The Poincaré-Bendixson Theorem and the Non-linear Cauchy-Riemann Equations

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Abstract

In [6] Fiedler and Mallet-Paret proved a version of the classical Poincaré-Bendixson Theorem for scalar parabolic equations. We prove that a similar result holds for bounded solutions of the non-linear Cauchy-Riemann equations. The latter is an application of an abstract theorem for flows with an (unbounded) discrete Lyapunov function.

1 Introduction

The classical Poincaré-Bendixson Theorem describes the asymptotic behavior of flows in the plane. The topology of the plane puts severe restrictions on the behaviour of limit sets. The Poincaré-Bendixson Theorem states for example that if the α- and the ω-limit set of a bounded trajectory of a smooth flow in $\mathbb{R}^2$ does not contain equilibria, then the limit set is a periodic orbit. Several generalizations of this theorem have appeared in the literature. For instance, there are generalizations of the Poincaré-Bendixson Theorem to two-dimensional manifolds, cf. [3]. In [7] an extension to continuous (two-dimensional) flows is obtained, and [4] provides a generalization to semi-flows. The remarkable result by Fiedler and Mallet-Paret [6] establishes an extension of the Poincaré-Bendixson Theorem to infinite

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dimensional dynamical systems with a positive Lyapunov function. They apply their result to *scalar* parabolic equations of the form

$$u_s = u_{xx} + f(x, u, u_x), \quad x \in S^1, f \in C^2. \tag{1.1}$$

In this paper we establish a version of the Poincaré-Bendixson Theorem for bounded orbits of the nonlinear Cauchy-Riemann equations in the plane. A bounded orbit of the nonlinear Cauchy-Riemann equation in the plane is a (smooth) bounded function $u\colon \mathbb{R} \times S^1 \to \mathbb{R}^2$, which satisfies the equation

$$u_s - J\big(u_t - F(t, u)\big) = 0, \tag{1.2}$$

with $u(s,t) = \big(p(s,t), q(s,t)\big)$, $s \in \mathbb{R}$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$. Here $F(t,u)$ is a smooth non-autonomous vector field on $\mathbb{R}^2$ and J is the symplectic matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We prove that the asymptotic behavior, as s goes to infinity, of bounded solutions of Equation (1.2) is as simple as the limiting behavior of flows in $\mathbb{R}^2$. Equation (1.2) arises in many different contexts, in particular in the Floer homology literature, where the vector field has the form $F(t,u) = F_H(t,u)$, i.e., F_H is *Hamiltonian*, cf. [9]. The latter implies that there exists a time-dependent Hamiltonian function $H(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}$, such that $F_H(t,u) = J\nabla H(t,u)$. In the Hamiltonian case the Cauchy-Riemann equations are the L^2-gradient flow of the Hamilton action and as such bounded solutions of (1.2) will, generically, be connections orbits between equilibria. The Hamilton action is an $\mathbb{R}$-valued Lyapunov function for the Cauchy-Riemann equations. In this paper we obtain a result about the asymptotic behavior of orbits for *general* vector fields F in the Cauchy-Riemann Equations.

The main result for the Cauchy-Riemann equations in this paper concerns the asymptotic behavior of bounded solutions. A bounded solution of the Cauchy-Riemann equations is a smooth function u with $|u(s,t)| \le C$. Let X be the set of solutions bounded by a fixed (but arbitrary) constant (in the present work we will always choose $C = 1$). Endowed with the compact-open topology X is a compact Hausdorff space. The translation invariance of the Cauchy-Riemann equations in the s-variable defines an induced flow on X by translating solutions in the s-variable. A bounded solution u can be identified with its orbit $\gamma(u)$, and $\alpha(u)$ and $\omega(u)$ are well-defined elements of X. In Section 2 we given a detailed account of the space X and the induced translation flow in the context of the Cauchy-Riemann equations.

Theorem 1.1. *Let u be a bounded solution of the Cauchy-Riemann Equations (1.2). Then, for the ω-limit set $\omega(u)$ the following dichotomy holds:*

(i) either $\omega(u)$ consists of exactly one periodic orbit, or

(ii) $\alpha(v) \subseteq E$ and $\omega(v) \subseteq E$, for every $v \in \omega(u)$,

where E denotes the set of 1-periodic solutions of the vector field $F(t,x)$. The same dichotomy holds for the α-limit set $\alpha(u)$.

As in the classical Poincaré-Bendixson Theorem, alternative (ii) allows for $\omega(u)$ (or $\alpha(u)$) to consist of homoclinic and/or heteroclinic solutions joining equilibria. An important reason why a generalization of the Poincaré-Bendixson holds for the Cauchy-Riemann equations is that there exists a continuous projection onto $\mathbb{R}^2$, which is defined as follows. Let $t_0 \in S^1$ be arbitrary, then define

$$\begin{array}{rccl} \pi_{t_0}: & X & \to & \mathbb{R}^2 \\ & u=(p,q) & \mapsto & \pi_{t_0}(u) = \big(p(0,t_0), q(0,t_0)\big). \end{array} \tag{1.3}$$

Theorem 1.2. *Under the assumptions of Theorem 1.1 the projection*

$$\pi_{t_0}\colon \omega(u) \to \pi_{t_0}\omega(u)$$

is a homeomorphism onto its image.

In general, if a flow allows a continuous Lyapunov function, then limit sets of orbits consist only of equilibria. Such flows are referred to as gradient-like flows. Theorem 3.1 in this paper gives an abstract extension of the Poincaré-Bendixson Theorem to flows that allow a *discrete* Lyapunov function. In particular Theorem 3.1 implies Theorem 1.1. Note that Theorem 1.2 together with the classical Poincaré-Bendixson Theorem also implies Theorem 1.1. Theorem 1.2 is proved in Section 5 as an abstract version.

The main differences between the results in [6] for parabolic equations and the results in this paper, are that the Cauchy-Riemann equations do not define a well-posed initial value problem and, more importantly, the discrete Lyapunov functions that are considered in this paper are *not* bounded from below. Furthermore, the results obtained in this paper do not assume differentiability of the flow, nor does the flow need to be defined on a Banach space. We believe that most of the results in this paper can be extended to semi-flows, cf. [4].

In Section 2 we analyze the main properties of the Cauchy-Riemann equations (1.2), with additional details given in Section 6. In Section 3, we set up an abstract setting which generalizes the properties of the Cauchy-Riemann equations. In Sections 4 and 5 a full proof of the Poincaré-Bendixson Theorem is given, adapted to the abstract setting introduced in Section 3.

2 The Cauchy-Riemann Equations

Since the initial value problem of Equation (1.2) is ill-posed, we restrict our attention to bounded solutions, i.e., functions $u \in C^1(\mathbb{R} \times S^1; \mathbb{R}^2)$ that satisfy Equation (1.2) and for which

$$|u(s,t)| < \infty, \quad \text{for all } (s,t) \in \mathbb{R} \times S^1. \tag{2.1}$$

Since we can consider each bounded solution separately, it suffices to consider the space X of functions $u \in C^1(\mathbb{R} \times S^1; \mathbb{R}^2)$ satisfying Equation (1.2), and for which

$$|u(s,t)| \leq C, \quad \text{for all } (s,t) \in \mathbb{R} \times S^1,$$

for some fixed arbitrary constant $C > 0$. Note that, without loss of generality, we can choose $C = 1$. On X we consider the compact-open topology, i.e.

$$u^n \xrightarrow{X} u \quad \Longleftrightarrow \quad u^n \xrightarrow{C^0_{\text{loc}}} u, \tag{2.2}$$

where the latter indicates uniform convergence on compact subsets of $S^1 \times \mathbb{R}$. Since $C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$, endowed with the compact-open topology, is Hausdorff (see [10, §47]), and $X \subset C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$, also X is a Hausdorff space.

Proposition 2.1. *The solution space X is a* compact *Hausdorff space.*

Proof. See Section 6. □

Identify the translation mapping $(s, t) \mapsto (s+\sigma, t)$ by $\sigma \in \mathbb{R}$ and consider the evaluation mapping

$$\mathbb{R} \times C^0(\mathbb{R} \times S^1; \mathbb{R}^2) \to C^0(\mathbb{R} \times S^1; \mathbb{R}^2), \quad (\sigma, u) \mapsto \phi^\sigma(u) = u \circ \sigma. \tag{2.3}$$

Lemma 2.2. *The evaluation mapping $(\sigma, u) \mapsto \phi^\sigma(u)$ is continuous with respect to the compact-open topology on $C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$.*

Proof. Since $\mathbb{R} \times S^1$ is a locally compact Hausdorff space, the composition of mappings

$$C^0(\mathbb{R} \times S^1; \mathbb{R} \times S^1) \times C^0(\mathbb{R} \times S^1; \mathbb{R}^2) \to C^0(\mathbb{R} \times S^1; \mathbb{R}^2),$$

is continuous with respect to the compact-open topologies on $C^0(\mathbb{R} \times S^1; \mathbb{R} \times S^1)$ and $C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$, see [10, §46]. The translation σ, as defined above, is a continuous mapping in $C^0(\mathbb{R} \times S^1; \mathbb{R} \times S^1)$, which proves the lemma. □

Since the Cauchy-Riemann Equations are s-translation invariant we have that $u \in X$ implies that $\phi^\sigma(u) \in X$. We therefore obtain a continuous mapping $\mathbb{R} \times X \to X$, again denote by $\phi^\sigma(u)$. Also,

$$\phi^\sigma\big(\phi^{\sigma'}(u)\big) = (u \circ \sigma') \circ \sigma = u \circ (\sigma + \sigma') = \phi^{\sigma+\sigma'}(u),$$

which shows that ϕ^σ defines a continuous flow on X. A continuous flow on X is a continuous mapping $(\sigma, u) \mapsto \phi^\sigma(u) \in X$, such that $\phi^0(u) = u$ and $\phi^{\sigma+\sigma'}(u) = \phi^\sigma(\phi^{\sigma'}(u))$, for all $\sigma, \sigma' \in \mathbb{R}$ and for all $u \in X$.

Consider the evaluation mapping $\iota : C^0(\mathbb{R} \times S^1; \mathbb{R}^2) \to C^0(S^1; \mathbb{R}^2)$, defined by

$$u(\cdot, \cdot) \mapsto u(0, \cdot).$$

By a similar argument as in Lemma 2.2 it follows that the mapping ι is a continuous mapping with respect to the compact-open topology on $C^0(S^1; \mathbb{R}^2)$.

Proposition 2.3. *The mapping $\iota : X \to \mathscr{X}$, with $\mathscr{X} = \iota(X)$, is a homeomorphism.*

Proof. See Section 6. □

For ϕ^σ we have the following commuting diagram:

$$\begin{tikzcd} \mathbb{R} \times X \arrow[r, "\phi^\sigma"] \arrow[d, "\mathrm{id}\times\iota"'] & X \arrow[d, "\iota"] \\ \mathbb{R} \times \mathscr{X} \arrow[r, "T^\sigma"] & \mathscr{X}, \end{tikzcd}$$

with $u(0,\cdot) \mapsto T^\sigma(u(0,\cdot)) = u(\sigma,\cdot)$, and T^σ defines a flow on $\mathscr{X}$.

The principal tool in the proof Theorem 1.1 is the existence of an unbounded, discrete Lyapunov function, which decreases along orbits of the flow ϕ^σ. Let $u^1, u^2 \in X$ be two solutions, with $u^1 \neq u^2$, such that the function $t \mapsto u^1(s,t) - u^2(s,t)$ is nowhere zero. Then define $w := u^1 - u^2 \in C^0(\mathbb{R} \times S^1; \mathbb{R}^2)$. The s-dependent winding number $\mathscr{W}$ of the pair (u^1, u^2) is defined as the winding number of w about the origin, i.e.

$$\mathscr{W}\big(u^1(s,\cdot), u^2(s,\cdot)\big) := \mathscr{W}(w(s,\cdot), 0) = \frac{1}{2\pi}\int_{S^1} w^*\theta, \tag{2.4}$$

where $\theta = \frac{-qdp+pdq}{p^2+q^2}$ is a closed one-form on $\mathbb{R}^2 \setminus \{0\}$, cf. [11]. A pair of solutions $(u^1, u^2) \in X \times X$ is said to be *singular*, if they belong to the "crossing" set defined by

$$\Sigma_X := \{(u^1, u^2) \in X \times X : \exists\, s \in \mathbb{R} \;:\; u^1(s,t) = u^2(s,t) \text{ for some } t \in S^1\},$$

and $W\colon (X \times X) \setminus \Sigma_X \to \mathbb{Z}$ is defined by

$$W(u^1, u^2) := \mathscr{W}\big(\iota(u^1), \iota(u^2)\big). \tag{2.5}$$

The Lyapunov function W is continuous on $(X \times X) \setminus \Sigma_X$ and constant on connected components. The set Σ_X is a closed in $X \times X$, since uniform convergence on compact sets implies point-wise convergence. The function W is a symmetric:

$$W(u^1, u^2) = W(u^2, u^1), \qquad \text{for all } (u^1, u^2) \notin \Sigma_X.$$

The *diagonal* in $X \times X$ is defined by

$$\Delta := \{(u^1, u^2) \in X \times X : u^1 = u^2\},$$

and $\Delta \subset \Sigma_X$. The flow ϕ^σ induces a product flow on $X \times X$, via $(u^1, u^2) \mapsto \big(\phi^\sigma(u^1), \phi^\sigma(u^2)\big)$, and the diagonal Δ is invariant for the product flow. For the action of the flow on W we have

$$
\begin{aligned}
W\big(\phi^\sigma(u^1), \phi^\sigma(u^2)\big) &= \mathscr{W}\big(\iota \circ \phi^\sigma(u^1), \iota \circ \phi^\sigma(u^2)\big) \\
&= \mathscr{W}\big(T^\sigma(\iota(u^1)), T^\sigma(\iota(u^2))\big) = \mathscr{W}(u^1(\sigma, \cdot), u^2(\sigma, \cdot)).
\end{aligned}
$$

In [11] it is proved that the set $\Sigma_X \setminus \Delta$ is "thin" in $X \times X$, which is the content of the following proposition.

Proposition 2.4 (see [11])**.** *For every singular solution pair $(u^1, u^2) \in \Sigma_X \setminus \Delta$, there exists an $\varepsilon_0 = \varepsilon(u^1, u^2) > 0$, such that $(\phi^\sigma(u^1), \phi^\sigma(u^2)) \notin \Sigma_X$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$.*

Orbits which intersect Σ_X "transversely" (and thus are not in the diagonal) instantly escape from Σ_X and the diagonal Δ is the maximal invariant set contained in Σ_X. The following proposition indicates W is a discrete Lyapunov function.

Proposition 2.5 (see [11])**.** *For every pair of singular solutions $(u^1, u^2) \in \Sigma_X \setminus \Delta$, there exists an $\varepsilon_0 = \varepsilon(u^1, u^2) > 0$, such that $W(\phi^\sigma(u^1), \phi^\sigma(u^2)) > W(\phi^{\sigma'}(u^1), \phi^{\sigma'}(u^2))$, for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$.*

For a given $u \in X$ define the α- and ω-limit sets as:

$$
\begin{aligned}
\omega(u) &:= \{w \in X : \phi^{\sigma_n}(u) \xrightarrow{X} w, \text{ for some } \sigma_n \to \infty\}, \\
\alpha(u) &:= \{w \in X : \phi^{\sigma_n}(u) \xrightarrow{X} w, \text{ for some } \sigma_n \to -\infty\}.
\end{aligned}
$$

The sets $\alpha(u)$ and $\omega(u)$ are closed invariant sets for the flow ϕ^σ, see [7, Lemma 4.6 Chapter IV]. Since X is compact, also $\alpha(u)$ and $\omega(u)$ are compact. Compactness of X also implies that $\alpha(u)$ and $\omega(u)$ are non-empty, see [7, Theorem 4.7 Chapter IV]. The Hausdorff property of X and the continuity of the flow ϕ^σ imply that $\alpha(u)$ and $\omega(u)$ are connected sets, see [7, Theorem 4.7 Chapter IV]. Define the equilibria of ϕ^σ by

$$
E := \{u \in X : \phi^\sigma(u) = u \text{ for all } \sigma \in \mathbb{R}\}.
$$

Equilibria are functions $u = u(t)$ which satisfy the stationary equation $u_t = F(t, u)$.

3 The abstract Poincaré-Bendixson Theorem

The concepts introduced so far can be embedded in a more abstract setting, which generalizes the work by Fiedler and Mallet-Paret in [6]. Let ϕ^σ be a continuous flow on a *compact* Hausdorff space X. In the case of the Cauchy-Riemann equations the flow ϕ^σ is defined in (2.3), where the space X is either the full solution space, or the space which consists of the closure of a single entire (bounded) orbit.

The notions of α- and ω-limit sets, defined in Section 2 remain unchanged, and $\alpha(u)$ and $\omega(u)$ are non-empty, compact, connected, invariant sets. We denote by $E \subset X$ the set of equilibria of ϕ^σ.

Let $\Delta = \{(u^1, u^2) \in X \times X\colon u^1 = u^2\}$ be invariant for the product flow induced by ϕ^σ. We assume that there exist a closed "thin" singular set Σ, with $\Delta \subset \Sigma \subset X \times X$, and functions $W\colon (X \times X) \setminus \Sigma \to \mathbb{Z}$ and $\pi : X \to \pi(X) \subset \mathbb{R}^2$, which satisfy the following five axioms:

(A1) the function $W : X \times X \setminus \Sigma \to \mathbb{Z}$ is continuous and symmetric;

(A2) the mapping $\pi : X \to \pi(X) \subset \mathbb{R}^2$ is a continuous projection onto its (compact) image;

(A3) the set $\{(u^1, u^2) \in X \times X\colon \pi(u^1) = \pi(u^2)\}$ is a subset of Σ;

(A4) for every $(u^1, u^2) \in \Sigma \setminus \Delta$, there exists an $\varepsilon_0 > 0$, depending on (u^1, u^2), such that $(\phi^\sigma(u^1), \phi^\sigma(u^2)) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$;

(A5) for every $(u^1, u^2) \in \Sigma \setminus \Delta$, there exists an $\varepsilon_0 > 0$, depending on (u^1, u^2), such that

$$W(\phi^\sigma(u^1), \phi^\sigma(u^2)) > W(\phi^{\sigma'}(u^1), \phi^{\sigma'}(u^2)),$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$.

Axioms (A1)-(A5) are modeled on the properties of the non-linear Cauchy-Riemann Equations discussed in Section 2, with $\pi = \pi_{t_0}$ defined in (1.3). The above axioms also generalize the conditions in the work of Fiedler and Mallet-Paret in [6]. Note that the function W is a priori unbounded in the present case and the flow ϕ^σ does not necessarily regularize. Under these assumptions we prove the following theorem.

Theorem 3.1 (Poincaré-Bendixson). *Let ϕ^σ be a continuous flow on a compact Hausdorff space X. Let Σ be a closed subset of $X \times X$, and let $W : (X \times X) \backslash \Sigma \to \mathbb{Z}$ and $\pi : X \to \pi(X) \subset \mathbb{R}^2$ be mappings as defined above, and which satisfy Axioms (A1)-(A5). Then, for $\omega(u)$ we have the following dichotomy:*

(i) either $\omega(u)$ consists of precisely one periodic orbit, or else

(ii) $\alpha(w) \subseteq E$ and $\omega(w) \subseteq E$, for every $w \in \omega(u)$.

The same dichotomy holds for $\alpha(u)$.

As in [6], the proof of Theorem 3.1 will be divided into the three Propositions listed below.

From this point on we assume the hypotheses of Theorem 3.1.

Proposition 3.2 (Soft version). *Let u be in X and let $w \in \omega(u)$, then $\omega(w)$ contains a periodic solution or an equilibrium. The same holds for $\alpha(w)$.*

Proposition 3.2 implies that, since $\omega(w)$ and $\alpha(w)$ are both subsets of $\omega(u)$, also $\omega(u)$ contains a periodic solution or an equilibrium.

Proposition 3.3. *Let u be in X and let $w \in \omega(u)$. Then either,*

(i) $\alpha(w)$ and $\omega(w)$ consist only of equilibria, or else

(ii) $\gamma(w)$ is a periodic orbit.

Proposition 3.4. *Let u be X. If $\omega(u)$ contains a periodic orbit, then $\omega(u)$ is a single periodic orbit.*

The proof of Proposition 3.2 is given in Section 4 and the proofs of Propositions 3.3 and 3.4 are carried out in Section 5.2. Section 5.2 also provides the proof of Theorem 1.2, with a formulation adapted to the abstract setting. Propositions 3.3 and 3.4 together imply Theorem 3.1, while Proposition 3.2 will be used to prove Proposition 3.3. Theorem 3.1 can be applied directly to the Cauchy-Riemann equations and therefore implies Theorem 1.1. Subsection 5.1 contains a number of technical lemmas. Finally, Section 6 provides the proofs of Propositions 2.1 and 2.3.

4 The soft version

This section deals with the proof of Proposition 3.2. The hypotheses of Section 3 will be assumed for the remainder of the paper.

Lemma 4.1. *For every pair* $(u^1, u^2) \in (X \times X) \setminus \Delta$*, the set*

$$A_{(u^1,u^2)} := \{\sigma \in \mathbb{R} \colon \big(\phi^\sigma(u^1), \phi^\sigma(u^2)\big) \in \Sigma\}$$

consists of isolated points only. Moreover, the mapping

$$\sigma \mapsto W(\phi^\sigma(u^1), \phi^\sigma(u^2)),$$

defined for $\sigma \in \mathbb{R} \setminus A_{(u^1,u^2)}$*, is a non-increasing function of* σ *and constant on the connected components of* $\mathbb{R} \setminus A_{(u^1,u^2)}$*.*

Proof. Suppose there exists an accumulation point $\sigma_n \to \sigma_*$, with $\sigma_n \in A_{(u^1,u^2)}$. By definition $\big(\phi^{\sigma_n}(u^1), \phi^{\sigma_n}(u^2)\big) \in \Sigma \setminus \Delta$, since Δ is invariant and $(u^1, u^2) \notin \Delta$. By the continuity of ϕ^σ we have that

$$\big(\phi^{\sigma_n}(u^1), \phi^{\sigma_n}(u^2)\big) \xrightarrow{n\to\infty} \big(\phi^{\sigma_*}(u^1), \phi^{\sigma_*}(u^2)\big) \in \Sigma,$$

since Σ is closed. This proves that $\sigma_* \in A_{(u^1,u^2)}$. The invariance of Δ implies that $\big(\phi^{\sigma_*}(u^1), \phi^{\sigma_*}(u^2)\big) \in \Sigma \setminus \Delta$. By Axiom (A4) there exists an $\varepsilon_0 > 0$, depending on $\big(\phi^{\sigma_*}(u^1), \phi^{\sigma_*}(u^2)\big)$, such that $(\phi^{\sigma_*+\varepsilon}(u^1), \phi^{\sigma_*+\varepsilon}(u^2)) \notin \Sigma$, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$. This contradicts the fact that σ_* is an accumulation point.

The set $A_{(u^1,u^2)}$ is a discrete and ordered set. Let $\sigma' < \sigma''$ be two consecutive points in $A_{(u^1,u^2)}$. By Axiom (A1), W is continuous and $\mathbb{Z}$-valued, and therefore $W(\phi^\sigma(u^1), \phi^\sigma(u^2))$ is constant on $\sigma \in (\sigma', \sigma'')$. The fact that W is non-increasing then follows from (A5), since $W(\phi^\sigma(u^1), \phi^\sigma(u^2))$ drops at points in $A_{(u^1,u^2)}$. $\square$

Lemma 4.2. *Let* $u \in X$ *and* $w \in \omega(u)$*. For every* $w^1, w^2 \in \mathrm{cl}\big(\gamma(w)\big)$ *with* $w^1 \neq w^2$*, it holds that* $(w^1, w^2) \notin \Sigma$*.*

Proof. We argue by contradiction. Suppose $(w^1, w^2) \in \Sigma \setminus \Delta$, then, by the Axioms (A4) and (A5), there exists an $\varepsilon_0 > 0$, such that $\big(\phi^\sigma(w^1), \phi^\sigma(w^2)\big) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^\sigma(w^1), \phi^\sigma(w^2)) > W(\phi^{\sigma'}(w^1), \phi^{\sigma'}(w^2)),$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$. Set $\sigma = -\varepsilon$ and $\sigma' = \varepsilon$, with $0 < \varepsilon < \varepsilon_0$. Since $w^1, w^2 \in \mathrm{cl}(\gamma(w))$, there exist $s_1, s_2 \in \mathbb{R}$, such that $\big(\phi^{s_1 \pm \varepsilon}(w), \phi^{s_2 \pm \varepsilon}(w)\big) \notin \Sigma$ and $\big(\phi^{s_1 \pm \varepsilon}(w), \phi^{s_2 \pm \varepsilon}(w)\big)$ is close to $\big(\phi^{\pm\varepsilon}(w^1), \phi^{\pm\varepsilon}(w^2)\big)$. The continuity of W (Axiom (A1)) then implies

$$\begin{aligned} W(\phi^{s_1+\varepsilon}(w), \phi^{s_2+\varepsilon}(w)) &= W(\phi^{\varepsilon}(w^1), \phi^{\varepsilon}(w^2)) \\ &< W(\phi^{-\varepsilon}(w^1), \phi^{-\varepsilon}(w^2)) \\ &= W(\phi^{s_1-\varepsilon}(w), \phi^{s_2-\varepsilon}(w)). \end{aligned} \tag{4.1}$$

Since $\gamma(w) \subset \omega(u)$ is an invariant subset of $\omega(u)$, the definition of ω-limit set and the continuity of ϕ^σ imply that there exists a sequence $\sigma_n \to \infty$, as $n \to \infty$, such that

$$\phi^{\sigma_n + s_1 - s_2 \pm \varepsilon}(u) \to \phi^{s_1 \pm \varepsilon}(w), \quad \text{and} \quad \phi^{\sigma_n \pm \varepsilon}(u) \to \phi^{s_2 \pm \varepsilon}(w). \tag{4.2}$$

Since σ_n is divergent, we may assume

$$\sigma_{n+1} > \sigma_n + 2\varepsilon, \quad \text{for all } n. \tag{4.3}$$

Inequality (4.1), the convergence in (4.2), Axiom (A1) (continuity) and the fact that W is locally constant (see Lemma 4.1), imply, for $\sigma_n \to \infty$, that

$$\begin{aligned} W(\phi^{\sigma_n+s_1-s_2+\varepsilon}(u), \phi^{\sigma_n+\varepsilon}(u)) &= W(\phi^{s_1+\varepsilon}(w), \phi^{s_2+\varepsilon}(w)) \\ &< W(\phi^{s_1-\varepsilon}(w), \phi^{s_2-\varepsilon}(w)) \\ &= W(\phi^{\sigma_n+s_1-s_2-\varepsilon}(u), \phi^{\sigma_n-\varepsilon}(u)). \end{aligned}$$

By combining the latter with (4.3) and the fact that W is non-increasing, we obtain

$$W(\phi^{\sigma_{n+1}+s_1-s_2-\varepsilon}(u), \phi^{\sigma_{n+1}-\varepsilon}(u)) < W(\phi^{\sigma_n+s_1-s_2-\varepsilon}(u), \phi^{\sigma_n-\varepsilon}(u)),$$

for all n. From this inequality we deduce that $\sigma \mapsto W(\phi^{\sigma+s_1-s_2}(u), \phi^\sigma(u))$ has infinitely many jumps and therefore

$$W(\phi^{\sigma+s_1-s_2}(u), \phi^\sigma(u)) \to -\infty, \quad \text{as} \quad \sigma \to \infty.$$

On the other hand, by continuity of W and (4.2) we have, for $\sigma_n \to \infty$, that

$$W(\phi^{\sigma_n+s_1-s_2+\varepsilon}(u), \phi^{\sigma_n+\varepsilon}(u)) = W(\phi^{s_1+\varepsilon}(w), \phi^{s_2+\varepsilon}(w)) > -\infty,$$

which is a contradiction. □

Lemma 4.3. *Let $u \in X$ and $w \in \omega(u)$, then*

$$\pi\colon\ \mathrm{cl}\,(\gamma(w)) \to \pi\,\mathrm{cl}(\gamma(w)) \subset \mathbb{R}^2$$

is a homeomorphism onto its image. Hence, $\pi \circ \phi^\sigma \circ \pi^{-1}$ is a continuous flow on $\pi\,\mathrm{cl}(\gamma(w))$.

Proof. By Axiom (A2), the projection $\pi\colon\ \mathrm{cl}\,(\gamma(w)) \to \pi\,\mathrm{cl}(\gamma(w))$ is continuous. Since $\mathrm{cl}\,(\gamma(w))$ is compact and $\pi\,\mathrm{cl}(\gamma(w))$ is Hausdorff, it is sufficient to show that π is bijective, see [10, § 26, Thm. 26.6]. The projection π is surjective and it remains to show that π is injective on $\mathrm{cl}(\gamma(w))$. Suppose π is not injective, then there exist $w^1, w^2 \in \mathrm{cl}(\gamma(w))$, such that $w^1 \neq w^2$ and $\pi(w^1) = \pi(w^2)$. Axiom (A3) then implies that $(w^1, w^2) \in \Sigma \setminus \Delta$. On the other hand, Lemma 4.2 implies that $(w^1, w^2) \notin \Sigma$, which is a contradiction. This establishes the injectivity of π. $\square$

For the projected flow on $\pi\,\mathrm{cl}(\gamma(w))$ we have the following commuting diagram:

$$\begin{tikzcd}
\mathbb{R} \times \mathrm{cl}(\gamma(w)) \arrow[r, "\phi^\sigma"] \arrow[d, "\mathrm{id}\times\pi"'] & \mathrm{cl}(\gamma(w)) \arrow[d, "\pi"] \\
\mathbb{R} \times \pi\,\mathrm{cl}(\gamma(w)) \arrow[r, "\psi^\sigma"] & \pi\,\mathrm{cl}(\gamma(w)),
\end{tikzcd} \tag{4.4}$$

where $\psi^\sigma = \pi \circ \phi^\sigma \circ (\mathrm{id} \times \pi)^{-1}$.

Corollary 4.4. *The equilibria of the planar flow $\psi^\sigma := \pi \circ \phi^\sigma \circ (\mathrm{id} \times \pi)^{-1}$ on $\pi\,\mathrm{cl}(\gamma(w))$ are in one-to-one correspondence with the equilibria of the flow ϕ^σ in $\mathrm{cl}(\gamma(w))$.*

Following the natural strategy in proving a Poincaré-Bendixson type result, we now would like to find a transverse curve at a non-equilibrium point and invoke a flow box theorem, ultimately leading to contradiction arguments involving the inside and outside of a Jordan curve made up of a flow line and the transversal. Transversals do exist for a continuous (but not necessarily smooth) flow in $\mathbb{R}^2$ [7, section VII.2]. However, our flow is defined on the closed invariant subset $\pi\,\mathrm{cl}(\gamma(w)) \subset \mathbb{R}^2$. This set has empty interior, and this prevents us from finding a section, as defined below, that is also a curve (i.e. a so-called transversal). Roughly speaking, we overcome this difficulty by adapting the usual Jordan curve arguments to a slightly "less local" version.

Let $(\sigma, x) \mapsto \psi^\sigma(x)$ be the (local) continuous flow on the subset $\mathscr{D} = \pi\,\mathrm{cl}(\gamma(w))$ of $\mathbb{R}^2$. A subset $\mathscr{C} \subset \mathscr{D}$ is a *section* for ψ^σ, if there is a $\delta > 0$, such that

$$\psi^{\sigma_1}(\mathscr{C}) \cap \psi^{\sigma_2}(\mathscr{C}) = \varnothing, \quad \text{for all } 0 \le \sigma_1 < \sigma_2 \le \delta.$$

The following lemma shows that for non-equilibrium points $x \in \mathscr{D}$ there exists a section for the flow in an ε-neighborhood of x.

Lemma 4.5. *Let $x \in \mathscr{D}$ be a non-equilibrium point of ψ^σ.*

(i) For sufficiently small $\delta > 0$ there exists a section $\mathscr{C}$ containing x such that the set

$$\mathscr{U} := \{\psi^\sigma(y) : y \in \mathscr{C}, \sigma \in [-\delta, \delta]\}$$

is homeomorphic to $\mathscr{C} \times [-\delta, \delta]$ via the map ψ, and for $\varepsilon > 0$ sufficiently small

- $B_\varepsilon(x) \cap \mathscr{D} \subset \mathscr{U}$,
- $h_\sigma(\overline{B_\varepsilon(x)} \cap \mathscr{U}) \in (-\delta, \delta)$.

where h_σ is the second components of the inverse homeomorphism $h : \mathscr{U} \to \mathscr{C} \times [-\delta, \delta]$, i.e., $h \circ \psi^\sigma(y) = (y, \sigma)$ for all $y \in \mathscr{C}, \sigma \in [-\delta, \delta]$.

(ii) For $\delta_0 < \delta$ sufficiently small the three balls $B^0 \equiv B_{\varepsilon_0}(x)$, $B^- \equiv B_{\varepsilon_0}(\psi^{-\delta_0}(x))$ and $B^+ \equiv B_{\varepsilon_0}(\psi^{\delta_0}(x))$ are, for $\varepsilon_0 < \varepsilon$ sufficiently small, disjoint subsets of $B_\varepsilon(x)$ such that $h_\sigma(B^- \cap \mathscr{U}) < -\frac{\delta_0}{2} < h_\sigma(B^0 \cap \mathscr{U}) < \frac{\delta_0}{2} < h_\sigma(B^+ \cap \mathscr{U})$. Furthermore, for $\varepsilon_1 < \varepsilon_0$ sufficiently small, we have $\psi^{\pm\delta_0}(y) \in B^\pm$ for all $y \in \mathscr{C}_0 \equiv \mathscr{C} \cap B_{\varepsilon_1}(x)$.

Proof. The first part follows from the construction of sections in [7, section VI.2]. The second part then follows from continuity of ψ and its inverse h. □

The situation described by Lemma 4.5 is illustrated in Figure 1.

Remark 4.6. In fact, as we will see later, we need to apply a variant of the above lemma to forward invariant closed subsets of the form

$$\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \ge \sigma_*\}),$$

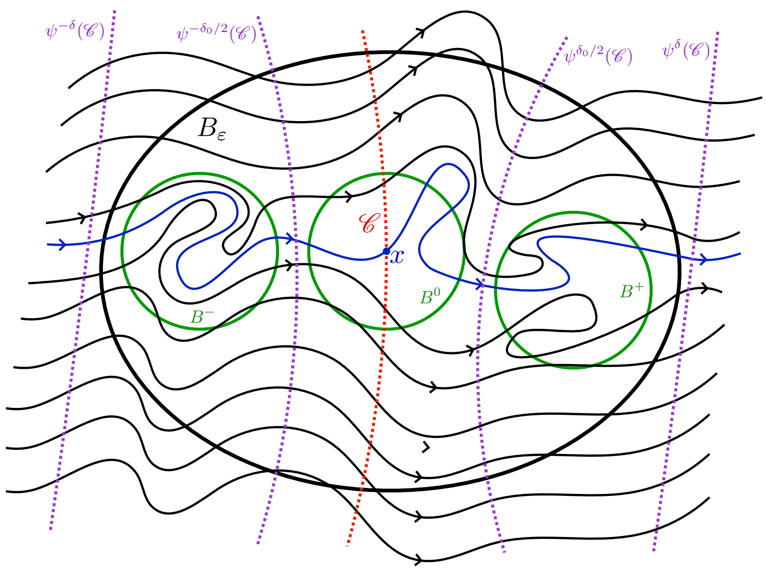

Figure 1: Sketch of the flow in $\mathscr{U}$ (and the subset $B_\varepsilon(x) \cap \mathscr{D}$) through the section $\mathscr{C}$. The time section $\sigma = \pm\delta_0/2$ separate the balls B^0 and $B^\pm$. We note that the section $\mathscr{C}$ (and its forward and backward translates in time) are not (necessarily) curves.

where $u \in X, w \in \omega(u), \sigma_* \in \mathbb{R}$. On $\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \geq \sigma_*\})$ we have a commuting diagram similar to (4.4). In order to have a bi-directional local flow, we define the slightly smaller set

$$\mathscr{V} := \pi\,\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \geq \sigma_* + 2\delta\}), \tag{4.5}$$

for $\delta > 0$ small. Then, if $x \in \mathscr{V}$ is not an equilibrium for ψ^σ, Lemma 4.5 continues to hold with $\mathscr{U}$ replaced by $\mathscr{V}$.

Remark 4.7. (i) The second part of Lemma 4.5 is used to construct a set that replaces the role of the transversal. Let y_1 and y_2 be two points in $\mathscr{C}_0$. Consider the line segment L_0 connecting y_1 and y_2. Then $L_0 \subset B^0$. It may happen that L_0 intersects the flow lines of $\psi^\sigma(y_{1,2})$ at some $\sigma \neq 0$, but this can be overcome by slightly varying σ. Indeed, let the line segment ℓ_0 be a subset of L_0 with endpoints $\psi^{\sigma_1^0}(y_1)$ and $\psi^{\sigma_2^0}(y_2)$ for some $\sigma_1^0, \sigma_2^0 \in (-\delta_0/2, \delta_0/2)$, such that ℓ_0 does not intersect the flow lines $\psi^\sigma(y_1)$ and $\psi^\sigma(y_2)$ at any other $\sigma \in [-\delta, \delta]$. We still have $\ell_0 \subset B^0$, see Figure 2.

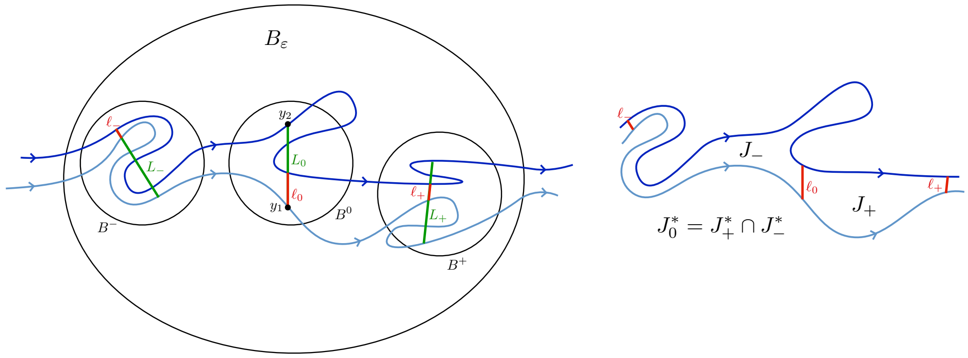

Figure 2: On the left an illustration of the construction of ℓ_0 and $\ell_\pm$. On the right a sketch of the interiors J_0 and $J_\pm$ of the Jordan curves $\mathscr{J}_0$ and $\mathscr{J}_\pm$ (as well as their exteriors J_0^* and $J_\pm^*$). Note that $J_0 = \operatorname{int}(\overline{J_- \cup J_+})$.

We repeat this construction in the balls B^- and B^+ to obtain line segments ℓ_- and ℓ_+, respectively, with one end point on each flow line and no other intersections with the flow lines.

Then we obtain three Jordan curves

$$
\begin{aligned}
\mathscr{J}_0 &= \{\psi^\sigma(y_1) : \sigma_1^- \le \sigma \le \sigma_1^+\} \cup \{\psi^\sigma(y_2) : \sigma_2^- \le \sigma \le \sigma_2^+\} \cup \ell_- \cup \ell_+,\\
\mathscr{J}_- &= \{\psi^\sigma(y_1) : \sigma_1^- \le \sigma \le \sigma_1^0\} \cup \{\psi^\sigma(y_2) : \sigma_2^- \le \sigma \le \sigma_2^0\} \cup \ell_- \cup \ell_0,\\
\mathscr{J}_+ &= \{\psi^\sigma(y_1) : \sigma_1^0 \le \sigma \le \sigma_1^+\} \cup \{\psi^\sigma(y_2) : \sigma_2^0 \le \sigma \le \sigma_2^+\} \cup \ell_0 \cup \ell_+,
\end{aligned}
$$

lying in $B_\varepsilon(x)$, see Figure 2. We denote the interior of $\mathscr{J}_j$ by J_j, and its exterior by J_j^*, $j \in \{-, 0, +\}$. Clearly, $J_\pm \subset J_0$ and $J_- \cap J_+ = \varnothing$.

(ii) By Lemma 4.5 any flow line in J_0 must leave J_0 in forward and backward time. By flow invariance of the other boundary components, a flow line can only enter or leave J_0 through ℓ_+ or ℓ_-. Moreover, *no* flow line can (in forward time) enter J_0 through ℓ_+ and then leave it through ℓ_-. In this sense, the set J_0 plays the role of a transversal. Analogous statements holds for J_+ and J_-. In particular, this implies a slightly stronger statement for the flow in $J_+ \subset J_0$: if a flow line is in J_+ then must leave J_0 through ℓ_+. Similarly, if a flow line is in J_- then must have entered J_0 through ℓ_-.

(iii) The flow lines $\{\psi^\sigma(y_{1,2}) : \sigma \in (\sigma_{1,2}^+, \delta]\}$ and $\{\psi^\sigma(y_{1,2}) : \sigma \in [-\delta, \sigma_{1,2}^-)\}$ lie in the exterior J_0^*. This follows from the fact that by construction they cannot cross ℓ_+ and ℓ_-, respectively, and $\psi^{\pm\delta}(y_{1,2})$ all lie in J_0^* by the second bullet of Lemma 4.5(i).

Proposition 4.8 (Soft version). *Let u be in X and $w \in \omega(u)$, then $\omega(w)$ contains a periodic orbit or an equilibrium. The same holds for $\alpha(w)$.*

Proof. Suppose $\omega(w)$ does not contain any equilibria. Choose $\zeta \in \omega(w)$ and $\zeta^* \in \omega(\zeta)$, then

$$\omega(\zeta) \subseteq \omega(\omega(w)) = \omega(w) \subseteq \omega(\gamma(w)) = \mathrm{cl}(\gamma(w)). \tag{4.6}$$

Since ζ^* is not an equilibrium, then $\pi(\zeta^*)$ is not an equilibrium for $\psi^\sigma = \pi \circ \phi^\sigma \circ (\mathrm{id} \times \pi)^{-1}$ by Corollary 4.4. According to Lemma 4.5 there exists a section $\mathscr{C}$ for ψ^σ through $x = \pi(\zeta^*)$. Since $\zeta^* \in \omega(\zeta)$, there exist times $\sigma_n \to \infty$, such that $\phi^{\sigma_n}(\zeta) \to \zeta^*$. By Lemma 4.5 these times can be chosen such that $\pi \circ \phi^{\sigma_n}(\zeta) \in \mathscr{C}_0$, as defined in Lemma 4.5(ii), for σ_n sufficiently large. Moreover, $\pi \circ \phi^\sigma(\zeta) \notin \mathscr{C}_0$ for $\sigma \in (\sigma_n, \sigma_{n+1})$. We consider two cases.

Case 1. For some $n \neq n'$, we have $\pi \circ \phi^{\sigma_n}(\zeta) = \pi \circ \phi^{\sigma_{n'}}(\zeta)$. Then, since π is a homeomorphism on $\mathrm{cl}\big(\gamma(w)\big)$ (see Lemma 4.3) and since $\omega(\zeta) \subset \mathrm{cl}\big(\gamma(w)\big)$ (see Equation (4.6)), it follows that $\phi^{\sigma_n}(\zeta) = \phi^{\sigma_{n'}}(\zeta)$, and thus $\phi^\sigma(\zeta)$ is a periodic orbit.

Case 2. All $\pi \circ \phi^{\sigma_n}(\zeta)$ are mutually distinct. Take n sufficiently large so that $y_1 \equiv \pi \circ \phi^{\sigma_{n+1}}(\zeta)$ and $y_2 \equiv \pi \circ \phi^{\sigma_{n+1}}(\zeta)$ both lie in $\mathscr{C}_0$. Denote $\widetilde{\sigma} = \sigma_{n+1} - \sigma_n$, so that $y_2 = \psi^{\widetilde{\sigma}}(y_1)$. Apply the construction of Remark 4.7(i) to these y_1 and y_2. In addition to $\mathscr{J}_0$ we obtain two more Jordan curves

$$\mathscr{G}_- = \{\psi^\sigma(y_1) : \sigma_1^- \leq \sigma \leq \widetilde{\sigma} + \sigma_2^-\} \cup \ell_-$$
$$\mathscr{G}_+ = \{\psi^\sigma(y_1) : \sigma_1^+ \leq \sigma \leq \widetilde{\sigma} + \sigma_2^+\} \cup \ell_+.$$

Both curves separate $\mathbb{R}^2$ into two open sets, say $A^1_\pm$ and $A^2_\pm$, see Figure 3. Here, to fix notation, we require that $J_0 \subset A^1_+$ and $J_0 \subset A^2_-$ (recall that J_0 is the interior of $\mathscr{J}_0$), so that $A^2_+ \cap A^1_- = \varnothing$. It follows from the property of J_0 described in Remark 4.7(ii) and flow invariance of $\{\psi^\sigma(y_1) : \sigma_1^- \leq \sigma \leq \widetilde{\sigma} + \sigma_2^+\}$ that once a flow line is in A^2_+ it can never enter A^1_- (in forward time).

Finally, we note that Remark 4.7(iii) implies that $x_1 = \psi^{-\delta}(y_1) = \pi \circ \phi^{\sigma_n - \delta}(\zeta)$ lies in A^1_-, while $x_2 = \psi^\delta(y_2) = \pi \circ \phi^{\sigma_{n+1}+\delta}(\zeta)$ lies in A^2_+. Now consider the orbit $\pi \circ \phi^\sigma(w)$. Since $\zeta \in \omega(w)$ and π is continuous, $\pi(\zeta)$ is an ω-limit point of $\pi(w)$ under ψ^σ. Consequently, the orbit $\pi \circ \phi^\sigma(w)$ keeps (in forward time) visiting arbitrarily small neighborhoods of $x_1 \in A^1_-$ and $x_2 \in A^2_+$. However, as argued above, once a flow line is in A^2_+ it can never enter A^1_-. This is a contradiction. $\square$

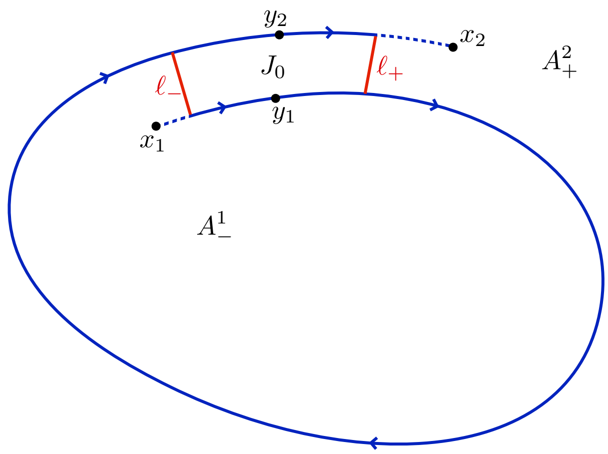

Figure 3: Sketch of the construction of A_+^2 and A_-^1. Whether A_-^1 is a bounded region and A_+^2 an unbounded one (as depicted here) or the other way around, is irrelevant for the argument.

Remark 4.9. In [6, Proposition 2] the “soft version” was proved using both smoothness of the flow and fact that there exists a non-negative discrete Lyapunov function. The extension given by Proposition 4.8 makes it applicable to the Cauchy-Riemann equations, for which a $\mathbb{Z}$-valued Lyapunov function exists.

5 The strong version

This section is subdivided into two subsections. In the first subsection we show some preliminary lemmas that will be used to prove the strong version of the Poincaré-Bendixson Theorem. The proof of Proposition 3.3 occupy the second subsection. Proofs are as in [6], but worked out in more details, and eventually adjusted to our setting.

5.1 Technical lemmas

Lemma 5.1. *Let $u \in X$, then for every $w \in \omega(u)$ there exists an integer $k(w)$, such that*

$$W(w^1, w^2) = k(w),$$

for all $w^1, w^2 \in \mathrm{cl}\big(\gamma(w)\big)$, *with* $w^1 \neq w^2$.

Proof. See also [6, Lemma 3.1]. Since we consider two distinct $w^1, w^2 \in \mathrm{cl}\big(\gamma(w)\big)$, we may exclude the case that w is an equilibrium. We therefore distinguish two cases: (i) $\gamma(w)$ is a periodic orbit, or (ii) $\sigma \mapsto \phi^\sigma(w)$ is injective. Lemma 4.2 implies that $(w^1, w^2) \notin \Sigma$, and therefore $(w^1, w^2) \mapsto W(w^1, w^2)$ is a continuous $\mathbb{Z}$-valued function on $\big(\mathrm{cl}(\gamma(w)) \times \mathrm{cl}(\gamma(w))\big) \setminus \Delta$.

(i) If $\gamma(w)$ is a periodic orbit, then, $\mathrm{cl}(\gamma(w)) = \gamma(w)$, which is homeomorphic to S^1, and $\gamma(w) \times \gamma(w)$ is therefore homeomorphic to the 2-torus $\mathbb{T}^2$. Therefore $(w^1, w^2) \mapsto W(w^1, w^2)$ induces a continuous $\mathbb{Z}$-valued function on $\mathbb{T}^2 \setminus S^1$. Since the latter is connected, it follows that W is constant on $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$.

(ii) If $\sigma \to \phi^\sigma(w)$ is injective, then $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$ has two connected components given by $(\phi^{\sigma_1}(w), \phi^{\sigma_2}(w))$, with $\sigma_1 > \sigma_2$, and $\sigma_1 < \sigma_2$, respectively. Since W is symmetric (Axiom (A1)) we conclude that W is constant on $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$. Note that $\big(\mathrm{cl}(\gamma(w)) \times \mathrm{cl}(\gamma(w))\big) \setminus \Delta$ is the closure of $\big(\gamma(w) \times \gamma(w)\big) \setminus \Delta$ in $(X \times X) \setminus \Delta$. Since W is continuous on $\big(\mathrm{cl}(\gamma(w)) \times \mathrm{cl}(\gamma(w))\big) \setminus \Delta$, it is also constant, which proves the lemma. $\square$

Lemma 5.2. *Assume that* $u \in X$ *and* $w \in \omega(u)$. *Let* $k(w)$ *be defined as in Lemma 5.1. If* $\alpha(w) \cap \omega(w) = \varnothing$, *then there exists a* $\sigma_* \geq 0$, *such that*

$$W(u^1, w^1) = k(w) \tag{5.1}$$

for every $u^1 \in \mathrm{cl}\{\phi^\sigma(u), \sigma \geq \sigma_*\}$ *and every* $w^1 \in \mathrm{cl}(\gamma(w))$, *such that* $u^1 \neq w^1$. *In particular, if* $\pi(u^1) = \pi(w^1)$ *for some* $u^1 \in \mathrm{cl}\{\phi^\sigma(u), \sigma \geq \sigma_*\}$ *and* $w^1 \in \mathrm{cl}(\gamma(w))$, *then* $u^1 = w^1$. *Hence*

$$\pi \circ \phi^\sigma(u) \notin \pi\, \mathrm{cl}(\gamma(w)) \ \text{ for all } \sigma \geq \sigma_*. \tag{5.2}$$

Proof. See [6, Lemma 3.2]. We start by observing that it is enough to prove that (5.1) holds for $u^1 \in \phi^\sigma(u), \sigma \geq \sigma_*$. Then by continuity of W, the statement follows for all $u^1 \in \mathrm{cl}\{\phi^\sigma(u), \sigma \geq \sigma_*\}$.

Suppose there exist sequences $\sigma_n \to \infty$, $w_n \in \mathrm{cl}(\gamma(w))$, with

$$\phi^{\sigma_n}(u) \neq w_n, \quad k_n := W(\phi^{\sigma_n}(u), w_n) \neq k(w).$$

We may assume, passing to a subsequence if necessary, that for all n we have that either $k_n > k(w)$ or $k_n < k(w)$. We will split the proof in two cases.

Case 1: $k_n < k(w)$. Again passing to a subsequence if necessary, we may assume that either $w_n \in \alpha(w)$ for all n or else $w_n \in \mathrm{cl}(\gamma(w)) \setminus \alpha(w)$ for all n. Since $\alpha(w)$ and $\omega(w)$ are disjoint by assumption, it follows that $\mathrm{cl}(\gamma(w)) \setminus \alpha(w) = \gamma(w) \cup \omega(w)$. Choose now $w^1 \in \omega(w)$ in case $w_n \in \alpha(w)$, and $w^1 \in \alpha(w)$ in case $w_n \in \gamma(w) \cup \omega(w)$. In both cases we have $w^1 \in \omega(u)$, hence we can choose a sequence $\tilde{\sigma}_n$ with $\tilde{\sigma}_n > \sigma_n$, for every n such that

$$w^1 := \lim_{n\to\infty} \phi^{\tilde{\sigma}_n}(u).$$

In case $w_n \in \gamma(w) \cup \omega(w)$ we may assume that $\tilde{\sigma}_n - \sigma_n$ is so large that $\phi^{\tilde{\sigma}_n - \sigma_n}(w_n) \in \mathrm{cl}\{\phi^\sigma(w), \sigma > 0\}$. For a further subsequence, we have convergence of $\phi^{\tilde{\sigma}_n - \sigma_n}(w_n)$. Call

$$w^2 := \lim_{n\to\infty} \phi^{\tilde{\sigma}_n - \sigma_n}(w_n).$$

Note that $w^1, w^2 \in \mathrm{cl}(\gamma(w))$, and $w^1 \neq w^2$ since $\alpha(w) \cap \omega(w) = \varnothing$. In fact, by construction it follows that either $w^1 \in \omega(w)$ and $w^2 \in \alpha(w)$, or else $w^1 \in \alpha(w)$ and $w^2 \in \mathrm{cl}\{\gamma(w), \sigma \geq 0\} = \{\phi^\sigma(w), \sigma \geq 0\} \cup \omega(w)$. By Lemma 5.1 there exists $k(w) \in \mathbb{Z}$ such that

$$W(w^1, w^2) = k(w).$$

Now, for n big enough, by continuity of W we obtain

$$\begin{aligned} k_n < k(w) &= W(w^1, w^2) = W(\phi^{\tilde{\sigma}_n}(u), \phi^{\tilde{\sigma}_n - \sigma_n}(w_n)) \\ &= W(\phi^{\sigma_n + (\tilde{\sigma}_n - \sigma_n)}(u), \phi^{\tilde{\sigma}_n - \sigma_n}(w^n)) \\ &\leq W(\phi^{\sigma_n}(u), w_n) = k_n, \end{aligned}$$

which is a contradiction.

The final assertion (5.2) follows from the following observation. Suppose, by contradiction that there exist a $u^1 = \phi^{\sigma_1}(u)$, for some $\sigma_1 \geq \sigma_*$ and $w^1 \in \mathrm{cl}(\gamma(w))$, such that $\pi(u^1) = \pi(w^1)$. By what we have just proved, we then have $u^1 = w^1$. Since $w^1 \in \mathrm{cl}(\gamma(w))$ and, by assumption, the sets $\alpha(w)$, $\gamma(w)$ and $\omega(w)$ are disjoint, there are only three different possibilities.

(a) $w^1 \in \omega(w)$. Then $\phi^{\sigma_1}(u) \in \omega(w)$. By invariance $\omega(u) \subseteq \omega(\omega(w)) = \omega(w)$. Since $\alpha(w) \subseteq \omega(u) \subseteq \omega(w)$, this contradicts $\alpha(w) \cap \omega(w) = \varnothing$.

(b) $w^1 \in \alpha(w)$. Then $\phi^{\sigma_1}(u) \in \alpha(w)$. By invariance $\omega(u) \subseteq \omega(\alpha(w)) = \alpha(w)$. Since $\omega(w) \subseteq \omega(u) \subseteq \alpha(w)$, this contradicts $\alpha(w) \cap \omega(w) = \varnothing$.

(c) $w^1 \in \gamma(w)$. Then $\phi^{\sigma_1}(u) \in \gamma(w)$. By invariance $\omega(u) = \omega(w)$. But $\alpha(w) \subseteq \omega(u) = \omega(w)$, again contradicting $\alpha(w) \cap \omega(w) = \varnothing$.

Case 2: $k_n > k(w)$. This case is analogous to the previous one. It is enough to exchange the roles of $\alpha(w)$ and $\omega(w)$. See [6, Lemma 3.2] for further details. $\square$

Remark 5.3. Lemma 5.2 implies that the commutative diagram (4.4) extends from $\mathrm{cl}(\gamma(w))$ to $\mathrm{cl}(\gamma(w) \cup \{\phi^\sigma(u), \sigma \geq \sigma_*\})$, if $\alpha(w) \cap \omega(w) = \varnothing$. Additionally, by Remark 4.6, the assertions of Lemma 4.5 hold for every $x \in \mathscr{V}$ (defined in (4.5)) that is not an equilibrium.

Lemma 5.4. *Let $u \in X$ and let γ_1 and γ_2 be (not necessarily distinct) stationary or periodic orbits in $\omega(u)$. Then, there exists a $k = k(\gamma_1, \gamma_2)$, $k \in \mathbb{Z}$, such that*

$$W(p^1, p^2) = k, \tag{5.3}$$

for every $p^j \in \gamma_j, p^1 \neq p^2$. In particular, the projections of disjoint periodic orbits are disjoint.

Proof. See [6, Lemma 3.3]. We consider the case where γ_1 and γ_2 are both periodic, the others are analogous or even simpler. We first claim that $W(p^1, p^2)$ is defined for every $p^1 \in \gamma^1$ and every $p^2 \in \gamma^2$ with $p^1 \neq p^2$. Suppose, by contradiction, that there exist $p^1 \in \gamma^1$ and $p^2 \in \gamma^2$ with $p^1 \neq p^2$ such that $(p^1, p^2) \in \Sigma \setminus \Delta$. Then, by Axiom (A4) and (A5) there exists an $\varepsilon_0 > 0$, such that $(\phi^\sigma(p^1), \phi^\sigma(p^2)) \notin \Sigma$ for every $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^{\sigma'}(p^1), \phi^{\sigma'}(p^2)) < W(\phi^\sigma(p^1), \phi^\sigma(p^2)), \tag{5.4}$$

for $\sigma' \in (0, \varepsilon_0)$ and $\sigma \in (-\varepsilon_0, 0)$. Set $\sigma' = \frac{\varepsilon_0}{2}$ and $\sigma = -\frac{\varepsilon_0}{2}$. By continuity of W there exists an $\eta \in (0, \frac{\varepsilon_0}{2})$ such that W is constant on the set

$$\mathcal{U} = \left\{ (\phi^{\sigma_1}(p^1), \phi^{\sigma_2}(p^2)) \mid -\tfrac{\varepsilon_0}{2} - \eta < \sigma_1, \sigma_2 < \tfrac{\varepsilon_0}{2} + \eta \right\}.$$

By periodicity of γ^1 and γ^2 there is a $\sigma_3 > \varepsilon_0$ such that $(\phi^{\sigma_3}(p^1), \phi^{\sigma_3}(p^2)) \in \mathcal{U}$ (both in the periodic and the quasi-periodic case). Now, by (5.4)

$$W(\phi^{\varepsilon_0/2}(p^1), \phi^{\varepsilon_0/2}(p^2)) < W(\phi^{-\varepsilon_0/2}(p^1), \phi^{-\varepsilon_0/2}(p^2)) = W(\phi^{\sigma_3}(p^1), \phi^{\sigma_3}(p^2)).$$

Since $\sigma_3 > \frac{\varepsilon_0}{2}$, this contradicts Lemma 4.1. Hence $(p^1, p^2) \notin \Sigma$ and $W(p^1, p^2)$ is well defined for every $p^1 \in \gamma^1$ and every $p^2 \in \gamma^2$, with $p^1 \neq p^2$.

This implies, by continuity of W, that the map

$$(p^1, p^2) \to W(p^1, p^2)$$

is locally constant on

$$\{(p^1, p^2) \in \gamma_1 \times \gamma_2 \mid p^1 \neq p^2\}.$$

This set is connected, which proves (5.3). $\square$

Lemma 5.5. *Let $u \in X$ and $e \in E$. For every $w \in \omega(u)$ with $w \neq e$ it holds $(w, e) \notin \Sigma$. If, furthermore, $e \neq \omega(u)$ then there exists a $\bar{\sigma} \in \mathbb{R}$ such that the map $\sigma \mapsto W(\phi^\sigma(u), e)$ is constant for $\sigma > \bar{\sigma}$.*

Proof. The proof resembles the one of Lemma 4.2. We repeat the argument. Let $w \in \omega(u)$. Since $w \neq e$, we can assume that $(w, e) \notin \Delta$. Suppose, by contradiction, that $(w, e) \in \Sigma \setminus \Delta$, then by Axioms (A4) and (A5), there exists an $\varepsilon_0 > 0$ such that $(\phi^\sigma(w), e) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^\sigma(w), e) > W(\phi^{\sigma'}(w), e),$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$. Set $\sigma = -\varepsilon$ and $\sigma' = \varepsilon$, with $0 < \varepsilon < \varepsilon_0$. Then we have

$$W(\phi^{-\varepsilon}(w), e) > W(\phi^{\varepsilon}(w), e). \tag{5.5}$$

By definition of the ω-limit set and the fact that ω is invariant, we have that there exists a sequence $\sigma_n \to \infty$, as $n \to \infty$ such that

$$\phi^{\sigma_n \pm \varepsilon}(u) \to \phi^{\pm\varepsilon}(w). \tag{5.6}$$

Since σ_n is divergent we assume that

$$\sigma_{n+1} > \sigma_n + 2\varepsilon, \quad \text{for all } n \in \mathbb{N}. \tag{5.7}$$

Inequality (5.5), convergence in (5.6) and Lemma 4.1 imply, for $\sigma_n \to \infty$, that

$$\begin{aligned} W(\phi^{\sigma_n+\varepsilon}(u), e) &= W(\phi^{+\varepsilon}(w), e) \\ &< W(\phi^{-\varepsilon}(w), e) \\ &= W(\phi^{\sigma_n-\varepsilon}(u), e). \end{aligned}$$

Combining the latter with (5.7) and the fact that W is non-increasing, we obtain

$$W(\phi^{\sigma_{n+1}-\varepsilon}(u), e) < W(\phi^{\sigma_n-\varepsilon}(u), e),$$

for all n. From this, we deduce that $\sigma \mapsto W(\phi^\sigma(u), e)$ has infinitely many jumps and therefore

$$W(\phi^\sigma(u), e) \to -\infty \quad \text{as} \quad \sigma \to \infty.$$

On the other hand, continuity of W and (5.6) imply, for $\sigma_n \to \infty$, that

$$W(\phi^{\sigma_n+\varepsilon}(u), e) = W(\phi^\varepsilon(w), e) > -\infty,$$

which is a contradiction.

To prove the final assertion, suppose, by contradiction, that such a $\bar{\sigma}$ does not exist. Then there exists a sequence $\sigma_n \to \infty$ such that $(\phi^{\sigma_n}(u), e) \in \Sigma$. Now choose a $w \in \omega(u) \setminus \{e\} \neq \varnothing$. There exists a sequence $\tilde{\sigma}_n \to \infty$ such that $\phi^{\tilde{\sigma}_n}(u) \to w$. By the first part of the lemma, $W(w, e) \in \mathbb{Z}$. We may choose $\tilde{\sigma}_n > \sigma_n$ without loss of generality. By continuity of W and axiom (A5) it follows that

$$W(w, e) = \lim_{n\to\infty} W(\phi^{\tilde{\sigma}_n}(u), e) = -\infty,$$

a contradiction. This concludes the proof. □

Lemma 5.6. *Let u be in X. There exists an integer $k_0 \in \mathbb{Z}$ such that*

$$W(w, e) = k_0 \tag{5.8}$$

for every $w \in \omega(u)$, and for every equilibrium $e \in \omega(u)$ such that $w \neq e$.

Proof. Fix $e \in E \cap \omega(u)$. Let $w \in \omega(u) \setminus \{e\}$. According to Lemma 5.5, $W(w, e)$ is well-defined. Since $\phi^{\sigma_n}(u) \to w$ for some $\sigma_n \to \infty$,

$$\begin{aligned} W(w, e) &= \lim_{n\to\infty} W(\phi^{\sigma_n}(u), e) \\ &= \lim_{\sigma\to\infty} W(\phi^\sigma(u), e) = k_e, \end{aligned}$$

where the second limit exists by Lemma 5.5. Since the above statement holds for any $w \in \omega(u) \setminus \{e\}$, this implies that $W(w, e)$ is independent of $w \in \omega(u) \setminus \{e\}$.

We still need to show that $W(w, e)$ is independent of $e \in E \cap \omega(u)$. Therefore let $e, \tilde{e} \in E \cap \omega(u), e \neq \tilde{e}$. Then, by Axiom (A1), by the fact that $e, \tilde{e} \in \omega(u)$, and by Lemma 5.5 it holds that

$$k_e = W(w, e) = W(\tilde{e}, e) = W(e, \tilde{e}) = W(w, \tilde{e}) = k_{\tilde{e}}.$$

This shows (5.8) and concludes the proof. □

5.2 Proof of the strong version

In this section we prove Propositions 3.3 and 3.4. This completes the proof of Theorem 3.1. We finish by proving Theorem 1.2 (in the abstract setting of Section 3).

Proof of Proposition 3.3. Let $u \in X$ and $w \in \omega(u)$. Suppose, *by contradiction*, that there is a non-equilibrium $w^* \in \omega(w)$ and that $\gamma(w)$ is not periodic. Lemma 4.3 implies that $\pi \circ \phi^\sigma$ is a planar flow on the set $\omega(w) \subseteq \mathrm{cl}(\gamma(w))$. By Corollary 4.4, the point $\pi(w^*)$ is not an equilibrium for $\pi \circ \phi^\sigma$. According to Lemma 4.5 there exist a section $\mathscr{C}$ through $\pi(w^*)$. Consider first $\pi \circ \phi^\sigma(w)$ and recall that by Lemma 4.3 the map $\sigma \to \pi \circ \phi^\sigma(w)$ is one-to-one since $\gamma(w)$ is not periodic. Let $\sigma_n \to \infty$ denote those positive times for which $\pi \circ \phi^{\sigma_n}(w) \in \mathscr{C}$. Note that $\{\pi \circ \phi^{\sigma_n}(w)\}_{n=1}^\infty$ are all distinct, and for n sufficiently large $y_1 = \pi \circ \phi^{\sigma_n}(w)$ and $y_2 = \pi \circ \phi^{\sigma_{n+1}}(w)$ both lie in $\mathscr{C}_0$. Denote $\widetilde{\sigma} = \sigma_{n+1} - \sigma_n$, so that $y_2 = \psi^{\widetilde{\sigma}}(y_1)$. We apply the construction of Remark 4.7(i) to these y_1 and y_2. In addition to $\mathscr{J}_0$ and $\mathscr{J}_\pm$ we obtain three more Jordan curves (the first two are the same as in the proof of Proposition 4.8)

$$\begin{aligned}
\mathscr{G}_- &= \{\psi^\sigma(y_1) : \sigma_1^- \leq \sigma \leq \widetilde{\sigma} + \sigma_2^-\} \cup \ell_- \\
\mathscr{G}_+ &= \{\psi^\sigma(y_1) : \sigma_1^+ \leq \sigma \leq \widetilde{\sigma} + \sigma_2^+\} \cup \ell_+ \\
\mathscr{G}_0 &= \{\psi^\sigma(y_1) : \sigma_1^0 \leq \sigma \leq \widetilde{\sigma} + \sigma_2^0\} \cup \ell_0.
\end{aligned}$$

These three curves separate $\mathbb{R}^2$ into two open sets, say A_j^1 and A_j^2, with $j \in \{-, 0, +\}$. To fix notation, we require that $J_0 \subset A_+^1$ and $J_0 \subset A_-^2$ and $J_+ \subset A_0^2$, see Figure 4. In particular, this implies that $A_+^2 \subset A_0^2 \subset A_-^2$ and $A_+^2 \cap A_-^1 = \varnothing$, as well as $A_+^2 \cap A_0^1 = \varnothing$ and $A_-^1 \cap A_0^2 = \varnothing$. It follows from the properties of J_0 and $J_\pm$ described in Remark 4.7(ii) and invariance of $\{\psi^\sigma(y_1) : \sigma_1^- \leq \sigma \leq \widetilde{\sigma} + \sigma_2^+\}$ that in *forward* time once a flow line is in A_+^2 it can never enter A_0^1, while in *backward* time once a flow line is in A_-^1 it can never enter A_0^2. We note that Remark 4.7(iii) implies that $x_1 = \pi \circ \phi^{\sigma_n - \delta}(w)$ lies in A_-^1, while $x_2 = \pi \circ \phi^{\sigma_{n+1}+\delta}(w)$ lies in A_+^2. We therefore conclude that, as illustrated in Figure 4, $\pi\omega(w) \subset A_0^2$, while $\pi\alpha(w) \subset A_0^1$, hence $\pi\omega(w) \cap \pi\alpha(w) = \varnothing$. We infer from Lemma 4.3 that $\alpha(w) \cap \omega(w) = \varnothing$.

Now consider the orbit of u. The assumptions of Lemma 5.2 are satisfied and hence there exists a time σ_*, such that the curve $\{\pi \circ \phi^\sigma(u) : \sigma \geq \sigma_*\}$ cannot cross the curve $\pi \circ \phi^\sigma(w)$. Furthermore, it follows from Remarks 4.6

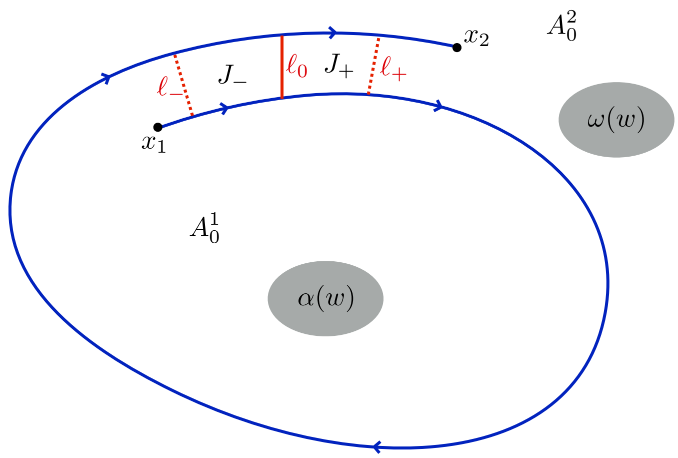

Figure 4: Sketch of the construction of A_0^1 and A_0^2. Note that $J_+ = A_-^2 \cap A_0^2$ and $J_- = A_+^1 \cap A_0^1$.

and 5.3 and the above construction, that once the flow line $\pi \circ \phi^\sigma(u)$ is in A_+^2 it can never enter A_0^1 (in forward time). Moreover, by Remark 4.7(ii), once a flow line is in A_0^2 then it must enter A_+^2 in forward time, after which it can no longer enter A_0^1. On the other hand, since both $\omega(w)$ and $\alpha(w)$ are contained in $\omega(u)$, the forward orbit $\pi \circ \phi^\sigma(u)$ will have ω-limit points when $\sigma \to \infty$ in both $\pi\alpha(w) \subset A_0^1$ and $\pi\omega(w) \subset A_0^2$. This is a contradiction. □

Proof of Proposition 3.4. See [6, Proposition 2]. Suppose that $\omega(u)$ strictly contains a periodic orbit $\gamma(p)$. Let $V \subseteq X$ be a closed tubular neighborhood of $\gamma(p)$. Choose V small enough such that it does not contain equilibria and such that $\omega(u)$ still has elements outside V. Since there are accumulation points (for $\phi^\sigma(u)$ when σ goes to infinity) both inside and outside V, then $\phi^\sigma(u)$ must enter and leave V infinitely often. Let $\sigma_n \to \infty$ be a sequence such that

$$p = \lim_{n\to\infty} \phi^{\sigma_n}(u)$$

and such that $\phi^\sigma(u)$ leaves V between any two consecutive times σ_n. Let $I_n := [\sigma_n - \alpha_n, \sigma_n + \beta_n]$ be the maximal time interval containing σ_n such that

$$\phi^\sigma(u) \in V \quad \text{for all } \sigma \in I_n.$$

Since ∂V is closed, we may assume convergence (passing to a subsequence, if necessary) of $\phi^{\sigma_n-\alpha_n}(u)$. Note that $\sigma_{n-1} < \sigma_n - \alpha_n$ thus $\sigma_n - \alpha_n \to \infty$. Let

$$q := \lim_{n\to\infty} \phi^{\sigma_n-\alpha_n}(u) \in \omega(u).$$

We have that $q \in \partial V$. Moreover we may assume that $\alpha_n + \beta_n \to \infty$ (at least for a subsequence) since $\omega(u)$ contains a periodic orbit in the interior of V. We have thus

$$\omega(q) \subseteq \mathrm{cl}(\phi^\sigma(q)) \subseteq V, \ \ \sigma > 0.$$

By Proposition 3.3 we have that $\gamma(q)$ is periodic. By construction $\gamma(q)$ and $\gamma(p)$ are distinct and $\gamma(q)$ is contained in V. By continuity of the flow and the projection π, and compactness of $V, \pi\gamma(p)$ and $\pi\gamma(q)$ are close to each other with the standard topology of $\mathbb{R}^2$, provided that we take the tubular neighborhood V sufficiently small. From this it follows that $\pi\gamma(q)$ and $\pi\gamma(p)$ are nested closed curves. Reducing V to separate $\gamma(p)$ from $\gamma(q)$, a periodic solution $\gamma(r)$ can be constructed in the same way. Note once more that $\pi\gamma(q), \pi\gamma(p)$ and $\pi\gamma(r)$ are nested closed curves. Applying Lemma 5.4 to the trajectories $\gamma(p)$ and $\gamma(q)$ we conclude that there exists a $k \in \mathbb{Z}$ such that

$$W(p^1, q^1) = k,$$

for all $p^1 \in \gamma(p)$ and $q^1 \in \gamma(q)$. By continuity of W (Axiom (A1)) this implies that

$$W(p^1, \phi^{\sigma_n-\alpha_n}(u)) = k,$$

for all $p^1 \in \gamma(p)$ when n is big enough, since $\phi^{\sigma_n-\alpha_n}(u) \to q \in \gamma(q)$. By Assumption (A5) we get $\pi \circ \phi^\sigma(u) \notin \pi\gamma(p)$ for every σ in the open interval with endpoints $\sigma_n - \alpha_n, \sigma_m - \alpha_m$, provided n, m are chosen large enough. Since $\sigma_m - \alpha_m \to \infty$, as $m \to \infty$, it follows that $\pi \circ \phi^\sigma(u) \notin \pi\gamma(p)$, for any σ large enough. In an analogous manner we can prove that, for σ large enough, the curve $\pi \circ \phi^\sigma(u)$ can never intersect $\pi\gamma(q)$ and $\pi\gamma(r)$, but this is a contradiction since $\pi \circ \phi^\sigma(u)$ has ω-limit points as $\sigma \to \infty$ in the three nested curves $\pi\gamma(p), \pi\gamma(q), \pi\gamma(r)$. $\square$

Proposition 5.7. *Let $u \in X$ then*

$$\pi : \omega(u) \to \pi(\omega(u))$$

is a homeomorphism onto its image. Hence $\pi \circ \phi^\sigma$ is a flow on $\pi(\omega(u))$.

Proof. See also [6, Theorem 2]. By Axiom (A5) it is enough to show that there exists a $k_0 \in \mathbb{Z}$ such that

$$W(w^1, w^2) = k_0, \tag{5.9}$$

for all $w^1, w^2 \in \omega(u), w^1 \neq w^2$. We now apply Theorem 3.1 (Poincaré-Bendixson). If $\omega(u)$ consists of a single periodic orbit, then (5.9) holds by Lemma 5.4. We may therefore assume for the remainder of the proof that for every $w \in \omega(u)$ we have $\alpha(w), \omega(w) \subseteq E$. If either w^1 or w^2 is an equilibrium then (5.9) holds with k_0 defined in Lemma 5.6. We may therefore assume that $w^1 \notin E$. Suppose now, by contradiction, that there exist $(w^1, w^2) \in \Sigma \setminus \Delta$. By Axioms (A4) and (A5), there exists an $\varepsilon_0 > 0$, such that $\big(\phi^\sigma(w^1), \phi^\sigma(w^2)\big) \notin \Sigma$, for all $\sigma \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ and

$$W(\phi^{\sigma'}(w^1), \phi^{\sigma'}(w^2)) < W(\phi^\sigma(w^1), \phi^\sigma(w^2))$$

for all $\sigma \in (-\varepsilon_0, 0)$ and all $\sigma' \in (0, \varepsilon_0)$. Set $\sigma = -\varepsilon$ and $\sigma' = \varepsilon$, with $0 < \varepsilon < \varepsilon_0$. Since $w^1 \in \omega(u)$, there exists $\sigma_n \to \infty$ such that

$$w^1 = \lim_{n\to\infty} \phi^{\sigma_n}(u),$$

and

$$0 < \sigma_{n+1} - \sigma_n \to \infty, \text{ as } n \to \infty.$$

Define $\hat{\sigma}_n := (\sigma_{n+1} - \sigma_n) \to \infty$ then, passing to a subsequence if necessary, the limits

$$e := \lim_{n\to\infty} \phi^{-\hat{\sigma}_n}(\phi^{-\varepsilon}(w^2)) \quad \text{and} \quad \tilde{e} := \lim_{n\to\infty} \phi^{\hat{\sigma}_n}(\phi^{\varepsilon}(w^2))$$

exist, and $e, \tilde{e} \in E$, since $\alpha(w^2) \subseteq E$ and $\omega(w^2) \subseteq E$. By Axiom (A1), Lemma 4.1, Lemma 5.6 and the fact that $w^1 \notin E$ we infer that, for n sufficiently large (slightly shifting ε if necessary to make W well-defined for all relevant pairs)

$$\begin{aligned}
W(\phi^{\varepsilon}(w^1), \phi^{\varepsilon}(w^2)) &< W(\phi^{-\varepsilon}(w^1), \phi^{-\varepsilon}(w^2)) \\
&= W(\phi^{\sigma_{n+1}-\varepsilon}(u), \phi^{-\varepsilon}(w^2)) \\
&\leq W(\phi^{\sigma_{n+1}-\hat{\sigma}_n-\varepsilon}(u), \phi^{-\hat{\sigma}_n-\varepsilon}(w^2)) \\
&= W(\phi^{\sigma_n-\varepsilon}(u), e) \\
&= W(\phi^{-\varepsilon}(w^1), e) \\
&= W(\phi^{-\varepsilon}(w^1), \tilde{e}) \\
&= W(\phi^{\varepsilon}(w^1), \tilde{e}) \\
&= W(\phi^{\sigma_{n+1}+\varepsilon}(u), \tilde{e}) \\
&= W(\phi^{\hat{\sigma}_n+\sigma_n+\varepsilon}(u), \phi^{\hat{\sigma}_n+\varepsilon}(w^2)) \\
&\leq W(\phi^{\sigma_n+\varepsilon}(u), \phi^{\varepsilon}(w^2)) \\
&= W(\phi^{\varepsilon}(w^1), \phi^{\varepsilon}(w^2)),
\end{aligned}$$

which is a contradiction. In the sixth and in the seventh equality we used Lemma 5.6. □

Since the Cauchy-Riemann Equations satisfy the Axioms (A1)-(A5) Theorem 1.2 follows from Proposition 5.7.

6 Proofs of Propositions 2.1 and 2.3

In this section we give the proofs of Propositions 2.1 and 2.3. Consider the operators

$$\partial = \partial_s - J\partial_t \text{ and } \overline{\partial} = \partial_s + J\partial_t,$$

and recall the following regularity estimates:

Lemma 6.1. *Let g be a function in $\in C_c^\infty(\mathbb{R} \times S^1; \mathbb{R}^2)$. For every $1 < p < \infty$, there exists a constant $C_p > 0$, such that*

$$||\nabla g||_{L^p(\mathbb{R}\times S^1)} \leq C_p ||\bar{\partial} g||_{L^p(\mathbb{R}\times S^1)}. \tag{6.1}$$

The same estimate holds for ∂ via $t \mapsto -t$.

Proof. See [1], [5] [8], [9, appendix B]. □

Proof of Proposition 2.1. For a solution $u \in X$, we can write

$$\overline{\partial} u = -JF(t, u) = f(s, t), \tag{6.2}$$

where F, and therefore f, are uniformly bounded since for every $u \in X$ we have $|u(s,t)| \leq 1$ for all $(s,t) \in \mathbb{R} \times S^1$, i.e. u satisfies the a priori estimate

$$||u||_{L^\infty(\mathbb{R}\times S^1)} \leq 1. \tag{6.3}$$

Extend f and u via periodic extension in the t-direction to a function on $\mathbb{R}^2$. By (6.3) we obtain the existence of a constant $M > 0$, such that

$$||f||_{L^\infty(\mathbb{R}^2)} \leq M. \tag{6.4}$$

We use (6.1) to obtain the interior regularity estimates for the Cauchy-Riemann operators.

Let K, L, G be compact sets contained in $\mathbb{R}^2$ such that $K \Subset L \Subset G \subset \mathbb{R}^2$, and let ε be positive such that $\varepsilon < \operatorname{dist}(L, \partial G)$. By compactness, L can be covered by finitely many open balls of radius $\varepsilon/2$:

$$L \subset \bigcup_{i=1}^{N_\varepsilon} B_{\varepsilon/2}(x_i).$$

Consider a partition of unity $\{\rho_{\varepsilon,x_i}\}_{i=1,\ldots,N_\varepsilon}$ on L subordinate to $\{B_\varepsilon(x_i)\}_{i=1,\ldots,N_\varepsilon}$. In particular the supports of ρ_{ε,x_i} are contained in $B_\varepsilon(x_i)$, for every $i = 1 \ldots N_\varepsilon$. Then, for every u, every small $\varepsilon > 0$ and every $i = 1 \ldots N_\varepsilon$, the function $v_{\varepsilon,i} := \rho_{\varepsilon,x_i} u$ belongs to $W_0^{k,p}(\mathbb{R}^2)$, for every $p \geq 1$, and every $k \in \mathbb{N}$. Using the Poincaré inequality and Lemma 6.1 we get (with C changing from line to line)

$$\begin{aligned} ||v_{\varepsilon,i}||_{W^{1,p}(\mathbb{R}^2)} &= ||v_{\varepsilon,i}||_{W^{1,p}(B_\varepsilon(x_i))} \leq C||v_{\varepsilon,i}||_{W_0^{1,p}(B_\varepsilon(x_i))} \\ &\leq C||\overline{\partial} v_{\varepsilon,i}||_{L^p(B_\varepsilon(x_i))} \\ &\leq C||\rho_{\varepsilon,x_i}\overline{\partial} u||_{L^p(B_\varepsilon(x_i))} + C||u\overline{\partial}\rho_{\varepsilon,x_i}||_{L^p(B_\varepsilon(x_i))} \\ &\leq C||\overline{\partial} u||_{L^p(G)} + C||u||_{L^p(G)}. \end{aligned} \tag{6.5}$$

As $\{\rho_{\varepsilon,x_i}\}_{i=1,\ldots,N_\varepsilon}$ is a partition of unity it follows that

$$||u||_{W^{1,p}(L)} = \left|\left|\sum_{i=1}^{N_\varepsilon} v_{\varepsilon,i}\right|\right|_{W^{1,p}(L)} \leq \sum_{i=1}^{N_\varepsilon} ||v_{\varepsilon,i}||_{W^{1,p}(B_\varepsilon(x_i))}. \tag{6.6}$$

By (6.5) and (6.6) we obtain

$$||u||_{W^{1,p}(L)} \leq C_{p,L,G}\left(||\overline{\partial} u||_{L^p(G)} + ||u||_{L^p(G)}\right). \tag{6.7}$$

Combining (6.7) with (6.2), (6.3) and (6.4) yields

$$||u||_{W^{1,p}(L)} \leq C_{p,L,G}\left(||f||_{L^p(G)} + ||u||_{L^p(G)}\right) \leq C^1_{p,L,G}, \tag{6.8}$$

where the constant $C^1_{p,L,G}$ depends on p, L, G, but not on u. By the Sobolev compact embedding $W^{1,p}(L) \hookrightarrow C^0(L)$, cf. [2], which implies that sequences $\{u^n\} \subset X$ have convergent subsequences in $C^0_{\text{loc}}(L)$. Since the latter holds for every $L \subset \mathbb{R}^2$, the convergence is in $C^0_{\text{loc}}(\mathbb{R}^2)$, and the limit u is a continuous function. It remains to show that the limit u solves Equation (1.2). Consider a partition of unity of $K \Subset L$, denoted $\{\rho_{\varepsilon,x_i}\}_{i=1,\dots N_\varepsilon}$, where $0 < \varepsilon < \operatorname{dist}(K, \partial L)$. On balls $B_\varepsilon(x_i)$ we obtain

$$\begin{aligned}
||\rho_{\varepsilon,x_i} u||_{W^{2,p}(B_\varepsilon)} &\leq C||\rho_{\varepsilon,x_i} u||_{W^{2,p}_0(B_\varepsilon(x_i))} \leq C||\overline{\partial}(\rho_{\varepsilon,x_i} u)||_{W^{1,p}(B_\varepsilon(x_i))} \\
&\leq C\left(||\rho_{\varepsilon,x_i}\overline{\partial}u||_{W^{1,p}(B_\varepsilon(x_i))} + ||u\overline{\partial}\rho_{\varepsilon,x_i}||_{W^{1,p}(B_\varepsilon(x_i))}\right) \\
&\leq C\left(||\overline{\partial}u||_{L^\infty(L)} + ||\overline{\partial}u||_{W^{1,p}(L)} + ||u||_{L^\infty(L)} + ||u||_{W^{1,p}(L)}\right).
\end{aligned}$$

As in (6.6), using (6.2), we obtain

$$||u||_{W^{2,p}(K)} \leq \tilde{C}_{p,K,L,G}\left(||f||_{L^\infty(L)} + ||f||_{W^{1,p}(L)} + ||u||_{L^\infty(L)} + ||u||_{W^{1,p}(L)}\right). \tag{6.9}$$

To estimate the three terms $||f||_{L^\infty(L)}$, $||u||_{L^\infty(L)}$ and $||u||_{W^{1,p}(L)}$ we use (6.3), (6.4), and (6.8). In order to control $||f||_{W^{1,p}(L)}$, differentiate the smooth vector field F:

$$\begin{aligned}
f_s(s,t) &= (F(t,u))_s = D_{t,u}X(t,u)(0,u_s) \\
f_t(s,t) &= (F(t,u))_t = D_{t,u}X(t,u)(1,u_t).
\end{aligned}$$

Both right-hand sides lie in $L^p(L)$ and hence $Df = (f_s, f_t)$ is in $L^p(L)$. By (6.9) there exists a constant $C^2_{p,K,L,G}$, independent of u, such that

$$||u||_{W^{2,p}(K)} \leq C^2_{p,K,L,G}.$$

By taking $p > 2$ the compact Sobolev embedding $W^{2,p}(K) \hookrightarrow C^1(K)$ implies that $u \in X$. □

Proof of Proposition 2.3. As in the proof of Lemma 4.3 if suffices to show that ι is injective. Suppose there exist $u_1, u_2 \in X$ such that $\iota(u_1) = \iota(u_2)$. By definition of ι we have

$$u_1(0,\cdot) = u_2(0,\cdot). \tag{6.10}$$

Define $v(s,t) := u_1(s,t) - u_2(s,t)$ for all $(s,t) \in \mathbb{R} \times S^1$. By (6.10) we have $v(0,t) = 0$ for all $t \in S^1$. By smoothness of the vector field F we can write

$$F(t,u_1) = F(t,u_2) + R(t,u_1,u_2-u_1)(u_2-u_1),$$

where R is a smooth function of its arguments. Upon substitution this gives

$$v_s - Jv_t + A(s,t)v = 0, \quad v(0,t) = 0 \quad \text{for all } t \in S^1, \tag{6.11}$$

and $A(s,t) = R(t,u_1(s,t),v(s,t))$ is (at least) continuous on $\mathbb{R} \times S^1$. Evaluating (6.11) at $t = 0$ we obtain,

$$v_s - Jv_t + A(s,t)v = 0, \quad v(0,0) = 0. \tag{6.12}$$

Introducing complex coordinates $z := s + it$, (6.12) becomes

$$\partial_{\overline{z}} v + A(z)v = 0, \quad v(0) = 0, \tag{6.13}$$

where the operator $\partial_{\overline{z}} := \partial_s - i\partial_t$ is the standard anti-holomorphic derivative. We used the identification between the complex structure J in $\mathbb{R}^2$ and i in $\mathbb{C}$. Multiplying (6.13) by $e^{\int_0^z A(\zeta)d\zeta}$ and defining

$$w(z) := e^{\int_0^z A(\zeta)d\zeta} v(z),$$

gives

$$\partial_{\overline{z}} w = 0, \quad w(0) = 0,$$

which implies that w is analytic. The latter yields that either 0 is an isolated zero for w, or there exists a $\delta > 0$, such that $w(z) = 0$, on $U_\delta := \{z \in \mathbb{C} : |z| \leq \delta\}$. By (6.11) we conclude that 0 cannot be an isolated zero for w, hence $w \equiv 0$ in $U_\delta := \{z \in \mathbb{C} : |z| \leq \delta\}$. Repeating these arguments we obtain that $w(s,t) = 0$ for all $(s,t) \in \mathbb{R} \times S^1$ and hence $v \equiv 0$. This implies $u_1 = u_2$, which concludes the proof. □

Remark 6.2. The same proof can be carried out in case J is a smooth map $\mathbb{R} \times S^1 \to \mathrm{Sp}(2,\mathbb{R})$ such that $J^2 = -\mathrm{Id}$. In this case one can prove that the equation $u_s - J(s,t)(u_t - F(t,u)) = 0$ can be transformed into (1.2) using [8, Theorem 12, Appendix A.6].

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