Homoclinic solutions for Swift-Hohenberg and suspension bridge type equations

D. Smets and J.B. van den Berg*

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Abstract: We establish the existence of homoclinic solutions for a class of fourth order equations which includes the Swift-Hohenberg model and the suspension bridge equation. In the first case, the nonlinearity has three zeros, corresponding to a double-well potential, while in the second case the nonlinearity is asymptotically constant on one side. The Swift-Hohenberg model is a higher order extension of the classical Fisher-Kolmogorov model. Its more complicated dynamics give rise to further possibilities of pattern formation. The suspension bridge equation was studied by Chen and McKenna in [4]; we give a positive answer to an open question raised by the authors.

1 Introduction

We investigate a class of fourth order equations possessing a variational structure. These are the Euler-Lagrange equations derived from second order Lagrangian principle. The Lagrangian densities that will be considered are of the form

$$L(u,u',u'') := \frac{1}{2}|u''|^2 - \frac{\beta}{2}|u'|^2 + V(u),$$

where $\beta \in \mathbb{R}$, and the potential V has to satisfy some appropriate conditions. Typical examples are the double-well potential $V(u) = \frac{1}{4}(u^2-1)^2$, the water wave model $V(u) = \frac{1}{3}u^3 - \frac{1}{2}u^2$, and the suspension bridge model $V(u) = e^u - u - 1$. When the parameter β is negative, the corresponding Euler-Lagrange equation

$$u'''' + \beta u'' + V'(u) = 0 \tag{1}$$

is called the extended Fisher-Kolmogorov (eFK) equation, whereas for positive β Equation (1) is referred to as the Swift-Hohenberg equation. Both are considered

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as models for studying nonlinear phenomena like phase transition in various fields: hydrodynamics [24], elasticity and solid mechanics [4], nonlinear optics [1], etcetera. For example, in the suspension bridge model, solutions of (1) with positive β correspond to travelling waves in the suspended structure of the bridge, which travel with speed $\sqrt{\beta}$.

In some sense, the Fisher-Kolmogorov situation ($\beta \leq 0$) is simpler to deal with, since all the terms in the Lagrangian density appear with a positive sign. Instead, the negative sign in front of the $|u'|^2$ term in the Swift-Hohenberg case is rather tricky to manage.

These equations have drawn much attention in recent years, and many different methods have proved to be successful. Concerning the eFK equation, the situation is rather deeply understood. Existence of heteroclinics [15, 9, 10], homoclinics [9, 10] and periodic solutions [18, 11] was proved together with additional features like multibump "chaotic" behaviour. For more background we also refer to the papers [3, 25, 12, 14, 16, 17].

Much less rigorous results exist for the Swift-Hohenberg case $\beta > 0$. Existence of (multibump) periodic solutions was proved by Peletier and Troy [19] in the suspension bridge model, and by Peletier, Troy and van den Berg [20] for the double-well potential, see also [26, 5]. An existence result concerning homoclinic solutions to the suspension bridge equation (corresponding to localised travelling waves in the bridge) was obtained by Chen and McKenna [4] (see also [13]). However, they need to assume rather restrictive conditions on V excluding for example $V(u) = e^u - u - 1$. Existence in this later case was raised as an open question by the authors. Finally, we refer to [21] for homoclinics found in a related constrained minimisation problem, where β acts as the Lagrange multiplier.

In this paper we will prove existence results for homoclinic solutions in Swift-Hohenberg type systems and for localised travelling waves in suspension bridge models. In particular, we will treat the cases $V(u) = e^u - u - 1$ and $V(u) = \frac{1}{4}(u^2-1)^2$.

The methods we use are variational in nature. The solutions are obtained by performing a mountain-pass procedure to the action functional

$$J(u) := \int_{\mathbb{R}} \frac{1}{2}|u''|^2 - \frac{\beta}{2}|u'|^2 + V(u).$$

The Palais-Smale condition is of course not satisfied by J, the first reason being translation invariance. The situation is much worse however, not only because of the negative term in the Lagrangian, but also due to the particular shape of V. Moreover, we have numerical evidence that homoclinics with negative energy (Lagrangian action) do exist, ruling out standard arguments often used. We will avoid these later homoclinics by using a modified problem, and the non-superquadratic shape will be overcome using the monotonicity trick of Struwe [22] (see also [23]). Nevertheless, as the nonlinearity $V'(u)$ in Equation (1) may have multiple zeros, abstract results like those developed by Jeanjean [8, 7] cannot be applied. Finally, we mention that in the special case where $V(u) = -\frac{1}{n}u^n + \frac{1}{2}u^2$ for some $n > 2$, the

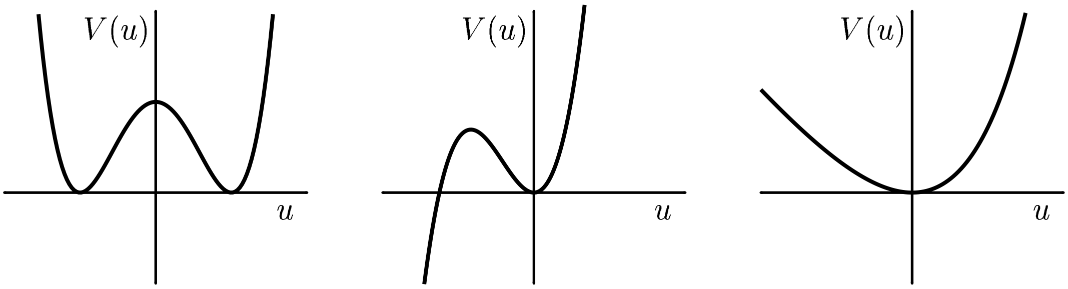

Figure 1: Typical shapes of potentials under consideration.

Palais-Smale condition can be established fairly directly, see [2], but this method seems restricted to this exceptional class of potentials, or at least to those for which $\sigma V(u) - V'(u)u$ has the right constant sign for some $\sigma > 2$ (this is sometimes called the Rabinowitz condition).

The linearised equation around the bottom well(s) of V plays a crucial role in passing to the limit. As expected, we restrict our analysis to the solutions which are homoclinic to an equilibrium of saddle-focus character, as centers will in general not allow homoclinic solutions. We treat nonlinearities $V'(u)$ with one, two and three zeros, typical examples of the shapes of the potentials being depicted in Figure 1.

In all cases we prove the existence of a homoclinic solution for almost all positive values of β for which the equilibrium point is of saddle-focus type. For a precise formulation of the results and the conditions on the potentials/nonlinearities we refer to Sections 4 to 6. Although our results give a clear insight in the generality of the existence of homoclinic solutions to saddle-foci, several questions remain open. First, one would like to fill the gap of measure zero in the set of β values for which existence has been established. Second, numerics show that there are multiple solutions and it would certainly be nice to be able to prove this, thus obtaining more understanding of the global picture.

In Section 2 we present the method of proof on the basis of the example $V(u) = \frac{1}{4}(u^2 - 1)^2$. The general statement of this result for nonlinearities with three zeros is formulated in Section 3, while Section 4 deals with double-well potentials where only one of the minima is a saddle-focus (and the other one is a center). Finally, in Sections 5 and 6 we consider nonlinearities with two zeros and one zero respectively.

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2 The Swift-Hohenberg equation

This section is devoted to the proof of existence of homoclinics for the classical Swift-Hohenberg equation. The result extends to equations with similar nonlinearities, as will be made precise in Theorems 7 and 8.

To begin with, we recall the equation to be solved, and we introduce the functional settings associated with it. The equation

$$u'''' + \beta u'' + V'(u) = 0, \qquad V(u) = \frac{1}{4}(u^2 - 1)^2, \tag{2}$$

has three stationary solutions: 0, 1 and -1. The last two are the bottom wells of the potential V. The potential energy has been normalised so that the bottom wells have zero energy. The solutions that will be found are homoclinic to either 1 or -1. By symmetry, it is sufficient to consider the case where the limit is 1. For convenience, we perform the change of variable $v \to u - 1$. In the new variable, the equation becomes

$$v'''' + \beta v'' + v^3 + 3v^2 + 2v = 0. \tag{3}$$

Numerics indicates that this equation possesses various families of homoclinics orbits, together with heteroclinic connections. Due to this complicated structure, we will need to modify the potential V in order to single out a particular family. Basically, the family that we obtain does not pass through the second bottom well at $v = -2$, so that we can assume that

$$V(v) := \begin{cases} \frac{1}{4}v^4 + v^3 + v^2 & \text{if } v > -2 \\ 0 & \text{otherwise.} \end{cases}$$

Note that this new potential $V(v)$ is of class $\mathcal{C}^1$. We introduce the action functional

$$J_\beta(v) := \int_{\mathbb{R}} \frac{(v'')^2}{2} - \beta\frac{(v')^2}{2} + V(v),$$

on the Hilbert space $H^2(\mathbb{R})$, equipped with the standard norm.

Lemma 1. *The functional J_β is of class $\mathcal{C}^1$ on $H^2(\mathbb{R})$, and if $v \in H^2(\mathbb{R})$, $v > -2$, is a critical point of J_β, then v is a classical solution of equation* (3) *such that*

$$v'v''' - \frac{1}{2}(v'')^2 + \frac{\beta}{2}(v')^2 + V(v) = 0 \tag{4}$$

identically.

Proof. The regularity part of the functional is quite standard, as well as the fact that any critical point is a classical solution of the Euler-Lagrange equation. The only thing that needs to be checked is Equation (4).

First notice that from Equation (3) we infer that for any critical point v, one has, for some $C > 0$,

$$|v''''|_{L^2}^2 \le C\left(|v''|_{L^2}^2 + |v|_{L^2}^2\right),$$

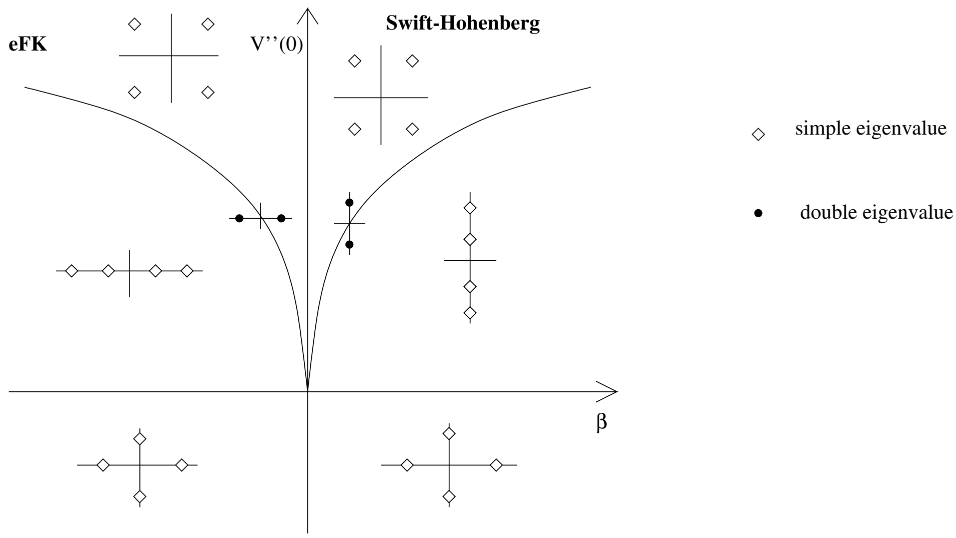

Figure 2: Eigenvalues for the linearisation.

so that in fact $v \in H^4(\mathbb{R})$. By the Sobolev embedding theorem, v, v', v'' and v''' tend to zero as t tends to $\pm\infty$.

Multiplying Equation (3) by v' yields:

$$\frac{d}{dt}\left(v'v''' - \frac{1}{2}(v'')^2 + \frac{\beta}{2}(v')^2 + V(v)\right) = 0, \tag{5}$$

so that the left hand side of Equation (4) is constant. Taking its limit as t tends to plus or minus infinity, one finds that this constant is zero. This ends the proof. ☐

Remark 2. *Notice that the left hand side of Equation* (4) *is nothing but the Hamiltonian for the Hamiltonian system corresponding to the Lagrangian density, and Equation* (5) *is the conservation law for the Hamiltonian.*

The linearisation of Equation (3) around each of the bottom wells gives

$$v'''' + \beta v'' + 2v = 0,$$

whose characteristic eigenvalues satisfy

$$z^2 = \frac{-\beta \pm \sqrt{\beta^2 - 8}}{2}.$$

Thus, the threshold $\beta = \sqrt{8}$ corresponds to the upper limit for saddle-focus equilibria. The complete picture of the linearised equation is depicted in Figure 2.

In the following, we will restrict to the case $0 < \beta < \sqrt{8}$.

Lemma 3. *There exist constants $\varepsilon > 0$ and $\delta > 0$ such that*

$$J_\beta(v) \geq \varepsilon \|v\|^2 \quad \textit{for} \quad \|v\| < \delta,$$

with a uniform lower bound on ε and δ for β in compact subsets of $[0, \sqrt{8})$.

Proof. Let $a > 0$ such that $\beta^2 < 4(2-a)$. There exists an $r > 0$ such that $|v| < r$ implies $V(v) > \frac{2-a}{2}|v|^2$. By the Sobolev embedding theorem, there exists a $\delta > 0$ such that

$$\|v\| < \delta \Rightarrow |v|_{L^\infty} < r.$$

Denote by $\mathcal{F}$ the Fourier transform. Then, if $\|v\| < \delta$, one has for some small $\varepsilon > 0$

$$\begin{aligned} J_\beta(v) &\geq \int_{\mathbb{R}} \frac{1}{2}(v'')^2 - \frac{\beta}{2}(v')^2 + \frac{2-a}{2}v^2\,dx \\ &= \int_{\mathbb{R}} \frac{1}{2}(\xi^4 - \beta\xi^2 + (2-a))(\mathcal{F}(v))^2\,d\xi \\ &\geq \int_{\mathbb{R}} \varepsilon(\xi^2+1)^2(\mathcal{F}(v))^2\,d\xi \\ &\geq \varepsilon\|v\|^2, \end{aligned}$$

where ε can be chosen independently of β as long as β does not approach $\sqrt{8}$. This is the required estimate. □

Lemma 4. *There exists an $e \in H^2(\mathbb{R})$, $\|e\| > \delta$, such that $J_\beta(e) < 0$. Moreover, e can be chosen independently of β for β in compact subsets of $(0, \sqrt{8}]$.*

Proof. Let $f \in C_0^\infty(\mathbb{R})$ be any nowhere positive function, $f \not\equiv 0$. Define $f_\lambda(x) := f(\lambda x)$ so that

$$\int_{\mathbb{R}} (f_\lambda'')^2 = \lambda^3 \int_{\mathbb{R}} (f'')^2 \qquad \text{and} \qquad \int_{\mathbb{R}} (f_\lambda')^2 = \lambda \int_{\mathbb{R}} (f')^2.$$

Thus, if $\lambda > 0$ is sufficiently small,

$$\int_{\mathbb{R}} \frac{1}{2}(f_\lambda'')^2 - \frac{\beta}{2}(f_\lambda')^2 := -a_0 < 0,$$

and, thanks to the cut-off of V, for fixed λ one has

$$J_\beta(Cf_\lambda) = -a_0\,C^2 + O(1), \quad \text{as } C \to \infty,$$

which ends the proof. Clearly, C and λ can be chosen independently of β as long as β does not tend to zero. □

Following Lemmas 3 and 4, we see that J_β has the so-called mountain pass geometry (see e.g. [27]). We thus define the mountain pass levels:

$$c_\beta := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_\beta(\gamma(t)),$$

where $\Gamma = \{\gamma : [0,1] \to H^2(\mathbb{R}) \;\text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = e\}$.
Clearly, the function $\beta \to c_\beta$ is positive and decreasing. It is thus almost everywhere differentiable. The derivative of c_β with respect to β is denoted by c_β'.

By a Palais-Smale sequence for J_β we mean a sequence w_n such that $J_\beta'(w_n) \to 0$ strongly, and such that $J_\beta(w_n)$ is bounded.

Proposition 5. *Let $0 < \beta < \sqrt{8}$ such that c'_β exists. Then there exists a Palais-Smale sequence (v_n) for J_β which satisfies the estimates:*

1. $\frac{1}{2}|v_n'|_{L^2}^2 \leq -c'_\beta + 1$;
2. $\frac{1}{2}|v_n''|_{L^2}^2 \leq c_\beta - \beta c'_\beta + \beta + o(1), \quad$ *as* $n \to \infty$;
3. $J_\beta(v_n) = c_\beta + o(1), \quad$ *as* $n \to \infty$.

Proof. First observe that by definition of J_β, estimate 2 is a direct consequence of estimates 1 and 3 and the positivity of V. Define

$$S := \left\{ u \in H^2(\mathbb{R}) \text{ such that } \frac{1}{2}|u'|_{L^2}^2 \leq -c'_\beta + \frac{1}{2} \right\}.$$

If the assertion is false, then clearly there exists $0 < \varepsilon < c_\beta/2$ such that

$$\forall u \in H^2(\mathbb{R}), \ \left(c_\beta - 2\varepsilon \leq J_\beta(u) \leq c_\beta + 2\varepsilon \text{ and } \operatorname{dist}(u, S) \leq \frac{1}{2} \right) \Rightarrow \|J'_\beta(u)\| > 32\varepsilon.$$

Indeed, otherwise, as ε goes to zero, one would obtain a Palais-Smale sequence satisfying 1,2 and 3.

Using the quantitative deformation lemma in [27] (Lemma 1.14 with $\delta = 1/4$), we obtain a flow $\eta \in \mathcal{C}([0,1] \times H^2(\mathbb{R}), H^2(\mathbb{R}))$ such that:

(a) $\eta(s, u) = u$ if $s = 0$ or if $u \notin J_\beta^{-1}(c_\beta - 2\varepsilon, c_\beta + 2\varepsilon)$ or if $\operatorname{dist}(u, S) > 1/2$;

(b) $J_\beta(\eta(s, u))$ is decreasing in s;

(c) $(u \in S$ and $J_\beta(u) \leq c_\beta + \varepsilon) \Rightarrow J_\beta(\eta(1, u)) \leq c_\beta - \varepsilon$.

Take an increasing sequence $\beta_n \nearrow \beta$ and choose n sufficiently large so that

- $\frac{c_{\beta_n} - c_\beta}{\beta - \beta_n} \leq -c'_\beta + \frac{1}{4}$,
- $c_{\beta_n} + \frac{1}{8}(\beta - \beta_n) \leq c_\beta + \varepsilon$.

Let $\gamma_n(\cdot)$ be a path in Γ (see above for the definition) satisfying:

$$\max_{t \in [0,1]} J_{\beta_n}(\gamma_n(t)) \leq c_{\beta_n} + \frac{1}{8}(\beta - \beta_n).$$

Such a path obviously exists by definition of c_{β_n}. If $t \in [0,1]$ is such that $J_\beta(\gamma_n(t)) \geq c_\beta - \frac{1}{8}(\beta - \beta_n)$ then (by the definition of J)

$$\frac{1}{2}|\gamma_n(t)'|_{L^2}^2 = \frac{J_{\beta_n}(\gamma_n(t)) - J_\beta(\gamma_n(t))}{\beta - \beta_n} \leq \frac{c_{\beta_n} - c_\beta}{\beta - \beta_n} + \frac{1}{4} \leq -c'_\beta + \frac{1}{2},$$

so that $\gamma_n(t) \in S$. Also,

$$J_\beta(\gamma_n(t)) \le J_{\beta_n}(\gamma_n(t)) \le c_{\beta_n} + \frac{1}{8}(\beta - \beta_n) \le c_\beta + \varepsilon.$$

Let $\overline{\gamma}_n(t) := \eta(1, \gamma_n(t))$. Clearly, $\overline{\gamma}_n(0) = \eta(1,0) = 0$ and $\overline{\gamma}_n(1) = \eta(1,e) = e$ because of $\varepsilon < c_\beta/2$ and property (a). Thus $\overline{\gamma}_n \in \Gamma$.
Using property (c) and the fact that $\frac{1}{8}(\beta - \beta_n) < \varepsilon$, we infer that

$$\max_{t\in[0,1]} J_\beta(\overline{\gamma}_n(t)) \le c_\beta - \frac{1}{8}(\beta - \beta_n),$$

which contradicts the definition of c_β. This ends the proof. □

We will now construct a homoclinic solution thanks to the Palais-Smale sequence constructed above.

First, observe that it is impossible for (v_n) to converge uniformly to 0. Otherwise, one would have

$$\begin{aligned} 0 &\le \frac{1}{2}|v_n''|_{L^2}^2 - \frac{\beta}{2}|v_n'|_{L^2}^2 + |v_n|_{L^2}^2 \\ &= J_\beta(v_n) + o(1)|v_n|_{L^2}^2 \quad \text{as } n \to \infty, \end{aligned}$$

and for $\beta < \sqrt{8}$ this would imply that (v_n) is bounded in $H^2(\mathbb{R})$. But then, since $J_\beta'(v_n) \to 0$,

$$\begin{aligned} 0 < c_\beta &= J_\beta(v_n) - \frac{1}{2}\langle J_\beta'(v_n), v_n\rangle + o(1) \\ &= \int_{\mathbb{R}} [V(v_n) - \frac{1}{2}V'(v_n)v_n] + o(1) \\ &= o(1), \quad \text{as } n \to \infty, \end{aligned}$$

which clearly is a contradiction.

Assume that (the other case will be dealt with shortly)

$$\limsup_{n\to\infty}(\min_{x\in\mathbb{R}} v_n(x)) \ge -1.$$

Then, as $V(v) \ge (v^2)/4$ for $v \ge -1$, we deduce from Proposition 5 that $\sup |v_n|_{L^2}^2 < \infty$, and (v_n) is bounded in $H^2(\mathbb{R})$. Let $\tau_n \in \mathbb{R}$ such that $|v_n(\tau_n)| = \max_{x\in\mathbb{R}} |v_n(x)|$ and define $w_n(x) := v_n(\tau_n + x)$. Clearly, (w_n) is a bounded Palais-Smale sequence, so that, going to a subsequence if necessary, we get $w_n \rightharpoonup w$, $w \in H^2(\mathbb{R})$. Since $J'(w) = 0$ (as weak convergence in H^2 implies uniform convergence on compact sets), we conclude that w is a nontrivial homoclinic solution of Equation (3). Notice that w is nontrivial because $w(0) = \lim_{n\to\infty} w_n(0) \neq 0$ as (v_n) does not uniformly converge to zero.

Hence we are left to study the case

$$\limsup_{n\to\infty}(\min_{x\in\mathbb{R}} v_n(x)) < -1.$$

Let $\tau_n \in \mathbb{R}$ be the points such that $v_n(\tau_n) = -1$ and $v_n(x) > -1$ for $x < \tau_n$. Then $w_n(x) := v_n(\tau_n + x)$ satisfies $w_n(0) = -1$. Besides, define $z_n(x) := w_n(x) + 1$. By Proposition 5, the sequence (z_n) is bounded in the Hilbert space

$$\mathcal{Z} := \left\{ z \in H^2_{loc}(\mathbb{R}) \text{ such that } z(0) = 0,\ z' \in L^2(\mathbb{R}),\ z'' \in L^2(\mathbb{R}) \right\}.$$

As in the previous case, we also know that (w_n) is bounded in $H^2(-\infty, 0)$. Thus, there exist a common subsequence (still denoted with n's) and $z \in \mathcal{Z}$ such that $z_n \rightharpoonup z$ in $\mathcal{Z}$ and $w_n \rightharpoonup z-1$ in $H^2(-\infty,0)$. Again, it is easily seen that $w := z-1$ is a solution of Equation (3). ¿From Equation (3), we infer that indeed, $w'''' \in L^2(-\infty, 0)$ so that $w \in H^4(-\infty, 0)$ and

$$w'w''' - \frac{1}{2}(w'')^2 + \frac{\beta}{2}(w')^2 + V(w) \equiv 0 \tag{6}$$

(to see this take the limit in $-\infty$ and use $H^4 \subset C^3_0$).

We assert that $w > -2$. Assume, by contradiction, that w has a local minimum below -2 at a certain t^*. Then by (6), $w'(t^*) = w''(t^*) = 0$. Since it is a minimum, necessarily $w'''(t^*) = 0$. By uniqueness of the Cauchy problem, this implies that $w \equiv w(t^*)$, which is a contradiction. Hence $w > -2$.

As weak convergence in $\mathcal{Z}$ implies uniform convergence on compact sets, for each $R > 0$ we have, V being non-negative,

$$\int_{-R}^{R} V(w) = \lim_{n\to\infty} \int_{-R}^{R} V(w_n) \le \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} V(w_n) < C$$

for some $C > 0$, hence

$$\int_{\mathbb{R}} V(w) < \infty.$$

Since V is non-negative and $\int V(w)$ and $\int |w'|^2$ are bounded, we conclude that either $w(t) \to 0$ as $t \to \infty$, in which case we are done, or $w(t) \to -2$. In the latter case, we infer from Equation (3) that $w + 2 \in H^4(\mathbb{R}^+)$, so that in phase space

$$(w(t), w'(t), w''(t), w'''(t)) \to (-2, 0, 0, 0) \text{ as } t \to \infty.$$

But because $(-2, 0, 0, 0)$ is an equilibria of saddle-focus type, from he Hartman-Grobman Theorem [6] on the conjugacy of linear and nonlinear flows, we get that there exists t^* such that $w(t^*) < -2$. This contradicts the previous claim that $w(t) > -2$ for all t.

Summarising, we have the following:

Theorem 6. *For almost every $\beta \in [0, \sqrt{8}]$, there exists a pair of solutions homoclinic to $+1$ and -1 respectively for the classical Swift-Hohenberg equation* (2). *The homoclinic to -1 does not pass through $+1$ and vice versa.*

3 The case of two saddle-foci

It appears from the proof above that the particular shape of V is not essential. Indeed, only the double well behaviour with two saddle-foci equilibria was used. With a very similar proof one thus obtains:

Theorem 7. *Assume that $V \in \mathcal{C}^2$ satisfies the following hypotheses:*

1. $V(0) = V'(0) = 0$ *and* $V''(0) = \alpha_0 > 0$;
2. $V(u^*) = V'(u^*) = 0$ *and* $V''(u^*) = \alpha_1 > 0$ *for some* $u^* < 0$;
3. $V(u) > 0$ *for each* $u \in (u^*, 0) \cup (0, \infty)$;
4. $\liminf_{u \to \infty} u^2 V(u) > 0$.

Then, for almost every $\beta \in [0, \beta^*]$, there exists a solution homoclinic to 0 for the generalised Swift-Hohenberg equation, where $\beta^* := \sqrt{4 \min(\alpha_0, \alpha_1)}$.

Hypotheses 1 and 2 ensure that the equilibrium points 0 and u^* are non-degenerate. Hypothesis 4 prevents functions with $\int u'^2$ and $\int V(u)$ bounded from tending to ∞. Of course the statement of the previous theorem is trivially adapted in case $u^* > 0$.

4 The case of two minima but only one saddle-focus

It turns out that for the existence of a homoclinic solution (say to 0), the type of the second equilibrium (u^* in the previous section) does not matter. The Hartman-Grobman Theorem concerning the conjugacy of the nonlinear flow with the linear one close to the equilibrium is no longer at hand. Nevertheless, a careful estimate will allow us to conclude as in the previous section.

Theorem 8. *Assume that $V \in \mathcal{C}^2$ satisfies the following hypotheses:*

1. $V(0) = V'(0) = 0$ *and* $V''(0) = \alpha > 0$;
2. $V(u^*) = V'(u^*) = 0$ *and* $V''(u^*) > 0$ *for some* $u^* < 0$;
3. $V(u) > 0$ *for each* $u \in (u^*, 0) \cup (0, \infty)$;
4. $\liminf_{u \to \infty} u^2 V(u) > 0$.

Then, for almost every $\beta \in [0, \beta^*]$, there exists a solution homoclinic to 0 for the generalised Swift-Hohenberg equation (1), where $\beta^* := \sqrt{4\alpha}$.

Proof. Everything goes the same way as in the proof of Theorem 6 until we used Hartman-Grobman Theorem. Using the notation of Section 2, we only have to prove that the alternative $w(t) \to -2$ as $t \to \infty$ (while $w(t) > -2$ for all t) is excluded in order to conclude the proof. This is done in the next lemma. □

Lemma 9. *Under the hypotheses of Theorem 8, there exists a neighbourhood* $\mathcal{U}$ *of* $(u^*, 0, 0, 0)$ *in phase space such that every solution* u *of the generalised Swift-Hohenberg equation that enters* $\mathcal{U}$ *satisfies* $\inf_{t\in\mathbb{R}} u(t) \leq u^*$.

Proof. The case of a saddle-focus was treated before, the only remaining one is then when $(u^*, 0, 0, 0)$ is a center. Let $\pm\sigma_1 i$ and $\pm\sigma_2 i$ be the eigenvalues of the linearised equation at $(u^*, 0, 0, 0)$. Consider the case $\sigma_1 \neq \sigma_2$. We choose a $T^* > 0$ such that $|\exp(i\sigma_1 T^*) + 1| \leq \varepsilon$ and $|\exp(i\sigma_2 T^*) + 1| \leq \varepsilon$ for some small ε that will be fixed later. The linearised equation around u^* can be rewritten in the form $\dot{X} = AX$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -V''(u^*) & 0 & -\beta & 0 \end{pmatrix}.$$

Let $\delta := (2T^* e^{\|A\|T^*})^{-1}$. There exists a constant $r > 0$ such that

$$|u - u^*| \leq r \Rightarrow |V'(u) - V''(u^*)(u - u^*)| \leq \delta |u - u^*|.$$

We claim that the ball $\mathcal{U} := B((u^*, 0, 0, 0), r/K)$ fulfils the assertion of the lemma, where we choose $K > 1$ so large that any orbit entering the ball $\mathcal{U}$ stays in the ball $B((u^*, 0, 0, 0), r)$ for at least a time $2T^*$ (such a K exists since u^* is an equilibrium point).

Let u be a solution of the generalised Swift-Hohenberg equation which enters $\mathcal{U}$ at, say, $t = 0$. Let $t^* \in [0, 2T^*]$ be a point such that $|u(t^*) - u^*| = \max_{t\in[0,2T^*]} |u(t) - u^*|$. In case $u(t^*) < 0$ there is nothing left to prove, so let us assume that $u(t^*) > 0$. Then, up to reversing the time, one has that the orbit stays in $B((u^*, 0, 0, 0), r)$ for t between t^* and $t^* + T^*$. After translation we may write $t^* = 0$. Clearly $|u(t) - u^*| < |u(0) - u^*|$ for $t \in [0, T^*]$.

Write $y := u - u^*$, and $Y = (y, y', y'', y''')$. Let W be the solution of the linear equation $\dot{W} = AW$ with $W(0) = Y(0)$. Define $Z := Y - W$, so that

$$\dot{Z}(t) = AZ(t) + C(t).$$

For $t \in [0, T^*]$ one has

$$|C(t)|_{\mathbb{R}^4} = |V'(u(t)) - V''(u^*)y(t)| \leq \delta |y(t)| \leq \delta |y(0)|.$$

Thus, for $0 \leq t \leq T^*$,

$$|Z(t)|_{\mathbb{R}^4} \leq \int_0^t \|A\| |Z(s)|_{\mathbb{R}^4}\, ds + \delta T^* |y(0)|,$$

and by Gronwall's inequality, for $0 \le t \le T^*$,

$$|Z(t)|_{\mathbb{R}^4} \le \delta T^* e^{T^*\|A\|}|y(0)| \le \frac{1}{2}|y(0)|.$$

Finally, using the fact that $w(0) = y(0)$, one obtains

$$\begin{aligned} y(T^*) &\le w(T^*) + |y(T^*) - w(T^*)| \le w(T^*) + |z(T^*)| \\ &\le -(1 - c(\varepsilon))w(0) + \frac{1}{2}w(0), \end{aligned}$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, choosing ε sufficiently small we conclude that $y(T^*)$ is negative.

The case $\sigma_1 = \sigma_2$ can be dealt with in an analogous manner, and we will not give the details here because, being a non-generic situation, it does not influence the statement of Theorem 8. □

5 The case of two equilibria

In this section we solve the case where possibly just two equilibrium points exists, one of which is a local minimum of the potential. More precisely, we assume that

1. $V(0) = 0$ and $V'(0) > 0$;
2. $V(u^*) = V'(u^*) = 0$ and $V''(u^*) = \alpha > 0$ for some $u^* > 0$;
3. $V(u) > 0$ for each $u \in (0, u^*) \cup (u^*, \infty)$;
4. $\liminf_{u\to\infty} u^2 V(u) > 0$.

We look for homoclinics to u^* (notice the change of notation compared to the previous sections). Again, the choice we make of the relative positions of the two zeros is completely arbitrary. We assume that u^* is an equilibrium of saddle-focus type, i.e. $0 < \beta < \beta^* := \sqrt{4\alpha}$. We are going to find solutions of the Equation (1) that are homoclinic to u^* for almost all values of β in this range.

Define the smooth cut-off function

$$W_\varepsilon(u) = \begin{cases} 0 & \text{if } u < -A_\varepsilon \\ \frac{B_\varepsilon}{2}(u + A_\varepsilon)^2 & \text{if } u \in [-A_\varepsilon, \varepsilon] \\ V(u) & \text{if } u > \varepsilon, \end{cases}$$

with

$$\frac{B_\varepsilon}{2}(\varepsilon + A_\varepsilon)^2 = V(\varepsilon) \quad \text{and} \quad B_\varepsilon(\varepsilon + A_\varepsilon) = V'(\varepsilon).$$

or explicitly:

$$A_\varepsilon = \frac{2V(\varepsilon)}{V'(\varepsilon)} - \varepsilon = \varepsilon + O(\varepsilon^2) \quad \text{and} \quad B_\varepsilon = \frac{V'(\varepsilon)^2}{2V(\varepsilon)} = \frac{V'(0)}{2\varepsilon} + O(1).$$

This way $W_\varepsilon \in \mathcal{C}^1$. By the results of the previous sections, for almost every $\beta \in [0, \beta^*]$ there exists a sequence u_ε (for some $\varepsilon \to 0$) of solutions of

$$u_\varepsilon'''' + \beta u_\varepsilon'' + W_\varepsilon'(u_\varepsilon) = 0,$$

with $u_\varepsilon(x) > -A_\varepsilon$ for all $x \in \mathbb{R}$. Without loss of generality we take a global minimum at the origin. If there exists an $\varepsilon > 0$ such that $u_\varepsilon(0) > \varepsilon$, then u_ε is a solution of the non-truncated equation and we have finished. If not, then

$$u_\varepsilon(0) \in (-A_\varepsilon, \varepsilon), \quad u_\varepsilon'(0) = 0, \quad u_\varepsilon''(0) = \sqrt{B_\varepsilon}(u_\varepsilon(0) + A_\varepsilon), \quad u_\varepsilon'''(0) \le 0.$$

The value of $u_\varepsilon''(0)$ comes from the energy, and we have $u_\varepsilon'''(0) \le 0$ after possibly inverting x. We will show that this leads to a contradiction by investigating the limit $\varepsilon \to 0$.

Define $v_\varepsilon = \frac{u_\varepsilon + A_\varepsilon}{\varepsilon + A_\varepsilon}$. Then v_ε satisfies

$$v'''' + \beta v'' + \frac{1}{A_\varepsilon + \varepsilon} f_\varepsilon(v) = 0,$$

where one calculates that

$$f_\varepsilon(v) = \begin{cases} 0 & \text{if } v < 0 \\ V'(\varepsilon)v & \text{if } v \in [0,1] \\ V'((A_\varepsilon + \varepsilon)v - A_\varepsilon) & \text{if } v > 1. \end{cases}$$

The initial conditions for v_ε are

$$v_\varepsilon(0) \in (0,1), \quad v_\varepsilon'(0) = 0, \quad v_\varepsilon''(0) = \sqrt{B_\varepsilon} v_\varepsilon(0), \quad v_\varepsilon'''(0) = \frac{u_\varepsilon'''(0)}{A_\varepsilon + \varepsilon} \le 0.$$

We distinguish two cases (possibly after taking a subsequence)

$$(i) \ \frac{u_\varepsilon'''(0)}{\sqrt[4]{A_\varepsilon + \varepsilon}} \text{ is bounded,} \qquad (ii) \ \frac{u_\varepsilon'''(0)}{\sqrt[4]{A_\varepsilon + \varepsilon}} \to -\infty \quad \text{as } \varepsilon \to 0.$$

Case (i): Use the rescaling

$$w_\varepsilon(y) = v_\varepsilon(x) \qquad \text{with} \qquad y = \frac{1}{\sqrt[4]{A_\varepsilon + \varepsilon}} x.$$

One obtains

$$w_\varepsilon'''' + \sqrt{A_\varepsilon + \varepsilon}\beta w_\varepsilon'' + f_\varepsilon(w_\varepsilon) = 0,$$

and (since $\sqrt{B_\varepsilon(A_\varepsilon + \varepsilon)} = \sqrt{V'(\varepsilon)}$)

$$w_\varepsilon(0) \in (0,1), \quad w_\varepsilon'(0) = 0, \quad w_\varepsilon''(0) = \sqrt{V'(\varepsilon)} w_\varepsilon(0), \quad w_\varepsilon'''(0) = \frac{u_\varepsilon'''(0)}{\sqrt[4]{A_\varepsilon + \varepsilon}} \in (-C, 0],$$

for some $C > 0$ (since we are in Case (i)). Taking a converging subsequence $w_\varepsilon \to w$ we obtain in the limit

$$w'''' + f(w) = 0, \qquad \text{with } f(w) = \begin{cases} 0 & \text{if } w < 0 \\ V'(0)w & \text{if } w \in [0,1] \\ V'(0) & \text{if } w > 1, \end{cases}$$

and

$$w(0) \in [0,1], \quad w'(0) = 0, \quad w''(0) = \sqrt{V'(0)}w(0), \quad w'''(0) \in (-C, 0].$$

If $w(0) \neq 0$ then, since $w'''' < 0$ as long as $w > 0$, this implies that w becomes negative, a contradiction (w is a limit of positive functions). If $w(0) = 0$ then w_ε stays close to 0 for very long times for small ε. On the other hand w_ε oscillates (since it is close to a saddle-focus) with a frequency of approximately $\sqrt{V'(0)/2}$ for small ε, so that w_ε becomes negative (cf. Section 2), a contradiction.

Case (ii): Use the rescaling

$$w_\varepsilon(y) = v_\varepsilon(x) \qquad \text{with} \qquad y = \left(\frac{|u'''(0)|}{A_\varepsilon + \varepsilon}\right)^{1/3} x.$$

Write $k_\varepsilon = \left(\frac{A_\varepsilon+\varepsilon}{|u_\varepsilon'''(0)|}\right)^{1/3}$, then $k_\varepsilon = \left(\frac{\sqrt[4]{A_\varepsilon+\varepsilon}}{|u_\varepsilon'''(0)|}\right)^{1/3} \sqrt[4]{A_\varepsilon + \varepsilon} \to 0$ as $\varepsilon \to 0$. One obtains

$$w_\varepsilon'''' + k_\varepsilon^2 \beta w_\varepsilon'' + k_\varepsilon^4 f_\varepsilon(w_\varepsilon) = 0,$$

and

$$w_\varepsilon(0) \in (0,1), \quad w_\varepsilon'(0) = 0, \quad w_\varepsilon''(0) = \sqrt{B_\varepsilon} k_\varepsilon^2 w_\varepsilon(0), \quad w_\varepsilon'''(0) = -1.$$

Taking a converging subsequence $w_\varepsilon \to w$ we obtain in the limit $w'''' = 0$, and, since by assumption (we are in Case (ii))

$$\sqrt{B_\varepsilon} k_\varepsilon^2 = \sqrt{B_\varepsilon(A_\varepsilon + \varepsilon)} \left(\frac{\sqrt[4]{A_\varepsilon + \varepsilon}}{|u'''(0)|}\right)^{2/3} = \sqrt{V'(\varepsilon)} \left(\frac{\sqrt[4]{A_\varepsilon + \varepsilon}}{|u'''(0)|}\right)^{2/3} \to 0,$$

one has

$$w(0) \in [0,1], \quad w'(0) = 0, \quad w''(0) = 0, \quad w'''(0) = -1.$$

Clearly w becomes negative, a contradiction.

Remark 10. *When $V(0) = V'(0) = 0$ and the potential has a zero on either side of 0, then the method developed in Sections 2 to 5 can be applied as well. To be precise, suppose there are points $u_1 < 0 < u_2$ such that $V(u_1) = V(u_2) = 0$ and $V(u) > 0$ on $(u_1, u_2) \setminus \{0\}$, and the non-degeneracy conditions $V'(u_1) > 0$ or $V''(u_1) > 0$ and $V'(u_2) < 0$ or $V''(u_2) > 0$ are satisfied. Then for almost all $\beta > 0$ for which 0 is of saddle-focus character there exists a homoclinic solution to 0.*

6 The suspension bridge equation

Travelling waves localised in space for the suspension bridge equation,

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4} + V'(u), \tag{7}$$

were studied by Chen and McKenna in [4]. The Ansatz $u(t,x) = v(x - ct)$ yields the profile equation:

$$v'''' + c^2 v'' + V'(v) = 0,$$

which has the same form as those studied in the previous section. Existence of solutions was proved in [4] under the the following assumptions:

1. $V \in \mathcal{C}^2(\mathbb{R}),\ V'(u) = \max\{0, u\} - 1 + g(u)$;
2. $g(1) = g'(1) = 0$;
3. $(u-1)g''(u) \leq 0$ for all $u \neq 0$;
4. $\exists K_0 > 0$ such that $|g''(u)| \leq K_0$ for all $u \neq 0$.

There, the authors asked if one could remove these limitations, as numerical evidence suggests. In particular, can one have $V(u) = e^u - u - 1$, which is the potential used in [4] for the numerics.

Indeed, the analysis of Section 2 still applies here. The absence of a second well even makes things simpler, since the bound on the L^2 norm for v_n readily follows from that on $|v_n'|_{L^2}$, $|v_n''|_{L^2}$ and J_β.

We obtain:

Theorem 11. *Assume that V satisfies the following hypotheses:*

1. $V(0) = V'(0) = 0$ *and* $V''(0) = \alpha > 0$;
2. $\limsup_{u \to -\infty} V(u)/|u|^2 = 0$;
3. $V(u) > 0$ *for each* $u \neq 0$;
4. $\liminf_{u \to \pm\infty} u^2 V(u) > 0$.

then for almost all $c \in [-\sqrt[4]{4\alpha}, \sqrt[4]{4\alpha}]$, there exists a travelling wave solution for Equation (7)*, with profile in $H^2(\mathbb{R})$.*

Proof. The proof follows the same lines as that of Theorem 6; just notice that the equivalent of Lemma 4 still holds thanks to assumption 2 above. □

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Current addresses:

SMETS DIDIER
UNIVERSITÉ CATHOLIQUE DE LOUVAIN, DÉPARTEMENT DE MATHÉMATIQUES, 2 CHEMIN DU CYCLOTRON, 1348 LOUVAIN-LA-NEUVE, BELGIUM.
E-mail address: smets@amm.ucl.ac.be

JAN BOUWE VAN DEN BERG
DIVISION OF THEORETICAL MECHANICS, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF NOTTINGHAM, NOTTINGHAM, NG7 2RD, UNITED KINGDOM
E-mail address: Jan.Bouwe@nottingham.ac.uk