A general method for computer-assisted proofs of periodic solutions in delay differential problems

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Abstract

In this paper we develop a general computer-assisted proof method for periodic solutions to delay differential equations. The class of problems considered includes systems of delay differential equations with an arbitrary number of (forward and backward) delays. When the nonlinearities include nonpolynomial terms we introduce auxiliary variables to first rewrite the problem into an equivalent polynomial one. We then apply a flexible fixed point technique in a space of geometrically decaying Fourier coefficients. We showcase the efficacy of this method by proving periodic solutions in the well-known Mackey-Glass delay differential equation for the classical parameter values.

1 Introduction

In many biological phenomena and engineering applications the dynamics of the system is determined in part by a feedback loop. When this feedback is delayed significantly compared to the time scale of the dynamics, such systems are often described by Delay Differential Equations (DDEs). The analysis of DDEs is considerably more difficult than that of ordinary differential equations (ODEs), since the phase space of the dynamics of DDEs is effectively infinite dimensional. Much progress has been made in studying DDEs, and we refer to [12, 15, 22, 23, 24, 33, 34, 41, 42, 55] for overviews and highlights. Nevertheless, it is fair to say that even for the study of relatively simple dynamic structures such as periodic solutions a great desire for new flexible, generally applicable techniques remains.

In this paper we develop computer-assisted techniques for finding (and proving) periodic solutions of DDEs. We will, in particular, focus on DDEs with finitely many discrete delays:

$$u'(t) = F\big(u(t-\tau_1), \dots, u(t-\tau_q)\big). \tag{1}$$

The unknown $u : \mathbb{R} \to \mathbb{R}^p$ is a periodic function with a priori unknown period. The delays are $\{\tau_j\}_{j=1}^q \subset \mathbb{R}$, and the nonlinearity F maps $(\mathbb{R}^p)^q$ to $\mathbb{R}^p$.

Ever since the first proof of the universality of the Feigenbaum constant [27], great strides have been made in the application of computer-assisted proofs in dynamical systems. By

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far the most progress has been achieved in ODE problems, with the existence of chaos in the Lorenz system [37, 48, 49] as one of the outstanding results. In the last decade these developments have also extended to partial differential equations. While it is beyond the scope of this paper to present a review of the literature, we remark that, roughly speaking, these efforts can be categorized in two categories. The first type of approach centers around developing rigorous integrators for (semi-)flows and combining these with intricate topological arguments to study a variety of dynamic behaviours. One could describe these as "*phase space*" methods, and we refer to [2, 4, 7, 10, 21, 35, 56, 57, 58] and the references therein for the tremendous advances made. The second, complementary viewpoint, is based on reformulating the question of locating and proving certain types of dynamic structures as a zero finding problem in an appropriate functional analytic setting. For a sample of such "*function space*" methods we refer to [1, 6, 8, 11, 13, 39, 52, 53] and the references therein.

The novel general approach to periodic orbits in systems of DDEs proposed here is of the *function space* type. In a nutshell, it is based on a careful consideration of a modified Newton iteration method. We study zeros of a function $\mathcal{F}(\boldsymbol{x})$ by looking for fixed points of a Newton-like map:

$$T(\boldsymbol{x}) = \boldsymbol{x} - \mathcal{A}\mathcal{F}(\boldsymbol{x}), \tag{2}$$

where $\mathcal{A}$ is a carefully chosen approximation of the inverse of $D\mathcal{F}(\boldsymbol{x})$. We thus construct a map $\mathcal{F}$, such that the zeros correspond to periodic solutions, and then show the existence of a zero by means of a fixed point argument applied to the map T. Such an approach was previously used in [28], which focussed on Wright's DDE

$$u'(t) = -\gamma u(t-1)[1+u(t)], \qquad \text{with } \gamma > 0. \tag{3}$$

We note that Equation (3) has received particular scrutiny in the context of computer-assisted proofs, see [3, 19, 20, 28, 51] More generally, additional *function space* efforts for DDEs include [16, 25], while concerning the *phase space* perspective on computer-assisted proofs we refer to [46, 47] for some applications to DDEs.

The main contribution of the present paper is a *function space* method that applies to periodic orbits of general systems of DDEs of the type (1). We stress that this class includes forward-backward problems (both positive and negative delays τ_j) for which there is no well-defined semi-flow. On a technical level, we propose a flexible zero finding formulation, together with an associated fixed point map T described by (2). The success of the method is intimately linked to our novel choice for $\mathcal{A}$, the approximate inverse of the Jacobian, see Section 4.4 for the details.

In order to illustrate the efficacy of our approach, we will consider several examples. The example that we will cover in most detail is the Mackey-Glass equation [32], a scalar DDE with a single delay and a nonpolynomial nonlinearity:

$$u'(t) = -\beta u(t) + \alpha \frac{u(t-\tau)}{1+u(t-\tau)^\rho}. \tag{4}$$

This DDE, which models the concentration of white blood cells in a subject, is one of the first scalar DDEs that was conjectured to exhibit chaotic behaviour. In this equation, α is the production rate of new cells and β is the rate at which the cells die. The delay parameter $\tau > 0$ models the time it takes for the subject's body to observe the concentration and

react, by either increasing or decreasing cell production. Finally, the positive real (i.e. not necessarily integer) parameter ρ models the assumption that the production of new cells will abruptly stop if the concentration is higher than the critical concentration. In the rescaled version (4), this critical concentration is given by $u = 1$.

Chaos has been observed numerically in those cases where the critical concentration lies close to an equilibrium solution, and where the cell-growth drops, parameterized by ρ, is sufficiently steep [32, 31]. We will therefore consider the case where $u = 1$ is an equilibrium of (4), which implies that $\alpha = 2\beta$. By rescaling time we can choose, without loss of generality, $\alpha = 2$ and $\beta = 1$. Furthermore, chaos only occurs for "large" values of ρ, in particular $\rho > 9$. Classically ρ is taken to be 10 or a non-integer value near 10.

The very first nonperturbative rigorous results on periodic solutions for the Mackey-Glass equation (4) were recently obtained in [47] using the *phase space* approach. It relies on carefully integrating the DDE to construct a Poincaré return map explicitly. This construction provides an existence proof of an attracting periodic orbit for both $\rho = 6$ and $\rho = 8$.

With the *function space* methods developed in the current paper we are able to complement these results by proving solutions for non-integer ρ, and for parameters values inside the chaotic regime. In particular, we will focus on two particular choices for ρ and τ. First, we will consider $\rho = 10$ and $\tau = 1.63$, which lies close to the chaotic regime (in the (ρ, τ)-plane). For these parameters, there is a long stable periodic orbit, which we will use as a basis to obtain two other periodic solutions. Subsequently, we will use these results to obtain solutions inside the chaotic regime, in particular for $\rho = 9.65$ and $\tau = 2$. Some example orbits can be found in Figure 1.

We note that these results complement and in some sense enhance the ones obtained previously in [47]. In particular, we prove two co-existing periodic solutions for classical (integer and non-integer) values of the parameter ρ, including in the chaotic regime. The periodic orbits are observed (although not yet proven) to be dynamically unstable, but our method is unaffected by this. We obtain tight (rigorous) uniform error bounds of $O(10^{-8})$ between the numerical approximation and the periodic solution, see Section 6.2 for the precise numbers. Rigorous error bounds on derivatives are also available, see Remark 4.7. Among other things, this implies that computing an eigenvalue of the linearized problem (around the solution) and proving it(s real part) is positive (and hence instability of the periodic orbit), is well within reach, although we did not pursue that in this paper. Moreover, rigorous parameterization of the associated unstable manifolds along the lines of [14] would be a natural next step.

The code to produce these results is available at [54].

Outline of the method and the paper

Let us for simplicity first consider *scalar* DDEs of the form (1), where F is nonlinear. When looking for periodic solutions of (delay) differential equations the classical method is to construct these solutions using Fourier series, i.e. we look for functions of the form

$$u(t) = \sum_{k \in \mathbb{Z}} a_k e^{ik\vartheta t},$$

where $2\pi/\vartheta$ is the a priori unknown period.

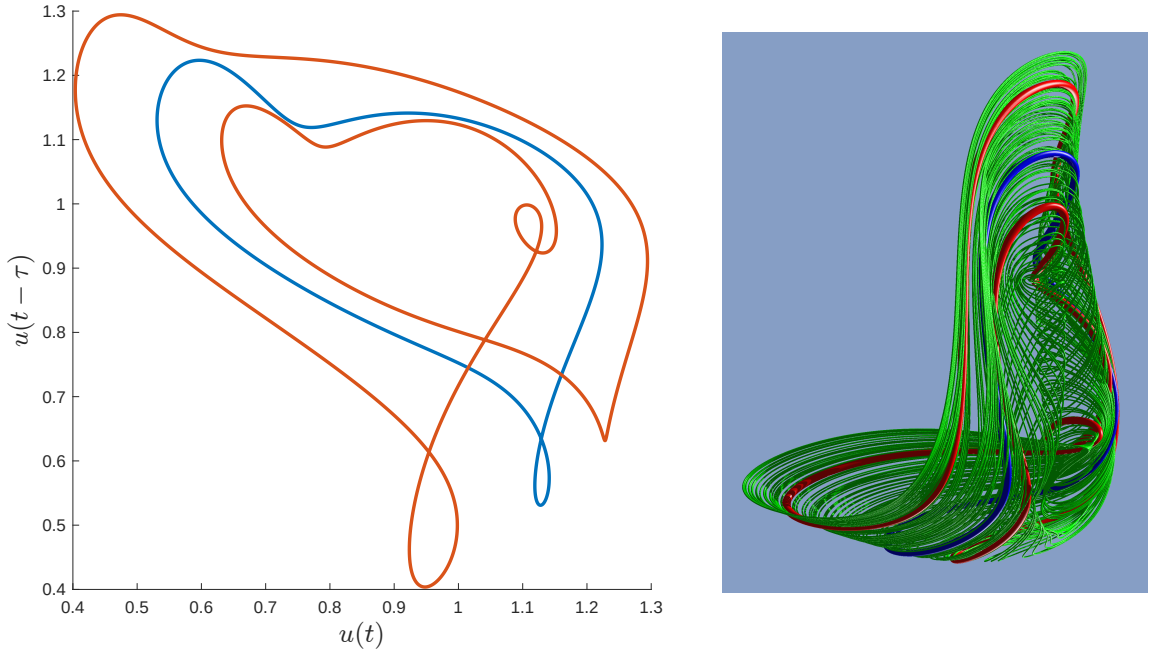

Figure 1: Two periodic solutions found in the chaotic regime, corresponding to $\alpha = 2$, $\beta = 1$, $\tau = 2$ and $\rho = 9.65$. Left: a parametric plot of $(u(t), u(t-\tau))$ of the two solutions. Right: a 3-dimensional rendering of $(u(t), u(t-\tau/2), u(t-\tau)))$ for the same two solutions. Also depicted is a (numerical) long orbit with random initial data, to illustrate the nature of the chaotic attractor.

Our first observation then is that both differential and delay operators have nice analogues in Fourier space:

$$u'(t) = \sum_{k\in\mathbb{Z}} (ik\vartheta a_k) e^{ik\vartheta t} \ , \qquad\qquad u(t-\tau_j) = \sum_{k\in\mathbb{Z}} (a_k e^{-ik\vartheta\tau_j}) e^{ik\vartheta t},$$

i.e. the Fourier coefficients of $u'(t)$ are given by $ik\vartheta a_k$ and those of $u(t-\tau_j)$ are given by $a_k e^{-ik\vartheta\tau_j}$.

Secondly, we observe that Fourier series have the property that pointwise multiplication translates to two-sided convolution products.

Definition 1.1 (Convolutions)**.** *Let $a, b \in \mathbb{C}^{\mathbb{Z}}$, then we define the convolution of a and b, denoted as $a * b$ as the bi-infinite sequence given by*

$$(a * b)_k = \sum_{k_1+k_2=k} a_{k_1} b_{k_2},$$

whenever the right-hand side converges.

Hence if the function $u(t)$ has Fourier coefficients a_k and $v(t)$ has coefficients b_k, then we

have that

$$u(t)v(t) = \sum_{k\in\mathbb{Z}} (a * b)_k e^{ik\vartheta t}.$$

Since the convolution product is associative, it is easy to see that *all* polynomial combinations of $2\pi/\vartheta$-periodic functions can be described by similar polynomial combinations of (possibly higher-order) convolutions.

In conclusion, if the function F from (1) is polynomial, one can naturally rewrite (1) as an equation on the space of Fourier coefficients. However, when a differential equation involves nonpolynomial nonlinearities, then finding a closed form expression for the Fourier coefficients can be at best tedious and in the worst case impossible. In Section 2 we introduce a method, inspired by automatic differentiation, as used in [30], to deal with a large class of nonpolynomial nonlinearities for delay equations. Our method is based on the observation that many "standard" functions can be constructed as solutions of (systems of) polynomial ODEs. In particular, when the nonlinearity F in (1) contains a nonpolynomial term, say described by some "standard" nonpolynomial scalar function f, then one can rewrite Equation (1) as a system of polynomial DDEs, by replacing a nonpolynomial nonlinearity with a new variable, and appending the differential equation that this nonlinearity solves.

In order to make this substitution precise, we must also add one (or more) scalar equations to the system. This is necessitated by the fact that in order to recover a nonlinear function from the polynomial ODE it satisfies, a specific boundary condition must be met. For instance, to obtain $f(x) = e^x$ from $f'(x) = f(x)$, we need to specify that $f(0) = 1$. The details on this can be found in Section 2.

This reformulation thus brings us to a situation where we have a system of DDEs with polynomial nonlinearities, appended with additional boundary conditions. These boundary conditions include a phase condition, which needs to be appended even if the original system was already polynomial (to lift the degeneracy of solutions due to time shift). For such systems we outline a computer-assisted method, that allows us to prove the existence of periodic solutions close to some numerically obtained approximate solution. The method that we will use, also called the method of radii polynomials, has in the last decade seen extensive use in a large range of dynamical systems problems. There is by now a large body of papers on this approach, see for instance [36, 17, 29] and the references therein. This computer-assisted method relies on the construction of a map T (on a sequence space) from a numerically obtained approximate solution. This map is constructed such that any fixed point of T corresponds to a solution of the system of DDEs. Subsequently we show the existence of a fixed point by means of a contraction argument, in particular by showing that it is contractive on a ball around a numerically calculated approximate solution. This method does not only provide a proof of existence, but the radius of the ball also allows us to give an explicit C^k-error bound on the numerical (approximate) solution used to find the exact solution (see Remark 4.7). The functional analytic framework needed to construct T and prove its contractivity are described in Section 4.

In Section 5, we outline in detail how to use the tools from Section 4 to obtain estimates necessary to prove the contractivity of T. Our guiding example there will be the Mackey-Glass equation (4), but since functional analytic foundations laid in Section 4 are comprehensive, the technique to derive the estimates works in full generality for any problem of the form (1)

for which we have a polynomial (re)formulation. We finish this paper by presenting several complicated periodic orbits of (4) in Section 6, both in and near the chaotic regime, which we succeeded in proving using the developed techniques. In Section 6.3 we discuss possible future work.

2 Polynomialization of nonlinearities

Our main goal is to construct methods to study nonpolynomial DDEs. In this section we will explain how to reformulate a nonpolynomial DDE as a system of polynomial DDEs. For the purposes of this paper we will call this procedure "polynomialization". The key idea is that many elementary nonpolynomial functions are solutions of polynomial differential equations. By appending this polynomial differential equation and solving for the transformed original variable, we can rewrite our nonpolynomial system as a (larger) polynomial system, see also [21, 26, 30, 50]

To see how one can apply this idea to a DDE, let us consider the following, instructive example.

Example 2.1 (Cooke's equation)**.** As a simple first example, we consider Cooke's equation, a nonpolynomial DDE arising from population dynamics [9], given by

$$u'(t) = -\beta u(t) + \alpha e^{-\gamma u(t-\tau)} u(t-\tau), \tag{5}$$

where $\alpha, \beta, \gamma > 0$. In this case, the nonpolynomial nonlinearity is given by

$$f(y) = e^{-\gamma y},$$

which satisfies

$$f'(x) = -\gamma f(x).$$

If we introduce a new variable $v = e^{-\gamma u}$, then $v'(t) = -\gamma v(t) u'(t)$, hence (5) becomes

$$\begin{aligned} u'(t) &= -\beta u(t) + \alpha u(t-\tau) v(t-\tau) \\ v'(t) &= -\gamma v(t)(-\beta u(t) + \alpha u(t-\tau) v(t-\tau)), \end{aligned} \tag{6}$$

where we substituted the equation for $u'(t)$ for the $u'(t)$ term appearing in the expression for $v'(t)$. For a pair (u, v) solving (6), it easily follows that $ve^{\gamma u}$ is constant, where a priori the constant is arbitrary. In order to make sure that the new variable v really satisfies $v = f(u)$, we need to impose boundary conditions.

Periodic solutions to (5) come in (at least) one parameter families, sine we can apply arbitrary time shifts to them. However, when we choose a c in the interior of the range of u, then we can impose $u(0) = c$ as a phase condition, if u is not locally constant near $t = 0$. This phase condition also gives us a natural boundary condition for v by imposing $v(0) = e^{-\gamma c}$. Motivated by numerical calculations, we will, for Cooke's equation, use $c = 1$.

If we want to treat u and v as coupled variables in an equation, then we have one final obstacle. The problem we have constructed thus far, consisting of finding periodic solutions u and v together with two phase conditions, is over-determined. In particular, we have two

phase conditions, but only one corresponding variable, namely the period. We solve this problem by adding an auxiliary variable η to (6) as follows:

$$\begin{aligned} u'(t) &= -\beta u(t) + \alpha u(t-\tau)v(t-\tau), & u(0) &= 1, \\ v'(t) &= -\gamma v(t)(-\beta u(t) + \alpha u(t-\tau)v(t-\tau)) + \eta, & v(0) &= e^{-\gamma}. \end{aligned} \tag{7}$$

We now wish to find periodic functions u, v solving (7) we well as two scalars, namely the frequency $\vartheta \in \mathbb{R}$ and the auxiliary variable $\eta \in \mathbb{R}$. If we can show the existence of such a solution, then it can be shown that η *must* be 0, and subsequently that $v = e^{\gamma u}$ and that u solves (5).

Remark 2.2. It should be noted that there are many ways in which we can add the auxiliary variable η to (6). Instead of adding it as a constant to the equation, we could also add $\eta v(t)$ instead. Some care has to be taken in how to choose the auxiliary variables, for instance, adding them as a constant does not work in the Ikeda equation, see Example 2.6.

We can summarise the relationship between (7) and (5) in the following proposition.

Proposition 2.3. *Let $\alpha, \beta, \gamma \in \mathbb{R}$, let $\eta \in \mathbb{R}$, and let (u, v) be a periodic solution of* (7)*. Then $\eta = 0$ and u is a periodic solution of Cooke's equation* (5)*.*

Proof. First note that v satisfies

$$v'(t) = -\gamma v(t)u'(t) + \eta.$$

If follows directly from this equation that $v'(t) = \eta$ whenever $v(t) = 0$. If $v(t_0) = 0$ for some t_0, and $\eta = 0$, then $v'(t_0) = v(t_0) = 0$, hence $v = 0$ everywhere by the uniqueness theorem for solutions to initial value problems, contradicting the assumption that v is periodic and $v(0) = e^{-\gamma}$. Hence it follows that if $v(t)$ is zero anywhere, then v crosses 0 in a direction determined by the sign of η. But η is a constant, hence if v crosses zero once, it cannot cross zero again, which contradicts the assumption that v is periodic. Therefore $v \neq 0$ everywhere. Since any solution must be continuous it follows from $v(0) > 0$ that $v > 0$ everywhere.

Since $v > 0$ everywhere, $\log v$ is also defined everywhere and

$$\frac{d}{dt}\left(\log v(t) + \gamma u(t)\right) = \frac{\eta}{v(t)}.$$

Since $v > 0$ the right-hand side has either constant sign or it vanishes. Moreover, $\log v + \gamma u$ is periodic, hence its derivative cannot have constant sign. We conclude that $\eta = 0$ and that $\log v + \gamma u$ is constant. Furthermore, the choice of boundary values in (7) implies that $v(0) = e^{-\gamma u(0)}$ and we conclude that $\log v + \gamma u = 0$, hence $v(t) = e^{-\gamma u(t)}$. In particular, $u(t)$ satisfies (5). □

We can apply the same ideas to the Mackey-Glass equation.

Example 2.4 (The Mackey-Glass equation)**.** Recall from the introduction that the Mackey-Glass equation is given by,

$$u'(t) = -\beta u(t) + \alpha \frac{u(t-\tau)}{1 + u(t-\tau)^\rho}, \tag{8}$$

where typically ρ is chosen large and possibly non-integer [31, 32]. In particular, this DDE involves a nonpolynomial nonlinearity,

$$f(y) = \frac{y}{1+y^\rho}.$$

The function f satisfies the non-autonomous, but polynomial, differential equation

$$f'(y) = (y^{-1} - \rho y^{\rho-2} f(y)) f(y). \tag{9}$$

This means that if we introduce a new variable

$$v(t) = \frac{u(t)}{1+u(t)^\rho} = f(u(t)),$$

then u satisfies

$$u'(t) = \alpha v(t-\tau) - \beta u(t).$$

However, (9) contains two new nonlinearities, namely y^{-1} and $y^{\rho-2}$. In order to handle these, we will introduce new functions: $x(t) = u(t)^{-1}$ and $w(t) = u(t)^{\rho-2}$. As in Example 2.1 we also introduce three auxiliary variables η_1, η_2 and η_3 together with the boundary conditions $u(0) = w(0) = x(0) = 1$ and $v(0) = 1/2$. We thus consider the following system of polynomial DDEs:

$$\begin{aligned}
u'(t) &= \alpha v(t-\tau) - \beta u(t), && u(0) = 1, && \text{(10a)}\\
v'(t) &= v(t)\left(x(t) - \rho v(t) w(t)\right)\left(\alpha v(t-\tau) - \beta u(t)\right) + \eta_1, && v(0) = 1/2, && \text{(10b)}\\
w'(t) &= (\rho-2) x(t) w(t)\left(\alpha v(t-\tau) - \beta u(t)\right) + \eta_2, && w(0) = 1, && \text{(10c)}\\
x'(t) &= -x(t)^2\left(\alpha v(t-\tau) - \beta u(t)\right) + \eta_3, && x(0) = 1. && \text{(10d)}
\end{aligned}$$

The relation between periodic solutions of (10) and (8) is guaranteed by the following proposition.

Proposition 2.5. *Let $\alpha, \beta, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$ and let (u, v, w, x) be a periodic solution of (10), then u is a periodic solution of the Mackey-Glass equation (8).*

Proof. First note that the x component of any periodic solution of (10) must satisfy

$$x'(t) = -x(t)^2 u'(t) + \eta_3.$$

Similar to the proof from Proposition 2.3, we can conclude that $x > 0$. It follows that we can write

$$\frac{d}{dt}\left(u(t) - x(t)^{-1}\right) = \frac{\eta_3}{x(t)^2}.$$

As in the proof of Proposition 2.3, $u(t) - x(t)^{-1}$ is periodic, hence $\eta_3 = 0$ and $u(t) - x(t)^{-1}$ must be constant. Since $x(0) = u(0) = 1$, $x(t)^{-1} = u(t)$. In a similar vein,

$$w'(t) = (\rho-2) x(t) w(t) u'(t) + \eta_2.$$

Since $w(0) > 0$, we again have by (10c) that $w > 0$, hence using that $x, w > 0$ and that $x(t) = u(t)^{-1}$, we find

$$\frac{d}{dt}\left(\log w(t) - (\rho - 2)\log u(t)\right) = \frac{\eta_2}{w(t)}.$$

By analogous arguments, we can then conclude that $\eta_2 = 0$ and $w(t) = u(t)^{\rho-2}$. Finally, combining (10a) and (10b) gives

$$v'(t) = v(t)\left(x(t) - \rho v(t)w(t)\right)u'(t) + \eta_1,$$

hence using the relations $x(t) = u(t)^{-1}$ and $w(t) = u(t)^{\rho-2}$ we conclude that

$$\frac{d}{dt}\left(\frac{u(t)}{v(t)} - u(t)^{\rho}\right) = -\eta_1 \frac{u(t)}{v(t)^2},$$

implying that $\eta_1 = 0$. Because $u(0) = 2v(0) = 1$, we now have that $\frac{u(t)}{v(t)} - u(t)^{\rho} = 1$, resulting in $v(t) = u(t)/(1 + u(t)^{\rho})$.

In conclusion, any periodic solution to (10) satisfies $u'(t) = -\beta u(t) + \alpha f(u(t-\tau))$, which is the Mackey-Glass equation (8). □

As a final example we consider the Ikeda equation, where the polynomialization is slightly more subtle.

Example 2.6 (The Ikeda equation)**.** The Ikeda equation [18, 45] is a simple DDE with a sinusoidal nonlinearity:

$$u'(t) = \sin(u(t-\tau)). \tag{11}$$

In this case we introduce *two* new variables, namely $v = \sin(u)$ and $w = \cos(u)$. These new variables satisfy $v' = wu'$ and $w' = -vu'$. As phase condition, we can put $u(0) = v(0) = 0$ and $w(0) = 1$. In this case we pass to the system

$$\begin{aligned} u'(t) &= v(t-\tau), & u(0) &= 0, \\ v'(t) &= w(t)v(t-\tau) + \eta_1 v(t) + \eta_2 w(t), & v(0) &= 0, \\ w'(t) &= -v(t)v(t-\tau) + \eta_1 w(t) - \eta_2 v(t), & w(0) &= 1. \end{aligned} \tag{12}$$

The introduction of the two linear terms with auxiliary variables η_1 and η_2 is justified by the next proposition.

Proposition 2.7. *Let $\eta_1, \eta_2 \in \mathbb{R}$ and let (u, v, w) be a periodic solution of* (12) *with period $2\pi/\vartheta$. If $|\eta_2| < \vartheta$, then u is a periodic solution of the Ikeda equation* (11)*.*

Proof. If we define $r^2(t) \stackrel{\text{def}}{=} v(t)^2 + w(t)^2$, then

$$\frac{1}{2}\frac{d}{dt}r^2 = vv' + ww' = \eta_1(v^2 + w^2) = \eta_1 r^2. \tag{13}$$

Clearly r^2 must be periodic, and $r^2(0) = 1$. This can only be the case if $\eta_1 = 0$, Hence (13) implies that $r^2(t) = 1$ for all t so that $(v(t), w(t))$ lies on the unit circle in $\mathbb{R}^2$. If $\phi(t)$ describes the angle of $(w(t), v(t)) = (\cos\phi(t), \sin\phi(t))$, then $\phi(0) = 0$ and

$$\phi' = v'w - w'v = (v^2 + w^2)(\eta_2 + u') = \eta_2 + u'.$$

While v and w are periodic, it does not follow directly that ϕ is periodic. However, by the above equation, it does follows that $(\phi - u)' = \eta_2$, a constant. Furthermore, since u, v and w are periodic, it also follows that this constant must be an integer multiple of ϑ. Since $|\eta_2| < \vartheta$, we must therefore have that $\eta_2 = 0$, hence $\phi' = u'$. Since $u(0) = \phi(0) = 0$, it follows that $u = \phi$, and therefore $v(t) = \sin u(t)$. In particular, u satisfies the Ikeda equation (11). □

3 The zero finding problem

The approach described in the previous section allows us to rewrite a large class of non-polynomial DDEs as polynomial ones, which means that we can take advantage of all the benefits of Fourier series, and of Banach algebra properties in particular (avoiding somewhat cumbersome interpolation and truncation estimates). In particular, we will use this to show that we can solve these DDEs by finding zeros of a map $\mathcal{F}$ on a suitably chosen sequence space of Fourier coefficients. In this section we construct this map for each of the examples covered above. Subsequently, we will dedicate the remainder of this paper to finding zeros of these maps.

We recall that the Fourier series associated with the constant function $u = 1$ is given by the standard Kronecker-δ_0, i.e. $\delta_0 \in \mathbb{C}^{\mathbb{Z}}$ satisfying

$$\delta_{0,k} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

As we will frequently work with derivatives and delayed versions of functions derived from Fourier series, it will be useful to introduce the following operators related to the derivatives and delays.

Definition 3.1. *Let $(c_k)_{k\in\mathbb{Z}}$ be a bi-infinite sequence, then we define the operator $K : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by*

$$(Kc)_k = kc_k.$$

For a given delay $\tau > 0$ and frequency $\vartheta \in \mathbb{R}$ we define the map $d_\tau(\vartheta) : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by

$$(d_\tau(\vartheta)c)_k = e^{-ik\vartheta\tau}c_k.$$

Remark 3.2. For $\vartheta \in \mathbb{R}$, the operator $d_\tau(\vartheta)$ is loosely related to the identity map. If we take the absolute value entry-wise, we have $|d_\tau(\vartheta)| = \mathrm{Id}$, as long as $\vartheta \in \mathbb{R}$. If ϑ is allowed to be complex valued, i.e. $\operatorname{Im}\vartheta \neq 0$, then $e^{ik\vartheta\tau}$ is unbounded in k. While it is not impossible to work with such an operator, many of the estimates to follow will simplify a great deal if we restrict ourselves to $\vartheta \in \mathbb{R}$. For this reason, we will assume throughout the paper that $\vartheta \in \mathbb{R}$. Since we wish to unify the way we treat the scalar unknowns (ϑ and $\eta_1, \ldots, \eta_{m-1}$), we will analogously assume that $\eta_j \in \mathbb{R}$.

The operators from Definition 3.1 allow us to write, without having to resort to indices, the Fourier series associated with the differentiated and delayed functions. Using this notation, we can concisely rewrite the example equations covered in Section 2. Each of the systems derived from those examples can be written as

$$\boldsymbol{x}'(t) = g(\eta_1, \ldots, \eta_{m-1}; \boldsymbol{x}(t); \boldsymbol{x}(t-\tau)) \qquad\qquad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

where $\boldsymbol{x}(t) = (x^0(t), \ldots, x^{m-1}(t)) \in \mathbb{R}^m$ and $\eta_1, \ldots, \eta_{m-1}$ are unknown, $\boldsymbol{x}_0 \in \mathbb{R}^m$ is given and $g : \mathbb{R}^{m-1} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is polynomial. Since we also wish to solve for the frequency ϑ, we will from now on set $\eta_0 = \vartheta$ and write

$$\eta \stackrel{\text{def}}{=} (\vartheta, \eta_1, \ldots, \eta_{m-1}) \in \mathbb{R}^m.$$

If we write $\boldsymbol{x}$ as a Fourier series with coefficients $\boldsymbol{c}_k = (c_k^0, \ldots, c_k^{m-1}) \in \mathbb{C}^m$, i.e.,

$$\boldsymbol{x}(t) = \sum_{k \in \mathbb{Z}} \boldsymbol{c}_k e^{ik\vartheta t},$$

then $\boldsymbol{x}'(t) = \sum_k i\vartheta(K\boldsymbol{c})_k e^{ik\vartheta t}$ and similarly $\boldsymbol{x}(t-\tau) = \sum_k (d_\tau(\vartheta)\boldsymbol{c})_k e^{ik\vartheta t}$. The DDE can then be written as

$$i\vartheta K\boldsymbol{c} = \hat{g}(\eta, \boldsymbol{c}),$$

where $\hat{g}(\eta, \boldsymbol{c})$ "equals" $g(\eta_1, \ldots, \eta_{m-1}; \boldsymbol{c}; d_\tau(\vartheta)\boldsymbol{c})$, with all multiplications of bi-infinite sequences interpreted as convolutions (see Definition 1.1).

Because of its dependence on $d_\tau(\vartheta)$, the map $\hat{g}$ is not truly polynomial. However, since $\hat{g}$ is only nonpolynomial in the scalar variables $\eta = (\vartheta, \eta_1, \ldots, \eta_{m-1})$, and not in the bi-infinite variables $\boldsymbol{c}$, this does not pose a problem.

Example 3.3 (Cooke's equation)**.** In the case of Cooke's equation from Example 2.1, we find that (7) is equivalent to

$$\mathcal{F}_{\text{Cooke}}(\eta, \boldsymbol{c}) \stackrel{\text{def}}{=} \begin{pmatrix} \sum_k c_k^0 - 1 \\ \sum_k c_k^1 - e^{-\gamma} \\ i\vartheta K c^0 + \beta c^0 - \alpha d_\tau(\vartheta)(c^0 * c^1) \\ i\vartheta K c^1 - \beta\gamma\, c^0 * c^1 + \alpha\gamma\, c^1 * d_\tau(\vartheta)(c^0 * c^1) - \eta_1 \delta_0 \end{pmatrix} = 0, \tag{14}$$

meaning that we can rewrite our problem as $\mathcal{F}(\eta, \boldsymbol{c}) = 0$, where $\mathcal{F} : \mathbb{R}^2 \times (\mathbb{C}^{\mathbb{Z}})^2 \to \mathbb{C}^2 \times (\mathbb{C}^{\mathbb{Z}})^2$ is given by (14).

Example 3.4 (The Mackey-Glass equation)**.** We can rewrite the Mackey-Glass equation from Example 2.4 as $\mathcal{F}(\eta, \boldsymbol{c}) = 0$, where $\mathcal{F} : \mathbb{R}^4 \times (\mathbb{C}^{\mathbb{Z}})^4 \to \mathbb{C}^4 \times (\mathbb{C}^{\mathbb{Z}})^4$ is given by

$$\mathcal{F}_{\text{MG}}(\eta, \boldsymbol{c}) \stackrel{\text{def}}{=} \begin{pmatrix} \sum_k c_k^0 - 1 \\ \sum_k c_k^1 - 1/2 \\ \sum_k c_k^2 - 1 \\ \sum_k c_k^3 - 1 \\ i\vartheta K c^0 - (\alpha d_\tau(\vartheta) c^1 - \beta c^0) \\ i\vartheta K c^1 - (c^1 * c^3 - \rho c^1 * c^1 * c^2) * (\alpha d_\tau(\vartheta) c^1 - \beta c^0) - \eta_1 \delta_0 \\ i\vartheta K c^2 - (\rho - 2) c^2 * c^3 * (\alpha d_\tau(\vartheta) c^1 - \beta c^0) - \eta_2 \delta_0 \\ i\vartheta K c^3 + c^3 * c^3 * (\alpha d_\tau(\vartheta) c^1 - \beta c^0) - \eta_3 \delta_0 \end{pmatrix}. \tag{15}$$

Example 3.5 (The Ikeda equation)**.** For the system derived from the Ikeda equation in Example 2.6, we see that the problem can again be written as $\mathcal{F}(\eta, \boldsymbol{c}) = 0$, where $\mathcal{F} : \mathbb{R}^3 \times (\mathbb{C}^{\mathbb{Z}})^3 \to \mathbb{C}^3 \times (\mathbb{C}^{\mathbb{Z}})^3$ is given by

$$\mathcal{F}_{\text{Ikeda}}(\eta, \boldsymbol{c}) \stackrel{\text{def}}{=} \begin{pmatrix} \sum_k c_k^0 \\ \sum_k c_k^1 \\ \sum_k c_k^2 - 1 \\ i\vartheta K c^0 - d_\tau(\vartheta) c^1 \\ i\vartheta K c^1 - c^2 * d_\tau(\vartheta) c^1 - \eta_1 c^1 - \eta_2 c^2 \\ i\vartheta K c^2 + c^1 * d_\tau(\vartheta) c^1 - \eta_1 c^2 + \eta_2 c^1 \end{pmatrix}. \tag{16}$$

In each of the examples above, we have constructed a map $\mathcal{F} : \mathbb{R}^m \times (\mathbb{C}^{\mathbb{Z}})^m \to \mathbb{C}^m \times (\mathbb{C}^{\mathbb{Z}})^m$. Note that the $\mathbb{C}^m$ component in the range of $\mathcal{F}$ is due to the fact that we do not, a priori, assume that our functions are going to be real-valued. The $\mathbb{R}^m$ component in the domain of $\mathcal{F}$ is due to Remark 3.2.

Finally, when our starting point is a vector-valued problem (1), i.e., $u : \mathbb{R} \to \mathbb{R}^p$ with $p > 1$, then we end up with a map $\mathcal{F} : \mathbb{R}^m \times (\mathbb{C}^{\mathbb{Z}})^{m+p-1} \to \mathbb{C}^m \times (\mathbb{C}^{\mathbb{Z}})^{m+p-1}$. In order not to burden the notation, we will put $p = 1$ in what follows, and we just remark that the methodology naturally extends to $p > 1$.

4 Functional analytic necessities

In this section we study the problem of finding solutions to $\mathcal{F}(\eta, \boldsymbol{c}) = 0$, with $\mathcal{F}$ as constructed in Section 3. In particular, we set up the outline of the computer-assisted proof of the existence (and local uniqueness) of these solutions. Since our goal is to provide a general framework to prove periodic solutions in general systems of DDEs, a fair bit of preparation is required to introduce convenient notions, so that the necessary bounds can be derived in a relatively compact manner in Section 5. In particular,

- in §4.1 we discuss the ℓ^1 spaces needed for the Fourier analysis;
- in §4.2 we set up the parameterized Newton-Kantorovich argument;
- in §4.3 we select a suitable finitely truncated problem and associated projections;
- in §4.4 we construct an ingenious approximate inverse of the Jacobian needed in the definition of the fixed point operator;
- in §4.5 we deal with symmetry invariance, which eventually provides us with a *real-valued* periodic solution;
- in §4.6 we introduce the concept of pseudo-convolutions, which capture the delay operators appearing in DDE problems in a remarkably convenient notation;
- in §4.7 we estimate the norms of the linear operators that are central to obtaining the bounds in Section 5.

4.1 Sequence spaces for analytic functions

It is a well known fact that analytic DDEs with constant delays have analytic periodic solutions [40]. Since the DDEs considered in this paper are analytic (after all, they are polynomial), we can *a priori* assume that the solutions that we will find are analytic as well. We exploit this a priori information in order to identify the Banach space on which to consider our zero finding problem.

Suppose that a periodic function $u : \mathbb{R} \to \mathbb{C}$ is given by $u(t) = \sum_k c_k e^{ik\vartheta t}$, then u is analytic if and only if the Fourier coefficients c_k decay exponentially. Hence there exists a $\nu > 1$ such that

$$\sum_k |c_k| \nu^{|k|} < \infty.$$

This motivates the following definition.

Definition 4.1. *We define the Banach space ℓ^1_ν as*

$$\ell^1_\nu \stackrel{\text{def}}{=} \left\{ c \in \mathbb{C}^{\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |c_k| \nu^{|k|} < \infty \right\}$$

equipped with the norm

$$\|c\|_{1,\nu} \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} |c_k| \nu^{|k|}.$$

Since we are not solving a single equation, but a system of m equations, the following definition will be useful:

Definition 4.2. *We define the Banach space $\ell^{1,m}_\nu$ as*

$$\ell^{1,m}_\nu \stackrel{\text{def}}{=} \left\{ \mathbf{c} = (c^0, \ldots, c^{m-1}) \in (\mathbb{C}^{\mathbb{Z}})^m : \sum_{j=0}^{m-1} \sum_{k\in\mathbb{Z}} |c^j_k| \nu^{|k|} < \infty \right\}$$

which is a Banach-space under the norm

$$\|\mathbf{c}\|_{1,\nu} \stackrel{\text{def}}{=} \sum_{j=0}^{m-1} \sum_{k\in\mathbb{Z}} |c^j_k| \nu^{|k|} = \sum_{j=0}^{m-1} \|c^j\|_{1,\nu}.$$

The maps $\mathcal{F}$ constructed in Section 3 also depend on m real parameters, and involve m complex boundary/phase conditions, which motivates the following definition.

Definition 4.3. *We define the Banach spaces $\mathcal{X}^{\mathbb{R},m}_\nu$ and $\mathcal{X}^{\mathbb{C},m}_\nu$ as*

$$\mathcal{X}^{\mathbb{R},m}_\nu \stackrel{\text{def}}{=} \mathbb{R}^m \times \ell^{1,m}_\nu \qquad\qquad \mathcal{X}^{\mathbb{C},m}_\nu \stackrel{\text{def}}{=} \mathbb{C}^m \times \ell^{1,m}_\nu,$$

where the norm of $\boldsymbol{x} = (\eta, \mathbf{c})$, for both $\boldsymbol{x} \in \mathcal{X}^{\mathbb{R},m}_\nu$ and $\boldsymbol{x} \in \mathcal{X}^{\mathbb{C},m}_\nu$, is given by

$$\|\boldsymbol{x}\| = \sum_{j=0}^{m-1} |\eta_j| + \sum_{j=0}^{m-1} \|c^j\|_{1,\nu}.$$

In the above definitions, we purposefully choose to take the sum of ℓ^1 norms, as opposed to the more "classical" method of taking the maximum of norms when constructing products of Banach spaces. This is because taking the sum of ℓ^1 norms results in a Banach space which is also ℓ^1, albeit with more indices. This makes calculating the norm of operators on these new Banach spaces much more elementary.

In order to obtain the numerical results that form the basis of our computer-assisted proof, we need some formalism that deals with truncated sequences and finite dimensional subspaces. To this end we will, throughout this paper, consider $\mathbb{C}^{2n+1}$ as a ($2n+1$ dimensional) subspace of ℓ^1_ν by setting $c_k = 0$ for $|k| > n$. We will abuse notation by implicitly identifying the finite dimensional space $\mathbb{C}^{2n+1}$ with this subspace of ℓ^1_ν. Thus, this gives rise to a natural projection $\pi^n : \ell^1_\nu \to \mathbb{C}^{2n+1} \subset \ell^1_\nu$ that is obtained by setting

$$(\pi^n c)_k = \begin{cases} c_k & \text{if } |k| \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The complementary projection we denote by $\pi^{n,\infty}$, i.e. $\pi^n + \pi^{n,\infty} = \mathrm{Id}$.

When $n_1 < n_2$, we similarly have a natural embedding $\mathbb{C}^{2n_1+1} \subset \mathbb{C}^{2n_2+1} \subset \ell^1_\nu$. In this case we will also use the projection on the complement of $\mathbb{C}^{2n_1+1}$ in $\mathbb{C}^{2n_2+1}$, denoted by π^{n_1,n_2}:

$$(\pi^{n_1,n_2} c)_k = \begin{cases} c_k & \text{if } n_1 < |k| \leq n_2 \\ 0 & \text{otherwise,} \end{cases}$$

hence it satisfies $\pi^{n_1} + \pi^{n_1,n_2} = \pi^{n_2}$.

Finally, we note that these projections naturally extend to the product space $\mathbb{C}^{m(2n+1)} \subset \ell^{1,m}_\nu$, by setting

$$\boldsymbol{\pi}^n \mathbf{c} = (\pi^n c^0, \ldots, \pi^n c^{m-1}).$$

The primary reason to focus on ℓ^1-spaces (as opposed to other ℓ^p-spaces), is that these spaces are Banach-algebras with respect to two-sided convolutions. This is important because as our system of delay equations is polynomial, the corresponding nonlinearities can be written as convolutions.

Lemma 4.4. *Let $a, b \in \ell^1_\nu$, then the two-sided convolution $a * b$, given by*

$$(a * b)_k = \sum_{k_1+k_2=k} a_{k_1} b_{k_2}$$

also satisfies $a * b \in \ell^1_\nu$. In particular

$$\|a * b\|_{1,\nu} \leq \|a\|_{1,\nu} \|b\|_{1,\nu}.$$

Finally, we can give explicit expressions for the norms of maps and of linear functionals in terms of their coefficients. Since ℓ^1_ν is a sequence space, we can also represent $\phi : \ell^1_\nu \to \mathbb{C}$ as a (bi-infinite) sequence.

Remark 4.5. Let the basis elements $e^j \in \ell^1_\nu$ for $j \in \mathbb{Z}$ be such that $e^j_k = \delta_{j,k}$ where δ is the standard Kronecker-δ. Then the explicit representation of $\phi \in (\ell^1_\nu)^*$ as a sequence is via $\phi_j = \phi(e^j)$ for all $j \in \mathbb{Z}$.

Furthermore, we have by Hölder's inequality that

$$|\phi(c)| = \left|\sum_k \phi_k c_k\right| \leq \left(\sup_k |\phi_k|\nu^{-|k|}\right)\left(\sum_k |c_k|\nu^{|k|}\right).$$

This gives us the following result.

Lemma 4.6. *The dual $\ell^\infty_\nu \cong (\ell^1_\nu)^*$ of ℓ^1_ν is given by*

$$\ell^\infty_\nu \stackrel{\text{def}}{=} \left\{\phi \in \mathbb{C}^{\mathbb{Z}} : \sup_k |\phi_k|\nu^{-|k|} < \infty\right\},$$

and the norm on the dual is given by

$$\|\phi\|_{\infty,\nu} \stackrel{\text{def}}{=} \sup_k |\phi_k|\nu^{-|k|}.$$

Remark 4.7. If we write $u(t) = \sum_k c_k e^{ik\vartheta t}$, then Lemma 4.6 allows us to calculate an explicit bound on the C^n-norm by means of the following inequality:

$$\begin{aligned}\|u\|_{C^n} &= \max_{0\leq j\leq n} \sup_{t\in[0,2\pi/\vartheta]} \left|\frac{d^j}{dt^j}u(t)\right| \\ &\leq \max_{0\leq j\leq n} \vartheta^j \sum_k |k|^j |c_k| \\ &\leq \|c\|_{1,\nu} \max_{0\leq j\leq n} \vartheta^j \sup_k |k|^j \nu^{-|k|}.\end{aligned}$$

Lemma 4.6 can be used to calculate the norm of operators $M : \ell^1_\nu \to \ell^1_\nu$ as follows.

Lemma 4.8. *Let $M : \ell^1_\nu \to \ell^1_\nu$ be a bounded linear map, represented by a bi-infinite matrix $(M_{k_1,k_2})_{k_1,k_2\in\mathbb{Z}}$, then*

$$\|M\| = \sup_{k_2} \nu^{-|k_2|} \sum_{k_1} |M_{k_1,k_2}|\nu^{|k_2|}.$$

We can apply this lemma to calculate the norm of the linear map $d_\tau(\vartheta)$ that we introduced in Definition 3.1. But, if we try to apply Lemma 4.8 to the operator K, we run into a problem, namely if $c \in \ell^1_\nu$, then it does not follow that $Kc \in \ell^1_\nu$. To remedy this, we introduce the following Banach space

Definition 4.9. *We define the Banach space $\ell^1_{\nu+}$ as*

$$\ell^1_{\nu+} \stackrel{\text{def}}{=} \left\{c \in \mathbb{C}^{\mathbb{Z}} : |c_0| + \sum_{|k|>1} |c_k|\frac{\nu^{|k|}}{|k|} < \infty\right\}$$

equipped with the norm

$$\|c\|_{1,\nu+} \stackrel{\text{def}}{=} |c_0| + \sum_{|k|>1} |c_k|\frac{\nu^{|k|}}{|k|}.$$

We define the Banach spaces $\ell^{1,m}_{\nu+}$, $\mathcal{X}^{\mathbb{R},m}_{\nu+}$ and $\mathcal{X}^{\mathbb{C},m}_{\nu+}$ in the same manner as $\ell^{1,m}_\nu$, $\mathcal{X}^{\mathbb{R},m}_\nu$ and $\mathcal{X}^{\mathbb{C},m}_\nu$ from Definition 4.2 and Definition 4.3.

Using this we can now also specify the range and norm of K.

Lemma 4.10. *Let $K : \ell^1_\nu \to \ell^1_{\nu+}$ and $d_\tau(\vartheta) : \ell^1_\nu \to \ell^1_\nu$ be given by*

$$\begin{aligned}(Kc)_k &\overset{\text{def}}{=} kc_k,\\ (d_\tau(\vartheta)c)_k &\overset{\text{def}}{=} e^{-ik\vartheta\tau}c_k.\end{aligned}$$

Then $\|K\| = 1$ and $\|d_\tau(\vartheta)\| = 1$.

Since we choose to use the sum of the norms on the product spaces, we have a similar (but due to indices rather technical) result on $\ell^{1,m}_\nu$, and $\mathcal{X}^{\mathbb{R},m}_\nu$ and $\mathcal{X}^{\mathbb{C},m}_\nu$. To write down this result, we observe that we can write each element $x \in \mathbb{C} \times \ell^1_\nu$ as vector $(x)_{i\in\mathcal{I}}$, where the index-set is given by the disjoint union

$$\mathcal{I} = \{0\} \sqcup \mathbb{Z}.$$

Using this notation, we have that $\|x\| = \sum_{k\in\mathcal{I}} |x_k|\nu^{|k|}$, allowing us to formulate the following result.

Corollary 4.11. *Let $\mathcal{X}$ be either $\mathcal{X}^{\mathbb{R},m}_\nu$ or $\mathcal{X}^{\mathbb{C},m}_\nu$ and let $\mathcal{M} : \mathcal{X} \to \mathcal{X}$, then $\mathcal{M}$ can be written as a block-matrix $\mathcal{M} = (M^{j_1,j_2})_{j_1,j_2\in\{0,\dots,m-1\}}$, where for each j_1, j_2, $M^{j_1,j_2} \in \mathbb{C}^{\mathcal{I}\times\mathcal{I}}$. Furthermore,*

$$\|\mathcal{M}\| = \max_{0\le j_2\le m-1} \sup_{k_2\in\mathcal{I}} \nu^{-|k_2|} \sum_{j_1=0}^{m-1} \sum_{k_1\in\mathcal{I}} |M^{j_1,j_2}_{k_1,k_2}|\nu^{|k_1|}.$$

While the notation here is rather cumbersome, it should be noted that the norm above is obtained by taking the $\mathcal{X}$ norm of every column in $\mathcal{M}$, and taking the dual norm of the resulting row.

Finally we observe that $\mathbb{R}^m \subset \mathbb{C}^m$, hence we have a natural inclusion $\mathcal{X}^{\mathbb{R},m}_\nu \subset \mathcal{X}^{\mathbb{C},m}_\nu$. Furthermore, every linear map $\mathcal{M} : \mathcal{X}^{\mathbb{R},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$ has, by linearity, a natural extension to a map $\overline{\mathcal{M}} : \mathcal{X}^{\mathbb{C},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$. If we denote the inclusion $\iota : \mathcal{X}^{\mathbb{R},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$, then $\mathcal{M} = \overline{\mathcal{M}}\iota$. Since $\|\iota\| = 1$, it then follows that

$$\|\mathcal{M}\| = \|\overline{\mathcal{M}}\iota\| \le \|\overline{\mathcal{M}}\|\|\iota\| = \|\overline{\mathcal{M}}\|.$$

We summarise this conclusion in the following Corollary.

Corollary 4.12. *, Let $\mathcal{M} : \mathcal{X}^{\mathbb{R},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$ and let $\overline{\mathcal{M}} : \mathcal{X}^{\mathbb{C},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$ be its natural extension. Then $\|\mathcal{M}\| \le \|\overline{\mathcal{M}}\|$.*

4.2 General radii polynomial approach

Our computer-assisted proof makes use of the so-called *radii polynomial* approach, which we formulate here. The method itself is based on the Banach fixed point theorem and combines numerical (computer-derived) calculations and analytical estimates to both construct a candidate fixed point operator and verify its contractivity. We first present this method for a

general class of zero finding problems and subsequently comment on the implementation in the case of periodic orbits of (systems of) delay equations.

Let us consider a map $\mathcal{F} : X \to X'$, between two Banach spaces. The candidate for our contracting map is derived from the classical Newton method, which is used to find zeroes of maps by iterating the Newton operator on X given by

$$\boldsymbol{x} \mapsto \boldsymbol{x} - D\mathcal{F}(\boldsymbol{x})^{-1}\mathcal{F}(\boldsymbol{x}).$$

Suppose now that we have constructed (numerically) an approximate solution $\hat{\boldsymbol{x}} \in X$, i.e. we have that $\mathcal{F}(\hat{\boldsymbol{x}}) \approx 0$. We then expect that the Newton map will be contracting on some neighbourhood of $\hat{\boldsymbol{x}}$. The problem is that when working with infinite-dimensional systems, $D\mathcal{F}$ can be hard to construct and inconvenient to invert. Hence instead of using $D\mathcal{F}(\boldsymbol{x})^{-1}$, we will use an approximate inverse of the derivative in the point $\hat{\boldsymbol{x}}$, that is, we take $\mathcal{A} : X' \to X$ such that

$$\mathcal{A} \approx D\mathcal{F}(\hat{\boldsymbol{x}})^{-1}$$

and we define $T : X \to X$ by

$$T(\boldsymbol{x}) \stackrel{\text{def}}{=} \boldsymbol{x} - \mathcal{A}\mathcal{F}(\boldsymbol{x}). \tag{17}$$

If $\mathcal{A}$ is injective, then $\mathcal{F}(\widetilde{\boldsymbol{x}}) = 0$ if and only if $T(\widetilde{\boldsymbol{x}}) = \widetilde{\boldsymbol{x}}$. Hence $\widetilde{\boldsymbol{x}}$ is a solution of $\mathcal{F}(\widetilde{\boldsymbol{x}}) = 0$ if and only if $\widetilde{\boldsymbol{x}}$ is a fixed point of T. If we can show that T is a contraction, then T must have a fixed point and consequently $\mathcal{F}$ must have a zero.

The expectation that this T is contracting is not unreasonable. After all, the Newton-like operator T can be seen as a perturbation of the classical Newton operator on a neighbourhood of $\hat{\boldsymbol{x}}$. Since Newton's method is so strongly contracting, it is reasonable to expect that small perturbations, and hence T, will still be contracting. However, for any particular choice of $\mathcal{A}$ this of course needs to be verified.

In order to show that T is indeed contracting on a ball of radius r around $\hat{\boldsymbol{x}}$ we will use a parameterized version of the Newton-Kantorovich theorem. This particular variant of the theorem is due to [5].

Theorem 4.13. *Let $T : X \to X$ be differentiable and let $\hat{\boldsymbol{x}} \in X$. Furthermore, suppose there exist constants Y, Z_1 and a function $Z_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|T(\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\| \leq Y \tag{18a}$$

$$\|DT(\hat{\boldsymbol{x}})\| \leq Z_1 \tag{18b}$$

and such that for every $r > 0$ and $\|\boldsymbol{y}\| \leq 1$,

$$\|DT(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - DT(\hat{\boldsymbol{x}})\| \leq Z_2(r)\|\boldsymbol{y}\|. \tag{18c}$$

If there exists a radius $\hat{r} > 0$ such that the following two inequalities hold:

$$Y + \left(Z_1 + \frac{1}{2}Z_2(\hat{r})\right)\hat{r} < \hat{r}, \tag{19a}$$

$$Z_1 + Z_2(\hat{r}) < 1, \tag{19b}$$

then $T : B_{\hat{r}}(\hat{\boldsymbol{x}}) \to B_{\hat{r}}(\hat{\boldsymbol{x}})$ is a contraction.

Proof. Suppose $\boldsymbol{z} \in B_r(\hat{\boldsymbol{x}})$, then

$$\begin{aligned}
\|T(\boldsymbol{z})-\hat{\boldsymbol{x}}\| &\leq \underbrace{\|T(\hat{\boldsymbol{x}})-\hat{\boldsymbol{x}}\|}_{\leq Y} + \|T(\boldsymbol{z})-T(\hat{\boldsymbol{x}})\| \\
&\leq Y + \left\| \int_0^1 DT(\hat{\boldsymbol{x}} + t(\boldsymbol{z}-\hat{\boldsymbol{x}}))(\boldsymbol{z}-\hat{\boldsymbol{x}}) dt \right\| \\
&\leq Y + \|\boldsymbol{z}-\hat{\boldsymbol{x}}\| \left(\int_0^1 \|DT(\hat{\boldsymbol{x}})\|\, dt + \int_0^1 \|DT(\hat{\boldsymbol{x}} + t(\boldsymbol{z}-\hat{\boldsymbol{x}})) - DT(\hat{\boldsymbol{x}})\|\, dt \right) \\
&\leq Y + \|\boldsymbol{z}-\hat{\boldsymbol{x}}\| \left(\int_0^1 Z_1 dt + \int_0^1 Z_2(r)\|\boldsymbol{z}-\hat{\boldsymbol{x}}\| \frac{t}{r} dt \right) \\
&\leq Y + \|\boldsymbol{z}-\hat{\boldsymbol{x}}\| \left(Z_1 + Z_2(r) \int_0^1 t\ dt \right) \\
&= Y + \left(Z_1 + \frac{1}{2} Z_2(r) \right) r,
\end{aligned}$$

hence if (19a) holds, then T maps $B_{\hat{r}}(\hat{\boldsymbol{x}})$ into itself. Furthermore, we have similarly for $\boldsymbol{x}, \boldsymbol{z} \in B_r(\hat{\boldsymbol{x}})$ that

$$\|T(\boldsymbol{z}) - T(\boldsymbol{x})\| \leq \int_0^1 \|DT(\boldsymbol{x} + t(\boldsymbol{z}-\boldsymbol{x}))\|\, \|\boldsymbol{z}-\boldsymbol{x}\| dt.$$

If we denote $\boldsymbol{x}_t = \boldsymbol{x} + t(\boldsymbol{z}-\boldsymbol{x})$, then by convexity $\boldsymbol{x}_t \in B_r(\hat{\boldsymbol{x}})$, i.e. $\|\hat{\boldsymbol{x}} - \boldsymbol{x}_t\| \leq r$ for all $t \in [0,1]$, hence we have that

$$\begin{aligned}
\|T(\boldsymbol{z}) - T(\boldsymbol{x})\| &\leq \|\boldsymbol{z}-\boldsymbol{x}\| \int_0^1 \|DT(\boldsymbol{x}_t)\|\, dt \\
&\leq \|\boldsymbol{z}-\boldsymbol{x}\| \int_0^1 \|DT(\hat{\boldsymbol{x}} - (\hat{\boldsymbol{x}} - \boldsymbol{x}_t)\|\, dt \\
&\leq \|\boldsymbol{z}-\boldsymbol{x}\| \left(\int_0^1 \|DT(\hat{\boldsymbol{x}})\|\, dt + \int_0^1 \|DT(\hat{\boldsymbol{x}} - (\hat{\boldsymbol{x}} - \boldsymbol{x}_t)) - DT(\hat{\boldsymbol{x}})\|\, dt \right) \\
&\leq \|\boldsymbol{z}-\boldsymbol{x}\| \left(Z_1 + Z_2(r) \right).
\end{aligned}$$

We conclude that when (19b) holds in addition to (19a), then T is a contraction on $B_{\hat{r}}(\hat{\boldsymbol{x}})$. □

Remark 4.14. In practice, the Z_2 bound is often chosen to be polynomial (or even constant) in r. Furthermore, the inequality in (19a) is clearly equivalent to

$$p(\hat{r}) \overset{\text{def}}{=} Y + \left(Z_1 + \frac{1}{2} Z_2(\hat{r}) - 1 \right) \hat{r} < 0.$$

This polynomial is called the *radii polynomial*. In the case where Z_2 is linear, the further requirement provided by (19b) is then equivalent to $p'(\hat{r}) < 0$.

When we interpret Theorem 4.13 in the context of the map $\mathcal{F}$ and the approximate inverse $\mathcal{A}$, we obtain the following corollary.

Corollary 4.15. *Let $\mathcal{F} : X \to X'$ be differentiable and let $\mathcal{A} : X' \to X$ be an injective linear map. If there exist an $\hat{\boldsymbol{x}} \in X$, bounds Y, Z_1 and Z_2, and a radius $\hat{r}$ such that the map $T(\boldsymbol{x}) = \boldsymbol{x} - A\mathcal{F}(\boldsymbol{x})$ satisfies the assumptions in Theorem 4.13, then there exists a unique $\widetilde{\boldsymbol{x}} \in B_{\hat{r}}(\hat{\boldsymbol{x}}) \subset X$ such that $\mathcal{F}(\widetilde{\boldsymbol{x}}) = 0$.*

Remark 4.16 (Interval Arithmetic)**.** The estimates appearing in Theorem 4.13 depend explicitly on a computer-obtained approximate numerical solution. Hence the Y and Z bounds will also have to be calculated using a computer. One problem is that such calculations are prone to rounding errors caused by floating-point operations. To make our computations rigorous, instead of just numerical, we therefore need a way to rigorously bound the rounding errors that accumulate during the calculation of Y and Z. The standard way to do this is by making use of an interval-arithmetic library [38, 50]. For the calculations done in this paper, the Intlab library for Matlab was used [44, 43].

From here on out, we will use the Banach spaces $\mathcal{X}_\nu^{\mathbb{R},m}$ and $\mathcal{X}_\nu^{\mathbb{C},m}$ that we introduced in Section 4.1, and solely focus on the maps $\mathcal{F} : \mathcal{X}_\nu^{\mathbb{R},m} \to \mathcal{X}_{\nu+}^{\mathbb{C},m}$ we obtained in Section 3. The first obstacle in using Theorem 4.13 is selecting an approximate inverse. Because of our restriction of $\mathcal{F}$ to the Banach space $\mathcal{X}_\nu^{\mathbb{R},m}$, the construction of the approximate inverse, in Section 4.4, will be slightly more subtle than described above. We will spend the following sections providing the necessary functional analytic setup to construct this approximate inverse and a version of Corollary 4.15, suitable for the type of maps described in Section 3. This result is formulated in Proposition 4.25.

The remainder of the paper will then be dedicated to providing the means to calculate the Y and Z bounds needed to apply Proposition 4.25.

4.3 The finite dimensional projection

In order to compute the approximate solution as well as the approximate inverse mentioned in the previous section, we will need to truncate the map $\mathcal{F}$. To this end, we observe that each of the examples covered in Section 3 can be decomposed in the following way:

$$\mathcal{F}(\boldsymbol{x}) = \mathcal{F}(\eta, \boldsymbol{c}) = \begin{pmatrix} H(\boldsymbol{c}) \\ G(\eta, \boldsymbol{c}) \end{pmatrix},$$

where $H : \ell_\nu^{1,m} \to \mathbb{C}^m$ represents the boundary/phase conditions and where $G : \mathcal{X}_\nu^{\mathbb{R},m} = \mathbb{R}^m \times \ell_\nu^{1,m} \to \ell_{\nu+}^{1,m}$ encodes the differential equations. In particular, the map H is affine, meaning that we can write $H(\boldsymbol{c}) = \boldsymbol{E}\boldsymbol{c} - h$, where $\boldsymbol{E} : \ell_\nu^{1,m} \to \mathbb{C}^m$ is linear and where $h \in \mathbb{C}^m$. Furthermore, each of the components of $G = (G^0, \dots, G^{m-1})$ with $G^j(\eta, \boldsymbol{c}) \in \ell_{\nu+}^1$ satisfies

$$G^j(\eta, \boldsymbol{c})_k = ik\vartheta c_k^j + \text{convolution terms}.$$

Note that the convolution terms will depend on η and might include zeroth convolution powers of c as well. This means that, especially for large $|\vartheta k|$, the (diagonal) $ik\vartheta c_k^j$ term will be dominant, which implies that for large $|\vartheta k|$ it is reasonable to approximate G with a diagonal operator. This motivates the following definitions.

Definition 4.17. *Let $K : \ell^1_\nu \to \ell^1_{\nu+}$ be as in Definition 3.1. We then define the operator $\boldsymbol{K} : \ell^{1,m}_\nu \to \ell^{1,m}_{\nu+}$ by*

$$\boldsymbol{K} \stackrel{\text{def}}{=} \operatorname{diag}(K, \dots, K).$$

Furthermore, we define for a fixed and given $\vartheta > 0$ the diagonal operators $\tilde{\Omega}_\vartheta : \ell^1_\nu \to \ell^1_{\nu+}$ and $\tilde{\boldsymbol{\Omega}}_\vartheta : \ell^{1,m}_\nu \to \ell^{1,m}_{\nu+}$ by

$$\tilde{\Omega}_\vartheta \stackrel{\text{def}}{=} i\vartheta K \qquad \text{and} \qquad \tilde{\boldsymbol{\Omega}}_\vartheta \stackrel{\text{def}}{=} i\vartheta \boldsymbol{K}.$$

Using this notation, we see that we can write

$$G = \tilde{\boldsymbol{\Omega}}_\vartheta + \text{convolution terms}.$$

Of course $\tilde{\Omega}$ is not invertible, but when restricting to the tail, i.e., those indices for which $|k| > n > 0$, it is. In the following we choose $n > 0$ to be the number of modes after which we wish to truncate $\mathcal{F}$.

Definition 4.18. *We define for a fixed and given $\vartheta > 0$ the diagonal operator $\Omega_\vartheta : \pi^{n,\infty}\ell^1_\nu \to \pi^{n,\infty}\ell^1_{\nu+}$ by*

$$(\Omega_\vartheta c)_k \stackrel{\text{def}}{=} ik\vartheta c_k, \qquad \textit{i.e.,} \qquad \Omega_\vartheta = i\vartheta K \pi^{n,\infty}.$$

We define the inverse operator $\Omega^{-1}_\vartheta : \pi^{n,\infty}\ell^1_{\nu+} \to \pi^{n,\infty}\ell^1_\nu$ by

$$(\Omega^{-1}_\vartheta c)_k \stackrel{\text{def}}{=} \frac{1}{ik\vartheta} c_k.$$

Finally, we define the extended operators $\boldsymbol{\Omega}_\vartheta : \boldsymbol{\pi}^{n,\infty}\ell^{1,m}_\nu \to \boldsymbol{\pi}^{n,\infty}\ell^{1,m}_{\nu+}$ and $\boldsymbol{\Omega}^{-1}_\vartheta : \boldsymbol{\pi}^{n,\infty}\ell^{1,m}_{\nu+} \to \boldsymbol{\pi}^{n,\infty}\ell^{1,m}_\nu$ by

$$\boldsymbol{\Omega}_\vartheta \stackrel{\text{def}}{=} \operatorname{diag}(\Omega_\vartheta, \dots, \Omega_\vartheta) \qquad \text{and} \qquad \boldsymbol{\Omega}^{-1}_\vartheta \stackrel{\text{def}}{=} \operatorname{diag}(\Omega^{-1}_\vartheta, \dots, \Omega^{-1}_\vartheta).$$

We apply traditional numerical techniques in order to calculate an approximate solution $\hat{\boldsymbol{x}} = (\hat{\eta}, \hat{\boldsymbol{c}}) \in \mathbb{R}^m \times \mathbb{C}^{m(2n+1)}$ satisfying

$$\begin{pmatrix} \boldsymbol{E}\hat{\boldsymbol{c}} - h \\ \pi^n G(\hat{\eta}, \hat{\boldsymbol{c}}) \end{pmatrix} \approx 0,$$

where we used the affine formulation of $H(\boldsymbol{c}) = \boldsymbol{E}\boldsymbol{c} - h$ to represent the phase condition. This approximate solution of the truncation of $\mathcal{F}$ is also an approximate solution of the full-dimensional approximation $\mathcal{F}^n : \mathbb{R}^m \times \ell^{1,m}_\nu \to \mathbb{C}^m \times \ell^{1,m}_{\nu+}$ of $\mathcal{F}$ given by

$$\mathcal{F}^n(\eta, \boldsymbol{c}) \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{E}\boldsymbol{c} - h \\ \boldsymbol{\pi}^n G(\eta, \boldsymbol{\pi}^n \boldsymbol{c}) \\ \boldsymbol{\Omega}_{\eta_0} \boldsymbol{\pi}^{n,\infty} \boldsymbol{c} \end{pmatrix}.$$

The benefit of working with this particular approximation $\mathcal{F}^n$ is that it is relatively easy to calculate its derivative and to invert this Jacobian approximately but accurately, as we discuss in the following section.

4.4 The approximate inverse

In this section we construct an operator $\mathcal{A} : \mathcal{X}_{\nu+}^{\mathbb{C},m} \to \mathcal{X}_{\nu}^{\mathbb{C},m}$ that approximates the inverse of $D\mathcal{F}$. We note that, since the domain of $\mathcal{F}$ was specifically chosen to be $\mathcal{X}_{\nu}^{\mathbb{R},m}$ (see Remark 3.2), we will need to make a few modifications to the theory behind Theorem 4.13 and Corollary 4.15 to obtain an analogous result for $\mathcal{A}$ and $\mathcal{F}$, see Section 4.5.

Remark 4.19. The diagonals $\Omega_{\hat{\vartheta}}, \Omega_{\hat{\vartheta}}^{-1}$ related to $\mathcal{F}^n$ all depend on the approximate frequency $\hat{\eta}^0 = \hat{\vartheta}$. We will from now on suppress the dependency of $\hat{\vartheta}$, except where needed.

We recall that in the examples covered in Section 3, the affine map $H(\boldsymbol{c}) = \boldsymbol{E}\boldsymbol{c} - h$ is always given by a simple summation over the coefficients of $\boldsymbol{c}$. When inverting the derivative of $\mathcal{F}^n$, we will see that this operator $\boldsymbol{E}$ will interact with $\Omega^{-1} = \Omega_{\hat{\vartheta}}^{-1}$. To deal with these interactions, we introduce the following operators.

Definition 4.20. *We define $E : \ell_{\nu}^1 \to \mathbb{C}$ by*

$$E : c \mapsto \sum_{k\in\mathbb{Z}} c_k.$$

Equivalently, see Remark 4.5, we can define $E \in \ell_{\nu}^{\infty}$ by $E_k = 1$ for all $k \in \mathbb{Z}$.

We define the row-operator $\omega^{-1} : \pi^{n,\infty}\ell_{\nu+}^1 \to \mathbb{C}$ by

$$\omega^{-1} = E\Omega^{-1}.$$

Note that $\omega^{-1} \in \pi^{n,\infty}\ell_{\nu+}^{\infty}$ is simply given by $\omega_k^{-1} = 1/ik\vartheta$ and hence $\Omega^{-1} = \operatorname{diag}(\omega^{-1})$.

Finally we define the operators $\boldsymbol{E} : \ell_{\nu}^{1,m} \to \mathbb{C}^m$ and $\boldsymbol{\omega}^{-1} : \boldsymbol{\pi}^{n,\infty}\ell_{\nu+}^{1,m} \to \mathbb{C}^m$ by

$$\boldsymbol{E} \overset{\text{def}}{=} \operatorname{diag}(E,\dots,E) \qquad \text{and} \qquad \boldsymbol{\omega}^{-1} = \operatorname{diag}(\omega^{-1},\dots,\omega^{-1}).$$

Based on the natural inclusion $\mathbb{C}^{m(2n+1)} \subset \ell_{\nu}^{1,m}$, we write $E = E\boldsymbol{\pi}^n + E\boldsymbol{\pi}^{n,\infty}$. With respect to this decomposition we will write $\boldsymbol{E}$ as a block row matrix by setting

$$\boldsymbol{E} = \begin{pmatrix} \boldsymbol{E}\boldsymbol{\pi}^n & \boldsymbol{E}\boldsymbol{\pi}^{n,\infty} \end{pmatrix}.$$

Here we interpret $\boldsymbol{E}\boldsymbol{\pi}^n$ as a finite $(m \times m(2n+1))$ matrix.

Using this notation the derivative of $\mathcal{F}^n$, can be written as a linear map $\mathcal{A}^\dagger : \mathcal{X}_{\nu}^{\mathbb{R},m} \to \mathcal{X}_{\nu+}^{\mathbb{C},m}$, given by

$$\mathcal{A}^\dagger \overset{\text{def}}{=} D\mathcal{F}^n(\hat{\boldsymbol{x}}) = \begin{pmatrix} 0 & \boldsymbol{E}\boldsymbol{\pi}^n & \boldsymbol{E}\boldsymbol{\pi}^{n,\infty} \\ V & A^\dagger & 0 \\ 0 & 0 & \boldsymbol{\Omega} \end{pmatrix},$$

where $A^\dagger$ and V are finite dimensional.

The operator $V : \mathbb{R}^m \to \boldsymbol{\pi}^n\ell_{\nu+}^{1,m}$ has a natural extension to an operator $\overline{V} : \mathbb{C}^m \to \boldsymbol{\pi}^n\ell_{\nu+}^{1,m}$ obtained by setting

$$\overline{V}(\boldsymbol{z}) = V\operatorname{Re}(\boldsymbol{z}) + iV\operatorname{Im}(\boldsymbol{z}).$$

This "complexification" of V then satisfies $V = \overline{V}\iota_m$, where $\iota_m : \mathbb{R}^m \to \mathbb{C}^m$ is the inclusion. We similarly have a natural extension of $\mathcal{A}^\dagger$ to an operator $\overline{\mathcal{A}^\dagger} : \mathcal{X}_\nu^{\mathbb{C},m} \to \mathcal{X}_{\nu+}^{\mathbb{C},m}$, which is given by

$$\overline{\mathcal{A}^\dagger} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \boldsymbol{E\pi}^n & \boldsymbol{E\pi}^{n,\infty} \\ \overline{V} & A^\dagger & 0 \\ 0 & 0 & \boldsymbol{\Omega} \end{pmatrix}, \tag{20}$$

satisfying $\mathcal{A}^\dagger = \overline{\mathcal{A}^\dagger}\iota$, where $\iota : \mathcal{X}_\nu^{\mathbb{R},m} \to \mathcal{X}_\nu^{\mathbb{C},m}$ is the inclusion.

If we then calculate (complex) matrices M, P, Q and A such that

$$\begin{pmatrix} M & P \\ Q & A \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{E\pi}^n \\ \overline{V} & A^\dagger \end{pmatrix} = J \approx \mathrm{Id}, \tag{21}$$

then it follows that $P\overline{V} \approx \mathrm{Id}_m$ (the $m \times m$ identity) and therefore $PV = P\overline{V}\iota_m \approx \iota_m$. Furthermore, we have that

$$\begin{pmatrix} M & P & -M\boldsymbol{E\pi}^{n,\infty}\boldsymbol{\Omega}^{-1} \\ Q & A & -Q\boldsymbol{E\pi}^{n,\infty}\boldsymbol{\Omega}^{-1} \\ 0 & 0 & \boldsymbol{\Omega}^{-1} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{E\pi}^n & \boldsymbol{E\pi}^{n,\infty} \\ \overline{V} & A^\dagger & 0 \\ 0 & 0 & \boldsymbol{\Omega} \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & \mathrm{Id} \end{pmatrix},$$

i.e., while the finite-dimensional part of $\overline{\mathcal{A}^\dagger}$ is inverted approximately, the infinite-dimensional part is inverted exactly.

Hence, if we assume that we have numerically calculated the matrices M, P, Q and A, then the approximate inverse of $\overline{\mathcal{A}^\dagger}$, denote by $\mathcal{A} : \mathcal{X}_{\nu+}^{\mathbb{C},m} \to \mathcal{X}_\nu^{\mathbb{C},m}$ is given by

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{pmatrix} M & P & -M\boldsymbol{\omega}^{-1} \\ Q & A & -Q\boldsymbol{\omega}^{-1} \\ 0 & 0 & \boldsymbol{\Omega}^{-1} \end{pmatrix}. \tag{22}$$

This construction ensures that $\mathcal{A}\overline{\mathcal{A}^\dagger} \approx \mathrm{Id}$, the identity on $\mathcal{X}_\nu^{\mathbb{C},m}$. Similarly, it follows that $\mathcal{A}\mathcal{A}^\dagger \approx \iota$.

Remark 4.21. This particular construction of the approximate inverse will allow us to obtain tight bounds on the terms originating from exact scalar equations. In particular, this method produces smaller bounds, and therefore improves on previous methods dealing with exact scalar equations [30, 52]. More generally, the approach described above is well suited to systems where exact phase-conditions are needed or where other quantities, like the energy level, need to be fixed, or where a (nontrivial) integral condition is appended. Hence, this approach could be useful when considering Hamiltonian systems where the energy needs to be fixed in order to isolate periodic orbits

4.5 Symmetry and real-valued solutions

Until now, we have solely focussed on using complex Fourier series to find solutions to our DDEs. What we have skimmed over thus far is the problem of making sure these solutions are real-valued. A function $u : \mathbb{R} \to \mathbb{C}$ defined by its Fourier series $c \in \ell_\nu^1$ takes on real values

if and only if c is symmetric in the sense that $c_{-k} = c_k^*$ for all $k \in \mathbb{Z}$, where given $z \in \mathbb{C}$, z^* denotes its complex conjugate. We therefore need an argument to show that the solution obtained from our computer assisted proof satisfies this symmetry. In practice, this result will follow from the following addendum to the Banach fixed point theorem.

Lemma 4.22. *Let X be a complete metric space and $B, S \subset X$ be closed. Let $T : X \to X$ be such that $T(B) \subseteq B$ and $T(S) \subseteq S$. If T is a contraction on B and $B \cap S \neq \emptyset$, then T has a unique fixed point in $B \cap S$.*

Proof. Clearly $T(B \cap S) \subseteq B \cap S$, hence T is also a contraction on $B \cap S$. The result then follows by direct application of the Banach fixed point theorem. □

From this it follows that we need to show that the space of symmetric solutions is closed and that the Newton-like operator constructed at the beginning of this section also respects this symmetry. The next couple of lemmas will enumerate the results needed to conclude this.

Let us denote the subset of symmetric sequences, $(\ell^1_\nu)^{\text{sym}} \subset \ell^1_\nu$ by

$$(\ell^1_\nu)^{\text{sym}} \stackrel{\text{def}}{=} \left\{ c \in \ell^1_\nu : c_{-k} = c_k^* \text{ for all } k \in \mathbb{Z} \right\}.$$

We then make the following observations.

Lemma 4.23. *Suppose $a, b \in (\ell^1_\nu)^{sym}$, then the following all hold.*

1. *$(\ell^1_\nu)^{sym}$ is a closed real subspace of ℓ^1_ν.*
2. *The convolution $a * b \in (\ell^1_\nu)^{sym}$.*
3. *If $L : \ell^1_\nu \to \ell^1_{\nu+}$ satisfies $L_{-j,-k} = L^*_{j,k}$ then $La \in (\ell^1_{\nu+})^{sym}$, hence L maps $(\ell^1_\nu)^{sym}$ to $(\ell^1_{\nu+})^{sym}$.*
4. *If $\phi \in \ell^\infty_\nu$ such that $\phi_{-k} = \phi^*_k$, then $\phi(a) \in \mathbb{R}$, hence ϕ maps $(\ell^1_\nu)^{sym}$ to $\mathbb{R}$.*
5. *For $\vartheta \in \mathbb{R}$, the diagonal operator $d_\tau(\vartheta)$ satisfies $d_\tau(\vartheta)a \in (\ell^1_\nu)^{sym}$, hence $d_\tau(\vartheta)$ maps $(\ell^1_\nu)^{sym}$ to itself.*
6. *The diagonal operators Ω and Ω^{-1} map $(\ell^1_\nu)^{sym}$ to $(\ell^1_{\nu+})^{sym}$ and $(\ell^1_{\nu+})^{sym}$ to $(\ell^1_\nu)^{sym}$, respectively.*

The symmetry on ℓ^1_ν extends naturally to a symmetry on $\ell^{1,m}_\nu$. The corresponding symmetric subspace we denote by $(\ell^{1,m}_\nu)^{\text{sym}} \subset \ell^{1,m}_\nu$. On this space we make the following observations.

Lemma 4.24. *Let $\mathcal{F}$ be as in Section 3 and let $\mathcal{A}$ be as in (22). Furthermore, let $T : \mathcal{X}^{\mathbb{R},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$ be given by $T(\boldsymbol{x}) = \boldsymbol{x} - \mathcal{A}\mathcal{F}(\boldsymbol{x})$. Then the following all hold.*

1. *$\mathcal{F}$ maps maps $\mathbb{R}^m \times (\ell^{1,m}_\nu)^{sym}$ to $\mathbb{R}^m \times (\ell^{1,m}_{\nu+})^{sym}$.*

2. *If we construct M such that $M \in \mathbb{R}^{m\times m}$, construct P, Q and A such that for $j_1, j_2 \in \{0,\dots,m-1\}$ and $|k_1|, |k_2| \le n$*

$$A^{j_1,j_2}_{-k_1,-k_2} = (A^{j_1,j_2}_{k_1,k_2})^*, \qquad P^{j_1,j_2}_{-k} = (P^{j_1,j_2}_{k})^* \qquad \text{and} \qquad Q^{j_1,j_2}_{-k} = (Q^{j_1,j_2}_{k})^*,$$

then $\mathcal{A}$ maps $\mathbb{R}^m \times (\ell^{1,m}_{\nu+})^{sym}$ to $\mathbb{R}^m \times (\ell^{1,m}_{\nu})^{sym}$.

3. *If $\mathcal{A}$ is as above, then $T(\boldsymbol{x}) = \boldsymbol{x} - \mathcal{A}\mathcal{F}(\boldsymbol{x})$ maps $\mathbb{R}^m \times (\ell^{1,m}_{\nu})^{sym}$ to $\mathbb{R}^m \times (\ell^{1,m}_{\nu})^{sym}$.*

We can now apply these observations, in combination with Lemma 4.22, to any solution found using the radii polynomials from Theorem 4.13, resulting in the following, modified version of Corollary 4.15.

Proposition 4.25. *Let $\mathcal{F} : \mathcal{X}^{\mathbb{R},m}_{\nu} \to \mathcal{X}^{\mathbb{C},m}_{\nu+}$ and $\mathcal{A} : \mathcal{X}^{\mathbb{C},m}_{\nu+} \to \mathcal{X}^{\mathbb{C},m}_{\nu}$ satisfy the symmetries from Lemma 4.23 and Lemma 4.24, and let $\mathcal{A}$ be injective. Let $T : \mathcal{X}^{\mathbb{R},m}_{\nu} \to \mathcal{X}^{\mathbb{C},m}_{\nu}$ be given by $T(\boldsymbol{x}) = \boldsymbol{x} - \mathcal{A}\mathcal{F}(\boldsymbol{x})$ and let $\hat{\boldsymbol{x}} \in \mathbb{R}^m \times (\ell^{1,m}_{\nu})^{sym}$. Furthermore suppose there exist $Y, Z_1 > 0$ and $Z_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|T(\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\|_{1,\nu} \le Y \tag{23a}$$

$$\|DT(\hat{\boldsymbol{x}})\| \le Z_1 \tag{23b}$$

and such that for every $r > 0$ and $\|\boldsymbol{y}\| \le 1$,

$$\|DT(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - DT(\hat{\boldsymbol{x}})\| \le Z_2(r)\|\boldsymbol{y}\|. \tag{23c}$$

If there exists a radius $\hat{r} > 0$ such that

$$Y + \left(Z_1 + \frac{1}{2}Z_2(\hat{r})\right)\hat{r} < \hat{r} \tag{24a}$$

$$Z_1 + Z_2(\hat{r}) < 1 \tag{24b}$$

then $\mathcal{F}$ has a unique zero in $\mathbb{R}^m \times (\ell^{1,m}_{\nu})^{sym}$.

Proof. Let us define $\mathcal{J} : \mathcal{X}^{\mathbb{C},m}_{\nu} \to \mathcal{X}^{\mathbb{R},m}_{\nu}$ by setting

$$\mathcal{J} = \begin{pmatrix} \mathrm{Re} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Let us now define $\tilde{T} : \mathcal{X}^{\mathbb{R},m}_{\nu} \to \mathcal{X}^{\mathbb{R},m}_{\nu}$ by setting $\tilde{T} = \mathcal{J}T$. Since $\|\mathcal{J}\| = 1$, it follows that

$$\|\tilde{T}(\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\|_{1,\nu} = \|\mathcal{J}T(\hat{\boldsymbol{x}}) - \mathcal{J}\hat{\boldsymbol{x}}\|_{1,\nu} \le \|\mathcal{J}\|\|T(\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\|_{1,\nu} = \|T(\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\|_{1,\nu},$$

where we used that $\mathcal{J}\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}$. Similarly, we have for every $\boldsymbol{x}$ that

$$\|D\tilde{T}(\boldsymbol{x})\| = \|\mathcal{J}DT(\boldsymbol{x})\| \le \|\mathcal{J}\|\|DT(\boldsymbol{x})\| = \|DT(\boldsymbol{x})\|.$$

Hence, if we have bounds Y and Z as above, then it follows that the same estimates hold for $\tilde{T}$. This implies, by Theorem 4.13, that $\tilde{T}$ is a contraction on some ball $B_r(\hat{\boldsymbol{x}})$ in $\mathcal{X}^{\mathbb{R},m}_{\nu}$, hence it must have a unique fixed point $\widetilde{\boldsymbol{x}} \in B_r(\hat{\boldsymbol{x}}) \subset \mathcal{X}^{\mathbb{R},m}_{\nu}$. Furthermore, since $\hat{\boldsymbol{x}} \in \mathbb{R}^m \times (\ell^{1,m}_{\nu})^{\mathrm{sym}}$ we have by Lemma 4.22 that $\widetilde{\boldsymbol{x}} \in \mathbb{R}^m \times (\ell^{1,m}_{\nu})^{\mathrm{sym}}$.

Now note that since $\tilde{T}(\widetilde{\boldsymbol{x}}) = \widetilde{\boldsymbol{x}}$ and since $\mathcal{J}\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}$, we must have that $\mathcal{J}\mathcal{A}\mathcal{F}(\widetilde{\boldsymbol{x}}) = 0$. By Lemma 4.24, $\mathcal{A}\mathcal{F}(\widetilde{\boldsymbol{x}}) \in \mathbb{R}^m \times (\ell^{1,m}_{\nu})^{\mathrm{sym}}$, hence it follows that $\mathcal{A}\mathcal{F}(\widetilde{\boldsymbol{x}}) = 0$. Because $\mathcal{A} : \mathcal{X}^{\mathbb{C},m}_{\nu} \to \mathcal{X}^{\mathbb{C},m}_{\nu}$ is injective, we conclude that $\mathcal{F}(\widetilde{\boldsymbol{x}}) = 0$. □

Since our map $\mathcal{F}$ preserves the symmetry, it follows that derivative does as well. Hence any "good" approximation of the derivative and its inverse should also preserve the symmetry. In practice we see that that when using standard methods of computing the (numerical parts of) the approximate inverse, we end up with a matrix that is indeed machine-precision (10^{-16}) close to a matrix satisfying the symmetry assumptions in Lemma 4.24. In other words, imposing the symmetry condition (also) on the numerical part of $\mathcal{A}$ does not "worsen" our approximate inverse in any significant way.

4.6 Pseudo-convolutions

As mentioned before, one of the main strengths of using ℓ^1 spaces when working with periodic solutions is the behaviour of the convolution product. As we will need to compute derivative of convolutions when computing the Z bounds, we will look in detail at some maps one can construct from convolutions.

Definition 4.26. *Let $a \in \ell^1_\nu$ be fixed. We define the convolution operator $C(a) : \ell^1_\nu \to \ell^1_\nu$ by*

$$C(a) : c \mapsto a * c.$$

The map $C(a)$ is clearly linear and bounded since $\|C(a)\| = \|a\|_{1,\nu}$. Furthermore, it can be represented by the matrix $C(a)_{j,k} = a_{j-k}$. In particular, if a has finite length, in the sense that $a \in \mathbb{C}^{2n+1} \subset \ell^1_\nu$, then $C(a)_{j,k} = 0$ whenever $|j-k| > n$.

As can be seen from Section 3, often these convolutions do not appear by themselves, but in expressions of the form

$$a * d_\tau(\vartheta)b.$$

In many ways, especially when it comes to ℓ^1 estimates, the map $(a,b) \mapsto a * d_\tau(\vartheta)b$ still behaves like a convolution. Based on this observation we introduce the notion of a pseudo-convolution and pseudo-convolution operators.

Definition 4.27. *Let $S \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ be such that $|S_{j,k}| \leq 1$ for all $j,k \in \mathbb{Z}$. We define for $a, b \in \ell^1_\nu$ the pseudo-convolution associated with S as*

$$(a *_S b)_k = \sum_{k_1+k_2=k} S_{k_1,k_2} a_{k_1} b_{k_2}.$$

For fixed $a \in \ell^1_\nu$, we then define the pseudo-convolution operator $\Gamma_S(a)$ by

$$\Gamma_S(a) : c \mapsto a *_S c.$$

It is immediately clear that if $S_{j,k} = 1$ for all $j,k \in \mathbb{Z}$, then $a *_S b = a * b$. Furthermore, the following results are easy to check.

Lemma 4.28. *Let $S \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ be such that $|S_{j,k}| \leq 1$ for all $j,k \in \mathbb{Z}$. Then the following all hold.*

- *For every $a, b \in \ell^1_\nu$, $a *_S b \in \ell^1_\nu$ and $\|a *_S b\|_{1,\nu} \leq \|a\|_{1,\nu}\|b\|_{1,\nu}$.*

- *For every* $a \in \ell^1_\nu$, $\Gamma_S(a) : \ell^1_\nu \to \ell^1_\nu$ *and* $\|\Gamma_S(a)\| \leq \|a\|_{1,\nu}$.
- *For every* $a \in \ell^1_\nu$ *and* $j, k \in \mathbb{Z}$ *we have* $\Gamma_S(a)_{j,k} = S_{j-k,k} a_{j-k}$.
- *For every* $a \in \ell^1_\nu$ *and* $j, k \in \mathbb{Z}$ *we have* $|\Gamma_S(a)_{j,k}| \leq |a_{j-k}|$.

The derivative and shift operators K and $d_\tau(\theta)$ also interact in a very elementary way with (pseudo)-convolutions. On the function space side of things, the derivative operator satisfies the Leibniz rule and the shift operator acts distributively on products. Hence we can expect similar identities to hold for the operators K and $d_\tau(\vartheta)$. This motivates the following result.

Lemma 4.29. *Let* $S \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$ *be such that* $|S_{j,k}| \leq 1$ *for all* $j, k \in \mathbb{Z}$. *Then for every* $a, b \in \ell^1_\nu$ *we have*

$$\begin{aligned} K(a *_S b) &= (Ka) *_S b + a *_S (Kb) \\ d_\tau(\vartheta)(a *_S b) &= (d_\tau(\vartheta)a) *_S (d_\tau(\vartheta)b). \end{aligned}$$

Equivalently, for every $a \in \ell^1_\nu$

$$\begin{aligned} K\Gamma_S(a) &= \Gamma_S(Ka) + \Gamma_S(a)K \\ d_\tau(\vartheta)\Gamma_S(a) &= \Gamma_S(d_\tau(\vartheta)a)d_\tau(\vartheta). \end{aligned}$$

Proof. These identities can be found by simply writing out their definitions. □

Finally, where more intricate terms are involved, it will be useful to have a notion of polynomials constructed from pseudo-convolutions.

Definition 4.30. *We say that* γ *is a pseudo-convolution polynomial if we can write* γ *as a finite linear combination of* $c^0, \ldots, c^m \in \ell^1_\nu$*:*

$$\gamma = \sum_{j=1}^{N} \zeta_j c^{\alpha_{0,j}} *_{S^{1,j}} \ldots *_{S^{k_j,j}} c^{\alpha_{k_j,j}},$$

where $\zeta_j \in \mathbb{C}$, $k_j \in \mathbb{N}$, $\alpha_{0,j}, \ldots, \alpha_{k_j,j} \in \{0, \ldots, m\}$ *and where each product* $*_{S^{1,j}}, \ldots, *_{S^{k_j,j}}$ *is a pseudo-convolution.*

Because the pseudo-convolution may change with every product in the polynomial, the above definition is rather cumbersome. What is more important is that we use the notion of a pseudo-convolution polynomial to generalise the Banach-algebra property.

Lemma 4.31. *Let* $c^0, \ldots, c^m \in \ell^1_\nu$ *and let* γ *be a pseudo-convolution polynomial in* $c^0, \ldots, c^m$. *Then* $\gamma \in \ell^1_\nu$ *and*

$$\|\gamma\|_{1,\nu} \leq \sum_{j=1}^{N} \prod_{k=0}^{k_j} |\zeta_j| \|c^{\alpha_{k,j}}\|_{1,\nu}.$$

4.7 Estimates of operators

We finish this section by listing a couple of lemmas that will prove essential in obtaining (relatively) sharp Y and Z bounds. Most of these estimates involve the projection operators, diagonal operators and row operators which we have defined before.

Lemma 4.32. *Let $\Omega^{-1} = \Omega_{\hat{\vartheta}}^{-1}$ be as in Definition 4.18 and let $K : \ell^1_\nu \to \ell^1_{\nu+}$ be as in Definition 3.1 (or Lemma 4.10). Let $\bar{\Omega}^{-1} : \ell^1_{\nu+} \to \ell^1_\nu$ be given by*

$$\bar{\Omega}^{-1} = \frac{1}{\hat{\vartheta}(n+1)}\pi^n + \Omega^{-1}\pi^{n,\infty},$$

then $\bar{\Omega}^{-1} : \ell^1_\nu \to \ell^1_\nu$ and $\bar{\Omega}^{-1}K : \ell^1_\nu \to \ell^1_\nu$ are bounded and

$$\|\bar{\Omega}^{-1}\| = \frac{1}{\hat{\vartheta}(n+1)}$$
$$\|\bar{\Omega}^{-1}K\| = \frac{1}{\hat{\vartheta}}.$$

Proof. These results follow directly from applying Lemma 4.8 to diagonal operators. □

Lemma 4.33. *Let $\bar{\Omega}^{-1}$ be as in the previous lemma, let $c \in \ell^1_\nu$ and let $\Gamma_S(c)$ be a pseudo-convolution operator. Then $\bar{\Omega}^{-1}\Gamma_S(c)K$ is bounded and for all $a \in \ell^1_\nu$*

$$\|\bar{\Omega}^{-1}\Gamma_S(c)Ka\| \leq \frac{C_\nu}{\hat{\vartheta}}\|a\|_{1,\nu}\|c\|_{1,\nu},$$

where the constant C_ν is given by

$$C_\nu \stackrel{\text{def}}{=} \begin{cases} \frac{\nu^{2n+2}}{e\log\nu^{2n+2}} & \text{if } \nu^{2n+2} < e \\ 1 & \text{if } \nu^{2n+2} \geq e. \end{cases}$$

Proof. First note that

$$\begin{aligned}
\|\bar{\Omega}^{-1}\Gamma_S(c)Ka\|_{1,\nu} &\leq \sum_{k'} \nu^{|k'|} \sum_k |\bar{\Omega}^{-1}_{k',k}| \sum_{k_1+k_2=k} |c_{k_1}||a_{k_2}||k_2| \\
&\leq \sum_{k'} \sum_{k_1,k_2} \nu^{|k'|}|\bar{\Omega}^{-1}_{k',k_1+k_2}||c_{k_1}||a_{k_2}||k_2| \\
&= \sum_{k_1,k_2} \sum_{k'} \nu^{|k'|-|k_1|-|k_2|}|\bar{\Omega}^{-1}_{k',k_1+k_2}||k_2|\nu^{|k_1|+|k_2|}|c_{k_1}||a_{k_2}| \\
&\leq \|a\|_{1,\nu}\|c\|_{1,\nu} \sup_{k_1,k_2} \sum_{k'} \nu^{|k'|-|k_1|-|k_2|}|\bar{\Omega}^{-1}_{k',k_1+k_2}||k_2|.
\end{aligned}$$

Since $|\bar{\Omega}^{-1}_{k',k}| = \delta_{k',k}\frac{1}{\hat{\vartheta}\max\{|k'|,n+1\}}$, hence we need to prove that C_ν is a bound for

$$\sup_{k_1,k_2\in\mathbb{Z}} \frac{\nu^{|k_1+k_2|-|k_1|-|k_2|}|k_2|}{\max\{|k_1+k_2|,n+1\}} = \sup_{k,m\in\mathbb{Z}} \frac{\nu^{|m|-|m-k|-|k|}|k|}{\max\{|m|,n+1\}}. \tag{25}$$

For notational convenience we introduce

$$\xi(k,m) \stackrel{\text{def}}{=} \frac{\nu^{|m|-|m-k|-|k|}|k|}{\max\{|m|,n+1\}}.$$

Since $\xi(-k,-m)=\xi(k,m)$ we may restrict attention to $m\geq 0$ and $k\in\mathbb{Z}$. We split the analysis in three cases: $k\leq 0\leq m$, $0<k<m$ and $k\geq m\geq 0$. In the first case ($k\leq 0\leq m$)

$$\xi(k,m)=\frac{\nu^{m-(m-k)+k}|k|}{\max\{m,n+1\}}=\frac{\nu^{2k}|k|}{\max\{m,n+1\}}\leq\frac{\sup_{\tilde{k}>0}\nu^{-2\tilde{k}}\tilde{k}}{\max\{m,n+1\}}.$$

Since

$$\sup_{\tilde{k}\geq 1}\nu^{-2\tilde{k}}\tilde{k}=\begin{cases}\frac{1}{2e\log\nu} & \text{if } \frac{1}{2\log\nu}\geq 1\\ \nu^{-2} & \text{if } \frac{1}{2\log\nu}<1,\end{cases}$$

we obtain for all $k\leq 0\leq m$ that

$$\xi(k,m)\leq\max\left\{\frac{1}{e\log\nu^{2n+2}},\frac{1}{\nu^2(n+1)}\right\}\leq C_\nu. \tag{26}$$

In the second case, $0<k<m$, we estimate

$$\xi(k,m)=\frac{\nu^{m-(m-k)-k}k}{\max\{m,n+1\}}=\frac{k}{\max\{m,n+1\}}\leq 1\leq C_\nu. \tag{27}$$

For the third case, $k\geq m\geq 0$, we have

$$\xi(k,m)=\frac{\nu^{2m-2k}k}{\max\{m,n+1\}}.$$

We now fix m and vary k:

$$\sup_{k\geq m}\nu^{-2k}k=\begin{cases}\frac{1}{2e\log\nu} & \text{if } \frac{1}{2\log\nu}\geq m\\ \nu^{-2m}m & \text{if } \frac{1}{2\log\nu}<m,\end{cases}$$

hence for all $k\geq m\geq 0$ we find

$$\xi(k,m)\leq\widehat{\xi}(m)\stackrel{\text{def}}{=}\begin{cases}\frac{\nu^{2m}}{2e\max\{m,n+1\}\log\nu} & \text{if } m\leq\frac{1}{2\log\nu}\\ 1 & \text{if } m>\frac{1}{2\log\nu}.\end{cases} \tag{28}$$

We now first consider the subcases $\nu^{2n+2}\geq e$ and $\nu^{2n+2}<e$ separately. The former corresponds to $n+1\geq\frac{1}{2\log\nu}$. Then, by varying over m in the right-hand side of (28) we obtain, since ν^{2m} is increasing in m,

$$\sup_{m\geq 0}\widehat{\xi}(m)\leq\max\left\{\frac{\nu^{2\frac{1}{2\log\nu}}}{2e(n+1)\log\nu},1\right\}=\max\left\{\frac{1}{\log\nu^{2n+2}},1\right\}=1. \tag{29}$$

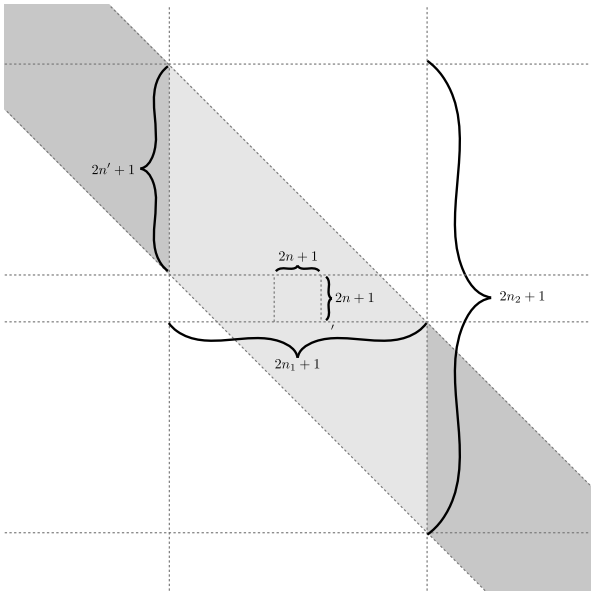

Figure 2: A sketch of the entries of the pseudo-convolution operator $\Gamma_S(a)$ as a $\mathbb{Z}\times\mathbb{Z}$ matrix. The light- and dark-grey part represents the nonzero entries of $\Gamma_S(a)$. The darker grey area represents the nonzero entries of the operator $\pi^{n,\infty}\Gamma_S(a)\pi^{n_1,\infty}$ for $a \in \mathbb{C}^{2n'+1} \subset \ell^1_\nu$ with $n < n'$ and $n_1 = n + n'$.

For the remaining subcase $\nu^{2n+2} < e$, i.e. $n+1 < \frac{1}{2\log\nu}$, we observe that $\frac{\nu^{2m}}{m}$ is decreasing in m for $m \le \frac{1}{2\log\nu}$. Hence

$$\sup_{m\ge 0} \widehat{\xi}(m) \le \max\left\{\frac{\nu^{2(n+2)}}{2e(n+1)\log\nu}, 1\right\} = \frac{\nu^{2n+2}}{e\log\nu^{2n+2}}. \tag{30}$$

The estimates (29) and (30) show that $\widehat{\xi}(m) \le C_\nu$, uniformly in m, hence (28) implies that $\xi(k,m) \le C_\nu$ for all $k \ge m \ge 0$. Combining this with (26) and (27) concludes the proof. □

Next we estimate several operators that appear in the calculation of the Z_1 bounds. These estimates involve cut-off versions of a pseudo-convolution operator $\Gamma_S(a)$. It may be helpful to consult Figure 2 for the role that the various indices play in this operator.

Lemma 4.34. *Let $n < n'$ and set $n_1 = n + n'$. Let $a \in \mathbb{C}^{2n'+1} \subset \ell^1_\nu$ be such that $|a_{-k}| = |a_k|$, let $\Gamma = \Gamma_S(a)$, with S as above and let $\Omega^{-1} = \Omega_{\hat\vartheta}^{-1}$ be as in Definition 4.18. Then $\Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty} : \ell^1_\nu \to \ell^1_\nu$ is bounded and*

$$\|\Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty}\| \le \frac{1}{\hat\vartheta}\sum_{k=-n'}^{n'} \frac{|a_k|}{n_1+1+k}\nu^k.$$

Proof. Let us write $M = \Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty}$, then clearly $M_{j,k} = 0$ whenever $|k| \le n_1$ or $|j| \le n$,

hence

$$\begin{aligned}\|M\| &= \sup_k \nu^{-|k|} \sum_j \nu^{|j|} \left|(\Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty})_{j,k}\right| \\ &= \sup_{|k|>n_1} \nu^{-|k|} \sum_{|j|>n} \nu^{|j|} \left|(\Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty})_{j,k}\right| \\ &= \sup_{|k|>n_1} \nu^{-|k|} \sum_{|j|>n} \frac{\nu^{|j|}}{\hat{\vartheta}|j|}|\Gamma_{j,k}| \\ &\leq \sup_{|k|>n_1} \frac{\nu^{-|k|}}{\hat{\vartheta}} \sum_{|j|>n} \frac{\nu^{|j|}}{|j|}|a_{j-k}|,\end{aligned}$$

where we used that $\Gamma_S(a)_{jk} = 0$ whenever $k > n_1$ and $j \leq n$. Since $|a_{j-k}| = |a_{k-j}|$, we can without loss of generality assume that $j > n$ and that $k > n_1 > n$, implying that

$$\begin{aligned}\|M\| &\leq \sup_{k>n_1} \frac{\nu^{-k}}{\hat{\vartheta}} \sum_{j>n} \frac{\nu^j}{j}|a_{j-k}| = \sup_{k>n_1} \frac{\nu^{-k}}{\hat{\vartheta}} \sum_{j=k-n'}^{k+n'} \frac{\nu^j}{j}|a_{j-k}| \\ &= \sup_{k>n_1} \frac{\nu^{-k}}{\hat{\vartheta}} \sum_{j=-n'}^{n'} \frac{\nu^{j+k}}{j+k}|a_j| = \sup_{k>n_1} \frac{1}{\hat{\vartheta}} \sum_{j=-n'}^{n'} \frac{\nu^j}{j+k}|a_j| \\ &= \frac{1}{\hat{\vartheta}} \sum_{j=-n'}^{n'} \frac{\nu^j}{j+n_1+1}|a_j|. \qquad \square\end{aligned}$$

Lemma 4.35. *Let $n < n'$ and set $n_1 = n + n'$. Let $a \in \mathbb{C}^{2n'+1} \subset \ell^1_\nu$ be such that $|a_{-k}| = |a_k|$, let $\Gamma = \Gamma_S(a)$, with S as above and let ω^{-1} be as in Definition 4.20. Let X be a Banach space and $V : \mathbb{C} \to X$ be a bounded linear operator. Then the map $V\omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty} : \ell^1_\nu \to X$ is bounded and*

$$\|V\omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty}\| \leq \|V\|\nu^{-(n+1)}\|\Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty}\|.$$

Proof. Note that since $\omega^{-1} = E\pi^{n,\infty}\Omega^{-1}$, we have

$$\|V\omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty}\| \leq \|V\|\|E\pi^{n,\infty}\|_{1,\infty}\|\Omega^{-1}\pi^{n,\infty}\Gamma\pi^{n_1,\infty}\|.$$

By combining this with $\|E\pi^{n,\infty}\|_{1,\infty} = \nu^{-(n+1)}$, the result follows. $\square$

5 Derivation of the Y and Z bounds for the Mackey-Glass equation

In this section we will explicitly construct all the bounds that are necessary to apply Theorem 4.13. In the following it will be helpful to recall from Section 4.3 that we consider $\mathcal{F} : \mathcal{X}_\nu^{\mathbb{R},m} \to \mathcal{X}_{\nu+}^{\mathbb{C},m}$ of the form

$$\mathcal{F}(\boldsymbol{x}) = \mathcal{F}(\eta, \boldsymbol{c}) = \begin{pmatrix} H(\boldsymbol{c}) \\ G(\eta, \boldsymbol{c}) \end{pmatrix},$$

where $H = (H^0, \dots, H^{m-1}) : \ell^{1,m}_\nu \to \mathbb{C}^m$ and $G = (G^0, \dots, G^{m-1}) : \mathbb{R}^m \times (\ell^1_\nu)^m \to (\ell^1_{\nu+})^m$. In particular, we have that H is given by the affine map $H : \boldsymbol{c} \mapsto \boldsymbol{E}\boldsymbol{c} - h$, while G can be written as

$$G(\eta, \boldsymbol{c}) = i\vartheta \boldsymbol{K}\boldsymbol{c} + \text{convolution terms},$$

where we use the notation $\eta_0 = \vartheta$, i.e., $\eta = (\vartheta, \eta_1, \dots, \eta_{m-1})$.

The methodology presented in this section results in general estimates for the class of functions described above, i.e., the bounds apply to arbitrary systems of polynomial DDEs (possibly after a reformulation as described in Section 2). For the parts of the estimates that are executed "term by term" (monomial by monomial in the pseudo-convolution polynomial, see Definition 4.30), we focus on deriving bounds for the Mackey-Glass equation from Example 3.4. Specifically, we are alluding here to the explicit expressions in the latter parts of Sections 5.2.3 and 5.2.4. The main simplification obtained from focusing on the Mackey-Glass system is that it only depends on one delayed variable (namely c^1). However, we stress that the methodology for obtaining the bounds, as presented in this section, is not at all restricted to this particular case.

We recall that for the Mackey-Glass system we have that $m = 4$ and using that $\eta = (\vartheta, \eta_1, \eta_2\eta_3)$, hence we can write the function $G = (G^0, \dots, G^3)$ as

$$G^j(\eta, \boldsymbol{c}) = i\vartheta K c^j + \Phi^j(\boldsymbol{c}) + \Psi^j(\boldsymbol{c}) * d_\tau(\vartheta)c^1 + L^j(\eta), \tag{31}$$

where $\Phi = (\Phi^0, \dots, \Phi^3)$ and $\Psi = (\Psi^0, \dots, \Psi^3)$ do not depend on η and where L^j is given by

$$L^j = \begin{cases} 0 & \text{if } j = 0, \\ -\eta_j & \text{if } j = 1, 2, 3. \end{cases}$$

Remark 5.1 (Other systems)**.** The construction of the bounds in following sections can be easily adjusted to suit the other examples from Section 2. In the case of Cooke's equation from Examples 2.1 and 3.3 we find that G is decribed by

$$G^j_{\text{Cooke}}(\eta, \boldsymbol{c}) = i\vartheta K c^j + \Phi^j(\boldsymbol{c}) + \Psi^j(\boldsymbol{c}) * d_\tau(\vartheta)(c^0 * c^1) + L^j(\eta),$$

where $\Phi = (\Phi^0, \Phi^1)$ and $\Psi = (\Psi^0, \Psi^1)$ do not depend on $\eta = (\vartheta, \eta_1)$ and where

$$L^j = \begin{cases} 0 & \text{if } j = 0, \\ -\eta_1 & \text{if } j = 1. \end{cases}$$

Similarly, for the Ikeda equation from Examples 2.6 and 3.5 we can describe G by

$$G^j_{\text{Ikeda}}(\eta, \boldsymbol{c}) = i\vartheta K c^j + \eta_1 \Phi^{1,j}(\boldsymbol{c}) + \eta_2 \Phi^{2,j}(\boldsymbol{c}) + \Psi^j(\boldsymbol{c}) * d_\tau(\vartheta)c^1,$$

where neither $\Psi = (\Psi^0, \Psi^1, \Psi^2)$ nor $\Phi^l = (0, \Phi^{l,1}, \Phi^{l,2})$ with $l = 1, 2$ depend on $\eta = (\vartheta, \eta_1, \eta_2)$.

5.1 The Y bounds

Recall from Theorem 4.13 that the Y bound must satisfy

$$\|T(\hat{\boldsymbol{x}}) - \hat{\boldsymbol{x}}\| = \|\mathcal{A}\mathcal{F}(\hat{\boldsymbol{x}})\| \leq Y,$$

where $T(\boldsymbol{x}) = \boldsymbol{x} - \mathcal{A}\mathcal{F}(\boldsymbol{x})$ as in (17).

Since $\hat{\boldsymbol{c}} \in \mathbb{C}^{m(2n+1)} \subset \ell^{1,m}_\nu$ and $G(\eta, \boldsymbol{c})$ is polynomial in $\boldsymbol{c}$ and $d_\tau(\vartheta)c^1$, there exists an $n' > 0$ such that $G^j(\hat{\eta}, \hat{\boldsymbol{c}})_k = 0$ for all $|k| > n'$ and $j = 0, 1, 2, 3$. In particular, for the Mackey-Glass equation, $n' = 4n$, since the Mackey-Glass system (10) is polynomial of degree 4.

From this it follows that $\mathcal{F}(\hat{\eta}, \hat{\boldsymbol{c}})$ only has *finitely* many nonzero terms, and using n' as above,

$$G(\hat{\eta}, \hat{\boldsymbol{c}}) = \boldsymbol{\pi}^{n'} G(\hat{\eta}, \hat{\boldsymbol{c}}) \in \mathbb{C}^{m(2n'+1)}.$$

Therefore all the nonzero elements of $\mathcal{A}\mathcal{F}(\boldsymbol{x})$ are given by

$$\begin{pmatrix} M & P & -M\boldsymbol{\omega}^{-1}\boldsymbol{\pi}^{n,n'} \\ Q & A & -Q\boldsymbol{\omega}^{-1}\boldsymbol{\pi}^{n,n'} \\ 0 & 0 & \boldsymbol{\pi}^{n,n'}\boldsymbol{\Omega}^{-1}\pi^{n,n'} \end{pmatrix} \begin{pmatrix} H(\hat{\boldsymbol{c}}) \\ \boldsymbol{\pi}^n G(\hat{\eta}, \hat{\boldsymbol{c}}) \\ \boldsymbol{\pi}^{n,n'} G(\hat{\eta}, \hat{\boldsymbol{c}}) \end{pmatrix} \in \mathbb{C}^m \times \mathbb{C}^{m(2n'+1)} \subset \mathcal{X}^{\mathbb{C},m}_\nu.$$

In conclusion, the computation of the Y-bound reduces to a finite number of computations, i.e., we can use interval arithmetic to compute Y such that

$$\left\| \begin{pmatrix} M & P & -M\boldsymbol{\omega}^{-1}\boldsymbol{\pi}^{n,n'} \\ Q & A & -Q\boldsymbol{\omega}^{-1}\boldsymbol{\pi}^{n,n'} \\ 0 & 0 & \boldsymbol{\pi}^{n,n'}\boldsymbol{\Omega}^{-1}\boldsymbol{\pi}^{n,n'} \end{pmatrix} \begin{pmatrix} H(\hat{\boldsymbol{c}}) \\ \boldsymbol{\pi}^n G(\hat{\eta}, \hat{\boldsymbol{c}}) \\ \boldsymbol{\pi}^{n,n'} G(\hat{\eta}, \hat{\boldsymbol{c}}) \end{pmatrix} \right\| \leq Y. \tag{32}$$

5.2 The Z bounds.

Recall from Theorem 4.13 that $Z_1 \geq 0$ must satisfy

$$\|DT(\hat{\boldsymbol{x}})\| = \|\iota - \mathcal{A}D\mathcal{F}(\hat{\boldsymbol{x}})\| \leq Z_1,$$

where $\iota : \mathcal{X}^{\mathbb{R},m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$ is the natural inclusion. Since we have constructed the approximate inverse $\mathcal{A}$ by means of an approximate derivative $\mathcal{A}^\dagger \approx D\mathcal{F}(\hat{\boldsymbol{x}})$, it is natural to decompose the Z_1 estimate as

$$\|\iota - \mathcal{A}D\mathcal{F}(\hat{\boldsymbol{x}})\| \leq \underbrace{\|\iota - \mathcal{A}\mathcal{A}^\dagger\|}_{\leq \mathcal{Z}^0} + \underbrace{\|\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}}) - \mathcal{A}^\dagger)\|}_{\leq \mathcal{Z}^1}. \tag{33}$$

Likewise, $Z_2 : \mathbb{R}_+ \to \mathbb{R}_+$ must satisfy, for all $\|\boldsymbol{y}\| \leq 1$,

$$\|DT(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - DT(\hat{\boldsymbol{x}})\| = \|\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - D\mathcal{F}(\hat{\boldsymbol{x}}))\| \leq Z_2(r)\|\boldsymbol{y}\|.$$

The most straightforward way to estimate this is by using a version of the mean-value theorem. This requires the computation of a second derivative of $\mathcal{F}$. In particular, this means that we will encounter terms involving the operator $\partial^2_\vartheta d_\tau(\vartheta) = -\tau^2 K^2 d_\tau(\vartheta)$, hence we need estimates

for terms involving $\|\Omega^{-1}(K^2 a) *_S b\|$, for unknown $a, b \in \ell^1_\nu$. Unfortunately, this cannot be done, since $\Omega^{-1}K^2$ is not bounded on ℓ^1_ν and the convolution does not help to temper this.

To circumvent this problem, we write $\boldsymbol{y} = (\lambda, \boldsymbol{a}) \in \mathcal{X}_\nu^{\mathbb{R},m}$ (with $\lambda \in \mathbb{R}^m$ and $\boldsymbol{a} \in \ell^{1,m}_\nu$), introduce the variable $\overline{\boldsymbol{y}} = (\lambda, 0) \in \mathcal{X}_\nu^{\mathbb{R},m}$, and decompose the Z_2 bound as

$$\begin{aligned}\|DT(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - DT(\hat{\boldsymbol{x}})\| \leq \\ \underbrace{\|\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - D\mathcal{F}(\hat{\boldsymbol{x}} + r\overline{\boldsymbol{y}}))\|}_{\leq \mathcal{Z}^2(r)} \|\boldsymbol{y}\| + \underbrace{\|\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\overline{\boldsymbol{y}}) - D\mathcal{F}(\hat{\boldsymbol{x}}))\|}_{\leq \mathcal{Z}^3(r)} \|\boldsymbol{y}\|. \end{aligned} \tag{34}$$

Note that the $\mathcal{Z}^2$ bound only requires differentiation with respect to η once. Conversely, in the $\mathcal{Z}^3$ bound we *do* differentiate with respect to η twice. However, as we will be made evident in Section 5.2.4, where we compute this bound, the K^2 operator that appears in this bound will solely act on the components of $\hat{\boldsymbol{c}}$, not on those of $\boldsymbol{a}$. Since $\hat{\boldsymbol{c}} = (\hat{c}^0, \dots, \hat{c}^{m-1})$ only has finitely many nonzero elements, we have that $K^2\hat{c}^j \in \ell^1_\nu$ for every $0 \leq j \leq m-1$, hence this term poses no problems.

We conclude this section by providing step-by-step estimates for each of the terms in (33) and (34).

5.2.1 The $\mathcal{Z}^0$ bound

This bound is by far the simplest Z bound to compute, hence we will be concise in our derivation. We first observe that, analogously to Corollary 4.12,

$$\|\iota - \mathcal{A}\mathcal{A}^\dagger\| = \|\iota - \mathcal{A}\overline{\mathcal{A}^\dagger}\iota\| \leq \|\operatorname{Id} - \mathcal{A}\overline{\mathcal{A}^\dagger}\|\|\iota\| \leq \|\operatorname{Id} - \mathcal{A}\overline{\mathcal{A}^\dagger}\|,$$

where $\overline{\mathcal{A}^\dagger} : \mathcal{X}_\nu^{\mathbb{C},m} \to \mathcal{X}_{\nu+}^{\mathbb{C},m}$ is the natural extension of $\mathcal{A}^\dagger : \mathcal{X}_\nu^{\mathbb{R},m} \to \mathcal{X}_{\nu+}^{\mathbb{C},m}$, see (20), and where Id is the identity on $\mathcal{X}_\nu^{\mathbb{C},m}$.

Next, we note that $\boldsymbol{\Omega}^{-1}\boldsymbol{\Omega} = \boldsymbol{\pi}^{n,\infty}$, hence we can write

$$\begin{aligned}\mathcal{A}\overline{\mathcal{A}^\dagger} &= \begin{pmatrix} M & P & -M\boldsymbol{\omega}^{-1} \\ Q & A & -Q\boldsymbol{\omega}^{-1} \\ 0 & 0 & \boldsymbol{\Omega}^{-1} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{E}\boldsymbol{\pi}^n & \boldsymbol{E}\boldsymbol{\pi}^{n,\infty} \\ V & A^\dagger & 0 \\ 0 & 0 & \boldsymbol{\Omega} \end{pmatrix} \\ &= \begin{pmatrix} P\overline{V} & M\boldsymbol{E}\boldsymbol{\pi}^n + PA^\dagger & 0 \\ A\overline{V} & Q\boldsymbol{E}\boldsymbol{\pi}^n + AA^\dagger & 0 \\ 0 & 0 & \operatorname{Id}_{n,\infty} \end{pmatrix},\end{aligned}$$

where $\operatorname{Id}_{n,\infty}$ denotes the identity on $\boldsymbol{\pi}^{n,\infty}\ell^{1,m}_\nu$. This means that the $\mathcal{Z}^0$ bound must satisfy

$$\|\operatorname{Id} - \mathcal{A}\overline{\mathcal{A}^\dagger}\| = \left\| \begin{pmatrix} \operatorname{Id}_m - P\overline{V} & -M\boldsymbol{E}\boldsymbol{\pi}^n - PA^\dagger & 0 \\ -A\overline{V} & \operatorname{Id}_{m(2n+1)} - Q\boldsymbol{E}\boldsymbol{\pi}^n - AA^\dagger & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\| \leq \mathcal{Z}^0.$$

In conclusion, we obtain a linear operator, whose nonzero part is given by a $m(2n+2) \times m(2n+2)$-matrix and whose norm can thus be calculated, with a finite computation, using Corollary 4.11. This norm provides us with the $\mathcal{Z}^0$ bound. In view of (21) this bound is tiny in practice.

5.2.2 The $\mathcal{Z}^1$ bound

We wish to find a bound $\mathcal{Z}^1$ satisfying

$$\|\mathcal{A}(DF(\hat{\boldsymbol{x}}) - \mathcal{A}^\dagger)\| \leq \mathcal{Z}^1.$$

Let us denote the derivative with respect to the Fourier coordinates by $D_{\boldsymbol{c}}$ and the derivative with respect to the scalar variables as D_η. Next, we introduce the shorthand

$$D_\eta G = D_\eta G(\hat{\eta}, \hat{\boldsymbol{c}})$$
$$D_{\boldsymbol{c}} G = D_{\boldsymbol{c}} G(\hat{\eta}, \hat{\boldsymbol{c}}).$$

We then consider

$$DF(\hat{\boldsymbol{x}}) - \mathcal{A}^\dagger = \begin{pmatrix} 0 & \boldsymbol{E}\boldsymbol{\pi}^n & \boldsymbol{E}\boldsymbol{\pi}^{n,\infty} \\ \boldsymbol{\pi}^n D_\eta G & \boldsymbol{\pi}^n D_{\boldsymbol{c}} G \boldsymbol{\pi}^n & \boldsymbol{\pi}^n D_{\boldsymbol{c}} G \boldsymbol{\pi}^{n,\infty} \\ \boldsymbol{\pi}^{n,\infty} D_\eta G & \boldsymbol{\pi}^{n,\infty} D_{\boldsymbol{c}} G \boldsymbol{\pi}^n & \boldsymbol{\pi}^{n,\infty} D_{\boldsymbol{c}} G \boldsymbol{\pi}^{n,\infty} \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{E}\boldsymbol{\pi}^n & \boldsymbol{E}\boldsymbol{\pi}^{n,\infty} \\ V & A^\dagger & 0 \\ 0 & 0 & \boldsymbol{\Omega} \end{pmatrix}.$$

We choose n_1 such that each component $G(\hat{\boldsymbol{x}})_k$ with $|k| \leq n$ only depends on those x_k with $|k| \leq n_1$. Furthermore, we choose $n_2 \geq n_1$ such that $G(\hat{\boldsymbol{x}})_k$ with $|k| \leq n_2$ depends only on x_k with $|k| \leq n_1$. For Mackey-Glass, $n_1 = n' + n = 5n$ and $n_2 = 2n' + n = 9n$, see also Figure 2.

Then we split the derivative into further blocks to obtain

$$DF(\hat{\boldsymbol{x}}) - \mathcal{A}^\dagger = \begin{pmatrix} \tilde{\boldsymbol{\Omega}}0 & 0 & 0 & 0 \\ \boldsymbol{\pi}^n D_\eta G - V & \boldsymbol{\pi}^n D_{\boldsymbol{c}} G \boldsymbol{\pi}^n - A^\dagger & \boldsymbol{\pi}^n D_{\boldsymbol{c}} G \boldsymbol{\pi}^{n,n_1} & 0 \\ \boldsymbol{\pi}^{n,n_2} D_\eta G & \boldsymbol{\pi}^{n,n_2} D_{\boldsymbol{c}} G \boldsymbol{\pi}^n & \boldsymbol{\pi}^{n,n_2} (D_{\boldsymbol{c}} G - \tilde{\boldsymbol{\Omega}}) \boldsymbol{\pi}^{n,n_1} & \boldsymbol{\pi}^{n,n_2} (D_x G - \tilde{\boldsymbol{\Omega}}) \boldsymbol{\pi}^{n_1,\infty} \\ 0 & 0 & 0 & \boldsymbol{\pi}^{n_2,\infty} (D_{\boldsymbol{c}} G - \tilde{\boldsymbol{\Omega}}) \boldsymbol{\pi}^{n_1,\infty} \end{pmatrix}.$$

Note that here we used the extended $\tilde{\boldsymbol{\Omega}}$, which satiesfies $\pi^{n,\infty} \tilde{\boldsymbol{\Omega}} = \boldsymbol{\Omega}$.

Hence, after multiplying with the approximate inverse, we can write this as

$$\mathcal{A}(DF(\hat{\boldsymbol{x}}) - \mathcal{A}^\dagger) = \left(\begin{array}{c|c} m(2n_2+2) \times m(2n_1+2) & \begin{matrix} -M\boldsymbol{\omega}^{-1} \boldsymbol{\pi}^{n,\infty} (D_{\boldsymbol{c}} G - \tilde{\boldsymbol{\Omega}}) \boldsymbol{\pi}^{n_1,\infty} \\ -Q\boldsymbol{\omega}^{-1} \boldsymbol{\pi}^{n,\infty} (D_{\boldsymbol{c}} G - \tilde{\boldsymbol{\Omega}}) \boldsymbol{\pi}^{n_1,\infty} \end{matrix} \\ \hline 0 & \boldsymbol{\Omega}^{-1} \boldsymbol{\pi}^{n,\infty} (D_{\boldsymbol{c}} G - \tilde{\boldsymbol{\Omega}}) \boldsymbol{\pi}^{n_1,\infty} \end{array}\right). \tag{35}$$

The block in the top-left corner is a finite matrix representing a map from $\mathbb{R}^m \times \mathbb{C}^{m(2n_1+2)}$ to $\mathbb{C}^m \times \mathbb{C}^{m(2n_2+2)}$, whose norm we can calculate using Corollaries 4.11 and 4.12. Let us denote this norm by $\mathcal{Z}^1_{\text{finite}}$. To analyse the block on the top-right, we write

$$W = \begin{pmatrix} M \\ Q \end{pmatrix},$$

where W is a $(m(2n+2) \times m)$ matrix and we denote $W^j \in \mathbb{C}^m \times \mathbb{C}^{m(2n+1)} \subset \mathcal{X}_\nu^{\mathbb{C},m}$ as the j-th "column" of W. Furthermore, we denote $\boldsymbol{\Gamma} = D_{\boldsymbol{c}} G - \tilde{\boldsymbol{\Omega}}$ and observe that, since the $i\vartheta K$

terms cancel exactly, we can view this operator as a $m \times m$ block-matrix, such that each of the blocks $\Gamma^{j,j'} : \ell^1_\nu \to \ell^1_\nu$ is a pseudo-convolution operator for $j, j' = 0, \dots, m-1$.

Then we can view the top-right block in (35) as a $1 \times m$ block-matrix, where each block is an operator given by

$$\left[W\boldsymbol{\omega}^{-1}\boldsymbol{\pi}^{n,\infty}\boldsymbol{\Gamma}\boldsymbol{\pi}^{n_1,\infty}\right]^{j'} = \sum_{j=0}^{m-1} W^j \pi^{n,\infty}\omega^{-1}\Gamma^{j,j'}\pi^{n_1,\infty} \qquad \text{for } j' = 0, \dots, m-1.$$

Similarly we can view the bottom-right block in (35) as a $m \times m$ block-matrix, where each block is an operator is given by

$$\left[\boldsymbol{\Omega}^{-1}\boldsymbol{\pi}^{n,\infty}\boldsymbol{\Gamma}\boldsymbol{\pi}^{n_1,\infty}\right]^{j,j'} = \Omega^{-1}\pi^{n,\infty}\Gamma^{j,j'}\pi^{n_1,\infty} \qquad \text{for } j, j' = 0, \dots, m-1.$$

When we combine this with the top-right operator, we find that we can estimate norm of the full right-block of (35) by

$$\max_{0 \le j' \le m-1} \left(\sum_{j=0}^{m-1} \left\| W^j \pi^{n,\infty}\omega^{-1}\Gamma^{j,j'}\pi^{n_1,\infty} \right\| + \sum_{j=0}^{m-1} \left\| \Omega^{-1}\pi^{n,\infty}\Gamma^{j,j'}\pi^{n_1,\infty} \right\| \right).$$

This expression can in turn be estimated by a constant $\mathcal{Z}^1_\infty$ that satisfies

$$\mathcal{Z}^1_\infty \geq \max_{0 \le j' \le m-1} \sum_{j=0}^{m-1} \Lambda^{j,j'},$$

where

$$\Lambda^{j,j'} \stackrel{\text{def}}{=} (1 + \|W^j\|\nu^{-(n+1)}) \left\| \Omega^{-1}\pi^{n,\infty}\Gamma^{j,j'}\pi^{n_1,\infty} \right\|. \tag{36}$$

Here we have applied Lemma 4.35. The final factor in (36) can be estimated by using Lemma 4.34. Hence we can calculate bounds $\mathcal{Z}^1_{\text{finite}}$ and $\mathcal{Z}^1_\infty$ such that

$$\|\mathcal{A}(DF(\hat{\boldsymbol{x}}) - \mathcal{A}^\dagger)\| \leq \mathcal{Z}^1 = \max\{\mathcal{Z}^1_{\text{finite}}, \mathcal{Z}^1_\infty\}.$$

5.2.3 The $\mathcal{Z}^2$ bound

Recall now that the $\mathcal{Z}^2$ bound requires us to estimate, for $\|\boldsymbol{y}\| = \|(\lambda, \boldsymbol{a})| \leq 1$,

$$\|\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - D\mathcal{F}(\hat{\boldsymbol{x}} + r\overline{\boldsymbol{y}}))\| = \|\mathcal{A}(D\mathcal{F}(\hat{\eta} + r\lambda, \hat{\boldsymbol{c}} + r\boldsymbol{a}) - D\mathcal{F}(\hat{\eta} + r\lambda, \hat{\boldsymbol{c}}))\|.$$

Let us also take $\boldsymbol{z} = (\mu, \boldsymbol{b}) \in \mathcal{X}^{\mathbb{R},m}_\nu$ and write

$$\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\boldsymbol{y}) - D\mathcal{F}(\hat{\boldsymbol{x}} + r\overline{\boldsymbol{y}}))\boldsymbol{z} = \int_0^r \frac{d}{ds}\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta} + r\lambda + t\mu, \hat{\boldsymbol{c}} + s\boldsymbol{a} + t\boldsymbol{b})\Big|_{t=0} ds.$$

Hence it suffices to obtain an estimate for all $\|\boldsymbol{y}\|, \|\boldsymbol{z}\| \leq 1$ and $0 \leq s \leq r$ of

$$\left\| \frac{d}{ds} \frac{d}{dt} \mathcal{A}\mathcal{F}(\hat{\eta} + r\lambda + t\mu, \hat{\boldsymbol{c}} + s\boldsymbol{a} + t\boldsymbol{b}) \Big|_{t=0} \right\|. \tag{37}$$

Now recall that we set $\mathcal{F} = (H, G)$, where H represents the phase/boundary conditions. The function H is linear, which implies that all second derivatives of H vanish. This means that we can, without loss of generality, look at how these second derivatives act on G only. In particular, this means that it suffices to look at the part of $\mathcal{A}$ that acts on ℓ^1_ν, i.e.,

$$\begin{pmatrix} P & -ME\boldsymbol{\pi}^{n,\infty}\boldsymbol{\Omega}^{-1} \\ A & -QE\boldsymbol{\pi}^{n,\infty}\boldsymbol{\Omega}^{-1} \\ 0 & \boldsymbol{\Omega}^{-1} \end{pmatrix}. \tag{38}$$

Observe that by defining $\mathcal{A}_0 : \ell^{1,m}_\nu \to \mathcal{X}^{\mathbb{C},m}_\nu$, $\tilde{\Omega} : \ell^1_{\nu+} \to \ell^1_\nu$ and $\tilde{\boldsymbol{\Omega}} : \ell^{1,m}_{\nu+} \to \ell^{1,m}_\nu$ such that

$$\begin{aligned} \mathcal{A}_0 &= \begin{pmatrix} \hat{\vartheta}(n+1)P & -ME\boldsymbol{\pi}^{n,\infty} \\ \hat{\vartheta}(n+1)A & -QE\boldsymbol{\pi}^{n,\infty} \\ 0 & \mathrm{Id} \end{pmatrix}, \\ \bar{\Omega}^{-1} &= \begin{pmatrix} \frac{1}{\hat{\vartheta}} \frac{1}{n+1} \mathrm{Id} & 0 \\ 0 & \Omega^{-1} \end{pmatrix}, \\ \bar{\boldsymbol{\Omega}}^{-1} &= \mathrm{diag}(\bar{\Omega}^{-1}, \dots, \bar{\Omega}^{-1}), \end{aligned}$$

the operator in (38) becomes equal to simply $\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}$. Writing M, P, Q and A as $m \times m$ block-matrices, we find using Corollary 4.11 that

$$\begin{aligned} \|\mathcal{A}_0\| = \max_{0 \leq j_2 \leq m-1} \Bigg\{ \hat{\vartheta}(n+1) \max_{|k_2| \leq n} \nu^{-|k_2|} \sum_{j_1=0}^{m-1} \left(|P^{j_1,j_2}_{k_2}| + \sum_{|k_1| \leq n} \nu^{|k_1|} |A^{j_1,j_2}_{k_1,k_2}| \right), \\ 1 + \nu^{-|(n+1)|} \sum_{j_1=0}^{m-1} \left(|M^{j_1,j_2}| + \sum_{|k_1| \leq n} \nu^{|k_1|} |Q^{j_1,j_2}_{k_1}| \right) \Bigg\}. \end{aligned}$$

Remark 5.2. The decomposition of the block-matrix from (38) into $\mathcal{A}_0$ and $\bar{\boldsymbol{\Omega}}^{-1}$ is actually optimal from a Banach-algebra perspective, i.e. $\|\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\| = \|\mathcal{A}_0\|\|\bar{\boldsymbol{\Omega}}^{-1}\|$. This is due to the fact that the sup (over the sums of the columns) appearing in Corollary 4.11, is attained in the inner-most columns of each block (i.e. in the $(n+1)$-th and $(-n-1)$-th column).

Let us now apply this notation to (37). We then have that

$$\begin{aligned} &\left\| \frac{d}{ds} \frac{d}{dt} \mathcal{A}\mathcal{F}(\hat{\eta} + r\lambda + t\mu, \hat{\boldsymbol{c}} + s\boldsymbol{a} + t\boldsymbol{b}) \Big|_{t=0} \right\| \\ &\qquad = \left\| \frac{d}{ds} \frac{d}{dt} \mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1} G(\hat{\eta} + r\lambda + t\mu, \hat{\boldsymbol{c}} + s\boldsymbol{a} + t\boldsymbol{b}) \Big|_{t=0} \right\|. \end{aligned}$$

Since we have constructed the norm on $\mathcal{X}_\nu^{\mathbb{C},m}$ by taking the sum of ℓ^1 norms, the above norms can now be decomposed as

$$\begin{aligned}&\left\|\frac{d}{ds}\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta}+r\lambda+t\mu,\hat{\boldsymbol{c}}+s\boldsymbol{a}+t\boldsymbol{b})\Big|_{t=0}\right\| \\ &\qquad\leq \|\boldsymbol{y}\| \max_{0\leq j,j'\leq m-1}\Big\{\left\|\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\partial_{\eta_j}\partial_{c^{j'}}G(\hat{\eta}+r\lambda,\hat{\boldsymbol{c}}+s\boldsymbol{a})\right\|, \\ &\qquad\qquad\left\|\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\partial_{c^j c^{j'}}G(\hat{\eta}+r\lambda,\hat{\boldsymbol{c}}+s\boldsymbol{a})\right\|\Big\},\end{aligned} \tag{39}$$

where we used that $\|\boldsymbol{z}\| = \|(\mu,\boldsymbol{b})\| \leq 1$ and that $\|\boldsymbol{a}\| \leq \|\boldsymbol{y}\| = \|(\lambda,\boldsymbol{a})\|$, and where we understand the norm of the bilinear operator $\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\partial_{c^j c^{j'}}G(\ldots) : \ell_\nu^{1,m}\times\ell_\nu^{1,m}\to\ell_\nu^{1,m}$ to be defined as

$$\|\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\partial_{c^j c^{j'}}G(\ldots)\| \stackrel{\text{def}}{=} \sup_{\|a'\|,\|b'\|\leq 1}\|\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\partial_{c^j c^{j'}}G(\ldots)(a',b')\|_{1,\nu}.$$

The Mackey-Glass equation

Let us now apply this to the Mackey-Glass system as given in (31). We first observe that the Mackey-Glass system only depends linearly on η_1,η_2 and η_3, hence all second partial derivatives involving differentiation to at least one of these variables will vanish. Only the $\eta_0=\vartheta$ term will be relevant.

Furthermore, we see that (39) has repeated occurrences of terms involving $\hat{\boldsymbol{c}}+s\boldsymbol{a}\in\ell_\nu^{1,4}$. To simplify some notation involving these terms, it will be useful to introduce the notation

$$\tilde{\boldsymbol{a}}_s = \hat{\boldsymbol{c}}+s\boldsymbol{a}.$$

While we do know now that $\|\tilde{\boldsymbol{a}}_s\|\leq\|\hat{\boldsymbol{c}}\|+s\|\boldsymbol{a}\|\leq\|\hat{\boldsymbol{c}}\|+r$, we do not a-priori know the value of r. In fact, r can only be determined *after* we have properly calculated all Y and Z bounds. However, we can make the assumption that $r\leq r^*$ for some $r^*>0$. If we then calculate all bounds using this assumption, and then verify that there exists an $\hat{r}\leq r^*$ satisfying Theorem 4.13, then the assumption that $r\leq r^*$ was justified. Hence, we will from now on assume the existence of such an $r^*>0$, and therefore $\|\tilde{\boldsymbol{a}}_s\|\leq\|\hat{\boldsymbol{c}}\|+r^*$. Note that for the components $\tilde{\boldsymbol{a}}_s=(\tilde{a}_s^0,\ldots,\tilde{a}_s^{m-1})\in\ell_\nu^{1,4}$, we then also know that $\|\tilde{a}_s^j\|_{1,\nu}\leq\|\hat{c}^j\|_{1,\nu}+r^*$. Likewise, we will denote

$$\tilde{\vartheta}_r = \hat{\vartheta}+r\lambda_0.$$

If we apply all of the above to the Mackey-Glass equation, as described in (31), and recall that we use $\eta_0=\vartheta$, then we see that (39) becomes

$$\begin{aligned}&\left\|\frac{d}{ds}\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta}+r\lambda+t\mu,\hat{\boldsymbol{c}}+s\boldsymbol{a}+t\boldsymbol{b})\Big|_{t=0}\right\| \leq \\ &\|\boldsymbol{y}\|\|\mathcal{A}_0\| \max_{0\leq j,j'\leq 3}\Big\{\sum_{l=0}^{3}\left\|\bar{\Omega}^{-1}\partial_\vartheta\partial_{c^{j'}}\left[i\tilde{\vartheta}_r K\tilde{a}_s^l+\Phi^l(\tilde{\boldsymbol{a}}_s)+\Psi^l(\tilde{\boldsymbol{a}}_s)*d_\tau(\tilde{\vartheta}_r)\tilde{a}_s^1\right]\right\|, \\ &\qquad\sum_{l=0}^{3}\left\|\bar{\Omega}^{-1}\partial_{c^j c^{j'}}\left[i\tilde{\vartheta}_r K\tilde{a}_s^l+\Phi^l(\tilde{\boldsymbol{a}}_s)+\Psi^l(\tilde{\boldsymbol{a}}_s)*d_\tau(\tilde{\vartheta}_r)\tilde{a}_s^1\right]\right\|\Big\}.\end{aligned}$$

We then find that for arbitrary $a', b' \in \ell^{1,4}_\nu$,

$$\partial_\vartheta \partial_{c^{j'}} \left[i\tilde{\vartheta}_r K \tilde{a}^l_s + \Phi^l(\tilde{\boldsymbol{a}}_s) \right] a' = iK\delta_{j',l} a',$$
$$\partial_{c^j c^{j'}} \left[i\tilde{\vartheta}_r K \tilde{a}^l_s + \Phi^l(\tilde{\boldsymbol{a}}_s) \right] b'a' = \partial_{c^j c^{j'}} \Phi^l(\tilde{\boldsymbol{a}}_s) b'a',$$

and

$$\begin{aligned}
\partial_\vartheta \partial_{c^{j'}} \left[\Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \right] a' &= -i\tau \big(\partial_{c^{j'}} \Psi^l(\tilde{\boldsymbol{a}}_s) a' \big) * d_\tau(\tilde{\vartheta}_r) K \tilde{a}^1_s - i\tau \delta_{1,j'} \Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) K a', \\
\partial_{c^j} \partial_{c^{j'}} \left[\Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \right] b'a' &= \big(\partial_{c^j c^{j'}} \Psi^l(\tilde{\boldsymbol{a}}_s) b'a' \big) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \\
&\quad + \delta_{1,j'} D_{c^j} \Psi^l(\tilde{\boldsymbol{a}}_s) b' * d_\tau(\tilde{\vartheta}_r) a' + \delta_{1,j} D_{c^{j'}} \Psi^l(\tilde{\boldsymbol{a}}_s) a' * d_\tau(\tilde{\vartheta}_r) b',
\end{aligned}$$

with all other derivatives, i.e., those with respect to η_1, η_2 and η_3, vanishing. This means that for $0 \le j, j', l \le 3$, there exist pseudo-convolution polynomials, in the sense of Definition 4.30, $\gamma^{jj'l}_{1\&4} = \gamma^{jj'l}_{1\&4}(\tilde{\boldsymbol{a}}_s, \boldsymbol{a}, \boldsymbol{b})$, $\gamma^{jj'}_2 = \gamma^{jj'}_2(\tilde{\boldsymbol{a}}_s, \boldsymbol{a})$, and $\gamma^l_3 = \gamma^l_3(\tilde{\boldsymbol{a}}_s)$, where only γ_4 involves actual pseudo-convolutions, such that

$$\begin{aligned}
\left\| \bar{\Omega}^{-1} \partial_\vartheta \partial_{c^{j'}} \left[i\tilde{\vartheta}_r K \tilde{a}^l_s + \Phi^l(\tilde{\boldsymbol{a}}_s) \right] a' \right\|_{1,\nu} &\le \delta_{j',l} \|\bar{\Omega}^{-1} K a'\|_{1,\nu}, \\
\left\| \bar{\Omega}^{-1} \partial_{c^j c^{j'}} \left[i\tilde{\vartheta}_r K \tilde{a}^l_s + \Phi^l(\tilde{\boldsymbol{a}}_s) \right] b'a' \right\|_{1,\nu} &\le \|\bar{\Omega}^{-1} \gamma^{jj'l}_1\|_{1,\nu}, \\
\left\| \bar{\Omega}^{-1} \partial_\vartheta \partial_{c^{j'}} \left[\Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \right] a' \right\|_{1,\nu} &\le \tau \left\| \bar{\Omega}^{-1} \Gamma_S(\gamma^{j'l}_2) K \tilde{a}^1_s \right\|_{1,\nu} + \tau \delta_{1,j'} \left\| \bar{\Omega}^{-1} \Gamma_S(\gamma^l_3) K a' \right\|_{1,\nu}, \\
\left\| \bar{\Omega}^{-1} \partial_{c^j} \partial_{c^{j'}} \left[\Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \right] b'a' \right\|_{1,\nu} &\le \|\bar{\Omega}^{-1} \gamma^{jj'l}_4\|_{1,\nu}.
\end{aligned}$$

Here Γ_S is the pseudo-convolution operator given by $\Gamma_S(a) : c \mapsto a * d_\tau(\tilde{\vartheta}_r) c$. When we take $\|a'\|, \|b'\| \le 1$ and repeatedly apply Lemmas 4.32 and 4.33, we obtain

$$\begin{aligned}
\left\| \bar{\Omega}^{-1} \partial_\vartheta \partial_{c^{j'}} \left[i\tilde{\vartheta}_r K \tilde{a}^l_s + \Phi^l(\tilde{\boldsymbol{a}}_s) \right] a' \right\|_{1,\nu} &\le \delta_{j',l} \frac{1}{\hat{\vartheta}}, \\
\left\| \bar{\Omega}^{-1} \partial_{c^j c^{j'}} \left[i\tilde{\vartheta}_r K \tilde{a}^l_s + \Phi^l(\tilde{\boldsymbol{a}}_s) \right] b'a' \right\|_{1,\nu} &\le \frac{1}{\hat{\vartheta}} \frac{1}{n+1} \|\gamma^{jj'l}_1\|_{1,\nu}, \\
\left\| \bar{\Omega}^{-1} \partial_\vartheta \partial_{c^{j'}} \left[\Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \right] a' \right\|_{1,\nu} &\le \tau \frac{C_\nu}{\hat{\vartheta}} \left(\|\gamma^{j'l}_2\|_{1,\nu} \|\tilde{a}^1_s\|_{1,\nu} + \delta_{1,j'} \|\gamma^l_3\|_{1,\nu} \right), \\
\left\| \bar{\Omega}^{-1} \partial_{c^j} \partial_{c^{j'}} \left[\Psi^l(\tilde{\boldsymbol{a}}_s) * d_\tau(\tilde{\vartheta}_r) \tilde{a}^1_s \right] b'a' \right\|_{1,\nu} &\le \frac{1}{\hat{\vartheta}} \frac{1}{n+1} \|\gamma^{jj'l}_4\|_{1,\nu},
\end{aligned}$$

where the norms of the (pseudo-)convolution polynomials γ can be computed using Lemma 4.31.

In conclusion, we can calculate $\mathcal{Z}^2$ by finding bounds such that $\|\gamma^{jj'l}_i\| \le \sigma^{jj'l}_i$, $\|\gamma^{jj'}_i\| \le \sigma^{jj'}_i$, and $\|\gamma^l_i\| \le \sigma^l_i$ and setting

$$\begin{aligned}
\mathcal{Z}^2(r) \ge r \frac{\|\mathcal{A}_0\|}{\hat{\vartheta}} \max_{0 \le j, j' \le 3} \Bigg\{ & 1 + \tau C_\nu \sum_{l=0}^{3} \left(\sigma^{j'l}_2 (\|\hat{c}^1\|_{1,\nu} + r^*) + \delta_{1,j'} \sigma^l_3 \right), \\
& \frac{1}{n+1} \sum_{l=0}^{3} \left(\sigma^{jj'l}_1 + \sigma^{jj'l}_4 \right) \Bigg\},
\end{aligned} \tag{40}$$

where the right-hand side of (40) depends linearly on r, and where we restrict our attention to $r \le r^*$.

5.2.4 The $\mathcal{Z}^3$ bound

Finally, we calculate the $\mathcal{Z}^3$ bound, which requires us to estimate

$$\|\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\overline{\boldsymbol{y}}) - D\mathcal{F}(\hat{\boldsymbol{x}}))\| = \|\mathcal{A}(D\mathcal{F}(\hat{\eta} + r\lambda, \hat{\boldsymbol{c}}) - D\mathcal{F}(\hat{\eta}, \hat{\boldsymbol{c}}))\|.$$

As many of the derivations are similar to those done for the $\mathcal{Z}^2$ bounds, we will be a little more succinct in deriving these bounds.

We again take $\boldsymbol{z} = (\mu, \boldsymbol{b}) \in \mathcal{X}_\nu^{\mathbb{R},m}$ and write

$$\mathcal{A}(D\mathcal{F}(\hat{\boldsymbol{x}} + r\overline{\boldsymbol{y}}) - D\mathcal{F}(\hat{\boldsymbol{x}}))\boldsymbol{z} = \int_0^r \frac{d}{ds}\,\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta} + s\lambda + t\mu, \hat{\boldsymbol{c}} + t\boldsymbol{b})\Big|_{t=0} ds.$$

Hence it suffices to obtain an estimate for all $\|\boldsymbol{y}\|, \|\boldsymbol{z}\| \leq 1$ and $0 \leq s \leq r$ of

$$\left\| \frac{d}{ds}\,\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta} + s\lambda + t\mu, \hat{\boldsymbol{c}} + t\boldsymbol{b})\Big|_{t=0} \right\|. \tag{41}$$

Again the phase/boundary conditions from the linear component H drop out, meaning that it suffices to look only at the G component in $\mathcal{F} = (H, G)$. If we again write $\mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}$ for the matrix in (38) then we find that (41) can be estimated as follows:

$$\begin{aligned}
&\left\| \frac{d}{ds}\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta} + s\lambda + t\mu, \hat{\boldsymbol{c}} + t\boldsymbol{b})\Big|_{t=0} \right\| \\
&\quad \leq \|\boldsymbol{y}\| \max_{0\leq j,j'\leq m-1} \left\{ \left\| \mathcal{A}_0\tilde{\boldsymbol{\Omega}}\partial_{\eta_j,\eta_{j'}} G(\hat{\eta} + s\lambda, \hat{\boldsymbol{c}}) \right\|, \left\| \mathcal{A}_0\bar{\boldsymbol{\Omega}}^{-1}\partial_{\eta_j}\partial_{c^{j'}} G(\hat{\eta} + s\lambda, \hat{\boldsymbol{c}}) \right\| \right\},
\end{aligned} \tag{42}$$

where we used that $|\lambda| \leq \|(\lambda, \boldsymbol{a})\| = \|\boldsymbol{y}\|$ and that $|\mu_j| \leq 1$.

The Mackey-Glass equation

Let us now, as with the $\mathcal{Z}^2$ bounds, focus on the Mackey-Glass problem (31). Then again the partial derivatives with respect to η_1, η_2 and η_3 vanish. Furthermore, we write $\tilde{\vartheta}_s = \hat{\vartheta} + s\lambda_0$ and conclude that $|\tilde{\vartheta}_s| \leq \hat{\vartheta} + r^*$. We see that (42) becomes

$$\begin{aligned}
&\left\| \frac{d}{ds}\frac{d}{dt}\mathcal{A}\mathcal{F}(\hat{\eta} + s\lambda + t\mu, \hat{\boldsymbol{c}} + t\boldsymbol{b})\Big|_{t=0} \right\| \\
&\quad \leq \|\boldsymbol{y}\|\|\mathcal{A}_0\| \max_{0\leq j'\leq 3} \left\{ \sum_{l=0}^{3} \left\| \bar{\Omega}^{-1}\partial_\vartheta^2 \left[i\tilde{\vartheta}_s K\hat{\boldsymbol{c}}^l + \Phi^l(\hat{\boldsymbol{c}}) + \Psi^l(\hat{\boldsymbol{c}}) * d_\tau(\tilde{\vartheta}_s)\hat{c}^1 \right] \right\|, \right. \\
&\qquad\qquad \left. \sum_{l=0}^{3} \left\| \bar{\Omega}^{-1}\partial_\vartheta\partial_{c^{j'}} \left[i\tilde{\vartheta}_s K\hat{\boldsymbol{c}}^l + \Phi^l(\hat{\boldsymbol{c}}) + \Psi^l(\hat{\boldsymbol{c}}) * d_\tau(\tilde{\vartheta}_s)\hat{c}^1 \right] \right\| \right\}.
\end{aligned} \tag{43}$$

When we write out the partial derivatives in the above estimate, we find, as in Section 5.2.3, that for arbitrary $b' \in \ell_\nu^{1,4}$,

$$\begin{aligned}
\partial_\vartheta^2 \left[i\tilde{\vartheta}_s K\hat{\boldsymbol{c}}^l + \Phi^l(\hat{\boldsymbol{c}}) \right] &= 0, \\
\partial_\vartheta\partial_{c^{j'}} \left[i\tilde{\vartheta}_s K\hat{\boldsymbol{c}}^l + \Phi^l(\hat{\boldsymbol{c}}) \right] b' &= i\delta_{j',l} Kb',
\end{aligned}$$

and likewise

$$\partial_\vartheta^2\left[\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)\hat{c}^1\right]=-\tau^2\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)K^2\hat{c}^1,$$
$$\partial_\vartheta\partial_{c^{j'}}\left[\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)\hat{c}^1\right]b'=-i\tau(\partial_{c^{j'}}\Psi^l(\hat{\boldsymbol{c}})b')*d_\tau(\tilde{\vartheta}_s)K\hat{c}^1-i\tau\delta_{1,j'}\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)Kb'.$$

Since $\hat{c}^1$ has only finitely many nonvanishing components, the new term involving the K^2 operator does not cause any problems.

Using the above identities, we see that there exist pseudo-convolution polynomials $\gamma_5^l=\gamma_5^l(\hat{\boldsymbol{c}})$, $\tilde{\gamma}_2^{j'l}=\gamma_2^{j'l}(\hat{\boldsymbol{c}},\boldsymbol{b})$ and $\tilde{\gamma}_3^l=\gamma_3^l(\hat{\boldsymbol{c}})$, such that the following inequalities hold:

$$\left\|\bar{\Omega}^{-1}\partial_\vartheta^2\left[i\tilde{\vartheta}_sK\hat{\boldsymbol{c}}^l+\Phi^l(\hat{\boldsymbol{c}})\right]\right\|_{1,\nu}=0,$$
$$\left\|\bar{\Omega}^{-1}\partial_\vartheta\partial_{c^{j'}}\left[i\tilde{\vartheta}_sK\hat{\boldsymbol{c}}^l+\Phi^l(\hat{\boldsymbol{c}})\right]b'\right\|_{1,\nu}\leq\delta_{j',l}\|\bar{\Omega}^{-1}Kb'\|_{1,\nu},$$
$$\left\|\bar{\Omega}^{-1}\partial_\vartheta^2\left[\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)\hat{c}^1\right]\right\|_{1,\nu}\leq\tau^2\|\bar{\Omega}^{-1}\Gamma_S(\gamma_5^l)K^2\hat{c}^1\|_{1,\nu},$$
$$\left\|\bar{\Omega}^{-1}\partial_\vartheta\partial_{c^{j'}}\left[\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)\hat{c}^1\right]b'\right\|_{1,\nu}\leq\tau\|\bar{\Omega}^{-1}\Gamma_S(\tilde{\gamma}_2^{j'l})K\hat{c}^1\|_{1,\nu}+\delta_{1,j'}\tau\|\bar{\Omega}^{-1}\Gamma_S(\tilde{\gamma}_3^l)Kb'\|_{1,\nu},$$

where Γ_S is the pseudo-convolution operator is given by $\Gamma_S(a):c\mapsto a*d_\tau(\tilde{\vartheta}_r)c$. If we then again apply Lemmas 4.32 and 4.33, we find that

$$\left\|\bar{\Omega}^{-1}\partial_\vartheta^2\left[i\tilde{\vartheta}_sK\hat{\boldsymbol{c}}^l+\Phi^l(\hat{\boldsymbol{c}})\right]\right\|_{1,\nu}=0,$$
$$\left\|\bar{\Omega}^{-1}\partial_\vartheta\partial_{c^{j'}}\left[i\tilde{\vartheta}_sK\hat{\boldsymbol{c}}^l+\Phi^l(\hat{\boldsymbol{c}})\right]b'\right\|_{1,\nu}\leq\delta_{j',l}\frac{1}{\hat{\vartheta}},$$
$$\left\|\bar{\Omega}^{-1}\partial_\vartheta^2\left[\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)\hat{c}^1\right]\right\|_{1,\nu}\leq\tau^2\frac{1}{\hat{\vartheta}}\frac{1}{n+1}\|\gamma_5^l\|_{1,\nu}\|K^2\hat{c}^1\|_{1,\nu},$$
$$\left\|\bar{\Omega}^{-1}\partial_\vartheta\partial_{c^{j'}}\left[\Psi^l(\hat{\boldsymbol{c}})*d_\tau(\tilde{\vartheta}_s)\hat{c}^1\right]b'\right\|_{1,\nu}\leq\tau\frac{1}{\hat{\vartheta}}\frac{1}{n+1}\|\tilde{\gamma}_2^{j'l}\|_{1,\nu}\|K\hat{c}^1\|_{1,\nu}+\delta_{1,j'}\frac{1}{\hat{\vartheta}}C_\nu\tau\|\tilde{\gamma}_3^l\|_{1,\nu}.$$

Here it should be noted that since $\hat{\boldsymbol{c}}\in\mathbb{C}^{4(2n+1)}$, we have that $K\hat{c}^j\in\ell^1_\nu$ and likewise $K^2\hat{c}^j\in\ell^1_\nu$, meaning that all norms above can be bounded explicitly. Specifically, if we compute bounds such that $\|\gamma_5^l\|=\sigma_5^l$, $\|\tilde{\gamma}_2^{j'l}\|\leq\tilde{\sigma}_2^{j'l}$ and $\|\tilde{\gamma}_3^l\|\leq\tilde{\sigma}_3^l$, then we can calculate $\mathcal{Z}^3$ by setting

$$\mathcal{Z}^3(r)\geq r\frac{\|\mathcal{A}_0\|}{\hat{\vartheta}}\max_{0\leq j'\leq 3}\left\{1+\tau\sum_{l=0}^{3}\left(\frac{\|K\hat{c}^1\|_{1,\nu}}{n+1}\tilde{\sigma}_2^{j'l}+\delta_{1,j'}C_\nu\tilde{\sigma}_3^l\right),\tau^2\frac{\|K^2\hat{c}^1\|_{1,\nu}}{n+1}\sum_{l=0}^{3}\sigma_5^l\right\},$$

where we note that the right-hand side depends linearly on r.

6 Numerics and results

In this final section of the paper, we exhibit the numerical solutions and bounds necessary to complete the existence proof for several parameter choices of the Mackey-Glass equation. In particular, we provide explicit values of the Y and Z bounds outlined in the previous section as well as discuss the limitations of this approach.

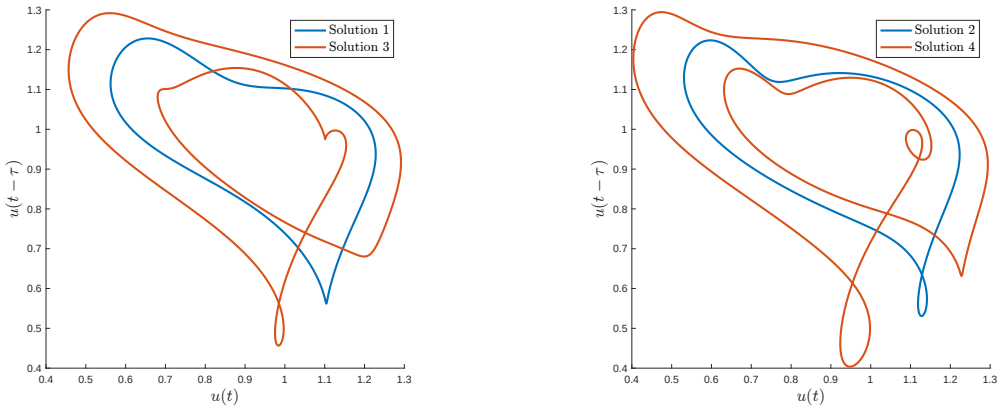

Figure 3: All four verified solutions plotted using delay embedding coordinates. Left: Solutions №1 and №3 from Table 1, corresponding to $\tau = 1.63$. Right: Solutions №2 and №4 from Table 1, corresponding to $\tau = 2$.

6.1 Control parameters

By means of the polynomialization scheme from Section 2, we were able to rewrite the Mackey-Glass equation as a system of equations outlined in (10). However, it is clear that the chosen polynomialization is not unique. In particular, we could have chosen to rescale the newly introduced auxiliary functions $v(t), w(t)$ and $x(t)$ by some factors. Doing so provides us with more choices when attempting to compute optimal Y and Z bounds for this problem, without fundamentally changing the problem. To this end, we rescale the v, w and x functions by a constant μ_1, μ_2 and μ_3 by simply setting $v = v_{\text{old}}/\mu_1$, $w = w_{\text{old}}/\mu_2$ and $x = x_{\text{old}}/\mu_3$.

Remark 6.1. Recalling that (u, v, w, x) correspond to the Fourier coefficients $(c^0, c^1, c^2, c^3) \in \ell^{1,4}_\nu$, this rescaling effectively changes the contribution of the c^1, c^2 and c^3 to the norm of $\boldsymbol{c}$. Hence, *equivalently*, we could have replaced the norm on $\ell^{1,4}_\nu$ by

$$\|\boldsymbol{c}\| = \|c^0\| + \mu_1\|c^1\| + \mu_2\|c^2\| + \mu_3\|c^3\|,$$

while leaving the functions v, w, x (and hence c^1, c^2, c^3) unchanged.

Using the new parameters μ_1, μ_2, μ_3, we obtain the following system, equivalent to (10):

$$\begin{aligned}
u'(t) &= \alpha\mu_1 v(t-\tau) - \beta u(t), && u(0) = 1,\\
v'(t) &= v(t)\left(\mu_3 x(t) - \rho\mu_1\mu_2 v(t)w(t)\right)\left(\alpha\mu_1 v(t-\tau) - \beta u(t)\right) + \eta_1, && v(0) = 1/2\mu_1,\\
w'(t) &= (\rho-2)\mu_3 x(t)w(t)\left(\alpha\mu_1 v(t-\tau) - \beta u(t)\right) + \eta_2, && w(0) = 1/\mu_2,\\
x'(t) &= -\mu_3 x(t)^2\left(\alpha\mu_1 v(t-\tau) - \beta u(t)\right) + \eta_3, && x(0) = 1/\mu_3.
\end{aligned} \tag{44}$$

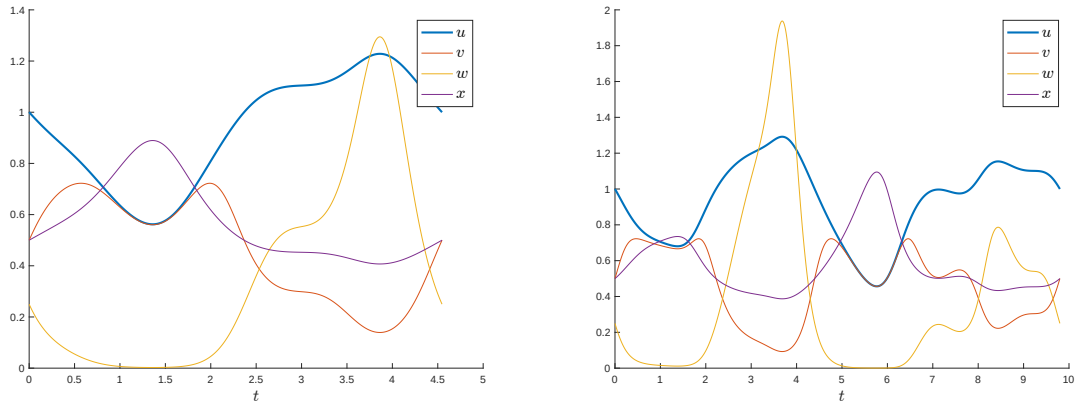

Figure 4: Two solutions of (44) for $\tau = 1.63$, $\rho = 10$, corresponding to solutions №1 with $\hat{\vartheta} = 1.3819$ (left) and №3 with $\hat{\vartheta} = 0.6412$ (right) from Table 1.

The corresponding map $\mathcal{F} : \mathcal{X}_\nu^{\mathbb{R},m} \to \mathcal{X}_\nu^{\mathbb{C},m}$ is then given by

$$\mathcal{F}(\eta, \boldsymbol{c}) \stackrel{\text{def}}{=} \begin{pmatrix} \sum_k c_k^0 - 1 \\ \sum_k c_k^1 - 1/2\mu_1 \\ \sum_k c_k^2 - 1/\mu_2 \\ \sum_k c_k^3 - 1/\mu_3 \\ i\vartheta K c^0 - (\alpha\mu_1 d_\tau(\vartheta) c^1 - \beta c^0) \\ i\vartheta K c^1 - (\mu_3 c^1 * c^3 - \rho\mu_1\mu_2 c^1 * c^1 * c^2) * (\alpha\mu_1 d_\tau(\vartheta) c^1 - \beta c^0) - \eta_1\delta_0 \\ i\vartheta K c^2 - (\rho - 2)\mu_3 c^2 * c^3 * (\alpha\mu_1 d_\tau(\vartheta) c^1 - \beta c^0) - \eta_2\delta_0 \\ i\vartheta K c^3 + \mu_3 c^3 * c^3 * (\alpha\mu_1 d_\tau(\vartheta) c^1 - \beta c^0) - \eta_3\delta_0 \end{pmatrix}. \quad (45)$$

It should be noted these new parameters are easily absorbed in the estimates from Section 5.

6.2 Results

In order to demonstrate the efficacy of our method, we prove the existence of some solutions both close to, and inside the chaotic regime of the Mackey-Glass equation. As we noted in the introduction (see also [31, 32]), chaos occurs when the equilibrium solution lies close to $u = 1$, hence we choose our parameters such that $u = 1$ is an equilibrium solution, which implies that $\alpha = 2\beta$. Since we can always rescale time (and the delay τ), we fix $\alpha = 2$ and $\beta = 1$.

We will now focus on two particular choices for ρ and τ. In order to find the numerical solutions around which we base our computer-assisted proof, we fix $\rho = 10$ and $\tau = 1.63$. Using the `dde23` integrator from MATLAB we observe a stable periodic orbit. This "*long*" stable solution is the product of two subsequent period doubling bifurcations when considering ρ as an increasing parameter, see [31]. We produce two more solutions, with periods equal to roughly a half and a quarter of the period of the stable orbit, by using Newton's method (on the truncated finite-dimensional problem). We call these the *medium* length and *short* periodic orbit, respectively. Via standard continuation methods we then continue the three

№	$\hat{\vartheta}$	ν	$\hat{n}+n_{\text{pad}}$	Y	$\mathcal{Z}^0$	$\mathcal{Z}^1$	$r^{-1}\mathcal{Z}^2(r)$	$r^{-1}\mathcal{Z}^3(r)$	$\hat{r}$
				$\times 10^{-10}$	$\times 10^{-11}$		$\times 10^{6}$	$\times 10^{5}$	$\times 10^{-7}$
1	1.3819	1.040	$70+0$	3.0774	6.6639	0.4213	2.8645	4.3377	1.6668
2	1.1932	1.040	$73+0$	4.2520	5.5870	0.4480	2.7036	3.9099	1.6945
				$\times 10^{-10}$	$\times 10^{-9}$		$\times 10^{8}$	$\times 10^{7}$	$\times 10^{-9}$
3	0.6412	1.010	$144+180$	2.8112	1.9989	0.3618	3.3467	4.5488	1.5947
4	0.5251	1.096	$177+130$	2.7836	2.5673	0.4612	2.8173	3.7215	1.6048

Table 1: The parameters, bounds and $\hat{r}$ that satisfy Proposition 4.25. Solutions №1 and №3 correspond to $\tau = 1.63$ and $\rho = 10$. Solutions №2 and №4 correspond to $\tau = 2$ and $\rho = 9.65$.

numerical approximate solutions to the chaotic parameter regime. In particular, we obtain approximate solutions for $\tau = 2$ and $\rho = 9.65$.

In both these cases, we are able to prove a pair of co-existing periodic solutions, corresponding to the short and medium length orbits. In particular, we prove the existence of a pair of solutions, corresponding to solutions №1 and №3 presented in Figure 4, with parameter values $\tau = 1.63$ and $\rho = 10$, and another pair, solutions №2 and №4 in Figure 4, with $\tau = 2$ and $\rho = 9.65$. In either case we are unable to verify the existence of the long stable periodic orbit with the current set of estimates. In Section 6.3 we discuss the cause of this issue, as well as possible ways to resolve it.

The following results were obtained for the Mackey-Glass equation as outlined in (44), where we chose the (experimentally obtained) rescaling factors: $\mu_1 = 1$, $\mu_2 = 4$ and $\mu_3 = 2$. In our numerical computations, we also have to decide how many Fourier coefficients we compute with. In practice, we choose n such that we can be reasonably sure that any coefficient of the solution outside of our truncation range is smaller than machine precision. That is, we use the largest n such that $\max_{j=0,1,2,3} |\hat{c}_n^j| \geq 10^{-16}$. This n may not be the optimal number of modes to run the computer-assisted proof. Especially the $\mathcal{Z}^1$ component of the Z_1 bound can usually only be made small by ensuring that n is big. We do this by padding $\hat{\boldsymbol{c}}$ with zeros until the desired length is obtained. In Table 1 we denote by $\hat{n}$ the number of Fourier modes used to compute $\hat{\boldsymbol{c}}$, and n_{pad} the number of zeros padded.

We will choose the a priori radius r^*, introduced in Section 5.2.3 and used in (40), such that $r^* = 10^{-6}$. As can be seen from Table 1, this is clearly larger than the computed $\hat{r}$, justifying this choice.

Next, we compute the approximate derivative in accordance with the restrictions outlined in Section 4.4 and 4.5. Using this, we compute the Y, Z_1 and Z_2 bounds as outlined in Section 5. It is at this step that the constant ν, appearing in the norm, is chosen experimentally such that both Y and Z_1 are suitably small. Finally, we compute $\hat{r}$ and verify that the inequalities from (19) hold. This, by Proposition 4.25, proves the existence of real valued periodic solutions near the numerical solutions from Figures 4 and 5. The Y and Z bounds, together with the appropriate choices for ν and the resulting $\hat{r}$ can be found in Table 1.

Note that solution №3 and №4 need respectively $n = 324$ and $n = 307$ elements to complete the proof. In the calculation of the $\mathcal{Z}^1$ bound, this results in having to multiply with a 9512×6056-matrix and a 10960×6712-matrix, respectively.

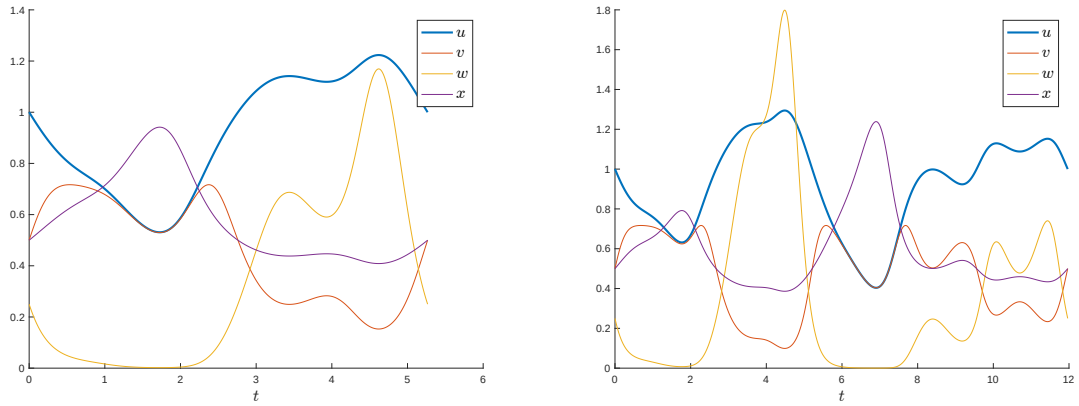

Figure 5: Two solutions of (44) for $\tau = 2$, $\rho = 9.65$, corresponding to solutions №2 with $\hat{\vartheta} = 1.1932$ (left) and №4 with $\hat{\vartheta} = 0.5251$ (right) from Table 1.

6.3 Discussion

While we are able to verify the existence of the short and medium length orbits, we were not able to verify the long periodic solution from which we obtained the shorter ones. In order to see where our analysis fails, consider that, for a linear $Z_2(r) = r\hat{Z}_2$,

$$Y + \left(Z_1 + \frac{1}{2}Z_2(r) - 1\right) r = Y + \left(Z_1 + \frac{1}{2}\hat{Z}_2 r - 1\right) r = 0$$

can only hold if $Y\hat{Z}_2 \leq 1/2$. However, as can be calculated from Table 1, $Y\hat{Z}_2 = Y(r^{-1}\mathcal{Z}^2(r) + r^{-1}\mathcal{Z}^3(r))$ becomes significantly bigger as the period increases. In particular, $\mathcal{Z}^2$ and $\mathcal{Z}^3$ grow two orders of magnitude between the short solutions (№1 and №2) and the medium length solutions (№3 and №4). When we consider the long orbits, this entails a similar increase in the order of magnitude of the Z_2 bound. Hence, we fail to verify these long solutions, not because of any problem with the Z_1 bound, but because of the respective magnitudes of Y and $Z_2(r)$. As adding extra modes does not significantly affect the residue bound Y, the only way to improve the Y bound is by lowering ν. However, the constant C_ν needed in the $\mathcal{Z}^2$ and $\mathcal{Z}^3$ bound does not scale nicely with ν. In particular, if we write $\nu = 1 + \epsilon$, then

$$C_\nu \sim 1 + \frac{1}{(n+1)\epsilon}.$$

Furthermore, since $\|\mathcal{A}_0\|$, as it appears in the Z_2-bounds, increases approximately linearly with n, this means that adding extra zeros or lowering ν will only worsen the $\mathcal{Z}^2$ and $\mathcal{Z}^3$ bounds.

Therefore, the only way to push the results further would be by either lowering the Y bound in another way, for instance by using multiple-precision methods, or by significantly altering the way the Z_2 bounds are estimated. One possible avenue would be to compute higher than second order derivatives for the Z_2 bounds, cf. (37) and (41).

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