# Uniqueness of solutions for the extended Fisher-Kolmogorov equation

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**Abstract**

We consider stationary solutions of the Extended Fisher-Kolmogorov (EFK) equation, a fourth-order model equation for bi-stable systems. We show that as long as the stable equilibrium points are real saddles, the paths in the $(u, u')$-plane of two bounded solutions do not cross. As a consequence we derive that the bounded solutions of the EFK equation correspond exactly to those of the classical Fisher Kolmogorov equation.

## Unicité de solutions de l'équation généralisée de Fisher-Kolmogorov

**Résumé**

On examine les solutions stationnaires de l'équation étendue de Fisher-Kolmogorov (EFK), une équation modèle du quatrième ordre pour des systèmes bi-stables. Nous montrons que tant que les points d'équilibre stables sont des 'real saddles', les trajectoires dans le $(u, u')$-plan de deux solutions bornées ne se croisent pas. Comme conséquence nous dérivons que les solutions bornées de l'équation EFK correspondent exactement à celles de l'équation classique de Fisher-Kolmogorov.

# Version française abrégée

Dans cette Note nous étudions des configurations spatiales pour un système bi-stable décrit par l'équation généralisée de Fisher-Kolmorgorov ('Extended Fisher-Kolmogorov equation' (EFK))

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0.$$

L'équation EFK a été proposée dans [3] et [4] comme généralisation de l'équation classique de Fisher-Kolmogorov (FK) $(\gamma = 0)$.

On s'intéresse ici aux solutions stationnaires, c'est-à-dire, aux solutions de l'équation

$$-\gamma u'''' + u'' + u - u^3 = 0, \qquad \gamma > 0. \tag{1}$$

On remarque que (1) possède une constante d'intégration. Quand l'équation (1) est multipliée par $u'$ et intégrée, on obtient *l'énergie* ou hamiltonien

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -\gamma u''' u' + \frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 + F(u) = E,$$

où $E$ est une constante si $u$ est une solution de (1), et $F(u) = -\frac{1}{4}(u^2-1)^2$ est le potentiel. La linéarisation au voisinage de $u = -1$ et $u = +1$ montre que le caractère des points d'équilibre dépend crucialement de la valeur de $\gamma$. Quand $0 < \gamma \leq \frac{1}{8}$, ils sont des *points selles* réels (valeurs propres réelles), tandis que quand $\gamma > \frac{1}{8}$ ce sont des *foyers* (valeurs propres complexes).

Comme l'équation EFK est une perturbation singulière de l'equation FK, il est naturel de chercher á savoir si l'équation (1) hérite des solutions de l'équation FK. Ceci peut être montré rigoureusement pour $\gamma$ petit en utilisant la théorie des perturbations singulières (voir [1], [5]). Les solutions bornées de l'équation FK sont trouvées directement dans le plan des phases. Le premier théorème montre que le plan $(u, u')$ conserve la propriété de l'unicité locale pour l'équation du quatrième ordre tant que $\gamma \in (0, \frac{1}{8}]$.

**Théorème 1** *Soient $u_1$ et $u_2$ des solutions bornées de (1) pour $\gamma \in [0, \frac{1}{8}]$. Alors les trajectoires de $u_1$ et $u_2$ dans le plan $(u, u')$ ne se croisent pas.*

Comme conséquence de ce théorème, on démontre que pour tout $\gamma \in (0, \frac{1}{8}]$, il y a une correspondance complète entre les solutions stationnaires bornées de l'équation EFK et celles de l'équation FK ($\gamma = 0$).

**Théorème 2** *Les seules solutions bornées de (1) pour $\gamma \in (0, \frac{1}{8}]$ sont les trois points d'équilibre, les deux solutions hétéroclines qui sont monotones et antisymétriques, et une famille de solutions périodiques, paramétrée par l'énergy $E \in (-\frac{1}{4}, 0)$.*

Notons que l'existence de ces solutions a été montrée dans [10] et [13].

La preuve de ces théorèmes est basée sur l'application répétée du principe du maximum. Dans [2] cette idée a servi à montrer l'unicité de l'orbite homocline pour une équation du quatrième ordre qui est intimement liée à (1). Nous soulignons que la méthode peut être appliquée à des non-linéarités plus générales que $u - u^3$, *tant que les points d'équilibre sont des points selles réels.* On remarque que l'invariance de (1) sous la transformation $u \to -u$ n'est pas utilisée dans la preuve de ces théorèmes (nous donnons ici des idées des démonstrations; les preuves complètes se trouvent dans [14]).

# 1 Introduction

In this Note we study spatial patterns in bi-stable systems described by the Extended Fisher-Kolmogorov (EFK) equation

$$\frac{\partial u}{\partial t} = -\gamma \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad \gamma > 0.$$

The EFK equation was proposed in [3] and [4] as a generalisation of the classical Fisher-Kolmogorov (FK) equation ($\gamma = 0$). We focus on the stationary solutions, i.e, solutions of the equation

$$-\gamma u'''' + u'' + u - u^3 = 0, \qquad \gamma > 0. \tag{1}$$

Note that (1) has a constant of integration. When we multiply (1) by $u'$ and integrate, we obtain the *energy* or Hamiltonian

$$\mathcal{E}[u] \stackrel{\text{def}}{=} -\gamma u''' u' + \frac{\gamma}{2}(u'')^2 + \frac{1}{2}(u')^2 + F(u) = E,$$

where $E$ is constant along solutions, and $F(u) = -\frac{1}{4}(u^2-1)^2$ is the potential (normalised so that $F(\pm 1) = 0$).

Linearisation around $u = -1$ and $u = +1$ shows that the character of these equilibrium points depends crucially on the value of $\gamma$. For $0 < \gamma \leq \frac{1}{8}$ they are emphreal saddles (real eigenvalues), whereas for $\gamma > \frac{1}{8}$ they are *saddle-foci* (complex eigenvalues). The behaviour of solutions of (1) is dramatically different in these two parameter regions.

For $\gamma \in (0, \frac{1}{8}]$ the solutions are very calm. It was proved in [10] that there exists a monotonically increasing heteroclinic solution (or *kink*) connecting $-1$ with $+1$ (by symmetry there is also a monotonically decreasing kink connecting $+1$ with $-1$). This solution is antisymmetric with respect to its (unique) zero. Moreover, it is unique in the class of monotone antisymmetric functions. In [13] it was shown that in every energy level $E \in (-\frac{1}{4}, 0)$ there exists a periodic solution, which is symmetric with respect to its extrema and antisymmetric with respect to its zeros. Remark that these solutions correspond exactly to the solutions of the FK equation.

In contrast, for $\gamma > \frac{1}{8}$ families of complicated heteroclinic solutions [6, 7, 11] and chaotic solutions [12] have been found. The outburst of solutions for $\gamma > \frac{1}{8}$ is due to the saddle-focus character of the equilibrium points $\pm 1$.

Since the EFK equation is a singular perturbation of the FK equation, it is natural to ask whether Equation (1) inherits solutions from the FK equation. This can be made rigorous for *small* $\gamma$ using singular perturbation theory (see [1, 5]). The bounded solutions of the FK equation are found directly from the phase-plane. Our first theorem states that the $(u, u')$-plane preserves the local uniqueness property for the fourth-order equation as long as $\gamma \in (0, \frac{1}{8}]$.

**Theorem 1** *Let $u_1$ and $u_2$ be bounded solutions of (1) for $\gamma \in [0, \frac{1}{8}]$. Then the paths of $u_1$ and $u_2$ in the $(u, u')$-plane do not cross.*

As a consequence of this theorem, we prove that for all $\gamma \in (0, \frac{1}{8}]$ there is a complete correspondence between the bounded stationary solutions of the EFK equation and those of the FK equation ($\gamma = 0$).

**Theorem 2** *The only bounded solutions of (1) for $\gamma \in (0, \frac{1}{8}]$ are the three equilibrium points, the two monotone antisymmetric kinks, and a one-parameter family of periodic solutions, parametrised by the energy $E \in (-\frac{1}{4}, 0)$.*

The proof of these theorems is based on repeated application of the maximum principle. In [2] this idea has been used to prove the uniqueness of the homoclinic orbit for a fourth-order equation that is closely related to (1). We stress that the method can be applied to other nonlinearities than $u - u^3$. For a class of nonlinearities it can serve to prove that the bounded solutions of the fourth order equation all correspond to those of the second order equation, *as long as the equilibrium points are real saddles.* Note that the invariance of (1) under the transformation $u \to -u$ is not used in the proof of these theorems (here, we will only give an outline of the proofs; full proofs will appear in [14]).

Among other things, Theorem 2 proves the conjecture in [10] that the kink for $\gamma \in (0, \frac{1}{8}]$ is unique (modulo translations). Moreover, using the invariance of (1) under $u \to -u$, we prove that this kink is a transversal intersection of the stable and unstable manifold. We mention that the uniqueness of the kink for $\gamma \in (0, \frac{1}{8}]$ can also be proved using so-called twist maps (see [8]).

## 2 The line of thought

If $\tilde{u}(x)$ is a solution of (1), then by the transformation

$$u(x) = \tilde{u}(\sqrt[4]{\gamma}\, x) \qquad \text{and} \qquad q = -\frac{1}{\sqrt{\gamma}}\,,$$

it is transformed to a solution of the equation

$$u'''' + qu'' - u + u^3 = 0, \qquad q < 0. \tag{2}$$

We examine the case where $q \le -\sqrt{8}$, corresponding to $\gamma \in (0, \frac{1}{8}]$. We choose $0 < \lambda \le \mu$ such that $\lambda\mu = 2$ and $\lambda + \mu = -q$, i.e.,

$$\lambda = -\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - 2} \qquad \text{and} \qquad \mu = -\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - 2}\,.$$

This is possible as long as $q \le -\sqrt{8}$, since $\lambda$ and $\mu$ are the squares of the eigenvalues of the linearisation around $\pm 1$ (see also [9]).

In the following we will assume that $q \le -\sqrt{8}$. To shorten notation, we will write $u(\infty)$ instead of $\lim_{x\to\infty} u(x)$.

We start by recalling a uniform bound that has been proved in [12].

**Lemma 1** *For any $q \le 0$, let $u(x)$ be a bounded solution of (2). Then $|u(x)| \le \sqrt{2}$.*

The shape of the potential $F$ determines this value of $\sqrt{2}$. To be more specific, the bound originates from the fact that $F(\pm\sqrt{2}) = F(0)$.

The following lemma shows that if a bounded solution has two minima below the line $u = 1$, then between these minima it stays below 1. By symmetry, the same is true for maxima above the line $u = -1$.

**Lemma 2** *For any $q \le -\sqrt{8}$, let $u(x)$ be a bounded solution of (2), and let $-\infty \le a < b \le \infty$. Suppose that $u(a)$, $u(b) \le 1$ and $u''(a)$, $u''(b) \ge 0$. Then either $u \equiv 1$ or $u(x) < 1$ on $(a, b)$.*

*Proof.* The proof makes use of the maximum principle. Let $v(x) \equiv u(x) - 1$. The function $v(x)$ obeys

$$v'''' + qv'' + 2v = u - u^3 + 2(u-1) = -(u+2)(u-1)^2 \le 0,$$

where the inequality follows from Lemma 1. Now we define $w(x) \equiv v''(x) - \lambda v(x)$ . From the definition of $\lambda$ and $\mu$, and by the hypotheses on $u$ in $a$ and $b$, we find that $w(x)$ obeys the system

$$\begin{cases} w'' - \mu w = -(u+2)(u-1)^2 \le 0, \\ w(a) = v''(a) - \lambda v(a) \ge 0, \\ w(b) = v''(b) - \lambda v(b) \ge 0. \end{cases}$$

From the maximum principle we conclude that $w(x) \ge 0$ on $(a, b)$. Finally, $v(x)$ obeys the system

$$\begin{cases} v'' - \lambda v \equiv w \ge 0, \\ v(a) = u(a) - 1 \le 0, \\ v(b) = u(b) - 1 \le 0. \end{cases}$$

By the strong maximum principle we obtain that either $v \equiv 0$, or $v(x) < 0$ on $(a, b)$. This proves Lemma 2.

For heteroclinic solutions the previous lemma (with $a = -\infty$ and $b = +\infty$) implies that every heteroclinic solution is uniformly bounded from above by 1 and by symmetry, from below by $-1$.

For the case of a general bounded solution, let us look at the consecutive extrema for $x > 0$ (and similarly for $x < 0$) of a bounded solution $u(x)$. Either $u(x)$ oscillates forever and thus it has an infinite number of extrema, or it has a finite number of extrema and goes to a limit monotonically as $x \to \infty$. It is not difficult to prove that this limit can only be one of the equilibrium points, and that the second derivative tends to zero.

Besides, in the same way as in the proof of Lemma 1, we are able to prove that a bounded solution cannot have two consecutive minima above the line $u = 1$. These considerations imply that for every bounded solution which does not tend to limits at $\pm\infty$, we can find arbitrarily large negative $a$ and arbitrarily large positive $b$, such that $u(a)$ and $u(b)$ are local minima, and thus the conditions in Lemma 2 are satisfied. We can now apply Lemma 2 to improve the result of Lemma 1.

**Lemma 3** *For any $q \le -\sqrt{8}$, let $u \not\equiv \pm 1$ be a bounded solution of (2). Then $|u(x)| < 1$.*

This bound is necessary to be able to apply the following comparison lemma.

**Lemma 4 (comparison principle)** *For any $q \le -\sqrt{8}$, let $u(x)$ and $v(x)$ be solutions of (2) for all $x \ge 0$, such that $-1 < u(x), v(x) < 1$. Suppose that*

$$\begin{aligned} u(0) &\ge v(0), & u'(0) &\ge v'(0), \\ u''(0) - \lambda u(0) &\ge v''(0) - \lambda v(0), & u'''(0) - \lambda u'(0) &\ge v'''(0) - \lambda v'(0). \end{aligned}$$

*Then $u(x) \equiv v(x)$.*

The proof proceeds along the same lines as in [2] and relies on the same factorisation of the differential equation as is made the proof of Lemma 2.

Theorem 1 is a consequence of Lemma 3. Suppose for contradiction that (after translation) $u_1(0) = u_2(0)$ and $u_1'(0) = u_2'(0)$. Without loss of generality we may assume that $u_1''(0) \ge u_2''(0)$. Now, if $u_1'''(0) \ge u_2'''(0)$, then by Lemma 4 we find that $u_1 \equiv u_2$. On the other hand, if $u_1'''(0) < u_2'''(0)$, then we define $\tilde{u}_1(x) = u_1(-x)$ and $\tilde{u}_2(x) = u_2(-x)$. Clearly $\tilde{u}_1$ and $\tilde{u}_2$ are also solutions of (2). We next apply Lemma 4 to $\tilde{u}_1$ and $\tilde{u}_2$, which concludes the proof of Theorem 1.

We now touch upon a lemma that gives a lot of information about the shape of bounded solutions.

**Lemma 5 (symmetry with respect to extrema)** *For any $q \le -\sqrt{8}$, let $u(x)$ be a bounded solution of (2) such that $u'(0) = 0$ (e.g. after translation). Then $u(x) = u(-x)$.*

*Proof.* By Lemma 3 we have that $u \equiv \pm 1$ (trivial cases) or $|u(x)| < 1$. Define $v(x) = u(-x)$. By symmetry $v(x)$ is also a solution of (2). Clearly

$$u(0) = v(0), \qquad u'(0) = v'(0), \qquad \text{and} \qquad u''(0) = v''(0).$$

Moreover, either $u'''(0) \ge v'''(0)$ or $u'''(0) < v'''(0)$. Suppose we have $u'''(0) \ge v'''(0)$ (in the other case we just exchange $u$ and $v$). Now we are clearly in the setting of Lemma 4, thus $u(x) \equiv v(x)$.

It follows from this lemma that the only possible bounded solutions are equilibrium points, homoclinic solutions with one extremum, monotone kinks and periodic orbits with a unique maximum and minimum.

To fill in the remaining details of the phase-plane picture we use the following lemma, which establishes an ordering in terms of the energy $E$ of the paths in the $(u, u')$-plane. By Lemma 5 we only have to consider the upper half of the $(u, u')$-plane.

**Lemma 6** *For any $q \leq -\sqrt{8}$, let $u_1(x)$ and $u_2(x)$ be bounded solutions of (2). Suppose that (after translation) $u_1(0) = u_2(0)$ and $u_1'(0) > u_2'(0) \geq 0$. Then $\mathcal{E}[u_1] > \mathcal{E}[u_2]$.*

The proof of this lemma relies on the fact that if by contradiction $\mathcal{E}[u_1] \leq \mathcal{E}[u_2]$, then we can find a point $x_0$ such that $u_1(x_0) = u_2(x_0)$ and $u_1''(x_0) = u_2''(x_0)$. The energy identity then ensures that we can apply Lemma 4 to $u_1$ and $u_2$, resulting in a contradiction.

A final observation is that every bounded solution (except $u \equiv \pm 1$) has a zero. This readily follows from the fact that a bounded solution cannot have a minimum in the range $[0, 1)$, which is proved by an argument similar to the proof of Lemma 1.

Lemma 6 excludes the possibility of more than one periodic solution per energy level. Finally, a systematic survey of the energy levels makes clear that Theorem 2 holds.

Remark that we did not use the invariance of (2) under the transformation $u \to -u$. This invariance can be used to obtain further information on the shape of bounded solutions.

**Lemma 7 (antisymmetry with respect to zeros)** *For any $q \leq -\sqrt{8}$, let $u(x)$ be a bounded solution of (2) such that $u(0) = 0$ (e.g. after translation). Then $u(x) = -u(-x)$.*

The proof is analogous to the proof of Lemma 5. A technique similar to the one in [2] can be applied to prove that the monotonically increasing heteroclinic solution is a transverse intersection of $W^u(-1)$ and $W^s(+1)$.

**Lemma 8 (transversality)** *For $q \leq -\sqrt{8}$ the unstable manifold of $-1$ and the stable manifold of $+1$ intersect transversely in the zero energy set.*

We recall how crucially these arguments depend on the real-saddle character of the equilibrium points. Both Lemmas 3 and 4 do not hold when $q > -\sqrt{8}$. The variety of solutions which exist for $q > -\sqrt{8}$, shows that this bound is sharp.

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