Stable patterns for fourth order parabolic equations

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ABSTRACT. We consider fourth order parabolic equations of gradient type. For sake of simplicity the analysis is carried out for the specific equation $u_t = -\gamma u_{xxxx} + \beta u_{xx} - F'(u)$, with $(t,x) \in (0,\infty) \times (0,L)$ and $\gamma,\beta > 0$, and where F(u) is a bi-stable potential. We study its stable equilibria as function of the ratio $\frac{\gamma}{\beta^2}$. As the ratio $\frac{\gamma}{\beta^2}$ crosses an explicit threshold value the number of stable patterns grows to infinity as $L \to \infty$. The construction of the stable patterns is based on a variational gluing method, which does not require any genericity conditions to be satisfied.

1. Introduction

Higher order parabolic equations may display a multitude of stable stationary states. A good way to describe this phenomenon is to start with fourth order parabolic equations. In order to keep the exposition of the results and the methods transparant we will mainly restrict to the following model equation

(1)
$$u_t = -\gamma u_{xxxx} + \beta u_{xx} - F'(u), \qquad (t, x) \in \mathbb{R}^+ \times (0, L),$$

with $\gamma > 0$, $\beta > 0$. Bare in mind that the results apply equally well to a much larger class of fourth order parabolic equations as will be explained below. Our goal is to study stable stationary states of (1) as function of the parameters γ , β , the potential F, the interval-length L and the boundary conditions at x = 0 and x = L. In doing so we develop a new variational gluing method for constructing stable stationary states. The most important characteristic of the method is that no generic properties for Equation (1), such as non-degeneracy of stationary patterns, will be required.

In our notation u is a function of the variables t and x, and u_t and u_x denote the partial derivatives. The initial state u(0, x) is denoted by u_0 . The function $F \in C^2$ is a double-well potential that satisfies

(2)
$$F(\pm 1) = F'(\pm 1) = 0$$
, $F''(\pm 1) > 0$ and $F > 0$ for $u \neq \pm 1$.

On the potential the following growth condition is imposed:

 $F(u) > -C_0 + C_1 u^2$ for some $C_0, C_1 > 0$, i.e. F grows super-quadratically¹. Parabolic equations with a potential as described above are often referred to as *bi-stable* equations. For the second order bi-stable model ($\gamma = 0$) the only candidates for stable equilibria are constant solutions: critical points of F. As we will see later on this behavior dramatically changes as the dynamical nature² of the constants states changes with the ratio $\frac{\gamma}{\beta^2}$.

In certain physical models (Swift-Hohenberg equation, extended Fisher-Kolmogorov equation, see e.g. [30, 31, 32, 33, 34, 35]), in which Equation (1) occurs, the boundary conditions

$$u_x(t,0) = u_{xxx}(t,0) = 0$$
 and $u_x(t,L) = u_{xxx}(t,L) = 0$

are often used. These boundary conditions are referred to as the Neumann boundary conditions. In this case $u \equiv \pm 1$ are stable equilibria for all $\gamma, \beta, L > 0$. It should be noted at this point that the Neumann boundary conditions that we impose on Equation (1) are by no means a restriction for the results presented here, and different conditions can be used. We will come back to this point later on (especially in Sect. 6).

Essential to our analysis is the property that (1) is the L^2 -gradient flow equation for the action

(3)
$$J_L[u] = \int_0^L \frac{\gamma}{2} |u_{xx}|^2 + \frac{\beta}{2} |u_x|^2 + F(u).$$

This variational structure allows our methods to be applicable to more general actions: $J_L[u] = \int_0^L j(u, u_x, u_{xx}) dx$, where $j \ge 0$ satisfies the convexity condition $\partial_{u_{xx}}^2 j \ge \delta > 0$, and j(u, 0, 0) replaces the potential F. In order to best explain the overall features of our methods we restrict ourselves here to actions of the form given in (3).

In [21, 22, 23] stationary solutionss of (1) were found by means of minimization of the associated action (3). In particular the results in [22] will be drawn upon to construct stable solutions of the parabolic equation. We carry out the construction of stable equilibria in the case of the Neumann boundary conditions, as other boundary conditions can be dealt with in exactly the same way. The natural function space for this case is

$$H_N^2 \stackrel{\text{def}}{=} \{ u \in H^2(0, L) \, | \, u_x(0) = u_x(L) = 0 \}.$$

Equation (1) has a compact attractor $\mathcal{A} = \mathcal{A}(L, \gamma, \beta, F)$ for all $0 < L < \infty$, $\gamma, \beta > 0$ and for all potentials F that satisfy the growth condition $\liminf_{|u|\to\infty} \frac{F'(u)}{u} > 0$; for $\beta < 0$ one needs that $\liminf_{|u|\to\infty} \frac{F'(u)}{u} > \frac{\beta^2}{4\gamma}$ (see e.g. [18, Sect. 4.3])³. If L is small enough then \mathcal{A} contains exactly two stable equilibria $(u \equiv \pm 1)$. The size of the attractor \mathcal{A} depends on L in the sense that if L grows larger the

¹This growth condition is taken such as to simplify estimates, but can be weakened in various directions.

²The dynamical nature of a constant solution $u(t) \equiv u_*$ of equation (1) is determined by the characteristic equation $\gamma \lambda^4 - \beta \lambda^2 + F''(u_*) = 0$.

³Note the difference with the earlier growth condition of F.

attractor also becomes larger and the number of equilibria in \mathcal{A} increases. It is not a priori clear whether new *stable* equilibria are created. This question brings us to the main result of this paper.

brings us to the main result of this paper. If $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$, then the nature of the equilibrium points $u = \pm 1$ changes from real saddle to saddle-focus. Our main result states that as soon as the equilibrium states $u = \pm 1$ are both saddle-foci, then a lower bound on the number of stable states of Equation (1) grows exponentially with the interval length L. Moreover, we describe the shape and the attracting sets of these stable equilibria.

Since we do *not* require stationary solutions to be either hyperbolic (generic) or isolated we need the more general notion of *stable* set:

DEFINITION 1. A set S of stationary solutions of Equation (1) is stable if for any $\epsilon > 0$ there exists an open neighborhood $U \subset B_{\epsilon}(S)$ such that for all $u_0 \in U$ it holds that $u(t, x) \in B_{\epsilon}(S)$ for all t > 0.

We want to identify various attracting sets, i.e., forwardly invariant sets, in which we can then find stable sets of equilibria.

THEOREM 2. Let the potential F satisfy the Hypotheses (2) and grow superquadratically. Suppose that $\beta > 0$ and $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. Then for any $n \in \mathbb{N}$ there exists a constant $L_n > 0$, such that for all $L \ge L_n$ Equation (1) (with Neumann boundary conditions) has at least n disjoint stable sets of stationary solutions.

The number of stable stationary states will grow rapidly as the interval length L goes to infinity. In the proof of Theorem 2 various a priori estimates are used. If some of these estimates are carried out more carefully one can actually find a lower bound on the number of stable equilibria as function of the interval length L. We prove that there are constants $a_1 > 0$ and $a_2 > 0$ such that

(4) $\#\{\text{disjoint stable sets of equilibria}\} > a_1 e^{a_2 L}.$

Hence the number of stable sets grows exponentially in L (see Sect. 5).

Each stable set in the above theorem consists of stationary solutions with a specific geometrical shape, which differs from set to set (see Sect. 4). Notice that this theorem holds under very mild conditions on the double-well potential F and that no non-degeneracy assumptions are made (the same theorem holds for other boundary conditions).

The method we use to construct stable sets is motivated by a novel gluing technique due to Buffoni and Séré [9]. Usually gluing techniques require certain transversality/non-degeneracy conditions to be satisfied. The method described in [9] uses analyticity to obtain isolation properties, which circumvents transversality. The technique developed here uses neither transversality/non-degeneracy nor analyticity, and is specifically suited for finding minimizers. The minimization procedure for finding homoclinic/heteroclinic connections to the saddle-focus constant states $u = \pm 1$ in all homotopy classes, which was devised in [22], allows one to obtain various isolation properties of homoclinic/heteroclinic connections in these homotopy classes (see Sect. 3). These

isolation properties in turn are used then to construct product neighborhoods from truncated homoclinic minimizers as found in [22] on which J attains its minimum in the interior (see also Sect. 2). The advantage of this variational approach is that no generic assumptions are needed and this gluing via minimization produces stable sets of equilibria of various geometric shapes (in all the homotopy classes, see Figure 2). It also gives us controle over the interval length L on which such stable states must exist, and allows for estimates on their number as function of L (see below). A key issue for obtaining the isolation properties in this paper and in [22] is that isolation can be achieved if the equilibrium points $u = \pm 1$ are of saddle-focus type, which explains the transition at $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$ — for $F(u) = \frac{1}{4}(u^2 - 1)^2$ this transition is sharp.

What the above results imply is that the dynamics near the attractor depends in a very subtle manner on the parameters γ and β . This behavior is not captured by, for example, the general slow motion results of [24]. This question initiates the second part of the paper. How do the above results fit it with the structure of the attractor, and how does the latter depend on $\frac{\gamma}{\beta^2}$?

For $\gamma = 0$ the attractor is well understood. In fact, when for instance $F(u) = \frac{1}{4}(u^2 - 1)^2$ then $u \equiv \pm 1$ are the only stable equilibria for all L > 0, and the attractor in this case can be characterized completely [1, 10, 19] (see also Sect. 7). For $0 < \frac{\gamma}{\beta^2} \leq \frac{1}{8}$ — we restrict to the special choice for the potential F to simplify the presentation — the following theorem, based on a general result in [27], gives a strong characterization of the attractor, relating it to the second order equation ($\gamma = 0$). We first introduce some notation. The semi-flow associated with (1) with Neumann boundary conditions is denoted by $\phi(L, \gamma, \beta)$. The first bifurcation of the homogeneous solution $u \equiv 0$ occurs at $L = L_0(\gamma, \beta) \stackrel{\text{def}}{=} \pi \sqrt{\frac{2\gamma}{\sqrt{\beta^2 + 4\gamma - \beta}}}$ (and $L_0(0, \beta) = \pi\beta$).

THEOREM 3. Let $F(u) = \frac{1}{4}(u^2 - 1)^2$ and suppose that $\beta > 0$ and $0 < \frac{\gamma}{\beta^2} \leq \frac{1}{8}$, then for all L > 0 there is a semi-conjugacy between the flow on the attractor of (1) (Neumann boundary conditions) and the corresponding flow for the second order equation ($\gamma = 0$). To be precise, there is a semi-conjugacy between $\phi(L, \gamma, \beta)|_{\mathcal{A}}$ and $\phi(L\frac{L_0(0,\beta)}{L_0(\gamma,\beta)}, 0, \beta)|_{\mathcal{A}}$. Moreover, the equilibria are in one-to-one correspondence with the equilibrium solutions of (1) for $\gamma = 0$, and are all hyperbolic (non-trivial ones).

In particular this theorem implies that for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$ and all L > 0 the only stable solutions are the homogeneous states $u \equiv \pm 1$. Another consequence is the existence of connecting orbits between various stationary states (see Sect. 7 for more details). The above theorem holds for a more general class of potentials F(u). For example, a sufficient condition is that F is even, satisfies (2) and $F'''(u) \geq 0$ for $u \geq 0$ (this condition can be somewhat relaxed) and the parameter range for which the theorem holds is then $\frac{\gamma}{\beta^2} \leq \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. An analogous theorem holds for Navier boundary conditions: $u(t, 0) = u_{xx}(t, 0) = 0$ and $u(t, L) = u_{xx}(t, L) = 0$.

The third part of the paper, Sect. 8, decribes the transition at $\frac{\gamma}{\beta^2} = \frac{1}{8}$ (i.e. with the choice of $F(u) = \frac{1}{4}(u^2 - 1)^2$). At this bifurcation point we give a precise decription of how the attractor changes for $\frac{\gamma}{\beta^2} = \frac{1}{8} + \epsilon$, $0 < \epsilon \ll 1$. In this case all stationary solutions are found — not just stable ones — and complete bifurcation diagram is given. Theorem 2 explains that most stable solutions persist for all $\frac{\gamma}{\beta^2} > \frac{1}{8}$.

CONTENTS

1.	Introduction	1
2.	Homoclinic and heteroclinic minimizers	5
3.	A priori estimates	8
4.	Stable equilibrium solutions	15
5.	Estimating the number of equilibria	19
6.	Different boundary conditions	23
7.	The semi-conjugacy	24
8.	The bifurcation	27
References		38

2. Homoclinic and heteroclinic minimizers

We start our investigation of Equation (1) with the Neumann boundary conditions $u_x(t,0) = u_{xxx}(t,0) = 0$ and $u_x(t,L) = u_{xxx}(t,L) = 0$ in the case that the equilibrium points are saddle-foci. By extending the solutions to $x \in \mathbb{R}$ by reflecting in x = 0 and x = L, one may regard equilibrium solutions u of (1) as a closed curves in (u, u_x) -plane by drawing the (u, u_x) -curve over one period. In [21] it was proved that, when we puncture the (u, u_x) -plane in $(\pm 1, 0)$, for all homotopy classes of closed curves in $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ there exist associated minimizers for J^4 . These minimizers lie on the energy level E = 0, where the energy is defined by (9). The periodic minimizers give rise to minimizers of J_L with Neumann boundary conditions, but the interval length is dictated by the homotopy type and thus they occur only for certain interval lengths L. Roughly speaking, when L is sufficiently large, the numbers $L \approx S_0 + nT_0 + m\omega_0, n, m \in \mathbb{N}$, occur as interval lengths, where S_0, T_0 and ω_0 are constants depending only on γ, β and F. The integer m can be written as $m = \sum_{i=1}^{n} m_i, m_i \in \mathbb{N}$ and for every *n*-tuple (m_1, \ldots, m_n) there exists at least one minimizer with interval length $L \approx S_0 + nT_0 + m\omega_0$. We will prove that for values of L in between one can also find minimizers. Such minimizers do not necessarily lie on E = 0.

Let us briefly explain the idea. Trying to fit two pieces of solution together one uses a gluing function which lives in a small neighborhood of the equilibrium point. In Figure 1 the dependence of the action J on the interval length s (on which the gluing takes place) is depicted for a saddle-focus equilibrium.

⁴For most homotopy classes when the evenness assumption on F is dropped.



FIGURE 1. Sketch of the dependence of J on the interval length s for a gluing function close to a saddle-focus equilibrium.



FIGURE 2. A heteroclinic solution with homotopy type g = (2, 4). On the right the projection $\Gamma(u)$ of the orbit onto the (u, u_x) -plane has been depicted (schematically).

The local minima and maxima correspond to solutions with energy E = 0. The minima have been found previously in [21], i.e., stable solutions are found for discrete values of the interval length. The intermediate solutions, although not local minima of the curve, can still be (local) minima of the action for *fixed* s. The gluing procedure can be made rigorous under transversality assumptions, see [9, 23] and Sect. 8. In the absence of a transversality assumption, we follow a different approach.

In order to construct attracting sets which contain stable equilibria we will use the heteroclinic and homoclinic minimizers that were found in [22]. Let us first summarize the results of [22]. Consider the punctured plane $\mathcal{P} = \mathbb{R}^2 \setminus \{P_1, P_2\}$, where $P_1 = (-1, 0)$ and $P_2 = (+1, 0)$. Let u be a heteroclinic or homoclinic solution of (1) and let $\Gamma(u) = (u, u_x) : \mathbb{R} \to \mathcal{P}$ with $\Gamma(u(x))|_{x=\pm\infty} \in \{P_1, P_2\}$ and define its homotopy type as follows. As x goes from $x = -\infty$ to $x = \infty$, Γ can intersect the lines $L_- = \{(u, u_x) \in \mathcal{P}, u = -1\}$ and $L_+ = \{(u, u') \in \mathcal{P}, u = +1\}$. The number of consecutive intersections of L_- and L_+ is always even. We do not count the intersections of L_\pm at start and finish. In between one obtains a finite sequence of even numbers denoted by $g = (g_1, \ldots, g_k)$, which we call the homotopy type g one still has the freedom of choosing the initial point to be either P_1 or P_2 . Whether Γ terminates at P_1 or P_2 then depends on \boldsymbol{g} .

If $F(u) = \frac{1}{4}(u^2 - 1)^2$ it follows from the results discussed in Sect. 7 that for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$ the only minimizers are the constant solutions $u \equiv \pm 1$ and two heteroclinic connections with trivial homotopy type. On the contrary, for $\frac{\gamma}{\beta^2} > \frac{1}{8}$ it is proved in [22] that for any homotopy type g of any length there exists a 'geodesic' $\Gamma(u)$. In other words by minimizing $J[u] = \int_{\mathbb{R}} j(u)$ over functions ufor which the associated curve $\Gamma(u)$ has homotopy type g, a minimizer is found in every homotopy class⁵. The minimization is carried out in classes of functions defined via the homotopy type, and are are denoted by $M(\boldsymbol{g}, P_{\nu})$, where $P_{\nu} \in \{P_1, P_2\}$ (i.e. $\nu \in \{1, 2\}$ and $(u, u_x)(-\infty) = P_{\nu}$ for all $u \in M(\boldsymbol{g}, P_{\nu})$). To be precise, let $\chi_0(x)$ be a smooth function such that $\chi_0(x) = 1$ for $x \ge 1$ and $\chi_0(x) = -1$ for $x \leq -1$. Let $\chi_1(x) \equiv -1$, and let $\chi_i = \chi_i \mod 2$ for $i \geq 2$. Then we define for all $m \ge 0$ and any $\boldsymbol{g} \in \mathbb{N}^m$ (see [22]):

DEFINITION 4. A function u is in $M(\boldsymbol{g}, P_{\nu})$ if $u - (-1)^{\nu} \chi_m \in H^2(\mathbb{R})$ and if there exist nonempty subsets $\{A_i\}_{i=0}^{m+1}$ of \mathbb{R} such that

- (1) $u^{-1}(\pm 1) = \bigcup_{i=0}^{m+1} A_i;$
- (2) $#A_i = g_i \text{ for } i = 1, \dots, m;$
- (3) $\max A_i < \min A_{i+1}$ for i = 0, ..., m; (4) $u(x) = (-1)^{\nu+i+1}$ for all $x \in A_i$;
- (5) $\{\max A_0\} \cup (\bigcup_{i=1}^m A_i) \cup \{\min A_{m+1}\} \text{ consists of transverse crossings of }$ $\pm 1.$

Under these conditions $M(\boldsymbol{g}, P_{\nu})$ is an open set in $(-1)^{\nu}\chi + H^2(\mathbb{R})$. The function class with m = 0 is denoted by $M((0), P_{\nu})$. We will use the notation $|\mathbf{g}| = m$ if $g \in 2\mathbb{N}^m$, and drop the implicit dependence of $\chi_{|\mathbf{g}|}$ on $|\mathbf{g}|$ from the notation.

Define

$$J(\boldsymbol{g}, P_{\nu}) = \inf_{u \in M(\boldsymbol{g}, P_{\nu})} J[u],$$

where in this case the domain of integration is the entire real line. Finally, the set of global minimizers of J over the function class $M(\boldsymbol{g}, P_{\nu})$ is denoted by

$$CM(\boldsymbol{g}, P_{\nu}) = \{ u \in M(\boldsymbol{g}, P_{\nu}) \mid J[u] = J(\boldsymbol{g}, P_{\nu}) \}.$$

Since $M(\boldsymbol{g}, P_{\nu})$ is an open set, minimizers $u \in CM(\boldsymbol{g}, P_{\nu})$ satisfy the Euler-Lagrange equation

(5)
$$-\gamma u_{xxxx} + \beta u_{xx} + F'(u) = 0.$$

In [22] the following theorem is proved:

THEOREM 5. Let $F \in C^2(\mathbb{R})$ satisfy (2) and grow super-quadratically. Suppose that $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. Then

(a) if F is even: $J(\boldsymbol{g}, P_{\nu})$ is attained for any \boldsymbol{g} .

⁵This result is actually proved for general even potentials F under the condition that $\frac{\gamma}{\beta^2} > \frac{1}{2F''(\pm 1)}$

(b) if F is not even: there exists a universal constant $N_0(F, \gamma, \beta) \in \mathbb{N}$ such that $J(\boldsymbol{g}, P_{\nu})$ is attained for any $\boldsymbol{g} = (g_1, \dots, g_m)$ with $g_i \in \{2\} \cup \{n \geq N_0\}$ for all $i = 1, \dots, m$.

The homotopy types \boldsymbol{g} selected in the above theorem are called *admissible* types. In the following we will always assume that F satisfies the assumptions in the above theorem, that $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$, and that \boldsymbol{g} is an admissible homotopy type.

It has been proved in [22] that all minimizers obtained in Theorem 5 are *normalised*, i.e., all crossings of ± 1 are transverse and between two consecutive crossings of ± 1 the function is either monotone or has exactly one local extremum.

As was already pointed out, in order to find stable solutions with respect to the Neumann boundary conditions we need to consider certain types of homoclinic connections found in [22]. Of particular interest are the symmetric types with an odd number of entries, i.e. $\boldsymbol{g} = (g_1, \ldots, g_{2n+1})$ with $g_i = g_{2n+2-i}$. It follows from the minimizing property that the curves Γ (and thus also the functions u) inherit the symmetry in \boldsymbol{g} , i.e., the functions u are symmetric with respect to the line $u_x = 0$. To be precise, given a minimizer u there exists a point $x = x_0$ such that $u(x_0 + x) = u(x_0 - x)$. Since the minimizers are invariant under translations in x one can choose a representative u such that $x_0 = 0$, and in particular we have $u_x(0) = u_{xxx}(0) = 0$. For the functions $u_{-} = u|_{\mathbb{R}^{-}}$ and $u_{+} = u|_{\mathbb{R}^{+}}$ one can define the *restricted* homotopy type as before by counting the number of intersections of $\Gamma(u)$ with L_{-} and L_{+} . Thus $g(u_{-}) = (g_1, \ldots, g_n, g_{n+1}/2)$ and $g(u_{+}) = (g_{n+1}/2, g_{n+2}, \ldots, g_{2n+1})$. Restricting to functions over \mathbb{R}^+ we still have the freedom of choosing the endpoint to be either P_1 or P_2 . Define for all (restricted) homotopy types $\boldsymbol{g} = (g_1, \ldots, g_m)$ with $\boldsymbol{g}_1 \in \mathbb{N}$ and $g_i \in 2\mathbb{N}$ for $i = 2, \ldots, m$,

$$M_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu}) = \{ u \in (-1)^{\nu} + H^2(\mathbb{R}^+) \mid u_x(0) = 0, \ \boldsymbol{g}(u) = (\boldsymbol{g}) \}.$$

LEMMA 6. The infima $J_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu}) = \inf_{u \in M_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu})} J_{\mathbb{R}^+}[u]$ are precisely attained by $u_+ = u|_{\mathbb{R}^+}$ with $u \in CM(\boldsymbol{g}^{-1}\boldsymbol{g}, P_{\nu})$, where $\boldsymbol{g}^{-1}\boldsymbol{g} = (g_m, \ldots, g_2, 2g_1, g_2, \ldots, g_m)$ (under the same assumptions as in Theorem 5).

The minimizers of $J_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu})$ in $M_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu})$ are denoted by $CM_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu})$. For periodic solutions one can set up the same construction (see [21]). The homotopy type is now determined over one period. The function classes and sets of global minimizers are denoted by $M_{\text{per}}(\boldsymbol{g}, P_{\nu})$ and $CM_{\text{per}}(\boldsymbol{g}, P_{\nu})$ respectively, and $J_{\text{per}}(\boldsymbol{g}, P_{\nu})$ is attained under the same assumptions as in Theorem 5.

3. A priori estimates

For the class of homoclinic and heteroclinic connections that were found in Theorem 5 we prove certain a priori estimates concerning their asymptotic behaviour. We assume throughout this section that for either F even or F not even, the homotopy types are admissible (see Theorem 5). Also assume that $\frac{\gamma}{\beta^2} > \max\left\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\right\}$.



FIGURE 3. The intervals I_i and ℓ_i are indicated for a function of type $\boldsymbol{g} = (6, 4)$.

For easy notation we lift the translation invariance of minimizers of J by defining $CM_*(\boldsymbol{g}, P_{\nu}) = CM(\boldsymbol{g}, P_{\nu})/\mathbb{R}$, represented by functions $u \in CM(\boldsymbol{g}, P_{\nu})$ with the property that $u(0) = (-1)^{\nu+1}$ and such that $(-1)^{\nu+1}u(x) < 1$ for all x < 0 (this corresponds to taking $\min(A_1) = 0$). For a minimizer $u \in CM_*(\boldsymbol{g}, P_{\nu})$ recall that the sets A_i represent the successive crossings of $(-1)^{\nu+i}$, $i = 1, ..., |\boldsymbol{g}|$ and define (see also Figure 3)

 $I_i \stackrel{\text{def}}{=} [\min A_i, \max A_i] \text{ and } \ell_i \stackrel{\text{def}}{=} [\max A_{i-1}, \min A_{i+1}].$

The a priori bounds on minimizers $u \in CM(\boldsymbol{g}, P_{\nu})$ obtained in this section will immediately carry over to minimizers on the half line on account of Lemma 6.

LEMMA 7. There exists constants C_1 , C_2 , $C_3 > 0$ such that for any admissible homotopy type \boldsymbol{g} and any $u \in CM_*(\boldsymbol{g}, P_\nu)$ it holds that

$$\|u\|_{W^{1,\infty}(\mathbb{R})} \le C_1,$$

and

$$\operatorname{dist}_{\mathbb{R}^2} \left(\Gamma(u|_{\ell_i}), ((-1)^{\nu+i}, 0) \right) \ge C_2 e^{-C_3 g_i}, \quad \text{for } i = 1, 2, \dots, |\boldsymbol{g}|$$

where $\ell_i = [\max A_{i-1}, \min A_{i+1}].$

Before proceeding with the proof of this lemma we first introduce the notion of covering spaces in the present context (see also [21]). The fundamental group of $\mathcal{P} = \mathbb{R}^2 \setminus \{P_1, P_2\}$ is isomorphic to the free group on two generators e_1 and e_2 which represent loops (traversed clockwise) around $P_1 = (-1, 0)$ and $P_2 = (1, 0)$ respectively with base-point (0, 0). Since \mathcal{P} represents the phase-plane, the curves corresponding to functions u only traverse the loops in the clockwise direction. Note that \mathcal{P} is homotopic to a bouquet of two circles $X = S_1 \vee S_1$. The universal covering of X, denoted by \widetilde{X} , can be represented by an infinite tree whose edges cover either e_1 or e_2 in X, see Figure 4. The universal covering of \mathcal{P} denoted by $\wp : \widetilde{\mathcal{P}} \to \mathcal{P}$ can then be viewed as a thickened version of \widetilde{X} so that $\widetilde{\mathcal{P}}$ is homeomorphic to an open disk in \mathbb{R}^2 . The origin of $\widetilde{\mathcal{P}}$ will be denoted by \mathcal{O} . Of course every point in \mathcal{P} has many lifts. To be able to fix notation we distinguish a particular lift \wp^{-1} of the



FIGURE 4. The universal covering \widetilde{X} of $X = S_1 \vee S_1$ is a tree. The universal covering $\widetilde{\mathcal{P}}$ of \mathcal{P} is a thickened version of \widetilde{X} . Its origin is denoted by \mathcal{O} . The single and double arrows indicate the two different generators e_1 and e_2 which can only be traversed in one direction.



FIGURE 5. All minimizers in any class are bounded in the (u, u_x) plane from the outside by $u \in CM_{per}(2, 2)$ and from the inside by $u \in CM((0), P_{\nu})$ (only $u \in CM((0), P_1)$ is depicted here). The dotted curve represents (part of) a minimizer.

line $\{(0, u_x) | u_x \in \mathbb{R}\} \subset \mathcal{P}$ by requiring that $\wp^{-1}((0, 0)) = \mathcal{O}$ and continuous extension. Denote $\wp^{-1}(\{(0, u_x)\})$ by $\mathcal{N} \subset \widetilde{\mathcal{P}}$.

We now turn to the proof of Lemma 7.

PROOF. The first estimate (the outer bound) is proved in Theorem 5.1 in [21]. It follows from the fact that all minimizers are bounded in the (u, u_x) by a minimizer of class $M_{per}(2, 2)$ (see Figure 5). We will show that the second estimate in Lemma 7 comes from a similar argument where minimizers of class $M((0), P_{1,2})$ take the the role of inner bounds. The proof is completely analogous to the first estimate when we lift the problem to the covering space. The idea is that all minimizers lie "outside" the simple heteroclinic minimizers of type $\boldsymbol{g} = ((0), P_1)$, i.e., they spiral towards P_{ν} slower than these simple minimizers.

Let $u \in CM_*(\boldsymbol{g}, P_{\nu})$ with $\boldsymbol{g} \neq (0)$. The idea now is to compare different lifts of $\Gamma(u)$ to $\widetilde{\mathcal{P}}$ with lifts of minimizers in $CM_*((0), P_{\nu})$. Fix the index *i* to be any of the numbers $1, \ldots, |\boldsymbol{g}|$. Choose $u_0 \in CM_*((0), P_1)$ if $\nu + i$ is even, and $u_0 \in CM_*((0), P_2)$ if $\nu + i$ is odd. Set $x_0 \stackrel{\text{def}}{=} \max\{x < \min(A_i) | u(x) = 0\}$. Now lift $\Gamma(u_0)$ and $\Gamma(u)$ to \mathcal{P} requiring that both $\wp^{-1}(\Gamma(u_0(0))) \in \mathcal{N}$ and $\wp^{-1}(\Gamma(u(x_0))) \in \mathcal{N}$.

We claim that the lifts $\wp^{-1}(\Gamma(u_0))$ and $\wp^{-1}(\Gamma(u))$ intersect at most once. Indeed, suppose they intersect twice in say y_0 and y_1 , then their action J between y_0 and y_1 is equal since they are both minimizers. This implies that if one can replace u_0 by u between y_0 and y_1 , and thus obtain another minimizer of the same homotopy type. Since all minimizers satisfy (5), this contradicts the uniqueness of the initial value problem, which proves our claim. In fact the same argument shows that, for i = 1, and $i = |\mathbf{g}|$, the lifts $\wp^{-1}(\Gamma(u_0))$ and $\wp^{-1}(\Gamma(u))$ do not intersect at all.

For the remaining indices i we assert that if $\wp^{-1}(\Gamma(u_0))$ and $\wp^{-1}(\Gamma(u))$ intersect, then they do not cross. That is, if the curves have a point in common (intersect), then this intersection can be removed by an arbitrarily small perturbation (the intersection is tangent). Indeed, if the curves would cross, then there would be a second intersection point contradicting the statement above. This is most easily seen from the left picture in Figure 6 since both limits of $\wp^{-1}(\Gamma(u))$ as $x \to \pm \infty$ lie on the same side of $\wp^{-1}(\Gamma(u_0))$. It is also follows that $\wp^{-1}(\Gamma(u))$ lies on the "outside" of $\wp^{-1}(\Gamma(u_0))$, that is to say the curve $\Gamma(u)$ spirals about P_1 or P_2 on ℓ_i outside the spiral of $\Gamma(u_0)$ (see Figure 6; right).

Finally, the set $CM_*((0), P_1)$ is ordered by their derivative at the origin, i.e. u'(0) (since two minimizers cannot intersect in $\widetilde{\mathcal{P}}$). Besides, $CM_*((0), P_{\nu})$ turns out to be compact (see Lemma 12). Hence there exists a smallest and a largest element of $CM_*((0), P_1)$ (measured by u'(0)). The smallest element $\Gamma(u_0)$ spirals exponentially towards $P_{1,2}$ as $x \to \pm \infty$. A similar argument holds for $CM_*((0), P_2)$ (especially because these are the same functions with inverted x). Since all other minimizers spiral outside these minimal elements the second (exponential) estimate of the lemma follows. \Box

Another way to prove Lemma 7 is to construct annuli as covering spaces as was done in [21].

REMARK 8. The proof also shows that the tails of any minimizer cannot spiral towards the equilibrium point faster than some fixed exponential rate.

In [22] the Uniform Separation Property was introduced. This property is closely related to the question which types are admissible. Here the following result from [22] is used:



FIGURE 6. On the left: the lifts of the simple heteroclinic $\wp^{-1}(\Gamma(u_0))$ of class $\boldsymbol{g} = ((0), P_1)$ and, as an example, a minimizer $\wp^{-1}(\Gamma(u))$ of class $\boldsymbol{g} = ((4, 2), P_2)$. On the right: the heteroclinic class $\boldsymbol{g} = ((0), P_1)$ and, as an example, (part of) a minimizer of class $\boldsymbol{g} = ((6), P_1)$.

LEMMA 9. There exist a constant $C_4 > 0$ such that for any admissible homotopy type \boldsymbol{g} and any $u \in CM_*(\boldsymbol{g}, P_{\nu})$ it holds that

$$|u(x) - (-1)^{\nu}| \ge C_4 \qquad \text{for all } x \in I_i,$$

where $I_i = [\min A_i, \max A_i]$

We now deduce a bound on the length of the interval between the tails.

LEMMA 10. There exists a constant $\delta_1 > 0$ so that for any admissible homotopy type \boldsymbol{g} and any $\delta \leq \delta_1$ there exists constants $T_{\delta}^- < 0$ and $T_{\delta}^+ > 0$ such that for any $u \in CM_*(\boldsymbol{g}, P_{\nu})$

$$\|u - (-1)^{\nu}\|_{W^{1,\infty}(-\infty,T_{\delta}^{-})} < \delta, \qquad \|u - (-1)^{\nu+|\boldsymbol{g}|-1}\|_{W^{1,\infty}(T_{\delta}^{+},\infty)} < \delta.$$

PROOF. First of all we analyse the tails. We choose $\delta_1 > 0$ so small that the local theory near the equilibrium points from Sect. 4 in [22] applies for all $\delta < \delta_1$. According to the local theory there exists a $0 < \delta_2 < \delta$ such that if a point $x_1 \in (-\infty, \min A_1)$ in the left tail of u is such that $|u(x_0) - (-1)^{\nu}| < \delta_2$ and $|u'(x_0)| < \delta_2$, then $||u||_{W^{1,\infty}}(-\infty, x_1) < \delta$. This expresses the fact that $\Gamma(u)$ spirals towards P_{ν} as $x \to -\infty$. Of course a similar statement holds for the right tail.

Now choose $\kappa = \min\{\delta_2, C_4, C_2 e^{-C_3 \max_{1 \le i \le |g|} g_i}\}$, where C_2, C_3 and C_4 are defined in Lemmas 7 and 9. We are going to estimate the measure of

 $K_{\kappa} \stackrel{\text{def}}{=} \{ x \in \mathbb{R} \, | \, \text{dist}_{\mathbb{R}^2} \big((u(x), u_x(x)), \{P_1, P_2\} \big) < \kappa \},$

or rather its complement K_{κ}^{c} . By Lemmas 7 and 9 the interval $[\min A_{1}, \max A_{m}]$ is contained in K_{κ}^{c} . We assert that there is a constant C > 0

such that

$$J[u|_{K^c_{\kappa}}] \ge C|K^c_{\kappa}|\kappa^2.$$

Namely, considering $u \ge 0$ and u < 0 separately, we obtain that, for some C > 0, the inequality $j(u) \ge C\kappa^2$ holds pointwise for all $x \in K_{\kappa}^c$ (since F has non-degenerate equilibria). Since $J[u|_{K_{\kappa}^c}] < J(\boldsymbol{g}, P_{\nu})$ it follows that $|K_{\kappa}^c|$ is smaller than $\frac{J(\boldsymbol{g}, P_{\nu})}{C\kappa^2}$. Hence choosing $|T_{\delta_1}^{\pm}| = \frac{J(\boldsymbol{g}, P_{\nu})}{C\kappa^2}$ we have proved the lemma.

Our next aim is to obtain compactness of the set of minimizers. To proceed we need to convert to functions on a finite interval.

The restriction of the minimizers in $CM_*(\boldsymbol{g}, P_\nu)$ to $[T^-, T^+]$ is denoted by $CM_*^T(\boldsymbol{g}, P_\nu)$. Let $H_*^2(T^-, T^+) = \{u \in H^2(T^-, T^+) | u(0) = (-1)^{\nu+1}\}$, then $CM_*^T(\boldsymbol{g}, P_\nu) \subset H_*^2(T^-, T^+)$. Functions in $CM_*^T(\boldsymbol{g}, P_\nu)$ can be mapped back to $CM_*(\boldsymbol{g}, P_\nu)$ as follows. Define the map $E_0: CM_*^T(\boldsymbol{g}, P_\nu) \to CM_*(\boldsymbol{g}, P_\nu)$:

(6)
$$E_0[u] = \begin{cases} \alpha \left(x - T_{\delta}^-, \left(u(T_{\delta}^-), u_x(T_{\delta}^-) \right) \right) & x \in (-\infty, T_{\delta}^-) \\ u(x) & x \in [T_{\delta}^-, T_{\delta}^+] \\ \omega \left(x - T_{\delta}^+, \left(u(T_{\delta}^+), u_x(T_{\delta}^+) \right) \right) & x \in [T_{\delta}^+, \infty) \end{cases}$$

where α and ω are unique minimizers of an appropriate functional, i.e., α is the unique minimizer (see e.g. [22]) for J over functions α in $(-1)^{\nu} + H^2(-\infty, 0)$ for which $(\alpha(0), \alpha_x(0)) = (u(T_{\delta}^-), u_x(T_{\delta}^-))$. A similar definition holds for $\omega \in (-1)^{\nu+|g|+1} + H^2(0, \infty)$. The map E_0 is well-defined for all $u \in H^2_*(T_{\delta}^-, T_{\delta}^+)$ for which $\operatorname{dist}_{\mathbb{R}^2}((u, u^+)(T_{\delta}^{\pm}), \{P_1, P_2\})$ is sufficiently small (see e.g. [9, 22]), say $\operatorname{dist}_{\mathbb{R}^2}((u, u^+)(T_{\delta}^{\pm}), \{P_1, P_2\}) < \delta_3$.

We now fix

$$\delta = \delta_0 \stackrel{\text{def}}{=} \frac{1}{2} \min\{\delta_1, \delta_3\},$$

where δ_1 is defined in Lemma 10. Also fix $T^{\pm} \stackrel{\text{def}}{=} T^{\pm}_{\delta_0}$ (see Lemma 10). The set

$$V_{\epsilon}(\boldsymbol{g}, P_{\nu}) = \{ u \in H^2_*(T^-, T^+) \, | \, \text{dist}_{H^2}(u, CM^T_*(\boldsymbol{g}, P_{\nu})) \le \epsilon \},\$$

is a bounded neighborhood of $CM^T_*(\boldsymbol{g}, P_{\nu})$. For every $u \in V_{\epsilon}$ there exists $v \in CM^T_*(\boldsymbol{g}, P_{\nu})$ such that $||u - v||_{H^2} \leq \epsilon$ and thus $||u - v||_{W^{1,\infty}} \leq \tilde{C}\epsilon$, where \tilde{C} is the Sobolev embedding constant. When $\tilde{C}\epsilon < \delta_0$ then the map E_0 is well-defined on V_{ϵ} . If we choose

$$\epsilon \le \epsilon_0 \stackrel{\text{def}}{=} \min\{\delta_0, C_4, C_2 e^{-C_3 \max_{1 \le i \le |\mathbf{g}|} g_i}\}/\tilde{C},$$

then by Lemmas 7 and 9 the set $U_{\epsilon} = E_0[V_{\epsilon}]$ is contained $M_*(\boldsymbol{g}, P_{\nu})$. Fix $\epsilon = \epsilon_0$ and write $V(\boldsymbol{g}, P_{\nu}) \stackrel{\text{def}}{=} V_{\epsilon_0}(\boldsymbol{g}, P_{\nu})$. Of course, when necessary one can choose a smaller value of ϵ .

COROLLARY 11. The map E_0 is well-defined for all $u \in V(\boldsymbol{g}, P_{\nu})$ and the sets $U \stackrel{\text{def}}{=} E_0[V(\boldsymbol{g}, P_{\nu})] \subset M_*(\boldsymbol{g}, P_{\nu})$.

One now obtains the following compactness result.

LEMMA 12. For any admissible homotopy type \boldsymbol{g} the set $CM_*(\boldsymbol{g}, P_{\nu})$ is compact.

PROOF. The set $CM_*(\boldsymbol{g}, P_{\nu}) \subset (-1)^{\nu} \chi + H^2(\mathbb{R})$ is closed and bounded (by Lemma 7). It remains to show that $CM_*(\boldsymbol{g}, P_{\nu})$ is precompact. Let $\{u_n\} \subset CM_*(\boldsymbol{g}, P_{\nu})$, then by Lemma 10 we have that

$$\operatorname{dist}_{\mathbb{R}^2}((u_n, u_{n,x})(x), \{P_1, P_2\}) \le \delta_0 \quad \text{for } x \in [T^-, T^+]^c.$$

Define the functional $J^T \stackrel{\text{def}}{=} J \circ E_0$ on the bounded sets V. Since the functions u_n are minimizers it holds that $dJ^T[u_n] = dJ \circ E_0[u_n] = 0$. This yields the relation $0 = u_n + K[u_n]$, where K is a compact operator (cf. [23, Theorem 3.2]). For the sequence $\{u_n\}$ this implies that (possibly along a subsequence) u_n converges in $H^2(T^-, T^+)$ to some function u. Let us denote the tails of u_n on the intervals $(-\infty, T^-]$ and $[T^+, \infty)$ by α_n and ω_n respectively. Since δ_0 is sufficiently small and all α_n and ω_n satisfy Equation (5) it follows from the local theory near the equilibria that the tails α_n and ω_n also converge to $E_0[u]$ in $H^2(-\infty, T^-]$ and $H^2[T^+, \infty)$ respectively. Indeed, F has non-degenerate equilibria and thus $(F'(u_1) - F'(u_2))(u_1 - u_2) \geq \frac{1}{2}F''(\pm 1)(u_1 - u_2)^2$ for u_1 and u_2 sufficiently close to ± 1 . Hence we obtain, using the differential equation, for some small C > 0

$$\gamma \int_{-\infty}^{T^{-}} |\alpha_{n,xx} - \alpha_{m,xx}|^{2} + \beta \int_{-\infty}^{T^{-}} |\alpha_{n,x} - \alpha_{m,x}|^{2} + C \int_{-\infty}^{T^{-}} |\alpha_{n} - \alpha_{m}|^{2} \leq -\gamma (\alpha_{n,xxx} - \alpha_{m,xxx})(\alpha_{n} - \alpha_{m})(T^{-}) + \gamma (\alpha_{n,xx} - \alpha_{m,xx})(\alpha_{n,x} - \alpha_{m,x})(T^{-}) \\ -\beta (\alpha_{n,x} - \alpha_{m,x})(\alpha_{n} - \alpha_{m})(T^{-}).$$

The right-hand side tends to 0 as $n, m \to \infty$ since $\alpha_n(-T)$ and $\alpha_{n,x}(-T)$ converge, and $\alpha_{n,xx}(-T)$ and $\alpha_{n,xxx}(-T)$ are bounded (this follows from regularity arguments). Therefore the sequence $\{u_n\}$ converges strongly, possibly along a subsequence, in $\chi + H^2(\mathbb{R})$, which concludes the proof.

For J^T we can derive the following geometric properties.

LEMMA 13. The set of all minimizers of J^T in $V(\boldsymbol{g}, P_{\nu})$ is given by $CM_*^T(\boldsymbol{g}, P_{\nu})$. Moreover, there exist constants $C_0 = C_0(\boldsymbol{g}, F, \gamma, \beta) > 0$ such that $J^T|_{\partial V} \geq J(\boldsymbol{g}, P_{\nu}) + C_0$.

PROOF. By definition $U = E_0[V]$ and thus $\inf_V J^T = \inf_U J \ge J(\boldsymbol{g}, P_{\nu})$. For $u \in CM^T_*(\boldsymbol{g}, P_{\nu}) \subset V$ it follows that $J^T[u] = J(\boldsymbol{g}, P_{\nu})$ and therefore $\inf_V J^T = J(\boldsymbol{g}, P_{\nu})$. Clearly, if $J^T[u] = J(\boldsymbol{g}, P_{\nu})$ for some $u \in V$ then $E_0[u] \in CM_*(\boldsymbol{g}, P_{\nu})$ which proves the first claim.

Suppose there exists no constants C_0 such that $J^T|_{\partial V} \geq J(\boldsymbol{g}, P_{\nu}) + C_0$. Then one can find a sequence $u_n \in \partial V$ such that $J^T[u_n] \to J(\boldsymbol{g}, P_{\nu})$. By Ekeland's variational principle [14] there exists a slightly different sequence \tilde{u}_n with $\|\tilde{u}_n - u_n\|_{H^2(T^-, T^+)} \to 0$ as $n \to \infty$, such that $dJ^T[\tilde{u}_n] \to 0$, and $J^T[\tilde{u}_n] \leq J^T[u_n]$.

Since V is bounded it follows that there exists a subsequence, again denoted by \tilde{u}_n , such that $\tilde{u}_n \rightarrow u$ in $H^2(T^-, T^+)$ and $u_n \rightarrow u$ in $W^{1,\infty}(T^-, T^+)$. By the weak lower-semicontinuity of J we obtain the estimate $J^T[u] \leq J(\boldsymbol{g}, P_{\nu})$.

From the fact that $dJ^T[\tilde{u}_n] \to 0$ it follows, arguing as in the proof of Lemma 12, that $\tilde{u}_n \to u$ strongly in $H^2(T^-, T^+)$, hence $u_n \to u$, implying



FIGURE 7. Two symmetric homoclinic minimizers which have to be glued together to produce a stable stationary solution on the interval [0, L] satisfying Neumann boundary conditions.

that $u \in \partial V \subset M_*(\boldsymbol{g}, P_{\nu})$. From the definition of $J(\boldsymbol{g}, P_{\nu})$ it follows that $J^T[u] \geq J(\boldsymbol{g}, P_{\nu})$. Together with the reversed inequality which was already obtained, this implies that $u \in \partial V$ is a minimizer, a contradiction. \Box

REMARK 14. The constant C_0 in the above lemma depends on the homotopy type \mathbf{g} . In Sect. 5 we will prove that when we the neighborhood $V(\mathbf{g}, P_{\nu})$ is defined in a different way, C_0 can be chosen independent of \mathbf{g} for a large class of homotopy types \mathbf{g} .

4. Stable equilibrium solutions

The a priori properties of minimizers can be used now to construct stable equilibria for Equation (1) via a minimization procedure partly based on techniques used in [9] and [23]. Our first goal is to construct stable equilibria for (1) that satisfy the Neumann boundary conditions.

We split two symmetric homoclinics and glue the two halves together by matching their tails (see Figure 7). The length of the plateau thus formed in the middle can be arbitrarily long. Since our initial homoclinic minimizers are not necessarily isolated we have to perform a careful gluing procedure in special subsets V of the function space, so that the infimum of J on V is strictly larger than infimum of J on ∂V , and hence the minimum is attained in the interior of V.

Another way to express 'splitting' of symmetric homoclinic minimizers is to take minimizers from $CM_{\mathbb{R}^{\pm}}(\boldsymbol{g}, P_{\nu})$. Minimisers in $CM_{\mathbb{R}^{\pm}}(\boldsymbol{g}, P_{\nu})$ are obtained from minimizers in $CM(\boldsymbol{g}^{-1}\boldsymbol{g}, P_{\nu})$ in the following way. Normalise functions in $CM(\boldsymbol{g}^{-1}\boldsymbol{g}, P_{\nu})$ by setting u(0) = 0 at the unique point of even symmetry. The sets $CM_{\mathbb{R}^{-}}(\boldsymbol{g}, P_{\nu})$ and $CM_{\mathbb{R}^{+}}(\boldsymbol{g}, P_{\nu})$ are then obtained by restricting to the intervals $(-\infty, 0]$ and $[0, \infty)$ respectively. For functions in $CM(\boldsymbol{g}^{-1}\boldsymbol{g}, P_{\nu})$, that are normalised as described above, we now have that the conclusions of Lemma 10 hold for $|x| > T = (T^{+} - T^{-})/2$. Define $CM_{\mathbb{R}^{-}}^{T}(\boldsymbol{g}, P_{\nu})$ and $CM_{\mathbb{R}^{+}}^{T}(\boldsymbol{g}, P_{\nu})$ as the restrictions of functions in $CM_{\mathbb{R}^{-}}(\boldsymbol{g}, P_{\nu})$ and $CM_{\mathbb{R}^{+}}(\boldsymbol{g}, P_{\nu})$ to the intervals [-T, 0] and [0, T] respectively. Let $H_{n}^{2}(0, T) = \{u \in$ $H^2(0,T) | u_x(0) = 0 \}$ and $H^2_n(-T,0) = \{ u \in H^2(-T,0) | u_x(0) = 0 \}$, then $CM^T_{\mathbb{R}^-}(\boldsymbol{g}, P_{\nu}) \subset H^2_n(-T,0)$ and $CM^T_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu}) \subset H^2_n(0,T)$. As in the previous section we can define the map $E_0: \widetilde{CM}_{\mathbb{R}^+}^T(\boldsymbol{g}, P_{\nu}) \to \widetilde{CM}_{\mathbb{R}^+}(\boldsymbol{g}, P_{\nu}):$

$$E_0^+[u] = \begin{cases} u(x) & x \in [0,T] \\ \omega(x - T, (u(T), u_x(T))) & x \in [T, \infty). \end{cases}$$

By the same token we define the map E_0^- : $CM_{\mathbb{R}^-}^T(\boldsymbol{g}, P_{\nu}) \to CM_{\mathbb{R}^-}(\boldsymbol{g}, P_{\nu})$. The functionals $J_{\mathbb{R}^-} \circ E_0^-$ and $J_{\mathbb{R}^+} \circ E_0^+$ are well-defined on $CM_{\mathbb{R}^-}^T(\boldsymbol{g}, P_{\nu})$ and $CM_{\mathbb{R}^+}^T(\boldsymbol{g}, P_{\nu})$ respectively. As in the previous section we can define ϵ neighborhoods of $CM_{\mathbb{R}^+}^T(\boldsymbol{g}, P_{\nu}) \subset H_n^2(0, T^+)$ and $CM_{\mathbb{R}^-}^T(\boldsymbol{g}, P_{\nu}) \subset H_n^2(0, T^-)$, which we indicate by V^+ and V^- respectively. The functionals J_T^{\pm} are welldefined on these neighborhoods if ϵ is small enough, say $\epsilon \leq \epsilon_0(\boldsymbol{g})$ (see Corollary 11). The following is an immediate consequence of Lemma 13.

LEMMA 15. The set of all minimizers of J_T^+ over V^+ is given by $CM_{\mathbb{R}^+}^T(\boldsymbol{g}, P_{\nu})$. Moreover, there exist constants $C_0 = C_0(\boldsymbol{g}, F, \gamma, \beta) > 0$ such that $J_{\mathbb{R}^+} \circ E_0^+|_{\partial V^+(\boldsymbol{g},P_\nu)} \geq J_{\mathbb{R}^+}(\boldsymbol{g},P_\nu) + C_0$. The same statement holds for $J_{\mathbb{R}^-} \circ E_0^-$.

We now use Lemma 15 to construct neighborhoods $V \subset H^2_N(0,L)$ such that $\inf_{\partial V} J > \inf_{V} J$. In order to do so we again invoke the local theory near the equilibrium points (see Theorems 4.1 and 4.2 in [22]). Take $\bar{y} = (y_1, y_2)$ and $\bar{z} = (z_1, z_2)$, with both $|\bar{y} - (\pm 1, 0)| < \delta_1$ and $|\bar{z} - (\pm 1, 0)| < \delta_1$ and δ_1 sufficiently small (in fact one can take the same value as in Lemma 10). Then the boundary value problem for Equation (5) on an interval of length s with left and right boundary conditions given by $(u, u')(0) = \overline{y}$ and $(u, u')(s) = \overline{z}$ has a unique global minimizer if s is larger than some constant, say $s > S_0 = S_0(F, \gamma, \beta, \delta_1)$. This minimizer is denoted by $g(x, \bar{y}, \bar{z}, s)$.

Let g^- and g^+ be two admissible homotopy types, i.e. $g^{\pm} = (g_1^{\pm}, .., g_{|g^{\pm}|}^{\pm})$, with $g_1^{\pm} \in \mathbb{N}$ and $g_i^{\pm} \in 2\mathbb{N}$ for $i = 2, ..., |\mathbf{g}^{\pm}|$. Furthermore, let $H_N^2(0, 2T + s) = \{u \in H^2(0, 2T + s) \mid u_x(0) = u_x(2T + s) = 0\}$. Define the map $E_2^s : CM_{\mathbb{R}^+}^T(\mathbf{g}^+, P_\nu) \times CM_{\mathbb{R}^-}^T(\mathbf{g}^-, P_\nu) \to H_N^2(0, 2T + s)$ as

follows:

$$E_2^s[u^+, u^-] = \begin{cases} u^+(x) & x \in [0, T] \\ g(x - T, (u^+(T), u_x^+(T)), (u^-(0), u_x^-(0)), s) & x \in [T, T+s] \\ u^-(x - 2T - s) & x \in [T+s, 2T+s] \end{cases}$$

Arguing as in Sect. 3, since $\delta_0 \leq \frac{1}{2}\delta_1$ it follows that when we choose $\epsilon = \min\{\epsilon_0(\boldsymbol{g}^+), \epsilon_0(\boldsymbol{g}^-)\}$, the functional $J_s^T \stackrel{\text{def}}{=} J_{2T+s} \circ E_2^s : V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-) \to \mathbb{R}$ is well-defined for any $s > S_0$.

The estimate of Lemma 15 carries over to the current situation.

LEMMA 16. There exist constants S_1 , $C_0(\mathbf{g}^-)$, and $C_0(\mathbf{g}^+)$ such that

$$\inf_{\partial(V^+(\boldsymbol{g}^+)\times V^-(\boldsymbol{g}^-))} J_s^T \ge \inf_{V^+(\boldsymbol{g}^+)\times V^-(\boldsymbol{g}^-)} J_s^T + \min(C_0(\boldsymbol{g}^+), C_0(\boldsymbol{g}^-))/2$$

for all $s \geq S_1$.

PROOF. For any pair $(u^+, u^-) \in V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)$ we have that $J_s^T[u^+, u^-] = \int_0^T j(u^+) + \int_0^s j(g) + \int_{-T}^0 j(u^-)$ $= J_{\mathbb{R}^+} \circ E_0^+[u^+] - \int_0^\infty j(\omega) + \int_0^s j(g) - \int_{-\infty}^0 j(\alpha) + J_{\mathbb{R}^-} \circ E_0^-[u^-]$ $= J_{\mathbb{R}^+} \circ E_0^+[u^+] + J_{\mathbb{R}^-} \circ E_0^-[u^-] + A(s),$

where $A(s) = -\int_0^\infty j(\omega) + \int_0^s j(g) - \int_{-\infty}^0 j(\alpha)$. The behaviour of A(s) is governed by the linear flow near a saddle-focus and we find that $A(s) = O(e^{-c_0s})$ for $s \to \infty$, where $c_0 = c_0(F, \gamma, \beta) > 0$. Indeed, $A(s) = \int_0^{s/2} [j(g) - j(\omega)] - \int_{s/2}^\infty j(\omega) + \int_{-s/2}^0 [j(g(x+s)) - j(\alpha)] - \int_{-\infty}^{-s/2} j(\alpha)$, and each integral decays exponentially in s. For the second and fourth term this follows from the linearisation of the flow near the non-degenerate equilibrium point. Besides, we obtain, in a similar manner as in the proof of Lemma 12, that $\|\omega - g\|_{H^2(0,s/2)}$ is bounded by boundary terms and hence is of order $O(e^{-c_1s})$ for some $c_1 > 0$. It then follows that $\int_0^{s/2} j(g) - j(\omega) = O(e^{-c_2s})$ for some $c_2 > 0$, since ω and g are close to the (non-degenerate) equilibrium point. An analogous argument deals with the term $\int_{-s/2}^0 j(g(x+s)) - j(\alpha)$.

We choose $S_1 \ge S_0$ such that $A(s) \le \min(C_0(g^+), C_0(g^-))/4$ for all $s \ge S_1$. Applying Lemma 15 now finishes the proof.

The information of Lemma 16 can be used to find minimizers for J_s^T in $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$ for all $s \geq S_1$. Indeed, let $(u_n^+, u_n^-) \in V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$ be a minimizing sequence for J_s^T , for $s \geq S_1$ fixed. Then $\|u_n^+\|_{H^2_n(0,T)} + \|u_n^-\|_{H^2_n(-T,0)}$ is bounded and thus $(u_n^+, u_n^-) \rightarrow (u^+, u^-) \in H^2_n(0, T) \times H^2_n(-T, 0)$. In exactly the same way as in the proof of Lemma 13 one obtains that in fact $(u_n^+, u_n^-) \rightarrow (u^+, u^-)$ strongly in $H^2_n(0, T) \times H^2_n(-T, 0)$. It follows that $(u^+, u^-) \in V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$, and since J_s^T is weakly lower-semicontinuous we derive that (u^+, u^-) is a minimizer of J_s^T on $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$. The fact that the sets $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$ contain minimizers for J_s^T does not necessarily imply that the functions $E_2^s[u^+, u^-]$ are solutions to Equation (5). However, since the minimizers (u^+, u^-) lie in the interior of $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$ one can prove that $E_2^s[u^+, u^-]$ are local minimizers for J and hence solutions of (5).

LEMMA 17. Let (u^+, u^-) be a minimizer of J_T^s in $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$. Then for all $\phi \in H^2_N(0, 2T + s)$ with $\|\phi\|_{H^2}$ sufficiently small it holds that $J_{2T+s}[E_2^s[u^+, u^-] + \phi] \geq J_{2T+s}[E_2^s[u^+, u^-]]$. Moreover, the function $v = E_2^s[u^+, u^-]$ satisfies Equation (5) with the Neumann boundary conditions $u_x(0) = u_{xxx}(0) = 0$ and $u_x(2T + s) = u_{xxx}(2T + s) = 0$.

PROOF. Since the minimizer $u = E_2^s[u^+, u^-]$ lies in $\operatorname{int}(V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-))$ one can find small open neighborhoods $N^+ \subset V^+(\boldsymbol{g}^+)$ and $N^- \subset V^-(\boldsymbol{g}^-)$ of u^+ and u^- respectively such that $J_s^T[u^+ + \phi^+, u^- + \phi^-] \ge J_s^T[u^+, u^-]$ for all $(u^+ + \phi^+, u^- + \phi^-) \in N^+ \times N^-$.

Let $N \subset H^2_N(0, 2T + s)$ be a small neighborhood of $u = E^s_2[u^+, u^-]$, i.e., $v \in N$ can be written as $v = u + \phi$, with $\phi \in H^2_N(0, 2T + s)$ and $\|\phi\|_{H^2}$ small. If the neighborhood N is small enough then $\phi^- = \phi|_{[T+s, 2T+s]} \in N^-$

and $\phi^+ = \phi|_{[0,T]} \in N^+$. The part in the middle, $\phi|_{[T,T+s]}$, is denoted by ϕ^0 We can write $v + \phi^0 = v + \hat{\phi}^0 + (\phi^0 - \hat{\phi}^0)$, where $v + \hat{\phi}^0$ is the unique minimizer of $J_{[T,T+s]}$ over functions with boundary conditions at x = T and x = T + s equal to $(u^+ + \phi^+, u_x^+ + \phi_x^+)(T)$ and $(u^- + \phi^-, u_x^- + \phi_x^-)(T+s)$ respectively, i.e., functions of the form $v|_{[T,T+s]} + \psi_0$ with $\psi^0 \in H^2_0(T, T+s)$.

We now have that

$$\begin{aligned} J_{2T+s}[E_2^s[u^+, u^-] + \phi] &= \int_0^T j(u^+ + \phi^+) + \int_T^{T+s} j(v + \phi^0) + \int_{T+s}^{2T+s} j(u^- + \phi^-) \\ &\geq \int_0^T j(u^+ + \phi^+) + \int_T^{T+s} j(v + \hat{\phi}^0) + \int_{T+s}^{2T+s} j(u^- + \phi^-) \\ &= J_{2T+s}[E_2^s[u^+ + \phi^+, u^- + \phi^-]] = J_s^T[u^+ + \phi^+, u^- + \phi^-] \\ &\geq J_s^T[u^+, u^-] = J_{2T+s}[E_2^s[u^+, u^-]]. \end{aligned}$$

This proves the first claim. From the fact that $u = E_2^s[u^+, u^-]$ is a local minimizer of J_{2T+s} one easily deduces that u satisfies Equation (5) and the Neumann boundary conditions.

The next step is to construct proper attracting neighborhoods in $H_N^2[0, 2T + s]$ for Equation (1) that contain the equilibria $E_2^s[u^+, u^-]$.

Let $\phi \in B_r(0) \subset H_0^2(T, T + s)$ and consider the triples $(u^+, u^-, \phi) \in V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-) \times B_r(0)$. Define the map $\mathcal{F}^s : V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-) \times B_r(0) \to H_N^2(0, 2T + s)$ as follows: $\mathcal{F}^s(u^+, u^-, \phi) = E_2^s[u^+, u^-] + \tilde{\phi}$, where $\tilde{\phi} \in H_0^2(0, 2T + s)$ is the extension by zero of ϕ . Set $Y \stackrel{\text{def}}{=} \mathcal{F}^s(V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-) \times B_r(0))$. We want to show that $\inf_{\partial Y} J_{2T+s} > \inf_Y J_{2T+s}$, and from Lemma 16 we see that the remaining problematic boundary of Y is $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-) \times \partial B_r(0)$. However, if we for example choose r sufficiently large then this problem is overcome and

$$J_{2T+s}[u] = J[E_2^s[u^+, u^-] + \tilde{\phi}] \ge \inf_{V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)} J_s^T + \min(C_0(\boldsymbol{g}^-), C_0(\boldsymbol{g}^+))/2$$

is satisfied for all $u \in \partial Y$.

Let \mathcal{S} be a the set of minimizers of J_s^T in $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$. As before $u \in Y$ is in \mathcal{S} if and only if there is a pair (u^+, u^-) which minimizes J_s^T on $V^+(\mathbf{g}^+) \times V^-(\mathbf{g}^-)$, with $u = \mathcal{F}^s(u^+, u^-, 0)$. We will now show that \mathcal{S} is stable. Let $\eta < \epsilon = \min(\epsilon_0(\mathbf{g}^-), \epsilon_0(\mathbf{g}^+))$, then $B_\eta(\mathcal{S}) = \{u \in H^2(0, 2T + s) | \operatorname{dist}_{H^2}(u, \mathcal{S}) < \eta\}$ is contained in Y (for r sufficiently large).

Via the same reasoning as in Lemma 13 we find that

$$a \stackrel{\text{def}}{=} \frac{1}{2} \left(\inf_{\partial B_{\eta}(\mathcal{S})} J_{2T+s} - \inf_{B_{\eta}(\mathcal{S})} J_{2T+s} \right) > 0.$$

Define $N^a_{\epsilon} = J^a_{2T+s} \cap B_{\eta}(\mathcal{S})$, where J^a_{2T+s} is the sub-level set

$$J_{2T+s}^{a} = \{ u \in H_{N}^{2}(0, 2T+s) \mid J_{2T+s}[u] \leq \inf_{B_{\eta}(\mathcal{S})} J_{2T+s} + a \}.$$

It follows that $J_{2T+s}|_{\partial N^a_{\epsilon}} = a$. Since Equation (1) is the L^2 -gradient flow equation of J, the quantity $J_{2T+s}[u(t,x)]$ decreases in t, and thus for initial data $u(0,x) = u_0(x) \in N^a_{\epsilon}$ it holds that $u(t,x) \in N^a_{\epsilon}$ for all t > 0. This proves that \mathcal{S} is a stable set for Equation (1). Since $s > S_1$ is arbitrary and this construction can be carried out for all admissible homotopy types g^+ and g^- , we obtain the following theorem (Theorem 2 of the introduction).

THEOREM 18. Let $\frac{\gamma}{\beta^2} > \max\{\frac{1}{4F''(-1)}, \frac{1}{4F''(+1)}\}$. Then for any $n \in \mathbb{N}$ there exists a constant $L_n > 0$ such that for all $L \ge L_n$ Equation (1) with the Neumann boundary conditions has at least n disjoint sets of stable equilibria (in the sense of Definition 1).

5. Estimating the number of equilibria

Some of the estimates obtained in Sects. 3 and 4 can be made uniform with respect to the homotopy type g. With such uniform estimates one can obtain a lower bound on the number of stable solutions of Equation (1) as function of L. The crucial constant in this context is the constant introduced in Lemma 13:

$$C_0 = \inf_{\partial V} J[u] - \inf_V J[u].$$

We recall from Sect. 3 that fixing γ , β and F, we have that ϵ_0 only depends $\max_{1 \leq i \leq |\mathbf{g}|} g_i$. The following lemma is a uniform analogue of Lemma 13 and shows that, with an appropriate choice of the neighborhood V the constant C_0 also depends only on $\max_{1 \leq i \leq |\mathbf{g}|} g_i$.

LEMMA 19. For all $N_* \in \mathbb{N}$ there exists positive constants C_0 , D_1 and D_2 such that for any admissible homotopy type \boldsymbol{g} with $g_i \leq 2N_*$ for all $i = 1, 2, \ldots, |\boldsymbol{g}|$, there exists a bounded neighborhood $V(\boldsymbol{g}, P_{\nu}) \subset H^2_*(T^-, T^+)$ of $CM^T_*(\boldsymbol{g}, P_{\nu})$ with $|T^{\pm}| \leq D_1 + D_2|\boldsymbol{g}|$, such that $E_0[V(\boldsymbol{g}, P_{\nu})] \subset M_*(\boldsymbol{g}, P_{\nu})$ and $\inf_{\partial V} J \circ E_0[u] - J(\boldsymbol{g}, P_{\nu}) > C_0$.

It should be clear that we need to restrict the magnitude g_i to get such a uniform estimate, since the higher g_i the closer $CM_*(\boldsymbol{g}, P_{\nu})$ gets to the boundary of the class $M_*(\boldsymbol{g}, P_{\nu})$, i.e., the more oscillations around one of the equilibrium points the closer the function approaches the equilibrium. Note however that the length $|\boldsymbol{g}|$ of the homotopy type is arbitrary. This is made possible by an appropriate choice of $V(\boldsymbol{g}, P_{\nu})$, which will be discussed later on.

Before we prove the lemma we will first explain how the lemma can be used to count the number of equilibria (or attracting sets) as $L \to \infty$. Our goal is to derive the exponential lower bound on the number of stable equilibria as function of L mentioned in Equation (4). Choosing $V(\boldsymbol{g}, P_{\nu})$ as in Lemma 19 it follows from the proof of Lemma 16 that S_1 depends on N_* only (since C_0 depends on N_* only). We now fix $N_* > 1$ and only consider \boldsymbol{g} when $g_i \leq 2N_*$.

One can now construct stable solutions of (1) as in Sect. 4 by using building blocks $(u^+, u^-) \in V^+(\boldsymbol{g}^+) \times V^-(\boldsymbol{g}^-)$ for which $g_i^-, g_j^+ \leq 2N_*$. The solutions are defined on intervals of length $L = T(\boldsymbol{g}^+) + T(\boldsymbol{g}^-) + s$ with $s \geq S_1$. Since $s \geq S_1$ can be chosen arbitrarily a stable solution of such type then exist for all interval lengths $L \geq T(\boldsymbol{g}^+) + T(\boldsymbol{g}^-) + S_1$. Since $T(\boldsymbol{g}^{\pm}) \leq D_1 + D_2 |\boldsymbol{g}^{\pm}|$ by Lemma 19 a stable solution thus exist for all interval lengths $L \geq 2D_1 + D_2(|\boldsymbol{g}^+| + |\boldsymbol{g}^-|) + S_1$. Hence we obtain a stable solutions on an interval of length L for every pair $(\boldsymbol{g}^+, \boldsymbol{g}^-)$ with $g_i^-, g_j^+ \leq 2N_*$ such that $|\boldsymbol{g}^+| + |\boldsymbol{g}^-| \leq (L - S_1 - 2D_1)/D_2$. The number of such pairs to grows as $(N_*)^{(L-S_1-2D_1)/D_2}$, i.e., exponentially in L. This proves Equation (4).

To prove Lemma 19 we first recall the Uniform Separation Property from [22] (see also Lemma 9) which holds for all admissible types g:

Uniform Separation Property: There exists a $\tilde{\delta} > 0$ and an $\tilde{\epsilon} > 0$ such that for all admissible homotopy types \boldsymbol{g} and all $u \in M(\boldsymbol{g}, P_{\nu})$ with $J[u] \leq J(\boldsymbol{g}, P_{\nu}) + \tilde{\delta}$ we have $|u(x) - (-1)^{i}| > \tilde{\epsilon}$ for all $x \in I_{i}, i = 0, \dots, |\boldsymbol{g}| + 1$.

Although in [22] $\tilde{\epsilon}$ depends on g and the Uniform Separation Property is only used for so-called normalised functions, the constant ϵ can in fact be chosen independent of g and in absence of normalisation.

The justification of the construction of the neighborhoods V needed in Lemma 19 is quite technical. First define

$$W_{\epsilon} \stackrel{\text{def}}{=} \{ u \in M_*(\boldsymbol{g}, P_{\nu}) \, | \, \text{dist}_{\mathbb{R}^2} \big(\Gamma(u|_{I_{\text{core}}}), P_i \big) > \epsilon \text{ for } i = 1, 2 \},$$

where $I_{\text{core}} = [\max A_0, \min A_{|g|}]$ is the *core* interval. Next define

$$U_{\epsilon,\delta} \stackrel{\text{der}}{=} \{ u \in W_{\epsilon} \, | \, J[u] < J(\boldsymbol{g}, P_{\nu}) + \delta \}.$$

By Lemmas 7 and 9 and the set W_{ϵ} is a neighborhood of $CM_*(\boldsymbol{g}, P_{\nu})$ for ϵ small enough and all \boldsymbol{g} with $g_i \leq 2N_*$. By the Uniform Separation Property we have $\overline{U}_{\epsilon,\delta} \subset M_*(\boldsymbol{g}, P_{\nu})$ for δ small enough.

In order to reduce to function on a finite interval, define

$$\mathcal{U}_{\eta}^{T^{\pm}} \stackrel{\text{def}}{=} \Big\{ u \in H^2_*(T^-, T^+) \mid \operatorname{dist}_{\mathbb{R}^2} \big(\Gamma(u(T^-)), P_\nu \big) < \eta, \\ \operatorname{dist}_{\mathbb{R}^2} \big(\Gamma(u(T^+)), P_{\nu+|\boldsymbol{g}|-1 \mod 2} \big) < \eta \Big\},$$

where η is chosen so small that E_0 (see Sect. 3) is well-defined on $\mathcal{U}_{\eta}^{T^{\pm}}$. In what follows η is fixed. The following lemma shows that $U_{\epsilon,\delta} \subset \mathcal{U}_{\eta}^{T^{\pm}}$ for T^{\pm} large enough.

LEMMA 20. There exist constants $\tilde{\delta}(\eta) > 0$, $\tilde{T}(\eta) > 0$ such that for any $\delta < \tilde{\delta}$ and any \boldsymbol{g} with $g_i \leq 2N_*$ for all $i = 1, 2, \dots |\boldsymbol{g}|$ (and η and ϵ small enough) it holds that when $u \in U_{\epsilon,\delta}$ then $u \in \mathcal{U}_{\eta}^{T\pm}$, with $T^- = C^-(\boldsymbol{g}) - \tilde{T}$ and $T^+ = C^+(\boldsymbol{g}) + \tilde{T}$, where the constants $C^{\pm}(\boldsymbol{g})$ can be chosen such that $C^{\pm} < \tilde{C}|\boldsymbol{g}|$ for some \tilde{C} independent of \boldsymbol{g} and η .

PROOF. The functions u in $U_{\epsilon,\delta}$ are uniformly bounded in $W^{1,\infty}$. Indeed, a function $u \in M_*(\boldsymbol{g}, P_{\nu})$ with large $W^{1,\infty}$ -norm can be easily modified to a function $\tilde{u} \in M_*(\boldsymbol{g}, P_{\nu})$ with $J[\tilde{u}] < J[u] - C$ for some $C > \delta$ (the appropriate estimates can be found for example in [22, Lemma 5.1]. This contradiction shows that such u (with large $W^{1,\infty}$ -norm) are not in $U_{\epsilon,\delta}$.

It follows from a test function argument (cf. [22, Sect. 4]) that there exists a constant C > 0, independent of g_i , such that $J[u|_{\ell_i}] < C$, and thus $J[u|_{I_{\text{core}}}] \leq C|\boldsymbol{g}|$. Since $u \in W_{\epsilon}$, i.e., $\Gamma(u)$ stays away from the equilibrium points $(\pm 1, 0)$, this implies that $|I_{\text{core}}| \leq \tilde{C}|\boldsymbol{g}|$ for some $\tilde{C} = \tilde{C}(\tilde{\epsilon}) > 0$.

After taking care of the core interval, we need to estimate the tails. The action of the tails is also uniformly bounded by a test function argument. For δ smaller than δ (defined in the Uniform Separation Property above) this implies that the norm $||u - (-1)^{\nu}||_{H^2(-\infty,\max(A_0))}$ of the left tail is uniformly bounded (and similarly for the right tail).

Taking $T = T(\tilde{\eta})$ large enough there exists a point $x_1 \in [\max(A_0) - \sum_{i=1}^{n} T(\tilde{\eta})]$ $T, \max(A_0)$] such that $\Gamma(u(x_1)) \in B_{\tilde{\eta}}(P_{\nu})$. From, again, a test function argument and the local behaviour near the equilibrium it follows that for $\tilde{\eta}$ small enough $J[u|_{(-\infty,x_1)}] \leq c_1 \tilde{\eta}^2 + \delta$ for some $c_1, c_2 > 0$. On the other hand, for $\Gamma(u)$ to go from $\partial B_{\eta/2}(P_{\nu})$ to $\partial B_{\eta}(P_{\nu})$, it costs at least an amount $c(\eta) > 0$ of action. Take $\tilde{\eta} < \eta/2$ and moreover choose $\tilde{\eta} = \tilde{\eta}(\eta)$ and $\delta = \delta(\eta)$ so small that $c_1 \tilde{\eta}^2 + \delta < c(\eta)$. This ensures that $\Gamma(u(x)) \in B_{\eta}(P_{\nu})$ for all $x \leq x_1$ (and $x_1 \in [\max(A_0) - T, \max(A_0)])$. Taking $T^{\pm} = \tilde{C}|\boldsymbol{g}| + \tilde{T}$ we obtain that $u \in \mathcal{U}_{\eta}^{T\pm}$.

Finally, we pick up the proof of Lemma 19. Let $\tilde{\delta}(\eta)$ and T^{\pm} be as in Lemma 20. We next define the neighborhoods V needed in Lemma 19:

$$V_{\epsilon,\delta}(\boldsymbol{g}, P_{\nu}) \stackrel{\text{def}}{=} \{ u \in \mathcal{U}_{\eta}^{T^{\pm}} \mid E_0[u] \in W_{\epsilon} \text{ and } J \circ E_0[u] < J(\boldsymbol{g}, P_{\nu}) + \delta \}.$$

This is a bounded neighborhood of $CM^T_*(\boldsymbol{g}, P)$. Moreover, the construction of V is such that ∂V consists of three parts, i.e., any $u \in \partial V$ satisfies one of the following possibilities (see also Figure 8):

- $J \circ E_0[u] = J(\boldsymbol{g}, P_{\nu}) + \delta;$ $\Gamma(u(T^{\pm})) \in \partial B_{\eta}(P_{\nu});$
- $E_0[u] \in \partial W_{\epsilon}$.

The first possibility is no problem, since we in fact want to show that $\inf_{\partial V} J \circ$ $E_0 - \inf_V J \circ E_0$ is bounded away from zero (uniformly $\inf_{\sigma} g$). The second possibility is excluded by choosing $\delta \leq \tilde{\delta}(\frac{\eta}{2})$ so that $u \in \mathcal{U}_{\eta/2}^{T^{\pm}}$ by Lemma 20. The third possibility is dealt with in the next lemma, which states that for such u we have $J \circ E_0[u] \geq J(\boldsymbol{g}, P_{\nu}) + C_0$ for some $C_0 > 0$ if δ and ϵ are sufficiently small. Taking $C_0 = \min\{\tilde{C}_0, \delta\}$ finishes the proof of Lemma 19.

The following lemma deals with the third of the three possibilities above.

LEMMA 21. There exist constants \tilde{C}_0 and ϵ_0 such that for δ sufficiently small and any \boldsymbol{g} with $g_i \leq 2N_*$ for all $i = 1, 2, \dots |\boldsymbol{g}|$ it holds that when $u \in V_{\epsilon_0,\delta}$ and $E_0[u] \in \partial W_{\epsilon_0}$, then $J \circ E_0[u] \ge J(\boldsymbol{g}, P_{\nu}) + \tilde{C}_0$.

PROOF. Assume by contradiction that such $\tilde{C}_{\underline{0}}$ and ϵ_0 do not exist. Thus, for all δ_0 and ϵ_0 there exist functions $u_n \in \overline{V}_{\epsilon_0,\delta_0}(\boldsymbol{g}^n, P_{\nu_n})$ and $u_n \in \partial W_{\epsilon_0}(\boldsymbol{g}^n, P_{\nu_n})$ such that $J[u_n] - J(\boldsymbol{g}^n, P_{\nu_n}) \to 0$ as $n \to \infty$. We will choose δ_0 and ϵ_0 later on.

By taking a subsequence we may take ν_n constant, say $\nu_n = 1$, and we will drop P_{ν} from our notation. Let $x_n \in I_{\text{core}}^n$ be points such that

$$\operatorname{dist}_{\mathbb{R}^2} \left(\Gamma(u_n(x_n)), ((-1)^{k_n}, 0) \right) = \epsilon_0.$$



FIGURE 8. The boundary of $V_{\epsilon,\delta}$ a priori consists of three parts, since it is the intersection of $\mathcal{U}_{\eta}^{T^{\pm}}$, the sub-level set J_{δ} and the set $E_0^{-1}(W_{\epsilon})$. When δ is sufficiently small then the (appropriate part of) the sub-level set is contained in $\mathcal{U}_{\eta}^{T^{\pm}}$.

Again taking a subsequence, we may assume that k_n is constant in the previous expression, to fix ideas say $k_n = 2$ for all n (the other case, $k_n = 1$, is analogous).

We now want to locate the points x_n , and for this purpose we define the following sets (see also Figure 3 for the definition of ℓ_i and I_i):

$$S_i \stackrel{\text{def}}{=} \begin{cases} \ell_i & \text{if } i \text{ is odd,} \\ I_i & \text{if } i \text{ is even,} \end{cases} \quad \text{for } i = 1 \dots |\boldsymbol{g}| - 1,$$

and

$$S_{|\boldsymbol{g}|} \stackrel{\text{def}}{=} \begin{cases} \ell_{|\boldsymbol{g}|} & \text{if } |\boldsymbol{g}| \text{ is odd,} \\ [\max A_{|\boldsymbol{g}|}, \min A_{|\boldsymbol{g}|+1}] & \text{if } |\boldsymbol{g}| \text{ is even.} \end{cases}$$

These sets cover the core interval, i.e., $I_{\text{core}} = \bigcup_{i=1}^{|g|} S_i$. The points x_n are in at least one of these sets S_i , say S_{i_n} . Taking a subsequence we may assume that one of the following three cases holds:

- (1) $1 < i_n < |g|$ for all *n*;
- (2) $i_n = 1$ for all n;
- (3) $i_n = |\boldsymbol{g}|$ for all n.

We will exclude each of these three possibilities by choosing ϵ_0 and δ_0 small enough.

We start with Case 1. Taking a subsequence one may assume that i_n either is odd for all n, or even for all n. In the latter case we easily reach a contradiction by choosing $\epsilon_0 < \tilde{\epsilon}$ and $\delta_0 < \tilde{\delta}$, where $\tilde{\epsilon}$ and $\tilde{\delta}$ are defined in the Uniform Separation Property above.

We now deal with the case that i_n is odd for all n, which is somewhat more complicated. Taking a subsequence we can assume that $g_{i_n}^n$ is constant, say $g_{i_n}^n = \tilde{g} \in 2\mathbb{N}$. Shift all u_n so that $x_n = 0$ for all n. We now take another subsequence such that $g_{i_n-1}^n$ and $g_{i_n+1}^n$ are independent of n as well, say $g_{i_n-1}^n = \tilde{g}_{l_n}$ and $g_{i_n-1}^n = \tilde{g}_r$.

say $g_{i_n-1}^n = \tilde{g}_l$ and $g_{i_n-1}^n = \tilde{g}_r$. Let $\mathcal{I}_n \stackrel{\text{def}}{=} [\max(A_{i_n-2}), \min(A_{i_n+2})]$. The functions u_n are uniformly bounded in $W^{1,\infty}$, as discussed in the proof of Lemma 20. By a test function argument it follows that $J[u_n|_{\mathcal{I}_n}]$ is bounded, which in turn (since $u \in \overline{W}_{\epsilon_0}$) implies that $|\mathcal{I}_n|$ and $||u_n||_{H^2(\mathcal{I}_n)}$ are bounded.

Take a weak limit (along a subsequence) in H^2_{loc} which converges to v weakly in H^2_{loc} and strongly in $W^{1,\infty}_{\text{loc}}$. We have that $\text{dist}_{\mathbb{R}^2}(\Gamma(v(0)), (1,0)) = \epsilon_0$. The intervals \mathcal{I}_n and S_{i_n} converge to intervals \mathcal{I}_v and S_v respectively. It holds that v(x) = 1 on $\partial \mathcal{I}_v$, and v(x) = -1 on ∂S_v . Besides, v(x) has on \mathcal{I}_v subsequently \tilde{g}_r crossings of -1, then \tilde{g} crossings of +1 (in fact these crossings occur in S_v), and finally \tilde{g}_l crossings of -1.

Moreover, it is not too difficult to conclude that $v|_{\mathcal{I}_v}$ is a minimizer of J in the sense of [21, Definition 2.1], i.e., among function with the same boundary conditions (i.e., matching to $(v, v')|_{\partial \mathcal{I}_v}$) and the same number of crossings of ± 1 , where the interval length is arbitrary. However, such minimizers satisfy the result of Lemma 7 on the interval S_v , i.e., $||v - 1||_{W^{1,\infty}(S_v)} > c_1 e^{-2c_2N_*}$ for some $c_1, c_2 > 0$. We now take $\epsilon_0 < c_1 e^{-2c_2N_*}$ to reach a contradiction, i.e. contradicting the fact that $\operatorname{dist}_{\mathbb{R}^2}(\Gamma(v(0)), (1, 0)) = \epsilon_0$. Hence, the possibility in Case 1 is excluded.

In Case 2 a very similar argument holds. Namely, arguing along the same lines we now define $\mathcal{I}_n = [T^-, \min(A_3)]$ (or $[T^-, T^+]$ if $|\boldsymbol{g}| = 1$). We again find a weak limit v and $v|_{\mathcal{I}_v}$ is a minimizer of $J \circ E_0$ in the same sense as above, i.e., $E_0[v]|_{(-\infty,\max(\mathcal{I}_v))}$ is a minimizer of J among function with the same boundary conditions (instead of a left boundary conditions one takes $v+1 \in H^2$) and the same number of crossings of ± 1 . A contradiction is reached as in the previous case.

Case 3 is completely analogous to Case 2, except that we now use Remark 8 to reach a contradiction.

Having reached a contradiction in all three cases, we have proved the lemma. $\hfill \Box$

6. Different boundary conditions

Theorem 2 states that Equation (1) has an arbitrary number of stable equilibria provided that the interval length L is large enough. In the previous section we proved this in the case of the Neumann boundary conditions. The result remains unchanged for various other types of boundary conditions.

In the case of the Neumann boundary conditions the stable solutions are constructed using minimizers defined on the half-spaces \mathbb{R}^+ and \mathbb{R}^- that satisfy the Neumann boundary conditions at x = 0. These minimizers are derived from the homoclinic minimizers found in [22].

Now consider Equation (1) with the so-called Navier boundary conditions: u(t,0) = u(t,L) = 0, $u_{xx}(t,0) = u_{xx}(t,L) = 0$. In order to construct stable equilibria we need to find minimizers on the half-spaces \mathbb{R}^+ and \mathbb{R}^- which satisfy the boundary conditions $u(0) = u_{xx}(0) = 0$. If the potential F is even such minimizers can be derived from the results in [22]. Indeed, consider heteroclinic minimizers with homotopy type $(g_m, ..., g_1, g_1, ..., g_m)$. From [21, 22] it then follows that that such minimizers are odd with respect to a unique point of odd symmetry. Due to translation invariance we can choose this point to be x = 0. The restriction of these minimizers to the intervals \mathbb{R}^+ and $\mathbb{R}^$ now satisfies the boundary conditions $u(0) = u_{xx}(0) = 0$. From this point on the construction of stable equilibria is identical to the construction carried out in the previous section. The statement of Theorem 2 for the case of the Navier boundary conditions remains unchanged. Although this construction can only be carried out when F is even, the result also holds when F is not even, as we will see momentarily.

Another set of boundary conditions that can be considered are *Dirichlet* boundary conditions. General Dirichlet boundary conditions for Equation (1) are $(u(t,0), u_x(t,0)) = \bar{y} = (y_1, y_2)$ and $(u(t,L), u_x(t,L)) = \bar{z} = (z_1, z_2)$. The minimizers on the half-spaces \mathbb{R}^+ and \mathbb{R}^- needed for the construction of stable equilibria cannot be found via the results in [22]. To obtain such minimizers on for example \mathbb{R}^+ , we minimize $J_{\mathbb{R}^+}[u]$ over functions u for which the induced curve $\Gamma(u)$ starts at \bar{y} and terminates at P_1 (or P_2), and which has a certain homotopy type \boldsymbol{g} . The homotopy \boldsymbol{g} is defined as before by counting the number of consecutive crossings of the lines u = -1 and u = 1 excluding the intersections in the tail. This leads to the homotopy vector $\boldsymbol{g} = (g_1, ..., g_m)$, with $g_1 \in \mathbb{N}$ and $g_i \in 2\mathbb{N}$ for i = 2, ..., m. The function classes of a given homotopy \boldsymbol{g} and initial point \bar{y} are denoted by $M_{\mathbb{R}^{\pm}}(\boldsymbol{g}, \bar{y})$. The potential F is not assumed to be even here. As in [22] (see also Theorem 5) there exists a universal constant $N_0(\bar{y})$ such that, for homotopy types g with $g_i \geq N_0$, the infima of $J_{\mathbb{R}^{\pm}}$ over $M_{\mathbb{R}^{\pm}}(\boldsymbol{g}, \bar{\boldsymbol{y}})$ are attained. These minimizers are again the building blocks for constructing stable solutions to the Dirichlet problem. Consequently the statement of Theorem 2 also holds for the Dirichlet boundary conditions.

Let us now come back to the Navier boundary conditions when the potential F is not even. In this case the minimizers on the half-spaces \mathbb{R}^{\pm} , needed for the construction of stable solutions, are found in function classes in the space $\{u \in H^2(\mathbb{R}^+) \mid u(0) = 0\}$. From the variational principle minimizers satisfy the second boundary condition $u_{xx}(0) = 0$.

The various boundary conditions discussed above are not the only possibilities. For example, one can also treat the non-homogeneous Neumann and the non-homogeneous Navier boundary conditions. Furthermore, one can consider various types of mixed boundary conditions. The bottom line is that as long as one considers boundary conditions for which Equation (5) has a variational principle, then the method in this paper applies and a variant of Theorem 2 can be obtained.

7. The semi-conjugacy

We commence with the study of the attractor of (1) with $F(u) = \frac{1}{4}(u^2 - 1)^2$ and Neumann boundary conditions for $\frac{\gamma}{\beta^2} \leq \frac{1}{8}$, i.e. (7)

$$\begin{cases} u_t = -\gamma u_{xxxx} + \beta u_{xx} + u - u^3 & \text{for } x \in (0, L), \ t > 0 \\ u_x(t, 0) = u_x(t, L) = u_{xxx}(t, 0) = u_{xxx}(t, L) = 0 & \text{for all } t > 0. \end{cases}$$



FIGURE 9. A sketch of the bifurcation diagram for $\gamma \in [0, \frac{1}{8}]$. The shape of the bifurcating solutions $\pm u_1$, $\pm u_2$ and $\pm u_3$ is indicated.

Without loss of generality we put $\beta = 1$ throughout this section. We first consider the set of stationary solutions. Clearly all stationary solutions can be extended to the real line by reflection in the points x = 0 and x = L, and therefore correspond to bounded solutions of

(8)
$$-\gamma u_{xxxx} + u_{xx} + u - u^3 = 0.$$

Solutions of (8) have a constant of integration, the *energy*:

(9)
$$\mathcal{E}[u] \stackrel{\text{def}}{=} \gamma u_{xxx} u_x - \frac{\gamma}{2} |u_{xx}|^2 - \frac{1}{2} |u_x|^2 + \frac{1}{4} (u^2 - 1)^2 = E,$$

where $E \in \mathbb{R}$ is constant along solutions of (8).

It was found in [?, 4] that for $\gamma \in (0, \frac{1}{8}]$ the bounded solutions of (8) are in 1-1 correspondence with the bounded solutions of the second order equation ($\gamma =$ 0). To be precise, for $\gamma \in (0, \frac{1}{8}]$ the only bounded solutions of (8) are the three homogeneous solutions $u \equiv 0$ and $u \equiv \pm 1$; two monotone heteroclinic solutions connecting $u = \pm 1$; and a family of periodic solutions which are symmetric with respect to their extrema and antisymmetric with respect to their zeros. These periodic solutions form a continuous family and can be parametrised either by their energy $E \in (0, \frac{1}{4})$, or by their period $\ell \in (0, 2\pi \sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}})$. Existence of these solutions can be proved either via a shooting method where the energy is used as a parameter [30], via a minimization method where the period is used as a parameter [35], or via continuation [4]. The bifurcation diagram for the stationary solutions of (7) is given by Figure 9. For small Lthe only bounded solutions are the three homogeneous states. At $L = L_0 \stackrel{\text{de}}{=}$ $\pi \sqrt{\frac{2\gamma}{\sqrt{1+4\gamma}-1}}$ two non-uniform stationary solutions bifurcate. These solutions $\pm u_1(x;L)$ are monotone and have exactly one zero. The bifurcation is a generic supercritical pitchfork bifurcation (see e.g. [19, Sect. 6.2])). More generally, the same type of bifurcation occurs at $L = nL_0$ for all $n \ge 2$. The bifurcating stationary solutions are just multiples of the primary bifurcating branch.



FIGURE 10. The attractor for $\gamma = 0$ for $0 < L \leq \pi$ on the left; for $\pi < L \leq 2\pi$ in the middle; and for $2\pi < L \leq 3\pi$ on the right.

For $\gamma = 0$ the attractor of problem (1) with Neumann boundary has been extensively studied (see [1, 10, 19]). For $0 < L < \pi$ the attractor consists of the three uniform states and their connecting orbits. For $\pi < L < 2\pi$ the attractor contains five equilibrium points, namely the three uniform states and two monotone non-uniform states $\pm u_1$. For $2\pi < L \leq 3\pi$ the attractor is threedimensional and consists of the equilibrium points $u \equiv 0, u \equiv \pm 1, \pm u_1$ and $\pm u_2$, and their connecting orbits. The situation is depicted in Figure 10. In general, for $n\pi < L < (n+1)\pi$ the attractor contains 2n+3 equilibrium points. The flow on the attractor can be described completely. In particular, for all L >0 the flow $\phi(L, 0, 1)$ on the attractor is conjugated to simple ODE (see [27]).

We now turn our attention back to the fourth order equation for $\gamma \in (0, \frac{1}{8}]$. Theorem 3 states that there exists a semi-conjugacy between the flow on the attractor of the fourth order equation and the corresponding flow for the second order equation with the same number of stationary solutions. This follows immediately from [27, Theorems 1.2 & 2.1], since our problem obeys the conditions required for the analysis presented there:

- The semi-flows $\phi(L, \gamma, 1)$ have compact global attractors.
- The equilibrium solutions are given by the bifurcation diagram of Figure 9. The zero solution undergoes generic supercritical pitchfork bifurcations and the equilibria $u \equiv \pm 1$ are stable.
- There exists a Lyapunov functional $J_L[u]$ (given by (3)).

We remark that the theorem implies that the dynamics on the attractor are at least those of the second order equation. When we denote the solution on the k-th bifurcation branch by u_k then there exists a connecting orbit going from u_k to u_l if and only if k < l (hence $J_L(u_k) < J_L(u_l)$ for k < l, which can also be derived directly from [35]). The semi-conjugacy does not completely determine the flow on the attractor (as a conjugacy would), since it is unknown whether the problem satisfies the Morse-Smale property. The following lemma shows that away from the bifurcation points the equilibrium points are hyperbolic. Thus, the information which is lacking in order be able to check the Morse-Smale property is a proof of the transversality of the intersection between unstable and stable manifolds of these equilibria (for the second order equation this follows from the lap-number theorem [1, 20, 26]).

LEMMA 22. The nontrivial equilibrium solutions are hyperbolic.

The proof of this lemma can be found in [5], Chap. 4.

Again, the results in this section hold for a more general class of potentials F(u). Analogous results also hold for the Navier boundary conditions $(u(t,0) = u_{xx}(t,0) = 0$ and $u(t,L) = u_{xx}(t,L) = 0)$, and for the mixed case of Navier boundary conditions on one boundary and Neumann boundary conditions on the other boundary.

8. The bifurcation

In this section we analyse the bifurcation that occurs at $\frac{\gamma}{\beta^2} = \frac{1}{8}$. In particular, for $\frac{\gamma}{\beta^2}$ slightly larger than $\frac{1}{8}$ we will completely describe the set of stationary solutions for all L > 0. Without loss of generality we set $\beta = 1$:

- (10a) $-\gamma u_{xxxx} + u_{xx} + u u^3 = 0$ for $x \in (0, L)$
- (10b) $u_x(0) = u_{xxx}(0) = u_x(L) = u_{xxx}(L) = 0$

We stress that the bifurcation analysis in the present section is the only part of this paper where we need transversality information.

8.1. The finite dimensional reduction. As discussed in Sect. 7 for $\gamma = \frac{1}{8}$ the bifurcation diagram is as depicted in Figure 9. The results of [4], which are used in Sect. 7, can also be applied to $\gamma > \frac{1}{8}$. One obtains the following: the only solutions of (10a) with $||u||_{\infty} \leq \frac{4\gamma+1}{12\gamma}$ (any $\gamma > 0$) are $u \equiv 0$ and a one parameter family of periodic solutions, symmetric with respect to their extrema and antisymmetric with respect to their zeros. This family of periodic solutions can be parametrised by the energy or by the period. Denote this continuous family, including $u \equiv 0$, by \mathcal{F}_{γ} . These solutions of (10) form the skeleton of the bifurcation diagram.

The additional solutions that appear in the bifurcation diagram for γ slightly larger than $\frac{1}{8}$ are all in a small neighborhood of the heteroclinic cycle. We denote the unique monotonically increasing heteroclinic solutions at $\gamma = \frac{1}{8}$ by u_0 , and we divide out the translational invariance by fixing $u_0(0) = 0$. Let the heteroclinic cycle in phase space be

$$\Delta = \{ (\pm 1, 0, 0, 0) \} \cup \{ \pm (u_0(x), u'_0(x), u''_0(x), u'''_0(x)) \mid x \in \mathbb{R} \},\$$

and define $B_{\epsilon}(\Delta)$ to be the ϵ -neighborhood of Δ in \mathbb{R}^4 .

LEMMA 23. There exists a constant $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists a $\delta_0 = \delta_0(\epsilon) > 0$ such that for all $\frac{1}{8} < \gamma < \frac{1}{8} + \delta_0$ any bounded solution of (10a) is either an element of \mathcal{F}_{γ} or its orbit is entirely contained in $B_{\epsilon}(\Delta)$.

PROOF. Suppose by contradiction that the assertion does not hold. Then there exists an $\epsilon > 0$ and sequence $\gamma_n \downarrow \frac{1}{8}$ with corresponding solutions u_n of (10a), such that $u_n \notin \mathcal{F}_{\gamma_n}$ and $(u_n, u'_n, u''_n, u'''_n)(x_n) \notin B_{\epsilon}(\Delta)$ for some $x_n \in \mathbb{R}$. After translation we may assume that $x_n = 0$ for all n. Since bounded solutions of (10a) are uniformly bounded in $W^{3,\infty}$ there exists a subsequence, again denoted by u_n , which converges in C_{loc}^3 on compact sets to some limit function u. This function u is a bounded solution of (10a) for $\gamma = \frac{1}{8}$. Since $(u, u', u'', u''')(0) \notin B_{\epsilon}(\Delta)$ we have that u is one of the solutions in $\mathcal{F}_{\frac{1}{8}}$ (this follows from the complete classification of bounded solutions at $\gamma = \frac{1}{8}$). Therefore $\mathcal{E}[u] \in (0, \frac{1}{4}]$ and $||u||_{\infty} < 1$. In particular $||u||_{\infty} < \frac{4\gamma_n + 1}{12\gamma_n}$ for n sufficiently large. We now assert that $||u_n||_{\infty} \to ||u||_{\infty}$, which implies that $u_n \in \mathcal{F}_{\gamma_n}$ for n sufficiently large, a contradiction. Indeed, we show that $u_n \to u$ in phase space, i.e. orbital convergence, which implies that $||u_n||_{\infty} \to ||u||_{\infty}$. First notice that $\mathcal{E}[u_n] \to \mathcal{E}[u]$, since this holds for x = 0. Let $B_{\epsilon}(u)$ be the ϵ -neighborhood of $\{(u(x), u'(x), u''(x), u'''(x)) \mid x \in \mathbb{R}\}$. Suppose now, by contradiction, that there exists a constant $\eta > 0$ such that $\operatorname{dist}_{\mathbb{R}^4}((u_n, u'_n, u''_n, u'''_n)(n), B_{\epsilon}(u)) > \eta$ for some points $x_n \in \mathbb{R}$. As before, taking a subsequence, we obtain that $u_n(x+x_n)$ converges in C^3 on compact sets to some limit function v. Again, v is a bounded solution of (10a) for $\gamma = \frac{1}{8}$ and $\operatorname{dist}_{\mathbb{R}^4}((v, v', v'', v''')(0), B_{\epsilon}(u)) \geq \eta$. On the other hand it follows that $\mathcal{E}[v] = \lim_{n\to\infty} \mathcal{E}[u_n] = \mathcal{E}[u]$. Since there is only one bounded solution of (10a) with $\gamma = \frac{1}{8}$ in each energy level $\mathcal{E} \in (0, \frac{1}{4}]$ we conclude that $u \equiv v$ modulo translation, a contradiction.

For $\gamma = \frac{1}{8}$ the heteroclinic orbit is the unique, transversal intersection of $W^u(-1)$ and $W^s(+1)$. For γ slightly larger than $\frac{1}{8}$ this transversal intersection persists. This enables us to glue the two heteroclinics (going from -1 to +1 and back) together to form multitransition solutions. In particular we can find, for γ sufficiently close to $\frac{1}{8}$, all solutions of (10) in a neighborhood of the heteroclinic cycle. This method has already been successfully applied in [23] to show that there is a countable infinity of heteroclinic solutions. Besides, in [36] the stability of multiple-pulse solutions converging to a saddle-focus was studied via a reduction to a finite-dimensional center manifold (when the pulses are far apart). Here we will use the transversality to find all solutions of (10) and their index.

Let u_0 be the unique monotonically increasing heteroclinic solution of (10a) at $\gamma = \frac{1}{8}$. The transversality implies that $d^2J[u_0]$ is an invertible operator on $H_c^2(\mathbb{R}) = \{u \in H^2(\mathbb{R}) \mid u(0) = 0\}$, where we have made the usual identification $(H^2)^* = H^2$. Moreover, since u_0 is a non-degenerate minimum of J one has $(d^2J[u_0]v, v) \geq C_0 ||v||^2$ for some $C_0 > 0$ and all $v \in H_c^2(\mathbb{R})$. As in Sect. 3 we consider the restriction of u_0 to a large finite interval [-T, T]. The tails can be recovered by an application of the extension map E_0 defined in (6). Note that E_0 also depends on γ . Taking T large enough this extension map $E_0^{\gamma}[u]$ is well-defined in a small neighborhood of u_0 in $H_c^2(-T, T)$ for γ close to $\frac{1}{8}$. A perturbation argument shows that there exists a $C_1 > 0$ such that $(d^2(J \circ E_0^{\gamma})[u]v, v) \geq C_1 ||v||^2$ for all u in a small η -neighborhood $U_{\eta}(u_0) \subset$ $H_c^2(-T, T)$ of u_0 , all $v \in H_c^2(-T, T)$ and for all γ sufficiently close to $\frac{1}{8}$ and Tsufficiently large.

28

To glue transitions from -1 to +1 and vice versa together we introduce several gluing functions, as in Sect. 4. Write \vec{u} for the pair (u, u'). For $\bar{y} = (y_1, y_2)$ and $\bar{z} = (z_1, z_2)$ close to $(\pm 1, 0)$ and for large s we define $g_l(x, \bar{y}, s)$, $g_r(x, \bar{y}, s)$ and $g(x, \bar{y}, \bar{z}, s)$ as the unique local solutions of (10a) near the equilibrium points $u = \pm 1$, such that

$$\begin{aligned} g_l'(0,\bar{y},s) &= 0, \quad g_l'''(0,\bar{y},s) = 0 \quad \text{and} \quad \vec{g}_l(s,\bar{y},s) = \bar{y}; \\ \vec{g}_r(0,\bar{y},s) &= \bar{y} \quad \text{and} \quad g_r'(s,\bar{y},s) = 0, \quad g_r'''(s,\bar{z},s) = 0; \\ \vec{g}(0,\bar{y},\bar{z},s) &= \bar{y} \quad \text{and} \quad \vec{g}(s,\bar{y},\bar{z},s) = \bar{z}. \end{aligned}$$

Here we have implicitly assumed that it will be clear from the context if these solutions are close to +1 or close to -1. The functions g are the unique solutions of the boundary value problem which lie entirely in a small neighborhood of the equilibrium point in phase space. On the other hand, in function space they are the unique global minimizers of the corresponding variational problem, and the unique critical points a neighborhood of ± 1 in H^2 . By symmetry one has $g_l(x, (y_1, y_2), s) = g(x + s, (y_1, -y_2), (y_1, y_2), 2s)$, and similarly for g_r . Note that the solutions g also depend on γ .

We now glue *n* transitions together. Let $S_k = (2k+1)T + \sum_{i=0}^k s_i$, and define for $n \ge 1$ the gluing maps $E_n^{\gamma} = E_n^{\gamma}[u_1, \dots, u_n; s_0, \dots, s_n]$ as

$$E_n^{\gamma} = \begin{cases} g_l(t, \vec{u}_0(-T), s_0) & \text{for } t \in [0, S_0 - T] \\ u_1(t - S_0) & \text{for } t \in [S_0 - T, S_0 + T] \\ g(t - S_0 - T, \vec{u}_1(T), \vec{u}_2(-T), s_1) & \text{for } t \in [S_0 + T, S_1 - T] \\ u_2(t - S_1) & \text{for } t \in [S_1 - T, S_1 + T] \\ \vdots \\ g(t - S_{n-2} - T, \vec{u}_{n-1}(T), \vec{u}_n(-T), s_{n-1}) & \text{for } t \in [S_{n-2} + T, S_{n-1} - T] \\ u_n(t - S_{n-1}) & \text{for } t \in [S_{n-1} - T, S_{n-1} + T] \\ g_r(t - S_{n-1} + T, \vec{u}_n(T), s_n) & \text{for } t \in [S_{n-1} + T, S_n - T]. \end{cases}$$

This gluing function is well-defined for (u_1, \ldots, u_n) in a product neighborhood $V_{\eta} = U_{\eta}(u_0) \times U_{\eta}(-u_0) \times \cdots \times U_{\eta}((-1)^{n-1}u_0)$ in $(H_c^2(-T,T))^n$, and for s_0, \ldots, s_n large enough. Note that $E_1^{\gamma}[u] \to E_0^{\gamma}[u]$ as $s_0, s_1 \to \infty$. Similarly $E_2^{\gamma}[u_1, u_2]$ tends to a concatenation of $E_0^{\gamma}[u_1]$ and $E_0^{\gamma}[u_2]$ as $s_0, s_1, s_2 \to \infty$, etcetera.

Introduce the notation $\boldsymbol{u} = (u_1, \ldots, u_n)$ and $\boldsymbol{s} = (s_0, \ldots, s_n)$. For fixed \boldsymbol{s} we can find the unique critical point of $J_L \circ E_n^{\gamma}$ in the product neighborhood V_{η} . This is easily seen by using the following fixed point argument. Consider the iteration (with $\mathbf{1}_n$ the unit matrix in \mathbb{R}^n)

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k - \left(d^2 (J \circ E_0^{\gamma}) [u_0] \, \boldsymbol{1}_n \right)^{-1} d_u (J_L \circ E_n^{\gamma}) [\boldsymbol{u}_k; \boldsymbol{s}].$$

This is a contraction on V_{η} for η sufficiently small (say $0 < \eta \leq \eta_0$) and $|\gamma - \frac{1}{8}| < \delta_1(\eta)$ and $\min(s) \stackrel{\text{def}}{=} \min_{0 \leq i \leq n} s_i > \sigma(\eta)$. Here $\delta_1(\eta)$ and $\sigma(\eta)$ are positive constants which, as a function of η , are, respectively, non-decreasing and non-increasing. For an explicit calculation of the derivative $d_u(J_L \circ E_n^{\gamma})$ we refer to [23]. The contraction thus has a unique fixed point $\mathbf{z}(s)$ which depends smoothly on s for $\min(s) > \sigma(\eta)$. Since $(d^2(J \circ E_0^{\gamma})[u]v, v) \geq C_1 ||v||^2$

it follows that $\boldsymbol{z}(\boldsymbol{s})$ is the minimizer of $J_L \circ E_n^{\gamma}$ on V_{η} . We substitute this vector into the action and obtain

$$\mathcal{K}_n(\boldsymbol{s}) \stackrel{\text{def}}{=} J_L \circ E_n^{\gamma}[\boldsymbol{z}(\boldsymbol{s}); \boldsymbol{s}].$$

The variational problem has thus been reduced to a finite dimensional setting. Solutions of (10) correspond to critical points of $\mathcal{K}_n(s)$ under the constraint $\sum_{i=0}^n s_i = L - 2nT$.

LEMMA 24. Let $\eta \leq \eta_0$, let $\gamma \in (\frac{1}{8}, \frac{1}{8} + \delta_1(\eta))$ and let \boldsymbol{s} with $\min(\boldsymbol{s}) > \sigma(\eta)$ be a critical point of \mathcal{K}_n under the constraint $\sum_{i=0}^n s_i = L - 2nT$. Then $E_n^{\gamma}[\boldsymbol{z}(\boldsymbol{s}); \boldsymbol{s}]$ is a solution of (10). The index of the critical point \boldsymbol{s} (under the constraint) is equal to the index of the solution $E_n^{\gamma}[\boldsymbol{z}(\boldsymbol{s}); \boldsymbol{s}]$.

PROOF. It is immediately clear that $u = E_n^{\gamma}[\boldsymbol{z}(\boldsymbol{s}), \boldsymbol{s}]$ is a piecewise solution of the differential equation. We assert that these pieces connect nicely to a solution on the whole interval. Let v be a function in H_N^2 in a small neighborhood of u. Then v has precisely n zeros, say at x_1, \ldots, x_n . Let $v_i(x + x_i) = v(x)|_{[x_i - T, x_i + T]}$ and $t_0 = x_1 - T$ and $t_i = x_{i+1} - x_i - 2T$, $1 \leq i \leq n-1$ and $t_n = L - x_n - T$. Then v can be written as $v = E_n^{\gamma}[v_1, \ldots, v_n; t_0, \ldots, t_n] + \sum_{i=0}^n \phi_i$ with $\phi_i \in H_0^2(\tau_i, \tau_I + t_i)$ for $1 \leq i \leq n-1$ where $\tau_i = 2iT + \sum_{k=0}^{i-1} t_k$, and $\phi_0 \in H_{n0}^2(0, t_0)$ and $\phi_n \in H_{0n}^2(L - t_n, L)$. Here $H_{n0}^2(0, t_0) = \{u \in H^2(0, t_0) \mid u'(0) = u(t_0) = u'(t_0) = 0\}$. This shows that all variations in H_N^2 are covered by the decomposition of the variational method, hence u is a solution on the whole interval [0, L]. The statement about the index follows from the fact that both $g(\cdot, \bar{y}, \bar{z}, s_i)$ and $\boldsymbol{z}(\boldsymbol{s})$ are non-degenerate minimizers, thus the unstable directions only come from variations in s_i .

The previous lemma describes all solutions in a small neighborhood $B_{\epsilon}(\Delta)$ of the heteroclinic cycle.

LEMMA 25. Let $\eta \leq \eta_0$. There exists a constants $\epsilon_1(\eta)$ such that when u is a solution of (10) for $\gamma \in (\frac{1}{8}, \frac{1}{8} + \delta_1(\eta))$ with u entirely contained in $B_{\epsilon_1}(\Delta)$ then for some $n \geq 1$ it holds that $u = E_n(\boldsymbol{z}(\boldsymbol{s}, \boldsymbol{s}))$, where \boldsymbol{s} is a critical point of \mathcal{K}_n under the constraint $\sum_{i=0}^n s_i = L - 2nT$ and $\min(\boldsymbol{s}) > \sigma(\eta)$.

PROOF. Let u be a solution of (10) which lies entirely in $B_{\epsilon}(\Delta)$. Since $u'(0) \neq 0$, it follows that for ϵ sufficiently small u has a finite number of zeros, say at $x_1, \ldots x_n$. Let $u_i(x + x_i) = u(x)|_{[x_i - T, x_i + T]}$. and $\psi_i(x + x_i + T) = u(x)|_{[\tau_i, \tau_i + s_i]}$, where $s_0 = x_1 - T$ and $s_i = x_{i+1} - x_i - 2T$, $1 \leq i \leq n - 1$ and $s_n = L - x_n - T$, and $\tau_i = 2kT + \sum_{k=0}^{i-1} s_k$. The orbit of u passes close to the equilibrium points ± 1 . If ϵ is small enough then the distance between two zeros is larger than $2T + \sigma(\eta)$, hence $s_i > \sigma(\eta)$.

Firstly, we infer that $\psi_i = g(\cdot, \vec{u}_i(T), \vec{u}_{i+1}(-T), s_i)$ since ψ_i is entirely contained in some small neighborhood of the equilibrium point ± 1 , and g is the unique local solution of the corresponding boundary value problem. Secondly, for ϵ sufficiently small $u_i \in U_\eta(u_0)$. Since \boldsymbol{z} are the unique critical points in V_η we have that $\boldsymbol{u} = \boldsymbol{z}(\boldsymbol{s})$ and thus $\boldsymbol{u} = E_n^{\gamma}[\boldsymbol{z}(\boldsymbol{s}); \boldsymbol{s}]$. Finally, since \boldsymbol{u} is a critical point in $H_N^2(0, L)$ it follows that \boldsymbol{s} must be a critical point of \mathcal{K}_n under the constraint $\sum_{i=0}^{n} s_i = L - 2nT$. Hence *u* is obtained from a critical point of \mathcal{K}_n .

It follows from the above lemma that $\epsilon_1(\eta)$ can be chosen to be a nondecreasing function of η . Hence for $\epsilon < \epsilon_1(\eta_0)$ there exists an $\eta_1(\epsilon) < \eta_0$ such that $\epsilon_1(\eta_1(\epsilon)) < \epsilon$. Combining with Lemmas 23–25 implies the following theorem:

THEOREM 26. Let $\epsilon < \epsilon_2 \stackrel{\text{def}}{=} \min\{\epsilon_0, \epsilon(\eta_0)\}$, and let $\delta_2(\epsilon) \stackrel{\text{def}}{=} \min\{\delta_0(\epsilon), \delta_1(\eta_1(\epsilon))\}$. When u is a solution of (10) for $\gamma \in (\frac{1}{8}, \frac{1}{8} + \delta_2(\epsilon))$ and $u \notin \mathcal{F}_{\gamma}$, then u is entirely contained in $B_{\epsilon}(\Delta)$ and u corresponds to a critical point s of \mathcal{K} with $\min(s) > \sigma(\eta_1(\epsilon))$.

For $\gamma \leq \frac{1}{8}$ the functions \mathcal{K}_n can also be defined, but its only critical points are the symmetric sequences $(s_0, 2s_0, 2s_0, \ldots, 2s_0, s_0)$, corresponding to the simple periodic solutions in \mathcal{F}_{γ} . For γ slightly larger than $\frac{1}{8}$ Theorem 26 implies that the additional solutions appearing in the bifurcation are *completely* determined by the *bifurcation function* $\mathcal{K}(s)$. Part of the bifurcation diagram is still formed by the solutions in \mathcal{F}_{γ} . The solutions corresponding to critical points of \mathcal{K}_n will fit exactly onto those in \mathcal{F}_{γ} , and they form all of the remainder of the bifurcation diagram.

In the following we fix $\epsilon < \epsilon_2$, write $\sigma = \sigma(\eta_1(\epsilon))$, and assume that $0 < \gamma - \frac{1}{8} < \delta_2(\epsilon)$.

8.2. Analysis of the bifurcation function. What remains is to determine the critical points of the bifurcation function \mathcal{K}_n for all $n \geq 1$. For easy notation we denote the n + 1 gluing functions by $g_0, g_1, \ldots, g_{n-1}, g_n$. Recall that, by symmetry, one has $g_l(x, (y_1, y_2), s) = g(x+s, (y_1, -y_2), (y_1, y_2), 2s)$ and similarly for g_r , so that all g_i can be dealt with on the same footing (taking care to correctly transform the variables). In the following we will only discuss those g_i which live in a neighborhood of +1, the other case being completely analogous. Calculating the partial derivatives one obtains that

$$\frac{\partial \mathcal{K}_n(\boldsymbol{s})}{\partial s_i} = \mathcal{E}[g_i(\cdot, \boldsymbol{z}(\boldsymbol{s}), s_i)],$$

where \mathcal{E} is the energy, see (9). This follows from an explicit calculation, see e.g. [23]. To investigate the partial derivatives we use the following characterisation due to Buffuni and Séré [9]. When $\gamma > \frac{1}{8}$ then the equilibria ± 1 are saddle-foci. Shift the equilibrium point to the origin and choose coordinates $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ such that the local stable and unstable manifolds are given by $W_{\text{loc}}^s = \{(\xi_1, \xi_2, 0, 0) | \xi_1, \xi_2 \text{ small}\}$ and $W_{\text{loc}}^u = \{(0, 0, \xi_3, \xi_4) | \xi_3, \xi_4 \text{ small}\}$. Denote $\xi_s = (\xi_1, \xi_2)$ and $\xi_u = (\xi_3, \xi_4)$. In a small neighborhood $B_4(\delta) = \{|\xi_s| < \delta, |\xi_u| < \delta\}$ of the origin. the flow is given by

(11)
$$\xi' = \begin{pmatrix} -\lambda & -\omega & 0 & 0\\ \omega & -\lambda & 0 & 0\\ 0 & 0 & \lambda & -\omega\\ 0 & 0 & \omega & \lambda \end{pmatrix} \xi + f(\xi),$$

where f(0,0) = 0, f'(0,0) = 0, $f_u(\xi_s,0) = 0$ and $f_s(0,\xi_u) = 0$. The parameters $\lambda > 0$ and $\omega > 0$ are the real and imaginary part of the eigenvalues of the linearised problem respectively. An important observation, to which we will

come back later, is that $\lambda \to 2$ and $\omega \to 0$ as $\gamma \downarrow \frac{1}{8}$. Introduce polar coordinates (r_s, θ_s) and (r_u, θ_u) : $x_1 = r_s \cos \theta_s$, $x_2 = r_s \sin \theta_s$, and $x_3 = r_u \cos \theta_u$, $x_4 = r_u \sin \theta_u$. Write the gluing function $g_i(x, \mathbf{z}(\mathbf{s}), s_i)$ in these polar coordinates coordinates: $(r_s, \theta_s, r_u, \theta_u)(x; \mathbf{s})$. One obtains the following characterisation [9, Lemma A.2] of the energy

(12a)
$$\mathcal{E}_{g}[s_{i}; \boldsymbol{s}] \stackrel{\text{def}}{=} \mathcal{E}[g_{i}(\cdot, \boldsymbol{z}(\boldsymbol{s}), s_{i}] = \frac{\partial \mathcal{K}_{n}(\boldsymbol{s})}{\partial s_{i}}$$
$$= \sqrt{\lambda^{2} + \omega^{2}} |\rho(s_{i}; \boldsymbol{s})|^{2} \cos(\varphi(s_{i}; \boldsymbol{s})) + O(|\rho(s_{i}; \boldsymbol{s})|^{3}),$$

where

(12b)
$$\rho(s_i; \mathbf{s}) = e^{-\lambda s_i/2} \sqrt{r_s(0; \mathbf{s}) r_u(s_i; \mathbf{s})} (1 + O(\delta)),$$

(12c)
$$\varphi(s_i; \boldsymbol{s}) = \omega s_i + \theta_s(0; \boldsymbol{s}) - \theta_u(s_i; \boldsymbol{s}) - \mu + O(\delta).$$

Here μ is a constant which tends to 0 as $\gamma \to \frac{1}{8}$. The terms $O(\delta)$ and $O(|\rho(s_i; \boldsymbol{s})|^3)$ are due to the fact that near the equilibrium point the flow is in fact non-linear, i.e., they represent the higher order terms in (11).

We first analyse the values of $r_s(0; \mathbf{s})$, $\theta_s(0; \mathbf{s})$, $r_u(s_i; \mathbf{s})$ and $\theta_u(s_i; \mathbf{s})$, which will turn out to depend only weakly on s_i , i.e., they are almost constant.

One should keep in mind that for γ close to $\frac{1}{8}$ we have $\omega \approx 0$ and $\lambda \approx 2$. However, the linearisation for $\gamma = \frac{1}{8}$ is not given by (11) with $\omega = 0$. This is caused by the change of coordinates necessary to convert to the above form. For $\gamma = \frac{1}{8}$ one can choose coordinates such that for $\zeta \in B_4(\tilde{\delta})$

$$\zeta' = \begin{pmatrix} -\lambda & -1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \zeta + f(\zeta).$$

Of course we choose T so large that $(u_0, u'_0, u''_0, u''_0)(T) \in B_4(\tilde{\delta})$ and that the gluing functions g_i are entirely contained in $B_4(\tilde{\delta})$. Before making the connection between the ζ - and ξ -coordinates, we briefly look at the picture in ζ -coordinates. All orbits in W^s , and in particular the heteroclinic solution u_0 , tend to the origin along the ζ_2 axis. In fact, in ζ -coordinates u_0 behaves as $\zeta_1(x)/\zeta_2(x) = O(1/x)$ for $x \to \infty$. For $\gamma = \frac{1}{8} + \delta_{\gamma}$, $0 < \delta_{\gamma} \ll 1$ the eigenvalues are $\pm 2(1 \pm i\sqrt{2\delta_{\gamma}} - 3\delta_{\gamma} + O(\delta_{\gamma}^{3/2}))$. And after an appropriate scaling in x we may assume that the real part $\pm \lambda$ of the eigenvalues is constant, i.e., the eigenvalues are of the form $\pm 2 \pm i\omega$ and we may take ω (or ω^2) as the parameter instead of γ . The choice of coordinates is such that W^s is always given by $\{\zeta_3 = \zeta_4 = 0\} = \{|\zeta_u| = 0\}$. As opposed to the ξ -coordinates, the ζ -coordinates are chosen to depend smoothly on ω for $\omega \downarrow 0$. The flow becomes

$$\zeta' = \begin{pmatrix} -2 & -1+O(\omega) & 0 & 0\\ \omega^2 + O(\omega^3) & -2 & 0 & 0\\ 0 & 0 & 2 & -1+O(\omega)\\ 0 & 0 & \omega^2 + O(\omega^3) & 2 \end{pmatrix} \zeta + f(\zeta).$$

The coordinate change to get from ζ to ξ is of the form $\xi_1 = \omega \zeta_1 + O(\omega^2)$ and $\xi_2 = \zeta_2$. Since $\theta = \arctan \frac{\xi_1}{\xi_2} = \arctan \frac{\omega \zeta_1 + O(\omega^2)}{\zeta_2}$ it follows that $\theta_s(0) = O(\omega/T)$ or $\theta_s(0) = \pi + O(\omega/T)$. To determine which of these to possibilities occurs,

we look at situation at the bifurcation point. For $\gamma = \frac{1}{8}$ the only solutions of (10) are the periodic solutions in $\mathcal{F}_{\frac{1}{8}}$, and they have energy $\mathcal{E} \in (0, \frac{1}{4}]$, see Sect. 7. For large periods these solutions can also be described by the present variational gluing method. Since these solutions are symmetric it corresponds to a critical point of the form $\mathbf{s}_* = (s_0, 2s_0, \ldots, 2s_0, s_0)$ for some $s_0 > \sigma$, and $\mathbf{z}(\mathbf{s}) = (u_1, -u_1, u_1, \ldots)$ for some $u_1 \in H_c^2$. Hence $\mathcal{E}_g[2s_0, \mathbf{s}_*] \in (0, \frac{1}{4}]$. By continuity, for small ω and $s > \sigma$ not too large the energy $\mathcal{E}_g[2s_0; \mathbf{s}_*]$ must be positive. Therefore the first of the above possibilities holds: $\theta_s(0; \mathbf{s}) = O(\omega/T)$, and similarly $\theta_u(s_i; \mathbf{s}) = O(\omega/T)$.

Choosing ϵ small in Theorem 26, it follows that δ_2 and η_1 are arbitrary small and σ is arbitrary large, it follows that we may restrict our attention to gluing functions g_i such that the point $(g_i, g'_i, g''_i, g''_i)(0, \mathbf{z}(\mathbf{s}), s_i)$ is arbitrary close to $(u_0, u'_0, u''_0, u'''_0)(T)$. Let $\delta_* = \text{dist}_{\mathbb{R}^4}((u_0, u'_0, u''_0, u'''_0)(T), (1, 0, 0, 0))$. One thus has, for some constant $0 < \epsilon_2 \ll \delta_*$, that $||\zeta_s| - \delta_*| < \epsilon_2$, and $|\zeta_u| < \epsilon_2$. Hence $r_s(0) = \delta_* + O(\epsilon_2)$ and similarly $r_u(s) = \delta_* + O(\epsilon_2)$.

Having obtained estimates on $r_s(0; \mathbf{s})$, $\theta_s(0; \mathbf{s})$, $r_u(s_i; \mathbf{s})$ and $\theta_u(s_i; \mathbf{s})$, we are ready to investigate the function $\mathcal{E}_g[s_i; \mathbf{s}]$. We will first concentrate on solutions with one transition. We thus look for critical points of the function $\mathcal{K}_1(s_0, s_1)$ under the constraint $s_0 + s_1 = L - 2T$, i.e., zeros of $\mathcal{E}_g[s_0; \mathbf{s}] - \mathcal{E}_g[L - 2T - s_0; \mathbf{s}]$ with min $(\mathbf{s}) > \sigma$, where $\mathbf{s} = (s_0, L - 2T - s_0)$. Since in the present case one has to think of the gluing functions g_l and g_r as half of an ordinary gluing function g, we define $s = 2s_0$ and $G(s) \stackrel{\text{def}}{=} \mathcal{E}_g[\frac{s}{2}; (\frac{s}{2}, \frac{L_0 - s}{2})] - \mathcal{E}_g[\frac{L_0 - s}{2}; (\frac{s}{2}, \frac{L_0 - s}{2})]$, where $L_0 = 2(L - 2T)$.

For L_0 not too large and ω small, there is only one solution of the equation G(s) = 0, since this solution necessarily belongs to \mathcal{F}_{γ} , hence $s = L_0/2$. It is immediately clear that for any $L_0 > 2\sigma$ there is a symmetric solution corresponding to $s = L_0/2$. More generally, looking for zeros of G(s) we consider the good approximation

$$G(s) \approx G_0(s) \stackrel{\text{def}}{=} \sqrt{\lambda^2 + \omega^2} \delta_*^2 \Big(e^{-\lambda s} \cos \omega s - e^{-\lambda(L_0 - s)} \cos \omega (L_0 - s) \Big).$$

The scaling $\tilde{s} = \omega s$ is useful as well, effectively setting $\omega = 1$ and $\lambda \to \infty$ as $\gamma \downarrow \frac{1}{8}$.

It follows that for small ω zeros of $G_0(s)$ only occur in the neighborhood of the lines (in the (s, L_0) -plane) $s = \frac{L_0}{2}$, and $s = \frac{(2k-1)\pi}{2\omega}$, $s < \frac{L_0}{2}$ for $k \in \mathbb{N}$, and $s = L_0 - \frac{(2k+1)\pi}{2\omega}$, $s > \frac{L_0}{2}$ for $k \in \mathbb{N}$, see Figure 11. The second and third case are related by symmetry. Next we consider the derivative of G_0 in the neighborhood of these lines:

$$\frac{G_0'(s)}{\sqrt{\lambda^2 + \omega^2} \,\delta_*^2} = -\lambda e^{-\lambda s} \cos \omega s - \omega e^{-\lambda s} \sin \omega s - \lambda e^{-\lambda (L_0 - s)} \cos \omega (L_0 - s) - \omega e^{-\lambda (L_0 - s)} \sin \omega (L_0 - s).$$

Firstly, in a neighborhood of the line $s = \frac{L_0}{2}$ it follows that $G'_0(s) \neq 0$ if (s, L_0) is away from the points $s = \frac{L_0}{2} = \frac{(2k-1)\pi}{2\omega}$, because there the first and third



FIGURE 11. Critical points of \mathcal{K}_1 can only occur in the grey regions, which are shown both in the (L_0, s) -plane and in the (s_0, s_1) -plane.

terms in $G'_0(s)$ are dominant. This means that for fixed $L_0 \not\approx \frac{(2k-1)\pi}{\omega}$ the only zero of $G_0(s)$ in a neighborhood of $s = \frac{L_0}{2}$ is at $s = \bar{s} = L_0$.

Secondly, in a neighborhood of the line $s = \frac{(2k-1)\pi}{2\omega}$, $s < \frac{L_0}{2}$ it follows that $G'_0(s) \neq 0$ if (s, L_0) is away from the point $s = \frac{L_0}{2} = \frac{(2k-1)\pi}{2\omega}$, because there the second term in $G'_0(s)$ is dominant. This implies that for fixed $L_0 > \frac{(2k-1)\pi}{\omega}$ there is exactly one zero of $G_0(s)$ in a neighborhood of $s = \frac{(2k-1)\pi}{2\omega}$.

We conclude that, away from the special points $s = \frac{L_0}{2} = \frac{(2k-1)\pi}{2\omega}$ the zeros of $G_0(s)$ are transverse and thus depend smoothly on L_0 . On the line $s = \frac{L_0}{2}$ there are bifurcation points s_* near $s = \frac{(2k-1)\pi}{2\omega}$ These points are characterised by the fact that $G'_0(s_*) = 0$. Interpreting G_0 as a function of s and the parameter L_0 one calculates that at these points $(s = s_*, L_0 = 2s_*)$ a forward pitchfork bifurcation takes place:

(13)
$$\frac{\partial G_0}{\partial L_0} = 0, \quad \frac{\partial^2 G_0}{\partial s^2} = 0, \quad \frac{\partial^2 G_0}{\partial L_0 \partial s} \cdot \frac{\partial^3 G_0}{\partial s^3} < 0.$$

Next we have to consider the difference between G(s) and $G_0(s)$. We have already obtained estimates on $r_s(0; \mathbf{s})$, $\theta_s(0; \mathbf{s})$, $r_u(s_i; \mathbf{s})$ and $\theta_u(s_i; \mathbf{s})$, but we also need estimates on their derivative with respect to s_i . For this purpose we first look at $\frac{\partial g_i(0, \mathbf{z}(\mathbf{s}), s_i)}{\partial s_i}$. Let us consider $\bar{g}(x; s_i) = g_i(x, \bar{y}, \bar{z}, s_i) - 1$, which is the solution of

$$\begin{cases} -\gamma \bar{g}'''' + \bar{g}'' - 2\bar{g} = 3\bar{g}^2 + \bar{g}^3\\ \bar{g}(0) = y_1, \ \bar{g}'(0) = y_2, \ \bar{g}(s_i) = z_1, \ \bar{g}'(s_i) = z_2. \end{cases}$$

Scaling $\tilde{x} = x/s_i$ we get for $\tilde{g}(\tilde{x}) = g(x)$:

$$\begin{cases} -\gamma \frac{1}{s_i^4} \tilde{g}'''' + \frac{1}{s_i^2} \tilde{g}'' - 2\tilde{g} = 3\tilde{g}^2 + \tilde{g}^3\\ \tilde{g}(0) = y_1, \ \tilde{g}'(0) = y_2 s_i, \ \tilde{g}(1) = z_1, \ \tilde{g}'(1) = z_2 s_i \end{cases}$$

For $\tilde{h}(\tilde{x}) = \frac{\partial \tilde{g}}{\partial s_i}$ we obtain:

$$\begin{cases} -\gamma \frac{1}{s_i^4} \tilde{h}'''' + \frac{1}{s_i^2} \tilde{h}'' - 2\tilde{h} = 6\tilde{h}\tilde{g} + 3\tilde{h}\tilde{g}^2 + \frac{4\gamma}{s_i^5}\tilde{g}'''' - \frac{2}{s_i^3}\tilde{g}''\\ \tilde{h}(0) = 0, \ \tilde{h}'(0) = y_2, \ \tilde{h}(1) = 0, \ \tilde{h}'(1) = z_2. \end{cases}$$

And finally for $h(x) = \tilde{h}(\tilde{x}) = \frac{\partial \bar{g}}{\partial s_i} = \frac{\partial g_i}{\partial s_i}$ one gets:

$$\begin{cases} -\gamma h'''' + h'' - 2h = 6h\bar{g} + 3h\bar{g}^2 + \frac{4\gamma}{s_i}\bar{g}'''' - \frac{2}{s_i}\bar{g}'' \\ h(0) = 0, \ h'(0) = y_2/s_i, \ h(s_i) = 0, \ h'(s_i) = z_2/s_i. \end{cases}$$

Since $|\bar{y}| < \delta$ and $|\bar{z}| < \delta$, we conclude that $||h||_{W^{3,\infty}(0,s_i)} = O(\delta/s_i)$. By differentiating the identity $d(J \circ E_n^{\gamma})[\boldsymbol{z}(\boldsymbol{s}); \boldsymbol{s}] = 0$, one finds that

$$d_u^2(J \circ E_n^{\gamma}) \frac{\partial \boldsymbol{z}(\boldsymbol{s})}{\partial s_i} = -\frac{\partial}{\partial s_i} d_u(J \circ E_n^{\gamma}) = O(\|\frac{\partial g_i}{\partial s_i}\|_{W^{3,\infty}(0,s_i)})$$

Here the last equality follows from an explicit calculation of $d_u(J \circ E_n^{\gamma})$. Combining this with the above estimate on $\|\frac{\partial g_i}{\partial s_i}\|_{W^{3,\infty}(0,s_i)}$, we obtain that $\frac{\partial \boldsymbol{z}(\boldsymbol{s})}{\partial s_i} = O(\delta/s_i)$, so that $\frac{\partial r_s(0;\boldsymbol{s})}{\partial s_i} = O(\delta/s_i)$, $\frac{\partial r_u(s_i;\boldsymbol{s})}{\partial s_i} = O(\delta/s_i)$, and $\frac{\partial \theta_s(0;\boldsymbol{s})}{\partial s_i} = O(\omega/s_i)$, $\frac{\partial \theta_u(s_i;\boldsymbol{s})}{\partial s_i} = O(\omega/s_i)$.

From the previous analysis it is clear that we are only interested in s which are larger than approximately $\frac{\pi}{2\omega}$, since for smaller s there will only be one critical point of \mathcal{K}_1 , which is of the form $(\frac{s}{2}, \frac{s}{2})$. Since δ is small it follows that for such values of s the dominant term in (12c) is ωs_i , so that the zeros of G(s)can again only occur near the lines $s = \frac{L_0}{2}$, and $s = \frac{(2k-1)\pi}{2\omega}$, $s < \frac{L_0}{2}$ for $k \in \mathbb{N}$, and $s = L_0 - \frac{(2k+1)\pi}{2\omega}$, $s > \frac{L_0}{2}$ for $k \in \mathbb{N}$, see Figure 11. To be able to carry over the analysis of $G'_0(s)$ to G'(s) we need that $\frac{1}{r_s} \frac{\partial r_s}{\partial s_i} \ll \lambda$, $\frac{\partial \theta_s}{\partial s_i} \ll \omega$, which is true by the estimates above for large s_i , i.e., ω sufficiently small. Moreover, we need estimates on the derivatives of the terms of order $O(\delta)$ in (12). Since these terms originate from the higher order terms in (11) one finds that they are of order $O(\frac{\partial z(s)}{\partial s_i}) = O(\delta/s_i)$. Therefore these terms are dominated by ω , for small δ and $s_i > \frac{\pi}{4\omega}$. Hence, as for $G'_0(s)$, we conclude that, away from the special points $s = \frac{L_0}{2} = \frac{(2k-1)\pi}{2\omega}$, the zeros are unique (near the fore-mentioned lines) and depend continuously on L_0 .

The analysis of the bifurcation points also carries over from $G_0(s)$ to G(s), since estimates on the higher order derivatives are found in a similar manner as before: $\frac{\partial^2 \mathbf{z}(s)}{\partial s_i^2} = O(\delta/s_i^2)$ and $\frac{\partial^3 \mathbf{z}(s)}{\partial s_i^3} = O(\delta/s_i^3)$. Thus, at the bifurcation points s_* , characterised by $G'(s_*) = 0$, the inequality of (13) holds (for G instead of G_0), while the equalities follow from the symmetry.

Finally, the index of a critical point $(\frac{s}{2}, \frac{L_0-s}{2})$ is easily calculated: it is 1 if G'(s) < 0, and it is 0 if G'(s) > 0. More explicitly, the index is 0 if either



FIGURE 12. A blow-up of the first branch of the bifurcation diagram for γ slightly larger than $\frac{1}{8}$. The branch consists of solutions of (10) with one zero (at which it has positive slope). The profile of solutions on different parts of the branch are depicted below (for large L). The index of the solutions is also shown.

 $s = \frac{L_0}{2}$ and $s \in \left(\frac{(4k-3)\pi}{2\omega} + \epsilon, \frac{(4k-1)\pi}{2\omega} - \epsilon\right), k \in \mathbb{N}$, or or $s \approx \frac{(4k-1)\pi}{2\omega}, s < \frac{L_0}{2} - \epsilon$ or $s \approx L_0 - \frac{(4k-1)\pi}{2\omega}, s > \frac{L_0}{2} + \epsilon$ for $k \in \mathbb{N}$. Here ϵ is some small positive number which tends to 0 as $\omega \to 0$. On the complementary (parts of) branches the index of the critical point is 0. The points where the index changes are of course precisely the bifurcation points. Because all this is much easier to understand from a picture, Figure 12 shows all solutions (and their index) on the first branch (containing of solutions with one zero) of the bifurcation diagram for γ slightly larger than $\frac{1}{8}$.

We now turn our attention to the solutions with more transitions/zeros. To find critical points one needs to solve $\frac{\partial \mathcal{K}}{\partial s_i} = \frac{\partial \mathcal{K}}{\partial s_j}$ for all $0 \leq i, j \leq n$. Since all partial derivatives are of the form (12) the analysis of the case n = 1 can be repeated for $n \geq 2$. To make notation easier we define $\tilde{s}_0 = 2s_0$ and $\tilde{s}_n = 2s_n$ and subsequently drop the tildes from the notation. The critical points of \mathcal{K}_n can only occur near the diagonal $\{s_0 = s_1 = \cdots = s_n\}$, and, for any permutation τ , any $0 \leq m \leq n-1$ and any sequence $\{k_i\}_{i=0}^m \subset \mathbb{N}$ with $k_i \leq k_{i+1}$, near the line (14)

$$\{s_{\tau(i)} = \frac{(2k_i - 1)\pi}{2\omega}, \ 0 \le i \le m\} \cap \{s_{\tau(i)} = s_{\tau(i+1)} \ge \frac{(2k_m - 1)\pi}{2\omega}, \ m+1 \le i \le n\}.$$

In words this means that some (but not all) of the s_i are fixed at an odd multiple In words this means that some (but not all) of the s_i are fixed at an odd multiple of $\frac{\pi}{2\omega}$, while the remaining s_i are all equal and larger than the maximum of the fixed s_i . This gives the complete bifurcation diagram; for fixed L one needs to restrict to $\sum_{i=0}^{n} s_i - \frac{s_0 + s_n}{2} = L - 2nT$. We are solving the (n + 1) equations $f_i \stackrel{\text{def}}{=} \frac{\partial \mathcal{K}}{\partial s_i} - \frac{\partial \mathcal{K}}{\partial s_{i+1}} = 0, \ 0 \le i \le n - 1$, and $f_n \stackrel{\text{def}}{=} \sum_{j=0}^{n} s_j - (L_0 - 2nT) = 0$. To conclude uniqueness (and continuous

dependence) of the solutions of these equations, one needs $det\left(\frac{\partial f_i}{\partial s_i}\right) \neq 0$. A direct calculation shows that

$$\det\left(\frac{\partial f_i}{\partial s_j}\right) = \sum_{i=0}^n \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{\partial \mathcal{E}_g[s_j; \boldsymbol{s}]}{\partial s_j} + \text{ other terms,}$$

where all other terms are small compared to the first term if δ , ω and $\frac{1}{\min(s)}$ are sufficiently small. As in the case n = 1 discussed above, good bookkeeping reveals the dominant term(s) in this expression when (s, L_0) is not close one of the exceptional points, and one concludes that $det\left(\frac{\partial f_i}{\partial s_j}\right) \neq 0$. The exceptional points are the points where two or more of the lines, which were defined above, meet.

The index of the critical equal to the number of negative eigenvalues of the $(n \times n)$ -matrix $\frac{\partial^2 \mathcal{K}_n(s_0, s_1, \dots, s_{n-1}, L - \sum_{k=0}^{n-1} s_k)}{\partial s_i \partial s_j}$. Since $\frac{\partial \mathcal{K}_n}{s_i} = \mathcal{E}_g[s_i; \mathbf{s}] - \mathcal{E}_g[L - \sum_{k=0}^{n-1} s_k; \mathbf{s}]$ and $\mathcal{E}_g[s_i; \mathbf{s}]$ is well approximated by $F(s_i) \stackrel{\text{def}}{=} Ce^{-\lambda s_i} \cos \omega s_i$, we get

$$\left(\frac{\partial^2 \mathcal{K}_n}{\partial s_i \partial s_j}\right) = \begin{pmatrix} F'(s_0) + F'(s_n) & F'(s_n) & \dots & F'(s_n) \\ F'(s_n) & F'(s_1) + F'(s_n) & \dots & F'(s_n) \\ \dots & \dots & \dots & \dots & \dots \\ F'(s_n) & F'(s_n) & \dots & F'(s_{n-1}) + F'(s_n) \end{pmatrix} + \text{small terms.}$$

On the diagonal $\{s_0 = s_1 = \cdots = s_n\}$ this reduces to

$$\left(\frac{\partial^2 \mathcal{K}_n}{\partial s_i \partial s_j}\right) \approx F'(s_0) \left(\begin{array}{ccc} 2 & 1 & \cdots & 1\\ 1 & 2 & \cdots & 1\\ \cdots & \cdots & \cdots & \cdots\\ 1 & 1 & \cdots & 2\end{array}\right).$$

Since the matrix is positive definite, the index of the critical point Since the matrix is positive definite, the index of the critical point $(\frac{s}{2}, s, s, \ldots, s, \frac{s}{2})$ is 0 if $s \in (\frac{(4k-3)\pi}{2\omega} + \epsilon, \frac{(4k-1)\pi}{2\omega} - \epsilon), k \in \mathbb{N}$ with $\epsilon > 0$ small. On the complementary part of the diagonal the index is n.

Working out the number of negative eigenvalues on the other branches of solutions we get the following. Near the line (14) and away from the bifurcation points the index of the critical point is equal to the number $\#\{0 \leq i \leq i \leq i\}$ $m \mid \frac{k_i+1}{2} \in \mathbb{N}$ raised by n-m-1 if $s_{\tau(m+1)} = \cdots = s_{\tau(n)} \in \left(\frac{(4j-1)\pi}{2\omega} + \epsilon, \frac{(4j-3)\pi}{2\omega} - \epsilon\right)$ for some $j \in \mathbb{N}$.

A full examination of these bifurcation points for $n \ge 2$ is beyond the scope of this paper. We remark that a (numerical) analysis for the model function $F_i = Ce^{-\bar{\lambda}s_j} \cos \omega s_j$ (instead of $\mathcal{E}_g[s_i; \mathbf{s}]$) already gives a lot of insight. Walking along one of the curves of solutions near the lines (14) branches bifurcate in the neighborhood of points where all s_i are equal to an odd multiple of $\frac{\pi}{2\omega}$. The number of bifurcating branches is (n-m)(n-m-1), which can be explained



FIGURE 13. A blow-up of part of the second branch of the bifurcation diagram for γ slightly larger than $\frac{1}{8}$. The branch consists of solutions of (10) with two zeros. The profile of solutions on different parts of the branch are depicted below (for large L). The index of the solutions is also shown.

as follows. The jump in the index along the primary curve is n - m - 1., while there is an (n - m)-fold symmetry which is broken upon bifurcation. We refer to [2, 15] for rigorous results on the multiplicity of bifurcating branches in the presence of symmetries. However, keep in mind that the symmetry is usually broken upon returning to $\mathcal{E}_g[s_i; s]$ instead of the model function F_i . As an illustration part of the branch of solutions of (10) for n = 2 (i.e., with two transitions/zeros) is shown in Figure 13.

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