+1} \max_{0 \leq k \leq m-1} \left\{ \frac{1}{8} |t_{k+1} - t_k|^2 \cdot \sup \left| \mathcal{E}_{[t_k, t_{k+1}]}(\hat{h}_1^i + \hat{h}_2^i) + \mathcal{F}_{[t_k, t_{k+1}]}(\hat{h}_3^i) \right| \right\},$$

then

$$\|\Pi_\infty(T(\hat{w}_m) - \hat{w}_m)\|_\infty \leq Z_\infty(r).$$

9. Computational aspects of the proof of Theorem 1.1. In Lemmas 7.1, 7.2, 8.1 and 8.2 we have provided estimates that we can use in order to calculate the radii polynomials. However, we still need to show that the resulting polynomials indeed become simultaneously negative for some $r > 0$. Note that in order to do so, it is crucial that the constant terms of these polynomials are small and that the coefficients of the linear terms are negative, i.e., the Y -bounds and the first term of the Z -bounds must be close to zero.

We first determine a suitable approximate solution \hat{w}_m . We have found it most convenient to apply a standard ODE shooting method to the equation for $\tilde{v}(t) = (1-t)w(t) = v(t)/t$, i.e., to $\tilde{v}'' + \frac{3}{t}\tilde{v}' = \frac{\tilde{v}^3}{\log t}$, see the mathematica file `Mesh.nb` in [20]. Via the change of variables $w(t) = \tilde{v}(t)/(1-t)$ this leads to an approximate solution of (2.3).

The next step in establishing our computer-aided proof is to construct an appropriate mesh, $\Delta_m = \{t_k\}_{0 \leq k \leq m}$. As it turns out, a uniform mesh of 700 points is enough to complete the proof. By increasing the number of points, a higher accuracy, and thus a smaller r , can be achieved at the cost of increased computing time. In the code [20] used for the proof of Theorem 1.1, we take 1001 mesh points, i.e. $m = 1000$.

In general, one may reduce the number of necessary mesh points by choosing a non-uniform grid. In practice, it is the Y_∞ bound, which is obtained via a bound on $g''(t)$ (see (7.3) and Lemma 7.2), that poses the largest obstacle to completing the proof. Hence, by choosing a non-uniform mesh that is finest for those parts of the domain where $|g''(t)|$ is largest, a similar accuracy could be obtained at a decreased computing time. However, a uniform grid works fine for the specific problem considered in the present paper.

Now that we have chosen a grid, we evaluate the approximation solution at the mesh points to find an approximate solution to $G^m(w) = 0$, see again the Mathematica file `Mesh.nb` [20]. Next, we refine the approximate solution using a standard Newton method `Newton.m` in [20], in order to find an approximate zero \hat{w}_m of G^m . This piecewise linear approximation \hat{w}_m has been stored in the file `data1001.mat` in [20]. The graph of \hat{w}_m , as well as the corresponding approximate solution to (1.6), can be found in Figure 2.

As mentioned earlier, we fix the parameter ω that we use to define our closed balls $B_\omega(r)$ in $C[0, 1]$. By means of trial and error, we have established that taking $\omega \in [0.012, 0.024]$ allows us to complete the proof. In particular, we choose $\omega = 0.02$.

The remainder of the proof is now given in the form of a MATLAB program `GLProof.m` that can be found in [20] (it uses the `Intlab` package [14]). In this code, $G^m(\hat{w}_m)$, $DG^m(\hat{w}_m)$, A_m^\dagger , Y_k and Y_∞ are determined using interval arithmetic, as are the coefficients of $Z_k(r)$ and $Z_\infty(r)$.

This program computes (guaranteed by interval arithmetic) that for $r = 1.88 \cdot 10^{-2}$ all radii polynomials are (simultaneously) negative. It follows from Theorem 4.2 that the first part of Theorem 1.1 holds and by Lemma 5.2 that the second part also holds. It should be noted that although the code chooses a relatively small r to verify the negativity of the radii-polynomials, this is not the only value of r for which the proof goes through. Indeed, all $r \in [1.87 \cdot 10^{-2}, 2.27 \cdot 10^{-2}]$ are suitable. A smaller value of r

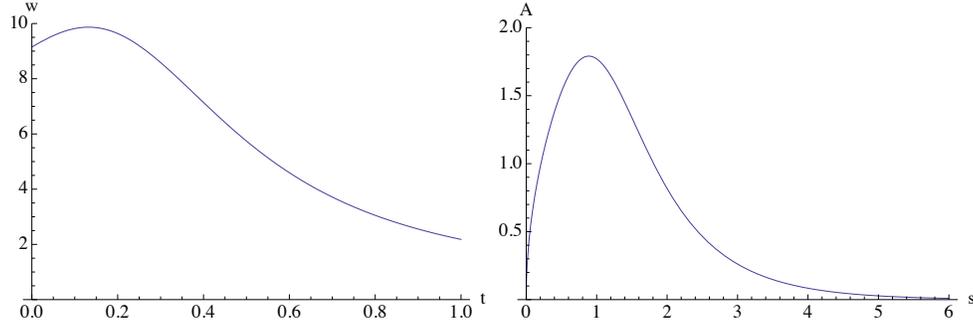


FIG. 2. On the left the graph of the approximate solution $\hat{w}_m \in S_m$ as a function of t . On the right the graph of the approximate solution of (1.6) corresponding to \hat{w}_m through (2.6).

yields slightly more information about how close the solution lies to the numerical, piecewise linear approximation.

Finally, we prove the final assertion of Theorem 1.1, i.e., we check that \hat{w} is positive by invoking Lemma 4.3 and by having the code verify that $\min \hat{w}_m > r(1 + \omega)$ using interval arithmetic.

Appendix. Exponential dichotomy.

For completeness we present a detailed proof of the exponential dichotomy for (5.4) in this Appendix. We will use that \hat{w} is bounded, hence $\hat{u}(s) \leq \hat{c}e^{-s}$ for some $\hat{c} > 0$ (in fact any rate of exponential decay suffices). We start with a preliminary lemma.

LEMMA A.1. *Let \hat{u} be bounded and let \tilde{u} be a solution of*

$$(A.1) \quad \tilde{u}''(s) = \tilde{u}(s) \left(1 - \frac{3\hat{u}^2}{s} \right), \quad s > 0.$$

Then either \tilde{u} is bounded as $s \rightarrow \infty$ and satisfies, for all $\epsilon > 0$,

$$\lim_{s \rightarrow \infty} |\tilde{u}(s)|e^{(1-\epsilon)s} = 0 \quad \text{and} \quad \lim_{s \rightarrow \infty} |\tilde{u}'(s)|e^{(1-\epsilon)s} = 0,$$

or for all $\epsilon > 0$ there exist $c > 0$ and $s_0 > 0$ such that

$$|\tilde{u}(s)| \geq ce^{(1-\epsilon)s} \quad \text{for all } s > s_0.$$

Proof. We fix $0 < \epsilon < 1$. First note that we can rewrite (5.4) as a two-dimensional first order system

$$(A.2) \quad x'(s) = M(s)x(s), \quad \text{where} \quad M(s) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ 1 - \frac{3\hat{u}(s)^2}{s} & 0 \end{pmatrix}.$$

Note that since $\frac{3\hat{u}(s)^2}{s} \rightarrow 0$ as $s \rightarrow \infty$, for all $0 < \epsilon < 1$ there exists an s_0 such that if $\lambda_{\pm}(s)$ denotes the (time-dependent) eigenvalues of $M(s)$, then

$$\lambda_+(s) > (1 - \epsilon) > 0 \quad \text{and} \quad \lambda_-(s) < -(1 - \epsilon) < 0,$$

for all $s > s_0$. Furthermore, we have for any matrix norm that $\|M(s)\| < C$ as $s \rightarrow \infty$ for some constant $C > 0$ (dependent on the norm used). It follows from [4,

proposition 1] that there exist $s'_0 > 0$, $0 < \epsilon' < \epsilon$, constants $k_1, k_2 > 0$ and a fundamental matrix $X(s)$ for (A.2), such that for all $s > s'_0$

$$\begin{aligned}\|X(s)PX^{-1}(s'_0)\| &\leq k_1 e^{-(1-\epsilon')(s-s'_0)} \\ \|X(s'_0)(1-P)X^{-1}(s)\| &\leq k_2 e^{-(1-\epsilon')(s-s'_0)},\end{aligned}$$

where $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. From the second inequality it follows that for $\xi \in \mathbf{R}^2$

$$\begin{aligned}|X(s'_0)(1-P)\xi| &= |X(s'_0)(1-P)X^{-1}(s)X(s)(1-P)\xi| \\ &\leq |X(s'_0)(1-P)X^{-1}(s)| |X(s)(1-P)\xi| \\ &\leq k_2 e^{-(1-\epsilon')(s-s'_0)} |X(s)(1-P)\xi|.\end{aligned}$$

We infer that for all $\xi \in \mathbf{R}^2$ there exist constants $c_1, c_2 \geq 0$ such that

$$\begin{aligned}|X(s)P\xi| &\leq c_1 e^{-(1-\epsilon')s}, \\ |X(s)(1-P)\xi| &\geq c_2 e^{(1-\epsilon')s},\end{aligned}$$

and if $(1-P)\xi \neq 0$, then $c_2 > 0$.

Since any solution of (A.2) corresponds to $X(s)\xi$ for some $\xi \in \mathbf{R}^2$, we see that any bounded (as $s \rightarrow \infty$) solution must correspond to a ξ that satisfies $(1-P)\xi = 0$. Hence, any such bounded solution satisfies $|x(s)|e^{(1-\epsilon)s} \leq c_1 e^{-(\epsilon-\epsilon')s}$ for some $c_1 > 0$ and all $s > s'_0$, therefore $\lim_{s \rightarrow \infty} |x(s)|e^{(1-\epsilon)s} = 0$. On the other hand, we see that unbounded (as $s \rightarrow \infty$) solutions to (A.2) must satisfy $|x(s)| \geq c_2 e^{(1-\epsilon)s} = 0$ for some $c_2 > 0$ and all $s > s'_0$. \square

This exponential dichotomy can be strengthened as follows.

LEMMA A.2. *Let $\hat{u}(s) \leq \hat{c}e^{-s}$ for some $\hat{c} > 0$, and let \tilde{u} be a solution of (A.1) such that for some $\epsilon_0 > 0$, $c_0 > 0$ and $s_0 > 0$*

$$(A.3) \quad |\tilde{u}(s)| \leq c_0 e^{(1-\epsilon_0)s} \quad \text{for all } s > s_0.$$

Then there exists a constant $C \in \mathbf{R}$ such that

$$\lim_{s \rightarrow \infty} \tilde{u}(s)e^s = C.$$

Proof. First we note that there exists a $k > 0$ such that $\frac{3\hat{u}(s)^2}{s} \leq ke^{-2s}$ for sufficiently large s . Define the (a priori possibly unbounded) function $p(s) = e^s \tilde{u}(s)$, which satisfies

$$\frac{d}{ds} e^{-2s} p'(s) = -e^{-2s} \frac{3\hat{u}(s)^2}{s} p(s).$$

Choose $0 < \epsilon < 1$ fixed, then by (A.3) and Lemma A.1 there exists a constant $c > 0$ such that $p(s) \leq ce^{\epsilon s}$. Moreover, by Lemma A.1, $\tilde{u}(s)e^{(1-\epsilon)s} \rightarrow 0$ and $\tilde{u}'(s)e^{(1-\epsilon)s} \rightarrow 0$ as $s \rightarrow \infty$. From the identity $p'(s) = \tilde{u}'(s)e^s + \tilde{u}(s)e^s$ we then infer that $p'(s)e^{-\epsilon s} \rightarrow 0$, hence $p'(s)e^{-2s} \rightarrow 0$ as $s \rightarrow \infty$. This allows us to write

$$e^{-2s} p'(s) = - \int_s^\infty \frac{d}{d\tilde{s}} e^{-2\tilde{s}} p'(\tilde{s}) d\tilde{s} = \int_s^\infty e^{-2\tilde{s}} \frac{3\hat{u}(\tilde{s})^2}{\tilde{s}} p(\tilde{s}) d\tilde{s},$$

hence

$$|p'(s)| \leq e^{2s} \int_s^\infty e^{-2\tilde{s}} \frac{3\hat{u}(\tilde{s})^2}{\tilde{s}} p(\tilde{s}) d\tilde{s} \leq e^{2s} \int_s^\infty e^{-2\tilde{s}} \cdot ke^{-2\tilde{s}} \cdot ce^{\epsilon\tilde{s}} d\tilde{s} \leq Ke^{(-2+\epsilon)s},$$

for some constant $K > 0$. Next, note that for $b > a$

$$|p(b) - p(a)| \leq \int_a^b |p'(\tilde{s})| d\tilde{s} \leq K \int_a^b e^{(-2+\epsilon)\tilde{s}} d\tilde{s} \leq \frac{K}{2-\epsilon} \left(e^{(-2+\epsilon)a} - e^{(-2+\epsilon)b} \right).$$

It follows that when s_n is a strictly increasing sequence with $s_n \rightarrow \infty$, then $p(s_n)$ is a Cauchy sequence with limit $C \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} p(s_n)$. Now let $\epsilon' > 0$ be arbitrary and choose $s_{\epsilon'}$ such that $\frac{K}{2-\epsilon} e^{(-2+\epsilon)s_{\epsilon'}} < \frac{1}{2}\epsilon'$, and n such that $s_n > s_{\epsilon'}$ and $|p(s_n) - C| \leq \frac{1}{2}\epsilon'$. Then we have that

$$|p(s) - C| \leq |p(s) - p(s_n)| + |p(s_n) - C| < \frac{K}{2-\epsilon} e^{(-2+\epsilon)s_n} + \frac{1}{2}\epsilon' < \epsilon',$$

for all $s > s_n > s_{\epsilon'}$. Since $\epsilon' > 0$ is arbitrary, the result follows. \square

REFERENCES

- [1] CAPD: *Computer assisted proofs in dynamics, a package for rigorous numerics*. <http://capd.ii.uj.edu.pl/>.
- [2] ZIN ARAI, HIROSHI KOKUBU, AND PAWEŁ PILARCZYK, *Recent development in rigorous computational methods in dynamical systems*, Japan J. Indust. Appl. Math., 26 (2009), pp. 393–417.
- [3] GIANNI ARIOLI AND HANS KOCH, *Computer-assisted methods for the study of stationary solutions in dissipative systems, applied to the Kuramoto-Sivashinski equation*, Arch. Ration. Mech. Anal., 197 (2010), pp. 1033–1051.
- [4] W. A. COPPEL, *Dichotomies in stability theory*, Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, Berlin-New York, 1978.
- [5] ANNIE CUYT, VIGDIS BREVIK PETERSEN, BRIGITTE VERDONK, HAAKON WAADELAND, AND WILLIAM B. JONES, *Handbook of continued fractions for special functions*, Springer, New York, 2008. With contributions by Franky Backeljauw and Catherine Bonan-Hamada, Verified numerical output by Stefan Becuwe and Cuyt.
- [6] SARAH DAY, JEAN-PHILIPPE LESSARD, AND KONSTANTIN MISCHAIKOW, *Validated continuation for equilibria of PDEs*, SIAM J. Numer. Anal., 45 (2007), pp. 1398–1424.
- [7] MARCIO GAMEIRO AND JEAN-PHILIPPE LESSARD, *Analytic estimates and rigorous continuation for equilibria of higher-dimensional PDEs*, J. Differential Equations, 249 (2010), pp. 2237–2268.
- [8] STUART P. HASTINGS AND J. BRYCE MCLEOD, *Classical methods in ordinary differential equations*, vol. 129 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2012. With applications to boundary value problems.
- [9] ALLAN HUNGRIA, JEAN-PHILIPPE LESSARD, AND JASON D. MIRELES JAMES, *Rigorous numerics for analytic solutions of differential equations: the radii polynomial approach*, 2014. Preprint.
- [10] OSCAR E. LANFORD, III, *A computer-assisted proof of the Feigenbaum conjectures*, Bull. Amer. Math. Soc. (N.S.), 6 (1982), pp. 427–434.
- [11] DAVID LLOYD AND BJÖRN SANDSTEDTE, *Localized radial solutions of the Swift-Hohenberg equation*, Nonlinearity, 22 (2009), pp. 485–524.
- [12] SCOTT G. MCCALLA AND BJÖRN SANDSTEDTE, *Spots in the Swift-Hohenberg equation*, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 831–877.
- [13] ZEEV NEHARI, *On a nonlinear differential equation arising in nuclear physics*, Proc. Roy. Irish Acad. Sect. A, 62 (1963), pp. 117–135 (1963).
- [14] S.M. RUMP, *INTLAB - INTerval LABoratory*, in Developments in Reliable Computing, Tibor Csendes, ed., Kluwer Academic Publishers, Dordrecht, 1999, pp. 77–104. <http://www.ti3.tuhh.de/rump/>.
- [15] GERALD H. RYDER, *Boundary value problems for a class of nonlinear differential equations*, Pacific J. Math., 22 (1967), pp. 477–503.

- [16] ARND SCHEEL, *Radially symmetric patterns of reaction-diffusion systems*, Mem. Amer. Math. Soc., 165 (2003), pp. viii+86.
- [17] J. SWIFT AND P.C. HOHENBERG, *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A, 15 (1977), pp. 319–328.
- [18] WARWICK TUCKER, *A rigorous ODE solver and Smale’s 14th problem*, Found. Comput. Math., 2 (2002), pp. 53–117.
- [19] JAN BOUWE VAN DEN BERG, ANDRÉA DESCHÊNES, JEAN-PHILIPPE LESSARD, AND JASON D. MIRELES JAMES, *Coexistence of hexagons and rolls*, 2014. Preprint.
- [20] JAN BOUWE VAN DEN BERG, CHRIS GROOTHEDDE, AND J.F. WILLIAMS, *Matlab and mathematica code for rigorous computation of a radially symmetric localised solution in a ginzburg-landau problem*. <http://www.few.vu.nl/~cge390/code/GinzburgLandau/>.
- [21] JAN BOUWE VAN DEN BERG AND JEAN-PHILIPPE LESSARD, *Chaotic braided solutions via rigorous numerics: chaos in the Swift-Hohenberg equation*, SIAM J. Appl. Dyn. Syst., 7 (2008), pp. 988–1031.
- [22] JAN BOUWE VAN DEN BERG, JASON D. MIRELES-JAMES, JEAN-PHILIPPE LESSARD, AND KONSTANTIN MISCHAIKOW, *Rigorous numerics for symmetric connecting orbits: even homoclinics of the Gray-Scott equation*, SIAM J. Math. Anal., 43 (2011), pp. 1557–1594.
- [23] NOBITO YAMAMOTO, *A numerical verification method for solutions of boundary value problems with local uniqueness by Banach’s fixed-point theorem*, SIAM J. Numer. Anal., 35 (1998), pp. 2004–2013 (electronic).