Validated Numerical Approximation of Stable Manifolds for Parabolic Partial Differential Equations

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Abstract

This paper develops validated computational methods for studying infinite dimensional stable manifolds at equilibrium solutions of parabolic PDEs. Our approach is constructive and combines the parameterization method with Lyapunov-Perron operators. More precisely, we decompose the stable manifold into three components: a finite dimensional slow component, a fast-but-finite dimensional component, and a strongly contracting infinite dimensional "tail". We employ the parameterization method in a finite dimensional projection to approximate the slow-stable manifold. We also parameterize attached invariant vector bundles describing the unstable and fast-but-finite dimensional stable directions in a tubular neighborhood of the slow stable manifold. Taken together the slow manifold parameterization and the attached invariant vector bundles provide a change of coordinates which largely removes the nonlinear terms in the slow stable directions. This facilitates application of the Lyapunov-Perron method in the resulting adapted coordinate system, leading to mathematically rigorous error bounds on the approximation errors. By using the parameterization method in the slow stable directions we obtain bounds valid in a larger neighborhood of the equilibrium than would be obtained using only the linear approximation given by the eigendirections. As a concrete example we illustrate the technique for a 1D Swift-Hohenberg equation.

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Contents

1 Introduction

When viewed as ODEs on Banach spaces, nonlinear parabolic PDEs are amenable to classical tools from dynamical systems theory. Theorems regarding the stability of equilibria, periodic orbits, and their attached invariant manifolds follow in analogy with the finite dimensional case. Connecting orbits between these invariant sets serve as a kind of a road map to the global dynamics, illuminating transitions between distinct regions of the phase space and signaling global bifurcations. Such orbits are main ingredients in forcing theorems like those of Smale and Shilnikov which guarantee the existence of rich dynamics, and are essential for defining geometric chain groups and boundary operators in the homology theories of Witten and Floer.

Precisely because of their global and nonlinear nature, connecting orbits are difficult to work with analytically. These difficulties are compounded in infinite dimensional settings. In specific applications researchers typically perform numerical calculations to gain insights into the properties of these important objects. Recent progress in computer-assisted methods of proof for infinite dimensional systems brings the mathematically rigorous quantitative study of connecting orbits for PDEs within the realm of possibility. For the case of a connection from a saddle to an attracting equilibrium, we refer to the work of [17, 46]. The works just mentioned study the finite dimensional unstable manifold of an equilibrium and develop mathematically rigorous tools for extending this manifold into a trapping neighborhood of a sink. In both studies the authors obtain explicit and mathematically rigorous bounds on the basin of attraction of the sink – an open set.

Controlling the asymptotic behavior of a connecting orbit requires an explicit description of the local stable and unstable manifolds of the limiting invariant sets. The major obstacle to extending the methods in [17, 46] to the general saddle-to-saddle case is obtaining an explicit description of the local stable manifold. It is worth mentioning that rigorous numerical integration of PDEs is a nontrivial task and invariably suffers from the so called wrapping effects resulting from the accumulation of numerical error. Consequently, in computer assisted arguments involving connecting orbits it is desirable to minimize integration time by absorbing as much of the connecting orbit in the local stable manifolds as possible.

We refer the interested reader also to the related work of [18], where saddle-to-saddle connections are established using topological methods based on Conley Index theory and its connection matrix. Being topological in nature these methods require much less in the way of C^1 information, resulting in a softer description of the dynamics. The challenge in applying these methods is the rigorous calculation of index information for macroscopic regions in the infinite dimensional phase space.

In this paper we develop a novel method for representing the *infinite dimensional* stable manifold of an equilibrium solution of a parabolic PDE. Our representation is valid in a large and explicitly prescribed neighborhood of the equilibrium, and our main goal is to obtain explicit a-posteriori bounds on all truncation and discretization errors. This goes far beyond using the stable subspace as a linear approximation of the manifold, by providing theorems that guarantee explicit estimates on the accuracy of the computed approximation.

A feature of the method is that we use approximations that improve on the tangent subspace in certain directions, namely those along which one expects connecting orbits to approach the equilibrium. We thus realize a description of the manifold that has improved accuracy in those directions which capture connecting orbits. The main difficulty to overcome is the nonuniformity of the decay in the stable manifold. Indeed, while finitely many eigenvectors of the linearized flow about an equilibrium may be approximated with high accuracy and precision, an infinitude shall always remain. We need to control the interaction of error terms corresponding to starkly different decay rates, which we accomplish by iteratively bootstrapping Gronwall's inequality for systems of inequalities.

The present work grows out of the thriving literature on methods of computer assisted proof for finite dimensional dynamical systems going back to the first proofs of the Feigenbaum conjectures [35, 36, 21, 20], the first proofs of chaotic motions in the Lorenz equations [39, 41, 40, 24] and for Chua's circuit [23], as well as the computer assisted resolution of Smale's 14th problem [50, 51]. In particular, we build on the solid foundations of computer assisted proofs for studying equilibrium solutions of parabolic PDEs and their stability. A thorough review is a task beyond the scope of the present work and we refer the reader to the work of [42, 44, 45, 64, 54, 25, 4, 1, 2, 57] for an introduction to the literature. We refer also to the book of [43] and to the recent review article [55].

A number of techniques for computer assisted proofs involving finite dimensional invariant manifolds have emerged from this literature. One family of methods for proving existence of unstable manifolds involves checking a number of geometric covering and cone conditions near the equilibrium in the same spirit as Fenichel theory [63, 11, 12]. Since time reversal is well defined for ODEs, equivalent bounds for stable manifolds follow as a trivial corollary. Applications of these methods to the study of stable manifolds for PDEs requires substantial modification and have – to the best of our knowledge – not yet appeared in the literature. We refer the interested reader to the recent work of [62] where, following [39, 41, 40, 24, 23], the authors bypass consideration of stable/unstable manifolds and provide a direct-computer assisted proof of the existence of a geometric horseshoe in the Kuramoto-Sivashinsky equation by studying covering relations in a Poincaré section.

Another technique for obtaining validated bounds on invariant manifolds which has been applied successfully in a number of finite dimensional setting is the parametrization method [8, 9, 10], see also to the book [28] for detailed discussions of the method and its applications. Briefly, the idea is to study a conjugacy equation between the dynamics on the manifold and the linear dynamics in an eigenspace. The conjugacy equation is reduced to a set of linear homological equations via recursive power matching, and one obtains a high order Taylor expansions for the manifold, as well as remainder estimates on the truncation errors in the tail of the series. This method recovers both the embedding of the manifold and the dynamics on it, and is very effective for representing invariant manifolds far beyond a small neighborhood of the equilibrium, periodic orbit, or invariant torus, where the linear approximation has validity. There is a substantial literature devoted to validated numerics based on the parameterization method for invariant manifolds of ODEs. We refer the interested reader to see the works of [3, 32, 6, 56, 13, 38] for more a complete discussion. Such methods have also been extended for studying finite dimensional invariant manifolds of infinite dimensional systems. The case of compact infinite dimensional maps is treated in [37], the case of PDEs is studied in [46], and DDEs are considered in [26, 29]. However, there is an obstruction to applying this technique to infinite dimensional manifolds in PDEs, which is that solving the homological equations requires certain non-resonance conditions between the eigenvalues of the equilibrium. There are techniques to deal with the case of having a finite number of resonant eigenvalues [8, 53]. Nonetheless, to describe an infinite dimensional manifold one will have an infinite number of resonance conditions to check, which seems to be a major obstruction.

There are principally two widespread approaches to the study of infinite dimensional invariant manifolds in Banach spaces: the graph transform method (e.g. see [5]) and the Lyapunov-Perron method (e.g. see [16]). We refer to [22, Section 1.4] for a comparison of these methods, but we mention that the graph transform method is most natural in a discrete time dynamical system. In [19] this method was applied in a computer assisted proof setting to study the dynamics of compact infinite dimensional maps generated by convolution against a smooth kernel, and was significant motivation for the present work. The graph transform method may be applied to continuous time systems by considering the implicitly defined time-1 map generated by the semi-flow. In the present work we have opted to work with the Lyapunov-Perron method, as it allows us to work with the vector field more explicitly.

Let us briefly outline our approach, which is somewhat involved due to our desire to track error control explicitly. While the abstract theory for invariant manifolds in Banach spaces has certainly been well developed, there are some obstacles preventing one from directly applying such theory to obtain a computer assisted proof. One complication stems from the fact that in a given application one generally does not have explicit formulas for the equilibrium nor the eigendecomposition of the linearized operator, but only approximations. A second difficulty concerns localizing the estimates, which is needed since the nonlinearities are not globally Lipschitz. Finally, applying generic functional analytic estimates usually leads to bounds that are valid in an inconveniently small neighborhood of the equilibrium only.

To overcome these barriers, before setting up the Lyapunov-Perron fixed point operator, we decompose the space into judiciously chosen subspaces, corresponding to approximate eigenspaces of the linearization at the equilibrium. In particular, we choose an approximation of the (finite dimensional) unstable subspace, and we split the approximate stable space into a finite dimensional stable part, corresponding to the small negative eigenvalues, and an infinite dimensional stable part. As a subtle refinement we will also consider a further decomposition of the finite dimensional stable part into a slow-stable and a fast-stable subspace.

The Lyapunov-Perron operator acts on functions α which map (an approximation of) the linear stable eigenspace to the (approximate) unstable subspace. The domains of the functions α are restricted to a product neighborhood of the equilibrium which respects the decomposition of the stable space. To localize our analysis we perform an explicit (potentially nonlinear) change of coordinates. This gives us flexibility in choosing the validity region of the parametrization.

To show that the Lyaponuv-Perron map is a contraction we need explicit bounds on the nonlinearities (or rather their projections onto the subspaces of the decomposition). This involves explicit tracking of the coordinate changes, and more importantly one needs to control errors due to finite truncation. Indeed, the coordinate transformations we work with are explicit finite dimensional approximations of inherently infinite dimensional objects. Moreover, to obtain effective bounds, i.e. ones that guarantee contraction for functions defined on a reasonably large neighborhood of the equilibrium, a naive Gronwall estimate does not suffice. Instead we take a more refined approach in which we bootstrap a system of Gronwall inequalities (roughly, decomposed along eigendirections) to benefit from the different decay rates in different directions. As mentioned before, our main aim for the future is for these local stable manifolds to act as stepping stones for computer-assisted proofs of connecting orbits. Since it is well known that $C^{1,1}$ bounds are usually needed for such constructions, this is the regularity class that we have chosen to work with.

After detailing the general framework, we illustrate its efficacy by applying the methodology to stable manifolds of nonhomogeneous equilibria of the Swift-Hohenberg PDE

$$u_t = -\beta_1 u_{xxxx} + \beta_2 u_{xx} + u - u^3, \tag{1}$$

posed on a one-dimensional spatial domain $x \in [0, \pi]$ with Neumann boundary conditions

$$u_x(0) = u_x(\pi) = 0 \qquad \text{and} \qquad u_{xxx}(0) = u_{xxx}(\pi) = 0.$$

The parameters of the problem are $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$. For comparison, we illustrate the use of our method after both a linear and a nonlinear change of variables. These are respectively centered about a linear and a nonlinear approximation of a particular equilibrium's stable manifold. As a result, we obtain stable manifold theorems of contrasting accuracy. For example, in Theorem 6.4 we prove that our linear approximation is 3.36×10^{-3} close to the true stable manifold, whereas our non-linear approximation in Theorem 7.1 is much more accurate: the size of the error is only 7.43×10^{-12} away. More details are provided in Section 7.6. In this particular example the manifold has co-dimension 1, but the method is applicable to (un)stable manifolds of any co-dimension.

The nonlinear change of variables is based on [52]. Namely, we compute a high order parameterization of the approximate slow stable manifold (see Figure 1) and attached invariant bundles (see Figure 2) to define the coordinate frame. It illustrates (even better than in the linear case) how the decomposition into product spaces and the associated bootstrap estimates lead to a description of the infinite dimensional local stable manifold that has a generous validity region.

The computational framework developed here is rather general, and can likely be used to describe invariant manifolds in a variety of circumstances. Typical examples we have in mind are (un)stable and center-(un)stable manifolds in delay differential equations and partial differential equations on domains in $\mathbb{R}^n$, as well as stable and unstable manifolds in strongly indefinite problems, where both the dimension and the co-dimension of the manifold are infinite dimensional (e.g. [14]). In [49] this methodology is used to construct part of a co-dimension 0 center-stable manifold of a homogeneous equilibrium in a complex-valued nonlinear heat equation. We are currently working on combining the description of the infinite dimensional stable manifold with a parametrization of unstable manifolds and a rigorous integrator to study transverse connecting orbits in parabolic PDEs. In future work, we also intend to extend our results to hyperbolic periodic orbits in PDEs.

The outline of the paper is as follows. In Section 2 we discuss the notation to be used in this paper, and the level of generality to be considered. Abstractly, we assume that our approximate (un)stable eigenspaces are decomposed into further subspaces, with (potentially) different time scales. This corresponds to our plan to develop distinct methods of approximation along the slow-stable, fast-but finite-stable, and infinite-stable eigenvalues. We intend to compute $C^{1,1}$ bounds on our manifold, and here we define a number of constants relating to our nonlinearity $\mathcal{N}$.

In Section 3 we discuss how we explicitly bootstrap Gronwalls inequality to get component-wise bounds on the exponential tracking problem. This iterative bootstrapping of Gronwall's inequality

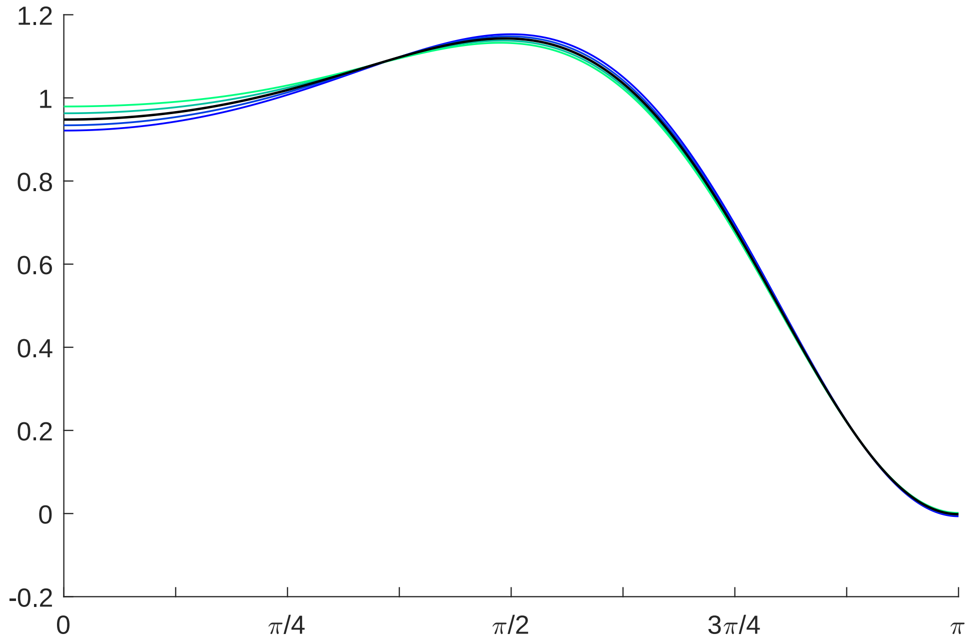

Figure 1: A verified (numerical approximation of an) unstable equilibrium (in black) for the Swift-Hohenberg PDE (1) with $\beta_1 = 0.05$ and $\beta_2 = -0.35$ and several (numerical approximations of) points along its verified slow stable manifold. Near this slow stable manifold we find a description of part of the full, co-dimension 1, stable manifold of the equilibrium. We remark that the validated bounds obtained near the slow-stable manifold are on the order of 10^{-12}. To obtain bounds this good using the linear approximation we would have to restrict to a neighborhood with size on the order of 10^{-6}. In a picture on the scale shown here it would have been impossible to distinguish the verified manifolds from the equilibrium. More details can be found in Sections 6.3 and 7.6.

is described in Algorithm 3.11. The approach is quite versatile, and we apply the same procedure several times in different scenarios. A general description for where this approach can be taken is described in Algorithm A.5.

In Section 4 we discuss the Lyapunov-Perron Operator Ψ, which is given in Definition 2.8. We formulate conditions for when Ψ maps a ball of $C^{0,1}$ functions into itself in Theorem 4.2, and for when Ψ maps a ball of $C^{1,1}$ functions into itself in Theorem 4.11.

In Section 5 we obtain the necessary estimates to show that the Lyapunov-Perron Operator is a contraction mapping. In Definition 5.2 we define a norm in which we wish to prove we have a contraction mapping. We then give conditions for when we have a contraction in Theorem 5.9, and the results of Sections 3–5 are summarized in Theorem 5.11.

In Section 6 we apply our results to the Swift-Hohenberg equation, obtaining the appropriate estimates for a linear change of variables at a nonlinear equilibrium. Finally in Section 7 we discuss how to get the estimates to work using a nonlinear change of coordinates at a nontrivial equilibrium. Computer assisted proofs of a stable manifold theorem using a linear approximation and a nonlinear approximation are given in Theorem 6.4 and Theorem 7.1 respectively, and the source code is available online [60].

2 Background and Notation

When studying an equilibrium of a differential equation, often the first thing one does is perform a change of coordinates taking the equilibrium to zero and, if possible, choose a basis aligned with the eigenvectors of the equilibrium's linearization. However, in general it is impossible to obtain an explicit representation of nontrivial equilibria. In ODEs, this may be due to the finite precision of a computer's floating point arithmetic. In PDEs, this may be due to only being able to compute finitely many terms in a series solution (or another finite dimensional truncation).

Nevertheless, by using an approximate solution and an approximate eigenbasis, we can make an explicit change of coordinates which well positions us for studying the dynamics about our equilibrium. Conceptually, we wish to begin our analysis *after* this change of coordinates, beginning with a set of coordinates where the origin is an approximate equilibrium, and our coordinates are an approximate eigenbasis. In Sections 6 and 7 we describe how to make this change of coordinates explicit in the 1D Swift-Hohenberg equation.

2.1 Parabolic PDEs and Semigroup Operators

Let us fix a Banach space X with norm $|\cdot| = |\cdot|_X$, and consider the differential equation

$$\dot{\mathrm{x}} = \tilde{\Lambda}\mathrm{x} + \tilde{\mathcal{N}}(\mathrm{x}), \tag{2}$$

where $\tilde{\Lambda} : \mathrm{Dom}(\tilde{\Lambda}) \subseteq X \to X$ is a densely defined linear map with bounded inverse on which we assume to have a fair amount of explicit control (to be discussed below), and $\tilde{\mathcal{N}} \in C^2_{\mathrm{loc}}(X, X)$ on which we have bounds that are explicit, in particular a bound on $D\tilde{\mathcal{N}}(0)$ and a local (uniform) bound on the second derivative(s), see Proposition 2.4 below. Let us suppose that there exists a locally unique hyperbolic equilibrium $\tilde{h} \in X$ to (2), where we think of $\tilde{h}$ as being small. We define a conjugate differential equation via the change of variables $\mathrm{x} \to \mathrm{x} + \tilde{h}$. That is,

$$\dot{\mathrm{x}} = \Lambda\mathrm{x} + L\mathrm{x} + \hat{\mathcal{N}}(\mathrm{x}). \tag{3}$$

where

$$\Lambda := \tilde{\Lambda}, \qquad L := D\tilde{\mathcal{N}}(\tilde{h}), \qquad \hat{\mathcal{N}}(\mathrm{x}) := \tilde{\mathcal{N}}(\tilde{h} + \mathrm{x}) - \tilde{\mathcal{N}}(\tilde{h}) - D\tilde{\mathcal{N}}(\tilde{h})\mathrm{x}. \tag{4}$$

This differential equation is constructed such that the origin is an equilibrium to (3), and both $\hat{\mathcal{N}}(0) = 0$ and $D\hat{\mathcal{N}}(0) = 0$.

As is usual, we decompose our space $X = X_s \times X_u$ into closed stable and unstable eigenspaces of the operator Λ. Moreover, we wish for a greater degree of granularity in these subspaces to take advantage of varying decay rates (and control of these). As such, we decompose X_s and X_u into further subspaces. Fix integers $m_s, m_u \in \mathbb{N}$ and consider the two indexing sets $I := \{1, 2, \dots, m_s\}$ and $I' := \{1', 2', \dots, m'_u\}$. For $i \in I$ and $i' \in I'$ fix closed subspaces $X_i \subseteq X_s$ and $X_{i'} \subseteq X_u$ such that:

$$X_s := \prod_{1 \leq i \leq m_s} X_i, \qquad X_u := \prod_{1' \leq i' \leq m'_u} X_{i'}.$$

We will always use a primed notation, such as i' or j', to index over X_u.

For the projections onto the subspaces X_i, $X_{i'}$, X_s and X_u we use the notation π_i, $\pi_{i'}$, π_s and π_u, respectively. Since these subspaces are closed, the projection maps are continuous linear operators, hence we may fix constants p_s, p_u, and $p_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I} := I \cup I'$ such that:

$$\|\pi_s\| \leq p_s \qquad \|\pi_u\| \leq p_u \qquad \|\pi_{\mathbf{i}}\| \leq p_{\mathbf{i}}. \tag{5}$$

We will be freely using the notation, $\mathrm{x}_{\mathbf{i}} = \pi_{\mathbf{i}}\mathrm{x}$, $\mathrm{x}_s = \pi_s \mathrm{x}$, etc, hence $\mathrm{x} = \mathrm{x}_s + \mathrm{x}_u$, $\mathrm{x}_s = \sum_{i \in I} \mathrm{x}_i$ and $\mathrm{x}_u = \sum_{i' \in I'} \mathrm{x}_{i'}$, as well as $\mathrm{x} = \sum_{\mathbf{i} \in \mathbf{I}} \mathrm{x}_{\mathbf{i}}$.

Assume that Λ is invariant along the subspaces X_i, $X_{i'}$. That is to say, assume that there exist $\Lambda_i : X_i \to X_i$ and $\Lambda_{i'} : X_{i'} \to X_{i'}$ such that

$$\Lambda \mathrm{x} = \sum_{i \in I} \Lambda_i \mathrm{x}_i + \sum_{i' \in I'} \Lambda_{i'} \mathrm{x}_{i'}.$$

Furthermore, we assume that there exist constants $\lambda_i < 0$ such that for $1 \leq i \leq m_s$

$$|e^{\Lambda_i t} \mathrm{x}_i| \leq e^{\lambda_i t} |\mathrm{x}_i|, \qquad t \geq 0, \mathrm{x}_i \in X_i, \tag{6}$$

and there exist constants $\lambda_{i'} > 0$ such that for $1' \leq i' \leq m_u'$

$$|e^{\Lambda_{i'} t} \mathrm{x}_{i'}| \leq e^{\lambda_{i'} t} |\mathrm{x}_{i'}|, \qquad t \leq 0, \mathrm{x}_{i'} \in X_{i'}. \tag{7}$$

In particular, this implies that the norm on X aligns well with flow of Λ on the subspaces X_i.

We decompose L along these subspaces in the following manner: for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$ we define bounded linear operators $L_{\mathbf{i}}^{\mathbf{j}} : X_{\mathbf{j}} \to X_{\mathbf{i}}$ such that

$$[L\mathrm{x}]_{\mathbf{i}} = \sum_{\mathbf{j} \in \mathbf{I}} L_{\mathbf{i}}^{\mathbf{j}} \mathrm{x}_{\mathbf{j}}.$$

We also introduce notation to denote the restrictions of Λ and L at the level of X_s and X_u. That is, we define the operators

$$\begin{aligned} &\Lambda_s \mathrm{x}_s : X_s \to X_s \qquad && L_s^s \mathrm{x}_s : X_s \to X_s \qquad && L_s^u \mathrm{x}_u : X_u \to X_s \\ &\Lambda_u \mathrm{x}_u : X_u \to X_u && L_u^s \mathrm{x}_s : X_s \to X_u && L_u^u \mathrm{x}_u : X_u \to X_u \end{aligned}$$

according to the following formulas:

$$\begin{aligned} \Lambda_s \mathrm{x}_s &:= \sum_{i \in I} \Lambda_i \mathrm{x}_i & L_s^s \mathrm{x}_s &:= \sum_{i,j \in I} L_i^j \mathrm{x}_j & L_s^u \mathrm{x}_u &:= \sum_{i \in I, j' \in I'} L_i^{j'} \mathrm{x}_{j'} \\ \Lambda_u \mathrm{x}_u &:= \sum_{i' \in I'} \Lambda_{i'} \mathrm{x}_{i'} & L_u^s \mathrm{x}_s &:= \sum_{i' \in I, j \in I} L_{i'}^j \mathrm{x}_j & L_u^u \mathrm{x}_u &:= \sum_{i' \in I', j' \in I'} L_{i'}^{j'} \mathrm{x}_{j'}. \end{aligned}$$

We assume that $-(\Lambda_u + L_u^u)$ and $(\Lambda_s + L_s^s)$ are negative operators, i.e., there are constants C_s, C_u and $\lambda_s < 0$ and $\lambda_u > 0$ such that:

$$|e^{(\Lambda_s + L_s^s)t} \mathrm{x}_s| \leq C_s e^{\lambda_s t} |\mathrm{x}_s|, \qquad t \geq 0, \mathrm{x}_s \in X_s, \tag{8}$$

$$|e^{(\Lambda_u + L_u^u)t} \mathrm{x}_u| \leq C_u e^{\lambda_u t} |\mathrm{x}_u|, \qquad t \leq 0, \mathrm{x}_u \in X_u. \tag{9}$$

See Section B for a general discussion of calculating these constants, and Section 6 for a specific example.

Remark 2.1. *For both the prime and non-prime spatial indices, we will use Einstein summation notation*

$$L_i^j \mathrm{x}_j \equiv \sum_{j \in I} L_i^j \mathrm{x}_j, \qquad \textit{and} \qquad L_i^{j'} \mathrm{x}_{j'} \equiv \sum_{j' \in I'} L_i^{j'} \mathrm{x}_{j'}.$$

For other summations, for example over $\mathbf{I} = I \cup I'$, we will write the summation explicitly.

We would like to write (3) as a coupled system of differential equations over the subspaces. Define the restriction of our nonlinearities in each subspace as $\hat{\mathcal{N}}_{\mathbf{i}} := \pi_{\mathbf{i}} \circ \hat{\mathcal{N}}(\mathrm{x})$ for $\mathbf{i} \in \mathbf{I}$. Similarly, we define $\hat{\mathcal{N}}_s(\mathrm{x}) := \pi_s \circ \hat{\mathcal{N}}(\mathrm{x})$ and $\hat{\mathcal{N}}_u(\mathrm{x}) := \pi_u \circ \hat{\mathcal{N}}(\mathrm{x})$. To consolidate notation, for $\mathbf{i} \in \mathbf{I}$ we define:

$$\mathcal{N}_{\mathbf{i}}(\mathrm{x}_s, \mathrm{x}_u) := L^j_{\mathbf{i}} \mathrm{x}_j + L^{j'}_{\mathbf{i}} \mathrm{x}_{j'} + \hat{\mathcal{N}}_{\mathbf{i}}(\mathrm{x}_s, \mathrm{x}_u). \tag{10}$$

We may rewrite (3) in each subspace as follows:

$$\dot{\mathrm{x}}_i = \Lambda_i \mathrm{x}_i + \mathcal{N}_i(\mathrm{x}_s, \mathrm{x}_u), \tag{11}$$
$$\dot{\mathrm{x}}_{i'} = \Lambda_{i'} \mathrm{x}_{i'} + \mathcal{N}_{i'}(\mathrm{x}_s, \mathrm{x}_u). \tag{12}$$

We also define

$$\mathcal{N}_s := \sum_{i \in I} \mathcal{N}_i, \qquad \mathcal{N}_u := \sum_{i' \in I'} \mathcal{N}_{i'}, \qquad \mathcal{N} := \mathcal{N}_s + \mathcal{N}_u.$$

As we plan to just work locally inside a neighborhood of 0, we define a ball about the origin.

Definition 2.2. *Fix positive vectors $r_s \in \mathbb{R}^{m_s}$ and $r_u \in \mathbb{R}^{m_u}$. We define closed balls $B_s(r_s) \subseteq X_s$ and $B_u(r_u) \subseteq X_u$ as follows:*

$$B_s(r_s) := \{\mathrm{x}_s \in X_s : |\mathrm{x}_i| \leq r_i \text{ for } i \in I\}$$
$$B_u(r_u) := \{\mathrm{x}_u \in X_u : |\mathrm{x}_{i'}| \leq r_{i'} \text{ for } i' \in I'\}.$$

When there is no ambiguity in the choice of vectors r_s, r_u, we abbreviate $B_s \equiv B_s(r_s)$ and $B_u \equiv B_u(r_u)$. Below we define bounds on our nonlinearity $\mathcal{N}$ over balls of fixed radius.

Definition 2.3. *Fix radii r_s, r_u. For $\mathrm{x}_s \in B_s(r_s)$, $\mathrm{x}_u \in B_u(r_u)$ we define for indices $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$*

$$\mathcal{N}^{\mathbf{i}}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u) := \frac{\partial}{\partial \mathrm{x}_{\mathbf{i}}} \mathcal{N}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u), \qquad \|\mathcal{N}^{\mathbf{i}}_{\mathbf{j}}\|_{(r_s, r_u)} := \sup_{\mathrm{x}_s \in B_s(r_s)} \sup_{\mathrm{x}_u \in B_u(r_u)} \|\mathcal{N}^{\mathbf{i}}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u)\|$$
$$\mathcal{N}^{\mathbf{ik}}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u) := \frac{\partial^2}{\partial \mathrm{x}_{\mathbf{i}} \partial \mathrm{x}_{\mathbf{k}}} \mathcal{N}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u), \qquad \|\mathcal{N}^{\mathbf{ik}}_{\mathbf{j}}\|_{(r_s, r_u)} := \sup_{\mathrm{x}_s \in B_s(r_s)} \sup_{\mathrm{x}_u \in B_u(r_u)} \|\mathcal{N}^{\mathbf{ik}}_{\mathbf{j}}(\mathrm{x}_s, \mathrm{x}_u)\|.$$

Using this notation we can compute bounds on L and $\mathcal{N}$ in terms of bounds on $\tilde{\mathcal{N}}$.

Proposition 2.4. *Suppose that $|\tilde{h}_{\mathbf{i}}| < \epsilon_{\mathbf{i}}$. Fix r_u, r_s, and constants $\tilde{D}^{\mathbf{i}}_{\mathbf{j}}$ and $\tilde{C}^{\mathbf{ik}}_{\mathbf{j}}$ satisfying*

$$\tilde{D}^{\mathbf{i}}_{\mathbf{j}} \geq \|\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0,0)\|, \qquad \tilde{C}^{\mathbf{ik}}_{\mathbf{j}} \geq \|\tilde{\mathcal{N}}^{\mathbf{ik}}_{\mathbf{j}}\|_{(r_s + \epsilon_s, r_u + \epsilon_u)}.$$

For $\mathbf{i}, \mathbf{j}, \mathbf{k} \in I \cup I'$ define constants $\hat{C}^{\mathbf{i}}_{\mathbf{j}}, D^{\mathbf{i}}_{\mathbf{j}}, C^{\mathbf{i}}_{\mathbf{j}}$, and $C^{\mathbf{ik}}_{\mathbf{j}}$ as below:

$$D^{\mathbf{i}}_{\mathbf{j}} := \tilde{D}^{\mathbf{i}}_{\mathbf{j}} + \tilde{C}^{\mathbf{i}l}_{\mathbf{j}} \epsilon_l + \tilde{C}^{\mathbf{i}l'}_{\mathbf{j}} \epsilon_{l'}, \qquad C^{\mathbf{ik}}_{\mathbf{j}} := \tilde{C}^{\mathbf{ik}}_{\mathbf{j}}$$
$$\hat{C}^{\mathbf{i}}_{\mathbf{j}} := \tilde{C}^{\mathbf{i}l}_{\mathbf{j}} r_l + \tilde{C}^{\mathbf{i}l'}_{\mathbf{j}} r_{l'} \qquad C^{\mathbf{i}}_{\mathbf{j}} := \hat{C}^{\mathbf{i}}_{\mathbf{j}} + D^{\mathbf{i}}_{\mathbf{j}}.$$

Then for L and $\hat{\mathcal{N}}$ defined in (4) and $\mathcal{N}$ defined in (10) we have the bounds

$$D^{\mathbf{i}}_{\mathbf{j}} \geq \|L^{\mathbf{i}}_{\mathbf{j}}\| \qquad C^{\mathbf{ik}}_{\mathbf{j}} \geq \|\mathcal{N}^{\mathbf{ik}}_{\mathbf{j}}\|_{(r_s, r_u)} \tag{13a}$$
$$\hat{C}^{\mathbf{i}}_{\mathbf{j}} \geq \|\hat{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}\|_{(r_s, r_u)} \qquad C^{\mathbf{i}}_{\mathbf{j}} \geq \|\mathcal{N}^{\mathbf{i}}_{\mathbf{j}}\|_{(r_s, r_u)}. \tag{13b}$$

The proof is left to the reader.

2.2 The Lyapunov-Perron Operator

In this paper, we aim to find a chart $\alpha : B_s \to X_u$ such that $\{(\xi, \alpha(\xi)) : \xi \in B_s\}$ is a local stable manifold of the origin for the differential equation (3). The radius of the ball B_s is one of our main computational parameters and will be denoted by ρ henceforth (whereas it was called r_s in Section 2.1). We aim to develop component-wise bounds on the Lipschitz constant of the derivative of such a map α. To that end, for a map $\alpha \in \mathrm{Lip}(B_s(\rho), X_u)$ we define the Lipschitz constant of α relative to the subspaces X_i and $X_{i'}$ as:

$$\mathrm{Lip}(\alpha)^i_{i'} := \sup_{\xi \in B_s} \sup_{\substack{0 \neq \zeta_i \in X_i \\ \xi + \zeta_i \in B_s}} \frac{|\alpha_{i'}(\xi + \zeta_i) - \alpha_{i'}(\xi)|}{|\zeta_i|}.$$

We note that if the Frechet derivative of α exists, then $\sup_{\xi \in B_s(\rho)} \|\alpha^i_{i'}(\xi)\| = \mathrm{Lip}(\alpha)^i_{i'}$. Below we define a ball of such functions.

Definition 2.5. *Fix positive tensors $\rho \in \mathbb{R}^{m_s}$, $P \in \mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u}$ and $\bar{P} \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_u}$. We define the following collections of functions:*

$$\begin{aligned} \mathcal{B}^{0,1}_{\rho,P} &:= \{\alpha \in C^{0,1}(B_s(\rho), X_u) : \alpha(0) = 0,\ \mathrm{Lip}(\alpha)^i_{i'} \leq P^i_{i'}\}, \\ \mathcal{B}^{1,1}_{\rho,P,\bar{P}} &:= \{\alpha \in C^{1,1}(B_s(\rho), X_u) : \alpha(0) = 0,\ \mathrm{Lip}(\alpha)^i_{i'} \leq P^i_{i'},\ \mathrm{Lip}(\partial_i \alpha)^j_{i'} \leq \bar{P}^{ij}_{i'}\}. \end{aligned}$$

Note that for all $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and $\xi, \zeta \in B_s$ we have: $|\alpha_{i'}(\xi) - \alpha_{i'}(\zeta)| \leq P^i_{i'}|\xi_i - \zeta_i|$. For a positive vector ρ and positive tensor P, the range of the $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ lies in a ball $B_u(r_u)$ with r_u given by $r_{i'} = P^i_{i'}\rho_i$. Given our interest in graphs $(\xi, \alpha(\xi))$ we find it convenient to make the following definition:

Definition 2.6. *Let the vector ρ and tensor P be as in Definition 2.5. Define r_u by $r_{i'} := P^i_{i'}\rho_i$. For constants C^i_j, $\hat{C}^i_j$ and D^i_j such that the bounds* (13) *hold with $r_s = \rho$, define positive tensors*

$$H^i_j := C^i_j + C^{i'}_j P^i_{i'}, \qquad H^i_{j'} := C^i_{j'} + C^{i'}_{j'} P^i_{i'}, \qquad \hat{H}^i_j := \hat{C}^i_j + (\hat{C}^{i'}_j + D^{i'}_j) P^i_{i'},$$

and the positive scalar:

$$\hat{\mathcal{H}} := \sup_{\alpha \in \mathcal{B}^{0,1}_{\rho,P}} \sup_{\mathrm{x}_s \in B_s(\rho)} \|\tfrac{\partial}{\partial \mathrm{x}_s} L^u_s \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_s} \hat{\mathcal{N}}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s))\|.$$

The tensor H will be used often in the following bound: if we fix ρ, P and $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, $\xi, \zeta \in B_s(\rho)$, then for each $\mathbf{j} \in \mathbf{I}$ we have:

$$|\mathcal{N}_{\mathbf{j}}(\xi, \alpha(\xi)) - \mathcal{N}_{\mathbf{j}}(\zeta, \alpha(\zeta))| \leq H^i_{\mathbf{j}}|\xi_i - \zeta_i|. \tag{14}$$

Depending on how the norm on X is defined, we can compute $\hat{\mathcal{H}}$ in terms of $\hat{H}$ as follows:

Proposition 2.7. *Fix ρ and P as in Definition 2.6. If the norm on X is defined such that $|\mathrm{x}| = \sum_{\mathbf{i} \in \mathbf{I}} |\mathrm{x}_{\mathbf{i}}|$ then $\hat{\mathcal{H}} \leq \max_{i \in I} \sum_{j \in I} \hat{H}^i_j$.*

Proof. Fix $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and $\mathrm{x}_s \in B_s(\rho)$. We may compute:

$$\begin{aligned} \left\|\tfrac{\partial}{\partial \mathrm{x}_i} L^u_s \alpha(\mathrm{x}_s)\right\| &= \left\|\sum_{j \in I} \tfrac{\partial}{\partial \mathrm{x}_i} L^{n'}_j \alpha^i_{n'}(\mathrm{x}_s)\right\| \leq \sum_{j \in I} D^{n'}_j P^i_{n'}, \\ \left\|\tfrac{\partial}{\partial \mathrm{x}_i} \hat{\mathcal{N}}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s))\right\| &\leq \left\|\sum_{j \in I} \hat{\mathcal{N}}^i_j(\mathrm{x}_s, \alpha(\mathrm{x}_s)) + \hat{\mathcal{N}}^{n'}_j(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \alpha^i_{n'}(\mathrm{x}_s)\right\| \leq \sum_{j \in I} \hat{C}^i_j + \hat{C}^{n'}_j P^i_{n'}. \end{aligned}$$

By the choice of our norm on X, it follows that $\|\pi_{\mathbf{i}}\| = 1$ for all $\mathbf{i} \in \mathbf{I}$. Hence, we can calculate:

$$\begin{aligned}\left\| \tfrac{\partial}{\partial \mathrm{x}_s} L_s^u \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_s} \hat{\mathcal{N}}_s\left(\mathrm{x}_s, \alpha(\mathrm{x}_s)\right) \right\| &= \sup_{u \in X_s, |u|=1} \left| \sum_{i \in I} \left(\tfrac{\partial}{\partial \mathrm{x}_i} L_s^u \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_i} \hat{\mathcal{N}}_s\left(\mathrm{x}_s, \alpha(\mathrm{x}_s)\right) \right) u_i \right| \\ &\leq \sup_{u \in X_s, |u|=1} \sum_{i,j \in I} \left(D_j^{n'} P_{n'}^i + \hat{C}_j^i + \hat{C}_j^{n'} P_{n'}^i \right) |u_i|. \end{aligned}$$

In the righthand side we recognize $\hat{H}_j^i$. We now estimate:

$$\sum_{i,j \in I} \hat{H}_j^i |u_i| = \sum_{i \in I} \Big(\sum_{j \in I} \hat{H}_j^i \Big) |u_i| \leq \sum_{i \in I} \Big(\max_{n \in I} \sum_{j \in I} \hat{H}_j^n \Big) |u_i| = \Big(\max_{i \in I} \sum_{j \in I} \hat{H}_j^i \Big) |u|. \tag{15}$$

By taking the sup over $u \in X_s, |u| = 1$, we obtain

$$\left\| \tfrac{\partial}{\partial \mathrm{x}_s} L_s^u \alpha(\mathrm{x}_s) + \tfrac{\partial}{\partial \mathrm{x}_s} \hat{\mathcal{N}}_s\left(\mathrm{x}_s, \alpha(\mathrm{x}_s)\right) \right\| \leq \max_{i \in I} \sum_{j \in I} \hat{H}_j^i. \qquad \square$$

2.3 Overview of the Lyapunov-Perron Approach

Now that we have introduced the bulk of our notation, we outline our approach in greater detail. Namely, we transform the problem of finding a stable manifold into the problem of finding a fixed point of the Lyapunov-Perron operator (for reference, see books [15, 30, 47]).

This operator is an endomorphism on charts $\alpha \in \mathcal{B}_{\rho,P}^{0,1}$. For such a map α we define $x(t, \xi, \alpha)$ to be the solution to the projected differential equation

$$\dot{\mathrm{x}}_s = \Lambda_s \mathrm{x}_s + \mathcal{N}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s)), \tag{16}$$

with the initial condition $\xi \in B_s(\rho)$ at time $t = 0$. In Section 3 we study solutions to (16). We show that if Λ_s sufficiently dominates the nonlinearity $\mathcal{N}_s$, then solutions to the projected system (16) do not blow up for any $\alpha \in \mathcal{B}_{\rho,P}^{0,1}$, and in fact, the solutions to the projected system limit to 0.

Considering the pair $(x(t, \xi, \alpha), \alpha(x(t, \xi, \alpha)))$, if equation (12) is satisfied for all $i' \in I'$, then by construction equation (11) is satisfied for all $i \in I$. Hence the pair $(x(t, \xi, \alpha), \alpha(x(t, \xi, \alpha)))$ is a solution to the full system (3), and moreover, the map $x \mapsto (x, \alpha(x))$ is a local chart for an invariant manifold of the equilibrium at the origin.

In order to find a map α such that equation (12) is satisfied for all $i' \in I'$, we use the variation of constants formula, and define the Lyapunov-Perron operator.

Definition 2.8. *Fix a positive vector $\rho \in \mathbb{R}^{m_s}$ and a positive tensor P. We define the Lyapunov Perron operator $\Psi : \mathcal{B}_{\rho,P}^{0,1} \to \mathrm{Lip}(B_s(\rho), X_u)$ for $\alpha \in \mathcal{B}_{\rho,P}^{0,1}$ as follows:*

$$\Psi[\alpha](\xi) := - \int_0^\infty e^{-\Lambda_u t} \mathcal{N}_u(x(t, \xi, \alpha), \alpha(x(t, \xi, \alpha))) dt. \tag{17}$$

The fixed point of Ψ is a local coordinate chart for an invariant manifold of the origin. Showing this is the stable manifold requires a small additional argument. Let $\mathbb{E}_s, \mathbb{E}_u \subseteq X$ denote the stable and unstable eigenspaces of the operator $\Lambda + L$. If either $\dim(X_s) = \dim(\mathbb{E}_s) < \infty$ or $\dim(X_u) = \dim(\mathbb{E}_u) < \infty$, then $\alpha = \Psi[\alpha]$ is a local chart for the stable manifold of the origin. In this case, it suffices to correctly count with multiplicity the finite number of stable/unstable eigenvalues of $\Lambda + L$. We consider this case in Sections 6 and 7. On the other hand, if both $\dim(\mathbb{E}_s) = \infty$ and $\dim(\mathbb{E}_u) = \infty$, then we may obtain the desired result by showing that the family of operators $\Lambda + sL$ does not have any eigenvalues crossing the imaginary axis for $s \in [0, 1]$. This is the approach taken in [59] and is of interest to studying strongly indefinite problems such as those typically appearing in elliptic problems, see e.g. [14].

In Section 4 we show that for an appropriate choice of constants, Ψ is an endomorphism on a ball $\mathcal{B}^{0,1}_{\rho,P}$, as well as an endomorphism on a ball $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$. In Section 5 we show that Ψ is a contraction map in a C^0-like norm (see Definition 5.2).

2.4 Good Coordinates: Parameterization of Slow Stable Manifolds and Attached Invariant Frame Bundles

Before applying the Lyapunov-Perron argument we first "flatten out" the nonlinearities in a neighborhood of the equilibrium by making a suitable change of coordinates. This change of coordinates exploits the fact that the dynamics in some directions are more important than others. In particular, we would like to describe the dynamics in the slow directions as accurately as possible, and will settle for only linear control in directions corresponding to fast dynamics.

Our approach is based on the parameterization method of [8, 9, 10], and especially on the notion of slow spectral sub manifolds discussed in the references just cited. See also the work of [52, 27, 48, 7, 33]. The theorem below is proved in [8, 10]. The version we state assumes that the eigenvalues are real and have geometric multiplicity one. These assumptions simplify the presentation and can be removed. Note that we will apply the parameterization method in a finite dimensional projection, and that this assumption cannot be removed. In slight abuse of notation, to align with the existing literature we use P te denote the parametrizaton of a slow stable manifold; this should not be confounded with the positive tensor denoted by the same symbol in previous subsection.

Theorem 2.9 (Slow-stable manifold parameterization)**.** *Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$ be a real analytic vector field, and $p_0 \in \mathbb{R}^d$ be a hyperbolic equilibrium point whose differential $DF(p_0)$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ denote the eigenvalues of $DF(p_0)$ and suppose that $\lambda_1, \ldots, \lambda_{m_{\text{slow}}}$ with $m_{\text{slow}} < d$ are the slow stable eigenvalues. Let $\xi_1, \ldots, \xi_{m_{\text{slow}}} \in \mathbb{R}^d$ denote associated slow stable eigenvectors. Write*

$$\Lambda_{\text{slow}} = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_{m_{\text{slow}}} \end{pmatrix}, \qquad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_d \end{pmatrix},$$

to denote respectively the $m_{\text{slow}} \times m_{\text{slow}}$ and $d \times d$ matrices of the slow stable eigenvalues and all the eigenvalues of $DF(p_0)$. Suppose that $P\colon [-1,1]^{m_{\text{slow}}} \to \mathbb{R}^d$ is a smooth solution of the invariance equation

$$F(P(\theta)) = DP(\theta)\Lambda_{\text{slow}}\theta, \qquad \theta \in [-1,1]^{m_{\text{slow}}}, \tag{18}$$

subject to the first order constraints $P(0) = p_0$ and $\partial_j P(0) = \xi_j$, $1 \le j \le m_{\text{slow}}$. Then P parameterizes the m_{slow} dimensional smooth slow manifold attached to p_0.

It can be shown that Equation (18) has analytic solution as long as for all $(m_1, \ldots, m_{\text{slow}}) \in \mathbb{N}$ with $m_1 + \ldots + m_{\text{slow}} \ge 2$, the non-resonance conditions $m_1\lambda_1 + \ldots + m_{\text{slow}}\lambda_{m_{\text{slow}}} \ne \lambda_j$ for $1 \le j \le d$, are satisfied. Observe that this is in fact only a finite number of conditions. Moreover the solution is unique up to the choice of the scalings of the eigenvectors $\xi_1, \ldots, \xi_{m_{\text{slow}}}$.

To control the fast dynamics we exploit the "slow manifold Floquet theory" developed in [52]. The idea is to study certain linearized invariance equations describing the stable/unstable bundles attached to the slow stable manifold. These invariant bundles describe the linear approximation of the full stable manifold near the slow stable manifold, and in addition they provide control over the normal and tangent directions. Combining the stable, unstable, and tangent bundles provides a frame bundle for the phase space in a tubular region surrounding the slow manifold – our "good coordinates". The idea is illustrated in Figure 2.

Computation of the invariant frame bundles is facilitated by the following theorem, the main result from [52]. Note that we apply this theorem only for the finite dimensional projection of our PDE.

Theorem 2.10 (Slow-stable manifold Floquet normal form)**.** *Let $F\colon \mathbb{R}^d \to \mathbb{R}^d$, $p_0 \in \mathbb{R}^d$, $DF(p_0)$, $\lambda_1, \ldots, \lambda_d$, $\xi_1, \ldots, \xi_d$, $m_{\text{slow}} < d$, Λ_{slow}, Λ, and $P\colon [-1,1]^{m_{\text{slow}}} \to \mathbb{R}^d$ be as in Theorem 2.9. Assume that for $1 \le j \le d$ the functions $q_j\colon [-1,1]^{m_{\text{slow}}} \to \mathbb{R}^d$ are smooth solutions of the equations*

$$DF(P(\theta))q_j(\theta) = \lambda_j q_j(\theta) + Dq_j(\theta)\Lambda_{\text{slow}}\theta, \tag{19}$$

for $\theta \in [-1,1]^{m_{\text{slow}}}$, subject to the constraints $q_j(0) = \xi_j$. Let $GL(\mathbb{R}^d)$ denote the collection of all non-singular $d \times d$ matrices with real entries. Define $Q\colon [-1,1]^{m_{\text{slow}}} \to GL(\mathbb{R}^d)$ by

$$Q(\theta) = [q_1(\theta)| \ldots |q_d(\theta)]\,.$$

Then

1. *For all $\theta \in [-1,1]^{m_{\text{slow}}}$ the collection of vectors $q_1(\theta)$, …, $q_d(\theta)$ span $\mathbb{R}^d$. That is, Q takes values in $GL(\mathbb{R}^d)$ and hence parameterizes a frame bundle.*
2. *For all $t \ge 0$ and for all $\theta \in [-1,1]^{m_{\text{slow}}}$, the derivative of the flow along the slow stable manifold factors as*
$$M(t) = Q(e^{\Lambda_{\text{slow}} t}\theta)e^{\Lambda t}Q^{-1}(\theta), \tag{20}$$
where $M(t)$ is the solution of the equation of first variation for F along $P(\theta)$:
$$M'(t) = DF(P(\theta))M(t), \qquad \text{for all } t \ge 0,$$
with $M(0)$ the identity matrix.

Considering (20) one column at a time gives that the frame bundles $q(\theta)_j$, $1 \le j \le d$ satisfy the invariance equation

$$M(t)q_j(\theta) = e^{\lambda_j t} q_j\left(e^{\Lambda_{\text{slow}} t}\theta\right), \qquad \text{for } \theta \in [-1,1]^{m_{\text{slow}}}.$$

This says that the flow along $P(\theta)$ leaves the direction of q_j invariant (maps the bundle into itself) but expands vectors at an exponential rate of λ_j. It follows that if $q_{m_{\text{slow}}+1}(\theta), \ldots, q_{m_s}(\theta)$ are the parameterized vector bundles associated with the stable eigenvalues which have not been designated as slow (the so called *fast stable* directions), then for each $\theta \in [-1,1]^{m_{\text{slow}}}$ these invariant bundles are the fastest contracting directions near $P(\theta)$, and hence they describe $W^s(p_0)$ near $P(\theta)$.

We now define a nonlinear change of coordinates which, to first order, diagonalizes the vector field F near $P(\theta)$. Let $d = m_{\text{slow}} + m_{\text{fast}} + m_{\text{unst}}$. Define the coordinate change $K\colon [-1,1]^{m_{\text{slow}}} \times [-\epsilon_f, \epsilon_f]^{m_{\text{fast}}} \times [-\epsilon_u, \epsilon_u]^{m_{\text{unst}}} \to \mathbb{R}^d$ by

$$K(\theta, \phi_f, \phi_u) := P(\theta) + Q_f(\theta)\phi_f + Q_u(\theta)\phi_u,$$

i.e. K is a diffeomorphism with $K(0,0,0) = p_0$ and $DK(0,0,0) = Q(0)$, the matrix of eigenvectors. Here θ is the coordinate in the slow stable manifold, Q_f and ϕ_f denote the fast stable directions, and Q_u and ϕ_u denote the unstable directions. Recall that the defining relations for P, Q_f and Q_u are

$$F(P(\theta)) = DP(\theta)\Lambda_{\text{slow}}\theta, \tag{21}$$
$$DF(P(\theta))Q_f(\theta) = DQ_f(\theta)\Lambda_{\text{slow}}\theta + Q_f(\theta)\Lambda_{\text{fast}}, \tag{22}$$
$$DF(P(\theta))Q_u(\theta) = DQ_u(\theta)\Lambda_{\text{slow}}\theta + Q_u(\theta)\Lambda_{\text{unst}}. \tag{23}$$

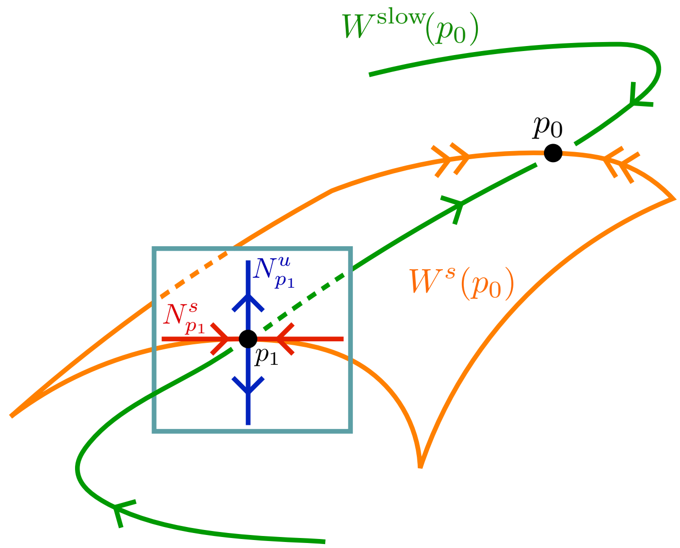

Figure 2: **Slow stable manifold and attached frame bundles:** the figure illustrates an equilibrium solution p_0 and its slow stable manifold in green. The orange surface illustrates the full stable manifold, of which the slow manifold is a subset. At each point on the slow manifold there are invariant stable/unstable normal bundles. The stable normal bundle describes the stable manifold near W^{slow}. Taking the stable, unstable, and tangent bundles gives a frame for the entire space. Theorem 2.10 provides an explicit method for computing these structures.

We use K to pull back the vector field $F\colon \mathbb{R}^d \to \mathbb{R}^d$. The resulting vector field has the form

$$\begin{pmatrix} \theta' \\ \phi_f' \\ \phi_u' \end{pmatrix} = DK^{-1}(\theta, \phi_f, \phi_u)\, F(K(\theta, \phi_f, \phi_u)) = \begin{pmatrix} \Lambda_{\text{slow}}\theta + N_\theta(\theta, \phi_f, \phi_u) \\ \Lambda_{\text{fast}}\phi_f + N_{\phi_f}(\theta, \phi_f, \phi_u) \\ \Lambda_{\text{unst}}\phi_u + N_{\phi_u}(\theta, \phi_f, \phi_u) \end{pmatrix},$$

where each of the $N_k(\theta, \phi_f, \phi_u)$ is *quadratic in* ϕ_f *and* ϕ_u, for $k = \theta, \phi_f, \phi_u$.

To see this, and to obtain explicitly the form of N_k, we first expand about $P(\theta)$:

$$\begin{aligned} F(K(\theta, \phi_f, \phi_u)) &= F(P(\theta) + Q_f(\theta)\phi_f + Q_u(\theta)\phi_u) \\ &= F(P(\theta)) + DF(P(\theta))\,[Q_f(\theta)\phi_f + Q_u(\theta)\phi_u] + R(\theta, \phi_f, \phi_u), \end{aligned} \tag{24}$$

where the remainder term R is quadratic in ϕ_f and ϕ_u. For the first two terms in (24) we use the defining relations for P, Q_f and Q_u as well as the definition of K to rewrite

$$\begin{aligned} F(P(\theta)) + DF(P(\theta))\,[Q_f(\theta)\phi_f + Q_u(\theta)\phi_u] &= DP(\theta)\Lambda_{\text{slow}}\theta \\ &\quad + DQ_f(\theta)(\Lambda_{\text{slow}}\theta, \phi_f) + Q_f(\theta)\Lambda_{\text{fast}}\phi_f \\ &\quad + DQ_u(\theta)(\Lambda_{\text{slow}}\theta, \phi_u) + Q_u(\theta)\Lambda_{\text{unst}}\phi_u \\ &= DK(\theta, \phi_f, \phi_u) \begin{pmatrix} \Lambda_{\text{slow}}\theta \\ \Lambda_{\text{fast}}\phi_f \\ \Lambda_{\text{unst}}\phi_u \end{pmatrix}. \end{aligned}$$

Then

$$DK^{-1}(\theta, \phi_f, \phi_u)\, F(K(\theta, \phi_f, \phi_u)) = \begin{pmatrix} \Lambda_{\text{slow}}\theta \\ \Lambda_{\text{fast}}\phi_f \\ \Lambda_{\text{unst}}\phi_u \end{pmatrix} + DK^{-1}(\theta, \phi_f, \phi_u)\, R(\theta, \phi_f, \phi_u),$$

hence

$$N(\theta,\phi_f,\phi_u) = DK(\theta,\phi_f,\phi_u)^{-1}R(\theta,\phi_f,\phi_u),$$

As R is quadratic in ϕ_f and ϕ_u, then so is N. Once again we refer to Figure 2 for the geometric idea behind the coordinate change.

Note that we do not actually have to solve the invariance equation (18) or the invariant bundle equations (19) exactly. Rather we approximately solve them in an appropriate finite dimensional projection of the PDE and incorporate the defects into our analysis. Our numerical approximations exploit formal power series methods which have been discussed in many places. In particular, we use the numerical schemes discussed in [52] freely throughout Section 7.

3 Exponential Tracking

Remark 3.1. *Throughout this section, $\rho \in \mathbb{R}^{m_s}$ denotes a positive vector (the radius of the domain of our charts) and $P \in \mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u}$ denotes a positive tensor (bounding the subspace-Lipschitz constants of our charts).*

To begin our analysis we first derive estimates on $x(t,\xi,\alpha)$, the solution of the projected system (16). A classic theorem we obtain by way of Gronwall's lemma is as follows:

Proposition 3.2. *Let $\xi,\zeta \in B_s(\rho)$. If $x(t,\xi,\alpha)$ and $x(t,\zeta,\alpha)$ stay inside B_s for all $t\in[0,T]$, then:*

$$|x(t,\xi,\alpha) - x(t,\zeta,\alpha)| \le C_s|\xi-\zeta|e^{(\lambda_s + C_s\hat{\mathcal{H}})t} \qquad \text{for all } t\in[0,T].$$

Proof of Proposition 3.2. Recall from (16) that

$$\dot{\mathrm{x}}_s = \Lambda_s \mathrm{x}_s + L_s^s \mathrm{x}_s + L_s^u \alpha(\mathrm{x}_s) + \hat{\mathcal{N}}_s(\mathrm{x}_s, \alpha(\mathrm{x}_s)).$$

To shorten our notation, we define $x(t) = x(t,\xi,\alpha)$ and $z(t) = x(t,\zeta,\alpha)$. By variation of constants, we have that

$$x(t) = e^{(\Lambda_s + L_s^s)t}\xi + \int_0^t e^{(\Lambda_s+L_s^s)(t-\tau)}\left(L_s^u\alpha(x(\tau)) + \hat{\mathcal{N}}_s\left(x(\tau),\alpha(x(\tau))\right)\right)d\tau.$$

From (8) we have that $|e^{(\Lambda_s+L_s^s)t}\xi_s| \le C_s|e^{\lambda_s t}\xi_s|$. When we define $U(t) = |x(t)-z(t)|$ then

$$\begin{aligned} e^{-\lambda_s t}U(t) \le C_s|\xi-\zeta| &+ \int_0^t C_s e^{-\lambda_s\tau}\left|L_s^u(\alpha(x(\tau)) - \alpha(z(\tau)))\right|d\tau \\ &+ \int_0^t C_s e^{-\lambda_s\tau}\left|\hat{\mathcal{N}}_s\left(x(\tau),\alpha(x(\tau))\right) - \hat{\mathcal{N}}_s\left(z(\tau),\alpha(z(\tau))\right)\right|d\tau. \end{aligned} \tag{25}$$

By our definition of $\hat{\mathcal{H}}$ in Definition 2.6 and the mean value theorem, it follows that

$$|L_s^u(\alpha(x(\tau)) - \alpha(z(\tau)))| + \left|\hat{\mathcal{N}}_s\left(x(\tau),\alpha(x(\tau))\right) - \hat{\mathcal{N}}_s\left(z(\tau),\alpha(z(\tau))\right)\right| \le \hat{\mathcal{H}}|x(\tau)-z(\tau)|.$$

We plug this bound into (25) and obtain

$$e^{-\lambda_s t}U(t) \le C_s|\xi-\zeta| + \int_0^t C_s\hat{\mathcal{H}}e^{-\lambda_s\tau}U(\tau)d\tau.$$

By Gronwall's inequality it follows that $e^{-\lambda_s t}U(t) \le C_s|\xi-\zeta|\exp\{C_s\hat{\mathcal{H}}t\}$, which we may rewrite as

$$U(t) \le C_s|\xi-\zeta|e^{(\lambda_s+C_s\hat{\mathcal{H}})t}. \qquad \square$$

Clearly, we need $\lambda_s + C_s\hat{\mathcal{H}} < 0$ in order for solutions to limit to zero. If so, and by taking $\zeta = 0$, this shows that points in $B_s(\frac{1}{C_s}\rho)$ stay in $B_s(\rho)$ for all time.

We would like to derive a sharper version of Proposition 3.2 taking into account the different subspaces of X_s. For example, consider a decomposition $X_s = X_{\text{slow}} \times X_{\text{fast}}$ and an initial condition $\xi = (\xi_{\text{slow}}, \xi_{\text{fast}}) \in X_{\text{slow}} \times X_{\text{fast}}$. From solving the linear system, we have our bound from (6) that $|e^{\Lambda_{\text{slow}} t}\xi_{\text{slow}}| \leq e^{\lambda_{\text{slow}} t}|\xi_{\text{slow}}|$ and $|e^{\Lambda_{\text{fast}} t}\xi_{\text{fast}}| \leq e^{\lambda_{\text{fast}} t}|\xi_{\text{fast}}|$. If $0 > \lambda_{\text{slow}} \gg \lambda_{\text{fast}}$, then we would expect that solutions to the differential equation (16) will have a component $x_{\text{fast}}(t,\xi,\alpha)$ that initially decreases very quickly. Below we define the characteristic "control" rates, arising from each subspace in the stable eigenspace, by which solutions to (16) grow/shrink. In addition, to describe the effect of coupling the various subspaces together, we define $\gamma_0 = \lambda_s + C_s\hat{\mathcal{H}}$ as the exponent derived in Proposition 3.2.

Definition 3.3. *Define constants γ_k for integers $0 \leq k \leq m_s$ as follows:*

$$\gamma_k := \begin{cases} \lambda_s + C_s\hat{\mathcal{H}} & \text{if } k = 0 \\ \lambda_k + H_k^k & \text{otherwise.} \end{cases}$$

We assume the ordering $\gamma_k > \gamma_{k+1}$.

In practice the ordering of γ_k is always satisfied by suitably (re)arranging the subspaces X. The strictness of the ordering indicates that on the balls chosen, the nonlinearities do not spoil the subspace splitting.

Using these exponential rates we estimate the components of $|x(t,\xi,\alpha)|$ using tensors $G^n_{j,k}$ of the following form:

Condition 3.4. *A tensor $G \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ satisfies Condition 3.4 on the interval $[0,T]$ if:*

$$|x_j(t,\xi,\alpha) - x_j(t,\zeta,\alpha)| \leq \sum_{\substack{n\in I \\ 0\leq k\leq m_s}} e^{\gamma_k t} G^n_{j,k}|\xi_n - \zeta_n|, \tag{26}$$

for all $t \in [0,T]$, all $\xi, \zeta \in B_s(\rho)$ and all $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$.

Remark 3.5. *Since $|x_j| \leq p_j|x|$, with p_j defined in* (5)*, by Proposition 3.2 the tensor*

$$\widehat{G}^n_{j,k} := \begin{cases} p_j C_s & \text{for } k = 0, \\ 0 & \text{for } k \neq 0, \end{cases}$$

satisfies Condition 3.4.

We note that while this tensor $\widehat{G}$ is non-negative, a generic tensor G satisfying Condition 3.4 can, and in practice will, have negative components.

Additionally, while this initial estimate is typically worse than just the bound from Proposition 3.2, by employing an explicit bootstrapping method we are able to obtain tighter component-wise bounds on solutions to (16). To do so, we apply variation of constants to (16) in each subspace, focusing on improving our bound on just a single component at a time. To begin, we first prove the following proposition.

Proposition 3.6. *Let $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and let $\xi, \zeta \in B_s(\rho)$. Define $u_i(t) := |x_i(t,\xi,\alpha) - x_i(t,\zeta,\alpha)|$ for $i \in I$. If $x(t,\xi,\alpha), x(t,\zeta,\alpha) \in B_s(\rho)$ for $t \in [0,T]$, then for each $j \in I$ and all $t \in [0,T]$ we have*

$$e^{-\lambda_j t}u_j(t) \leq |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j \tau} \sum_{i\in I} H^i_j u_i(\tau) d\tau. \tag{27}$$

Proof. By variation of constants

$$x_j(t,\xi,\alpha) = e^{\Lambda_j t}\xi_j + \int_0^t e^{\Lambda_j(t-\tau)}\mathcal{N}_j\left(x(\tau,\xi,\alpha),\alpha(x(\tau,\xi,\alpha))\right)d\tau.$$

Note that we have

$$|\mathcal{N}_j(x(t,\xi,\alpha),\alpha(x(t,\xi,\alpha))) - \mathcal{N}_j(x(t,\zeta,\alpha),\alpha(x(t,\zeta,\alpha)))| \le H^i_j u_i(t) \qquad \text{for all } t \ge 0.$$

Together with the estimate $|e^{\Lambda_j t}\xi_j| \le e^{\lambda_j t}|\xi_j|$ for $t \ge 0$ we obtain

$$e^{-\lambda_j t}u_j(t) \le |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j \tau}\sum_{i\in I} H^i_j u_i(\tau)d\tau. \qquad \square$$

From this, we show that we can use a tensor G satisfying Condition 3.4 to derive sharper component-wise estimates as described in the following theorem.

Theorem 3.7. *Let $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$ and let $\xi,\zeta \in B_s(\rho)$. Suppose G satisfies Condition 3.4, and fix $j \in I$. If $G^n_{i,j} = 0$ for all $n \in I$ and $i \in I - \{j\}$, then we have*

$$|x_j(t,\xi,\alpha) - x_j(t,\zeta,\alpha)| \le |\xi_j - \zeta_j|e^{\gamma_j t} + \sum_{\substack{n,i\in I, i\neq j\\ 0\le m\le m_s, m\neq j}} \frac{e^{\gamma_m t} - e^{\gamma_j t}}{\gamma_m - \gamma_j} H^i_j G^n_{i,m}|\xi_n - \zeta_n|. \tag{28}$$

In other words, for $j \in I$ we define $\mathcal{T}_j : (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1} \to \mathbb{R}^{m_s} \otimes \mathbb{R}^{m_s+1}$ to be the map

$$[\mathcal{T}_j(G)]^n_k := \begin{cases} \displaystyle\sum_{n,i\in I, i\neq j} (\gamma_k - \gamma_j)^{-1} H^i_j G^n_{i,k} & \text{if } k \neq j, \\ \delta^n_k - \displaystyle\sum_{\substack{n,i\in I, i\neq j\\ 0\le m\le m_s, m\neq j}} (\gamma_m - \gamma_j)^{-1} H^i_j G^n_{i,m} & \text{if } k = j. \end{cases} \tag{29}$$

Then G satisfies Condition 3.4 again if we replace $G^n_{j,k}$ by $[\mathcal{T}_j(G)]^n_k$.

Before we prove this proposition, we state two lemmas first (and prove the second one).

Lemma 3.8 (see [34, p.4])**.** *Let $u, V, h \in C^0\big([0,\infty),[0,\infty)\big)$ and suppose that*

$$u(t) \le V(t) + \int_0^t h(s)u(s)ds.$$

If V is differentiable, then

$$u(t) \le V(0)\exp\left\{\int_0^t h(s)ds\right\} + \int_0^t V'(s)\exp\left\{\int_s^t h(\tau)d\tau\right\}ds.$$

Lemma 3.9. *Fix constants $c_0, c_1, c_2 \in \mathbb{R}$ with $c_1, c_2 \ge 0$ and define $\mu_0 = c_0 + c_2$. For constants μ_k, a_k with $\mu_k \neq \mu_0$ for $k = 1, \ldots, K$, we set*

$$v(s) = \sum_{k=1}^K e^{\mu_k s}a_k.$$

If $v(t) \geq 0$ for $t \geq 0$, and if we have the inequality

$$e^{-c_0 t} u_0(t) \leq \left(c_1 + \int_0^t e^{-c_0 s} v(s) ds \right) + \int_0^t c_2 e^{-c_0 s} u_0(s) ds,$$

then

$$u_0(t) \leq c_1 e^{\mu_0 t} + \sum_{k=1}^{K} \frac{a_k}{\mu_k - \mu_0} \left(e^{\mu_k t} - e^{\mu_0 t} \right). \tag{30}$$

Furthermore the sum in the righthand side is non-negative for all $t \geq 0$.

Proof. By Lemma 3.8, it follows that

$$\begin{aligned} e^{-c_0 t} u_0(t) &\leq c_1 e^{c_2 t} + \int_0^t e^{-c_0 s} v(s) e^{c_2 (t-s)} ds. \\ &= c_1 e^{c_2 t} + e^{c_2 t} \int_0^t \sum_{k=1}^{n} a_k e^{(\mu_k - c_0 - c_2) s} ds \\ &= c_1 e^{c_2 t} + e^{c_2 t} \sum_{k=1}^{n} \frac{a_k}{\mu_k - \mu_0} \left(e^{(\mu_k - \mu_0) t} - 1 \right). \end{aligned} \tag{31}$$

After multiplying each side by $e^{c_0 t}$ we obtain the desired inequality (30). Since $v(t)$ is nonnegative, so is the integrand. Hence the sum in the righthand side of (31) is non-negative for all $t \geq 0$. □

We now turn to the proof of Theorem 3.7.

Proof of Theorem 3.7. We fix $j \in J$ and rewrite (27) as:

$$e^{-\lambda_j t} u_j(t) \leq |\xi_j - \zeta_j| + \sum_{i \in I, i \neq j} \int_0^t e^{-\lambda_j s} H_j^i u_i(s) ds + \int_0^t e^{-\lambda_j s} H_j^j u_j(s) ds. \tag{32}$$

Since G satisfies Condition 3.4 we compute:

$$\begin{aligned} \sum_{i \in I, i \neq j} H_j^i u_i(t) &\leq \sum_{i \in I, i \neq j} H_j^i \sum_{\substack{n \in I \\ 0 \leq m \leq m_s}} e^{\gamma_m t} G_{i,m}^n |\xi_n - \zeta_n| \\ &= \sum_{0 \leq m \leq m_s} e^{\gamma_m t} \sum_{n, i \in I, i \neq j} H_j^i G_{i,m}^n |\xi_n - \zeta_n| \\ &= \sum_{0 \leq m \leq m_s, m \neq j} e^{\gamma_m t} \sum_{n, i \in I, i \neq j} H_j^i G_{i,m}^n |\xi_n - \zeta_n|, \end{aligned} \tag{33}$$

where the final equality follows from the assumption that $G_{i,j}^n = 0$ whenever $i \neq j$. If we define:

$$v(s) = \sum_{0 \leq m \leq m_s, m \neq j} e^{\gamma_m s} a_m, \qquad \text{with} \qquad a_m := \sum_{n, i \in I, i \neq j} H_j^i G_{i,m}^n |\xi_n - \zeta_n|,$$

then by combining (32) and (33) we obtain:

$$\begin{aligned} e^{-\lambda_j t} u_j(t) &\leq |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j s} \sum_{0 \leq m \leq m_s, m \neq j} e^{\gamma_m s} a_m ds + \int_0^t e^{-\lambda_j s} H_j^j u_j(s) ds. \\ &= |\xi_j - \zeta_j| + \int_0^t e^{-\lambda_j s} v(s) ds + \int_0^t H_j^j e^{-\lambda_j s} u_j(s) ds. \end{aligned}$$

We now want to apply Lemma 3.9 with $u_0 = u_j$, $c_0 = \lambda_j$, $c_1 = |\xi_j - \zeta_j|$, $c_2 = H_j^j$ and using the re-indexing $\{\mu_k\}_{1\le k\le K} = \{\gamma_m\}_{0\le m\le m_s, m\neq j}$. Since $\gamma_m \neq \lambda_j + H_j^j = \gamma_j$ for $m \neq j$ by the strict ordering assumption in Definition 3.3, the assumption in Lemma 3.9 is satisfied. Hence applying Lemma 3.9 is justified and leads to the result (28). □

Theorem 3.7 allows us to pick a $j \in I$ and then replace a bound of the form (26) with the same bound but with $G^n_{j,k}$ replaced by $[\mathcal{T}_j(G)]^n_k$, which will hopefully produce a sharper bound. In Theorem 3.7, one assumption we impose on G is that for a fixed $j \in I$, we have $G^n_{i,j} = 0$ for all $n \in I$ and $i \in I - j$. Without this assumption we would end up with terms of the form $te^{\gamma_j t}$ in (28), which we choose to avoid as we prefer to work with a finite set of exponentially decaying functions as the basis for our estimates.

However, we will also need to deal with the case $G^n_{i,j} \neq 0$ for some $i \neq j$ and some $n \in I$. We handle this problem by modifying such an "ill-conditioned" G before replacing it with $\mathcal{T}_j(G)$. Namely, if $G^n_{i,j} \neq 0$ then, depending on the sign of $G^n_{i,j}$ we estimate $(G^n_{i,j})e^{\gamma_j t}$ from above by either $G^n_{i,j}e^{\gamma_{j-1}t}$ or $G^n_{i,j}e^{\gamma_{j+1}t}$ for all $t \geq 0$, where we use the ordering $\gamma_0 > \cdots > \gamma_{m_s}$ asserted in Definition 3.3. To be precise, for any fixed $j \in I$ we define the modified tensor

$$[\mathcal{Q}_j(G)]^n_{i,k} := \begin{cases} 0 & \text{if } k = j \\ G^n_{i,k} + G^n_{i,j} & \text{if } k = j-1, \text{and } G^n_{i,j} > 0 \\ G^n_{i,k} + G^n_{i,j} & \text{if } k = j+1, \text{and } G^n_{i,j} < 0 \\ G^n_{i,k} & \text{otherwise.} \end{cases} \tag{34}$$

Note that if $j = m_s$ and $G^n_{i,j} < 0$, then we are effectively employing the estimate $G^n_{i,j}e^{\gamma_{m_s}t} < 0$. The following lemma summarizes the preceeding discussion.

Lemma 3.10. *Fix $j \in I$. If G satisfies Condition 3.4, then $\mathcal{Q}_j(G)$ satisfies Condition 3.4.*

Thus, starting from an initial bound of the form (26) with tensor $\widehat{G}$ given in Remark 3.5, we can improve the bound iteratively by the following algorithm:

Algorithm 3.11. *Let $N_{bootstrap} \in \mathbb{N}$ be a computational parameter.*

$G \leftarrow \widehat{G}$
for $1 \le i \le N_{bootstrap}$ ***do***
  for $1 \le j \le m_s$ ***do***
    $G^n_{j,k} \leftarrow [\mathcal{T}_j \circ \mathcal{Q}_j(G)]^n_k$
  end for
end for
return G

In practice Algorithm 3.11 quickly converges to a fixed tensor G; it suffices to take $N_{\text{bootstrap}} \le 5$.

Theorem 3.12. *Let $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$. Let coefficients $G^n_{j,k}$ be the output of Algorithm 3.11 and fix initial conditions $\xi, \zeta \in B_s(\rho)$. If $x(\tau,\xi,\alpha)$ and $x(\tau,\zeta,\alpha)$ stay inside $B_s(\rho)$ for all $t \in [0,T]$, then we have:*

$$|x_j(t,\xi,\alpha) - x_j(t,\zeta,\alpha)| \le \sum_{\substack{n\in I \\ 0\le k\le m_s}} e^{\gamma_k t} \cdot G^n_{j,k}\,|\xi_n - \zeta_n| \qquad \text{for all } t \in [0,T]. \tag{35}$$

Furthermore if α is differentiable, then $\left\|\frac{\partial}{\partial \xi_n} x_j(t,\xi,\alpha)\right\| \le \sum_{0\le k\le m_s} e^{\gamma_k t} G^n_{j,k}$ for all $t \in [0,T]$.

The proof of Theorem 3.12 is done by induction on $N_{\text{bootstrap}}$, with Proposition 3.2 taking care of the base case ($N_{\text{bootstrap}} = 0$), and Theorem 3.7 taking care of the inductive step. The details are left to the reader.

Using Algorithm 3.11 we are able to derive sharper bounds iteratively. In Proposition 3.2 we could merely show that if $\gamma_0 < 0$, then only points $\xi \in B_s(C_s^{-1}\rho)$ have solutions to (16) which are guaranteed to stay in $B_s(\rho)$ for all $t \geq 0$. The following proposition gives conditions which extend this result to all points $\xi \in B_s(\rho)$.

Proposition 3.13. *Suppose that $\gamma_0 < 0$ and suppose $G^n_{j,k}$ is the output of Algorithm 3.11. If*

$$\rho_j \geq \sum_{\substack{n \in I \\ 0 \leq k \leq m_s}} e^{\gamma_k t} G^n_{j,k} \rho_n, \tag{36}$$

for all $t \geq 0$, then for all $\xi \in B_s(\rho)$ and $t \geq 0$ we have $x(t,\xi,\alpha) \in B_s(\rho)$ for all $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$.

Proof. Fix $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$, fix $0 < \epsilon < 1$, and fix $\xi \in B_s(\epsilon\rho)$. Define $T = \sup\{t \geq 0 : x(t,\xi,\alpha) \in B_s(\rho)\}$. Assume that $T < +\infty$; we prove by contradiction that in fact $T = +\infty$.

Since $x(0,\xi,\alpha) \in B_s(\epsilon\rho)$ and $x(t,\xi,\alpha)$ is continuous in t, it follows that $T > 0$. By Proposition 3.12 we have for all $t \in [0,T)$ that

$$|x_j(t,\xi,\alpha)| \leq \sum_{0 \leq k \leq m_s} e^{\gamma_k t} G^n_{j,k} \, |\xi_n| \leq \epsilon \sum_{0 \leq k \leq m_s} e^{\gamma_k t} G^n_{j,k} \rho_n \leq \epsilon \, \rho_j.$$

Hence $x(t,\xi,\alpha) \in B_s(\epsilon\rho)$ for all $t \in [0,T)$ and so by continuity $x(T,\xi,\alpha) \in B_s(\epsilon\rho)$. Since $x(T,\xi,\alpha)$ is in the interior of $B_s(\rho)$, then the solution to (16) starting at $x(T,\xi,\alpha)$ will stay inside the ball $B_s(\rho)$ for some positive amount of time. But this contradicts our definition of T as the supremum of $\{t \geq 0 : x(t,\xi,\alpha) \in B_s(\rho)\}$. Hence, if $0 < \epsilon < 1$ and $\xi \in B_s(\epsilon\rho)$, then $x(t,\xi,\alpha) \in B_s(\rho)$ for all $t \geq 0$.

By continuity of solutions this result extends to initial conditions on the boundary of $B_s(\rho)$. ☐

Remark 3.14. *In practice we verify the hypothesis of Proposition 3.13 in three steps:*

1. *For some $T_2 > 0$, we check that $\rho_j > \sum_{n \in I, 0 \leq k \leq m_s} e^{\gamma_k T_2} |G^n_{j,k}| \rho_n$, and hence (36) is satisfied for all $t \geq T_2$.*

2. *For some $0 < T_1 < T_2$, we use interval arithmetic to verify the inequality (36) for $T_1 \leq t \leq T_2$.*

3. *To prove inequality (36) for $t \in [0,T_1]$, we both prove that the inequality holds at $t = 0$ (explained below), and we show using interval arithmetic that the derivative of the right-hand side of (36) is negative:*

$$\sum_{\substack{n \in I \\ 0 \leq k \leq m_s}} \gamma_k e^{\gamma_k t} G^n_{j,k} \rho_n < 0 \qquad \text{for } t \in [0,T_1].$$

To prove that inequality (36) holds at $t = 0$, we fix $j \in I$. If G is the final output of Algorithm 3.11, then there is a tensor $\widetilde{G} \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ for which $G^n_{j,k} \leftarrow \left[\mathcal{T}_j \circ \mathcal{Q}_j(\widetilde{G})\right]^n_k$, and it is assigned at step j of the inner for-loop of the algorithm, and step $N_{bootstrap}$ of the outer for-loop. Let $\bar{G} := \mathcal{Q}_j(\widetilde{G})$, then it follows from the definition of $\mathcal{T}_j$ in (29) that

$$\sum_{\substack{n \in I \\ 0 \leq k \leq m_s}} e^{\gamma_k t} G^n_{j,k} |\xi_n| = |\xi_j| e^{\gamma_j t} + \sum_{\substack{n,i \in I, i \neq j \\ 0 \leq k \leq m_s, k \neq j}} \frac{e^{\gamma_k t} - e^{\gamma_j t}}{\gamma_k - \gamma_j} H^i_j \bar{G}^n_{i,k} |\xi_n|.$$

Evaluating at $t = 0$, we then have:

$$|x_j(0,\xi,\alpha)| = |\xi_j| = \sum_{0 \leq k \leq m_s} G^n_{j,k} |\xi_n|.$$

By then taking $|\xi_n| = \rho_n$ for all $n \in I$, it follows that $\rho_j = \sum_{0 \leq k \leq m_s} G^n_{j,k} \rho_n$. Hence (36) is satisfied at $t = 0$ for all $j \in I$.

Remark 3.15. *When inequality* (36) *fails to be true, then we cannot be sure that all solutions to* (16) *stay inside the ball* $B_s(\rho)$ *for all time. There are two common reasons for why this happens: first, the nonlinearity may be too large and solutions leave the ball never to return; second, solutions to* (16) *may temporarily leave the ball, reenter, and then converge to zero.*

If inequality (36) *fails to be true because of the first reason, then* ρ *should be made smaller. If inequality* (36) *fails to be true because of the second reason, it is often because* $B_s(\rho)$ *is too wide in one direction and too thin in another. If we suspect this to be the case, then, in order to better align the box with the flow, we iteratively select a new value of* ρ *using the map* $\rho_j \mapsto \sup_{0\le t\le T} \sum_k e^{\gamma_k t} G^n_{j,k}\rho_n$*. In practice, this heuristic is often effective at finding a value of* ρ *for which* (36) *is satisfied.*

Algorithm 3.11 can be applied in more general situations. The two fundamental conditions necessary to construct such an algorithm are Condition 3.4 and Proposition 3.6. These are all generalized in Appendix A so that we can apply such an algorithm in Section 4.2 to obtain bounds on $\frac{\partial}{\partial \xi_i} x(t,\xi,\alpha)$, and in Section 5 to construct bounds on $|x(t,\xi,\alpha)-x(t,\xi,\beta)|$ for charts $\alpha,\beta \in \mathcal{B}^{0,1}_{\rho,P}$.

4 Lyapunov-Perron Operator

In Section 3 we derived conditions on $\mathcal{B}^{0,1}_{\rho,P}$ for showing that solutions to (16) stay within the ball $B_s(\rho)$ and exist for all positive time. As a consequence the integrand in (17), and moreover the Lyapunov-Perron operator $\Psi[\alpha]$, is well defined. In Section 5 we aim to prove that Ψ is a contraction mapping on an appropriately defined domain. But first, in Section 4, we prove that Ψ is an endomorphism on balls $\mathcal{B}^{0,1}_{\rho,P}$ and $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ for appropriately chosen constants.

Remark 4.1. *Throughout this section, we fix a positive vector* $\rho \in \mathbb{R}^{m_s}$ *and a positive tensor* $P \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$*, and fix* $G \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ *as the output of Algorithm 3.11 taken with* $N_{bootstrap} \ge 1$*. Furthermore, we assume that the hypotheses of Proposition 3.13 are satisfied, in particular inequality* (36) *holds for all* $t \ge 0$*. Hence* G *satisfies Condition 3.4 on the interval* $[0,\infty)$*.*

Throughout this section we adopt Einstein summation convention for indices in I and I'.

4.1 Endomorphism on $\mathcal{B}^{0,1}_{\rho,P}$

In Section 3 we did the hard work of bounding solutions to (16). We may now use these estimates to compute a straightforward bound on $\mathrm{Lip}(\Psi[\alpha])$ for $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$.

Theorem 4.2. *Define* $\tilde{P} \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$ *component-wise by:*

$$\tilde{P}^n_{i'} := \sum_{0\le k\le m_s} (\lambda_{i'} - \gamma_k)^{-1} H^i_{i'} G^n_{i,k}.$$

If $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$*, then* $\mathrm{Lip}(\Psi[\alpha])^n_{i'} \le \tilde{P}^n_{i'}$*. If* $\tilde{P}^j_{j'} \le P^j_{j'}$ *then* $\Psi : \mathcal{B}^{0,1}_{\rho,P} \to \mathcal{B}^{0,1}_{\rho,P}$ *is well defined.*

Proof. Fix a map $\alpha \in \mathcal{B}^{0,1}_{\rho,P}$. Fix $\xi,\zeta \in B_s(\rho)$ and define $x(t) := x(t,\xi,\alpha)$ and $z(t) := x(t,\zeta,\alpha)$. We aim to prove that $|\Psi[\alpha]_{i'}(\xi) - \Psi[\alpha]_{i'}(\zeta)| \le \tilde{P}^n_{i'}\,|\xi_n - \zeta_n|$. From the definition of Ψ we have

$$\Psi[\alpha](\xi) - \Psi[\alpha](\zeta) = -\int_0^\infty e^{-\Lambda_u t}\left[\mathcal{N}_u(x(t),\alpha(x(t))) - \mathcal{N}_u(z(t),\alpha(z(t)))\right] dt.$$

By using the bound (14) and the fact that G satisfies Condition 3.4 on $[0,\infty)$, we obtain

$$\begin{aligned}|\Psi[\alpha]_{i'}(\xi)-\Psi[\alpha]_{i'}(\zeta)| &\le \int_0^\infty e^{-\lambda_{i'}t}H^i_{i'}|x_i(t)-z_i(t)|dt\\ &\le \int_0^\infty e^{-\lambda_{i'}t}\sum_{0\le k\le m_s} e^{\gamma_k t}H^i_{i'}G^n_{i,k}\,|\xi_n-\zeta_n|\,dt\\ &= \sum_{0\le k\le m_s}(\lambda_{i'}-\gamma_k)^{-1}H^i_{i'}G^n_{i,k}\,|\xi_n-\zeta_n|\,.\end{aligned}$$

For $\tilde{P}^n_{i'}$ defined above, it follows that

$$|\Psi[\alpha]_{i'}(\xi)-\Psi[\alpha]_{i'}(\zeta)|\le \tilde{P}^n_{i'}\,|\xi_n-\zeta_n|\,.$$

Hence $\mathrm{Lip}(\Psi[\alpha])^n_{i'}\le \tilde{P}^n_{i'}$. Since $\mathcal{N}(0)=0$ then direct evaluation reveals that $\Psi[\alpha](0)=0$, hence $\Psi[\alpha]\in\mathcal{B}^{0,1}_{\rho,P}$. □

Remark 4.3. *Ideally, we would like to choose a tensor P as small as possible while still satisfying the inequality $\tilde{P}^j_{i'}\le P^j_{i'}$. In practice, we are often able to find a nearly optimal P by iteratively mapping $P^j_{i'}\mapsto\tilde{P}^j_{i'}$. This has the effect that if $\tilde{P}^j_{i'}\le P^j_{i'}$, then the new value of P will be smaller. Since the bounds for H and G improve with smaller P, then the inequality $\tilde{P}^j_{i'}\le P^j_{i'}$ will likely be satisfied for the new P. On the other hand, if P is too small and $\tilde{P}^j_{i'}\le P^j_{i'}$ is not satisfied, then the new value of P will be larger, so the inequality will hopefully be satisfied the next time around.*

Note that the definitions of H and G depend on P, and so these constants need to be recomputed every time. Nevertheless, this iterative process provides an effective, algorithmic method for selecting appropriate $P^j_{i'}$.

By using second derivative bounds on $\mathcal{N}_u$, we can sharpen Theorem 4.2 as below.

Proposition 4.4. *Define $\tilde{P}\in\mathbb{R}^{m_u}\otimes\mathbb{R}^{m_s}$ component-wise by:*

$$\begin{aligned}\tilde{P}^n_{i'} := &\left(D^i_{i'}+D^{j'}_{i'}P^i_{j'}\right)\sum_{0\le k\le m_s}(\lambda_{i'}-\gamma_k)^{-1}G^n_{i,k}\\ &+\left(\hat{C}^{ij}_{i'}+\hat{C}^{j'j}_{i'}P^i_{j'}\right)\sum_{0\le k_1,k_2\le m_s}(\lambda_{i'}-\gamma_{k_1}-\gamma_{k_2})^{-1}G^m_{j,k_1}G^n_{i,k_2}\rho_m.\end{aligned}$$

If $\alpha\in\mathcal{B}^{0,1}_{\rho,P}$, then $\mathrm{Lip}(\Psi[\alpha])^n_{i'}\le\tilde{P}^n_{i'}$. If $\tilde{P}^j_{j'}\le P^j_{j'}$ then $\Psi:\mathcal{B}^{0,1}_{\rho,P}\to\mathcal{B}^{0,1}_{\rho,P}$ is well defined.

Proof. By the mean value theorem we have (recall that $\mathcal{N}^i_{i'}=\frac{\partial}{\partial x_i}\mathcal{N}_{i'}$)

$$|\mathcal{N}_{i'}(x,\alpha(x))-\mathcal{N}_{i'}(z,\alpha(z))|\le\left[\sup_{\substack{y\in B_s(\rho),\ j\in I\\ |y_j|\le\max\{|x_j|,|z_j|\}}}\|\mathcal{N}^i_{i'}(y,\alpha(y))\|\right]|x_i-z_i|.$$

We can estimate $\max\{|x_j(t)|,|z_j(t)|\}$ using the tensor G (which satisfies Condition 3.4), and since $\max\{|\xi_m|,|\zeta_m|\}\le\rho_m$ we obtain the following:

$$\begin{aligned}\sup_{\substack{y\in B_s(\rho),\ j\in I\\ |y_j|\le\max\{|x_j(t)|,|z_j(t)|\}}}\|\mathcal{N}^i_{i'}(y,\alpha(y))\| &\le D^i_{i'}+D^{j'}_{i'}P^i_{j'}+(\hat{C}^{ij}_{i'}+\hat{C}^{j'j}_{i'}P^i_{j'})\max\{|x_j(t)|,|z_j(t)|\}\\ &\le D^i_{i'}+D^{j'}_{i'}P^i_{j'}+(\hat{C}^{ij}_{i'}+\hat{C}^{j'j}_{i'}P^i_{j'})\sum_{0\le k\le m_s}e^{\gamma_k t}G^m_{j,k}\rho_m.\end{aligned}$$

Thus, once again using Condition 3.4, we obtain the following estimate:

$$|\mathcal{N}_{i'}(x,\alpha(x)) - \mathcal{N}_{i'}(z,\alpha(z))| \leq \left(D^i_{i'} + D^{j'}_{i'} P^i_{j'}\right) \sum_{0\leq k\leq m_s} e^{\gamma_k t} G^n_{i,k} \, |\xi_n - \zeta_n| + \left(\hat{C}^{ij}_{i'} + \hat{C}^{j'j}_{i'} P^i_{j'}\right) \sum_{0\leq k_1,k_2\leq m_s} e^{(\gamma_{k_1}+\gamma_{k_2})t} G^m_{j,k_1} G^n_{i,k_2} \rho_m \, |\xi_n - \zeta_n| \, .$$

We then obtain the desired result by integration:

$$\begin{aligned} |\Psi[\alpha]_{i'}(\xi) - \Psi[\alpha]_{i'}(\zeta)| &\leq \int_0^\infty e^{-\lambda_{i'} t} |\mathcal{N}_{i'}(x,\alpha(x)) - \mathcal{N}_{i'}(z,\alpha(z))| \, dt \\ &\leq \left(D^i_{i'} + D^{j'}_{i'} P^i_{j'}\right) \sum_{0\leq k\leq m_s} (\lambda_{i'} - \gamma_k)^{-1} G^n_{i,k} \, |\xi_n - \zeta_n| \\ &\quad + \left(\hat{C}^{ij}_{i'} + \hat{C}^{j'j}_{i'} P^i_{j'}\right) \sum_{0\leq k_1,k_2\leq m_s} (\lambda_{i'} - \gamma_{k_1} - \gamma_{k_2})^{-1} G^m_{j,k_1} G^n_{i,k_2} \rho_m \, |\xi_n - \zeta_n| \, . \end{aligned}$$

□

4.2 Endomorphism on $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$

We aim to bound the Lipschitz constant of the derivative of a stable manifold. To do this, we show that Ψ maps $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$, a ball of functions with Lipschitz derivative, into itself. Hence, if there are any fixed points $\Psi[\alpha] = \alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$, then by Definition 2.5 they satisfy $\mathrm{Lip}(\partial_i \alpha)^j_{i'} \leq \bar{P}^{ij}_{i'}$. To show that $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ we first derive bounds on the difference $\frac{\partial}{\partial \xi_i} x_j(t,\eta,\alpha) - \frac{\partial}{\partial \xi_i} x_j(t,\zeta,\alpha)$ for $i,j \in I$. In particular, we are interested in finding a tensor K satisfying the following condition:

Condition 4.5. *Define* $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_k\}_{k=0}^{m_s} \cup \{\gamma_{k_1} + \gamma_{k_2}\}_{k_1,k_2=0}^{m_s}$. *A tensor* $K \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ *is said to satisfy Condition 4.5 if*

$$\left\| \frac{\partial}{\partial \xi_i} x_j(t,\eta,\alpha) - \frac{\partial}{\partial \xi_i} x_j(t,\zeta,\alpha) \right\| \leq \sum_{k=1}^{N_\mu} e^{\mu_k t} K^{il}_{j,k} |\eta_l - \zeta_l|,$$

for all $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ *and* $\eta, \zeta \in B_s(\rho)$ *and* $i,j \in I$.

We obtain such bounds using an approach analogous to the one used in Section 3. Since we use this approach in Sections 3, 4, and 5, we present in Appendix A a generalization which encompasses all cases. In Proposition 4.6 we define a tensor S somewhat analogous to H given in Definition 2.6. In Proposition 4.7 we derive an *a priori* bound, constructing an initial tensor K satisfying Condition 4.5 (cf. Proposition 3.2). In Proposition 4.9 we derive a system of integral inequalities (cf. Proposition 3.6 and Condition A.2). From this, as described in Theorem 4.10, we can apply Algorithm A.5 (cf. Algorithm 3.11) to bootstrap Gronwall's inequality, and obtain successively sharper tensors K satisfying Condition 4.5. Finally, in Proposition 4.11, we give conditions guaranteeing that $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ is a well defined map.

Proposition 4.6. *Let* $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ *and* $\eta,\zeta \in B_s(\rho)$. *Define* $x = x(t,\eta,\alpha)$ *and* $z = x(t,\zeta,\alpha)$. *Define* $x^i_j = \frac{\partial}{\partial \xi_i} x_j(t,\eta,\alpha)$ *and likewise for* z^i_j. *We fix* $\mathbf{j} \in \mathbf{I}$ *and define*

$$S^{nm}_{\mathbf{j}} := (C^{nm}_{\mathbf{j}} + C^{nm'}_{\mathbf{j}} P^m_{m'}) + C^{n'}_{\mathbf{j}} P^{nm}_{n'} + (C^{n'm}_{\mathbf{j}} + C^{n'm'}_{\mathbf{j}} P^m_{m'}) P^n_{n'}.$$

Then we have

$$\left\| \frac{\partial}{\partial \xi_i} \left(\mathcal{N}_{\mathbf{j}}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}(z,\alpha(z)) \right) \right\| \leq S^{nm}_{\mathbf{j}} |x_m - z_m| \, \|z^i_n\| + H^n_{\mathbf{j}} \|x^i_n - z^i_n\|.$$

Proof. We have

$$\frac{\partial}{\partial \xi_i}\mathcal{N}_{\mathbf{j}}(x,\alpha(x)) = \Big(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\Big)\cdot x_n^i. \tag{37}$$

We may split the estimate into four parts as follows:

$$\begin{aligned}\frac{\partial}{\partial \xi_i}\Big(\mathcal{N}_{\mathbf{j}}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}(z,\alpha(z))\Big) = {} & \left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n}(z,\alpha(z))\right)\cdot z_n^i \\ & + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\left(\alpha_{n'}^{n}(x) - \alpha_{n'}^{n}(z)\right) z_n^i \\ & + \Big(\mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n'}(z,\alpha(z))\Big)\alpha_{n'}^{n}(z) z_n^i \\ & + \Big(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\Big)\cdot (x_n^i - z_n^i).\end{aligned}$$

We bound each term seperately:

$$\begin{aligned}\left(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n}(z,\alpha(z))\right)\cdot z_n^i &\le (C_{\mathbf{j}}^{nm} + C_{\mathbf{j}}^{nm'}P_{m'}^{m})|x_m - z_m|\,\|z_n^i\|, \\ \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\left(\alpha_{n'}^{n}(x) - \alpha_{n'}^{n}(z)\right) z_n^i &\le C_{\mathbf{j}}^{n'}P_{n'}^{nm}|x_m - z_m|\,\|z_n^i\|, \\ \Big(\mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x)) - \mathcal{N}_{\mathbf{j}}^{n'}(z,\alpha(z))\Big)\alpha_{n'}^{n}(z) z_n^i &\le (C_{\mathbf{j}}^{n'm} + C_{\mathbf{j}}^{n'm'}P_{m'}^{m})P_{n'}^{n}|x_m - z_m|\,\|z_n^i\|, \\ \Big(\mathcal{N}_{\mathbf{j}}^{n}(x,\alpha(x)) + \mathcal{N}_{\mathbf{j}}^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\Big)(x_n^i - z_n^i) &\le (C_{\mathbf{j}}^{n} + C_{\mathbf{j}}^{n'}P_{n'}^{n})\,\|x_n^i - z_n^i\|.\end{aligned}$$

The result follows by collecting all terms. ☐

Proposition 4.7. *Define a tensor* $\widetilde{K} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes (\mathbb{R}^{m_s+1})^{\otimes 2}$ *as*

$$\widetilde{K}_{j,k_1k_2}^{il} = (\gamma_{k_1} + \gamma_{k_2} - \gamma_0)^{-1} C_s p_j S_j^{nm} G_{m,k_1}^{l} G_{n,k_2}^{i}.$$

Then we have

$$\left\|\tfrac{\partial}{\partial \xi_i}x(t,\eta,\alpha) - \tfrac{\partial}{\partial \xi_i}x(t,\zeta,\alpha)\right\| \le \sum_{\substack{0\le k_1,k_2\le m_s \\ j\in I}} \Big(e^{(\gamma_{k_1}+\gamma_{k_2})t} - e^{\gamma_0 t}\Big)\,\widetilde{K}_{j,k_1k_2}^{il}|\eta_l - \zeta_l|,$$

for all $\alpha \in \mathcal{B}_{\rho,P,\bar{P}}^{1,1}$ *and* $\eta,\zeta \in B_s(\rho)$ *and* $i \in I$.

The indices in tensor notation $\widetilde{K}_{j,k_1k_2}^{il}$ may be interpreted as follows. The superscripts correspond to derivatives, the subscript to the left of the comma corresponds to subspace projections, while the subscript to the right of the comma correspond to exponentials.

Proof. Define $x = x(t,\eta,\alpha)$ and $z = x(t,\zeta,\alpha)$. Let $x^i = \frac{\partial}{\partial \xi_i}x(t,\eta,\alpha)$ and likewise for z^i. By variation of constants, we have that

$$\begin{aligned}x^i(t) - z^i(t) = {} & \int_0^t e^{(\Lambda_s + L_s^s)(t-\tau)}\frac{\partial}{\partial \xi_i}L_s^u\big(\alpha(x(\tau)) - \alpha(z(\tau))\big)d\tau. \\ & + \int_0^t e^{(\Lambda_s + L_s^s)(t-\tau)}\frac{\partial}{\partial \xi_i}\Big(\hat{\mathcal{N}}_s(x(\tau),\alpha(x(\tau))) - \hat{\mathcal{N}}_s(z(\tau),\alpha(z(\tau)))\Big)\,d\tau.\end{aligned} \tag{38}$$

We expand the partial derivatives appearing in (38), dropping the τ dependence in the notation in the right hand side:

$$\begin{aligned}\frac{\partial}{\partial \xi_i}L_s^u\alpha(x(\tau)) &= \sum_{j\in I} L_j^{n'}\alpha_{n'}^{n}(x)x_n^i \\ \frac{\partial}{\partial \xi_i}\hat{\mathcal{N}}_s\big(x(\tau),\alpha(x(\tau))\big) &= \sum_{j\in I}\Big(\hat{\mathcal{N}}_j^{n}(x,\alpha(x)) + \hat{\mathcal{N}}_j^{n'}(x,\alpha(x))\alpha_{n'}^{n}(x)\Big)\cdot x_n^i.\end{aligned}$$

In Proposition 4.6 we demonstrated how the tensor S offers a $C^{1,1}$ bound on $\mathcal{N}_{\mathbf{j}} = L^s_{\mathbf{j}} + L^u_{\mathbf{j}} + \hat{\mathcal{N}}_{\mathbf{j}}$, for $\mathbf{j} \in \mathbf{I}$. By using (8) we obtain, in analogy with the proof of Proposition 4.6,

$$e^{-\lambda_s t}\|x^i - z^i\| \le \int_0^t C_s e^{-\lambda_s \tau} \sum_{j\in I} p_j S_j^{nm} |x_m - z_m|\, \|z^i_n\| d\tau + \int_0^t e^{-\lambda_s\tau} C_s \hat{\mathcal{H}} \|x^i - z^i\| d\tau.$$

It then follows from Proposition 3.12 that

$$e^{-\lambda_s t}\|x^i - z^i\| \le \int_0^t C_s e^{-\lambda_s \tau} \sum_{\substack{0\le k_1,k_2\le m_s\\ j\in I}} e^{(\gamma_{k_1}+\gamma_{k_2})\tau} p_j S_j^{nm} G^l_{m,k_1} G^i_{n,k_2} |\eta_l - \zeta_l| d\tau$$
$$+ \int_0^t e^{-\lambda_s\tau} C_s \hat{\mathcal{H}} \|x^i - z^i\| d\tau.$$

By Lemma 3.9 we infer that

$$\|x^i - z^i\| \le \sum_{\substack{0\le k_1,k_2\le m_s\\ j\in I}} \frac{e^{(\gamma_{k_1}+\gamma_{k_2})t} - e^{\gamma_0 t}}{\gamma_{k_1}+\gamma_{k_2}-\gamma_0} C_s p_j S_j^{nm} G^l_{m,k_1} G^i_{n,k_2} |\eta_l - \zeta_l|. \qquad \square$$

Remark 4.8. *Define* $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_{k_1}\}_{k_1=0}^{m_s} \cup \{\gamma_{k_1}+\gamma_{k_2}\}_{k_1,k_2=0}^{m_s}$*, with* $N_\mu = (m_s+1)(m_s+4)/2$*. Let* $\widetilde{K}$ *be defined as in Proposition 4.7, and define a tensor* $\widehat{K} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ *by*

$$\widehat{K}^{il}_{j,k} := \begin{cases} p_j \sum_{m\in I} \widetilde{K}^{il}_{m,k_1k_2} + \widetilde{K}^{il}_{m,k_2k_1} & \text{if } \mu_k = \gamma_{k_1}+\gamma_{k_2} \text{ for } 0\le k_1,k_2\le m_s,\\ -p_j \sum_{m\in I}\sum_{0\le k_1,k_2\le m_s} \widetilde{K}^{il}_{m,k_1k_2} + \widetilde{K}^{il}_{m,k_2k_1} & \text{if } \mu_k = \gamma_0,\\ 0 & \text{if } \mu_k = \gamma_{k_1}, \text{ for } 1 \le k_1 \le m_s. \end{cases}$$

It follows from Proposition 4.7 that $\widehat{K}$ *satisfies Condition 4.5.*

We now establish componentwise Lipschitz bounds on the derivatives.

Proposition 4.9. *Let* $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ *and define* $x(t) = x(t,\eta,\alpha)$ *and* $z(t) = z(t,\zeta,\alpha)$ *for some* $\eta,\zeta \in B_s(\rho)$*. Let* $x^i_j(t) = \frac{\partial}{\partial \xi_i} x_j(t,\eta,\alpha)$ *and likewise for* $z^i_j(t)$*. Then*

$$e^{-\lambda_j t}\|x^i_j - z^i_j\| \le \int_0^t e^{-\lambda_j\tau} \sum_{0\le k_1,k_2\le m_s} e^{(\gamma_{k_1}+\gamma_{k_2})\tau} S_j^{nm} G^l_{m,k_1} G^i_{n,k_2} |\eta_l - \zeta_l| d\tau$$
$$+ \int_0^t e^{-\lambda_j\tau} H^n_j \|x^i_n - z^i_n\| d\tau.$$

Proof. By variation of constants, we have that

$$x^i_j(t) = e^{\Lambda_j t}\delta^i_j + \int_0^t e^{\Lambda_j(t-\tau)} \left(\frac{\partial}{\partial \xi_i}\mathcal{N}_j(x(\tau),\alpha(x(\tau)))\right) d\tau,$$

where δ^i_j is the Kronecker delta. Taking the difference $x^i_j - z^i_j$ we obtain

$$x^i(t) - z^i(t) = \int_0^t e^{\Lambda_s(t-\tau)} \frac{\partial}{\partial\xi_i}\Big(\mathcal{N}_j(x(\tau),\alpha(x(\tau))) - \mathcal{N}_j(z(\tau),\alpha(z(\tau)))\Big) d\tau.$$

From Proposition 4.6 we then have

$$e^{-\lambda_j t}\|x^i_j - z^i_j\| \le \int_0^t e^{-\lambda_j\tau} S_j^{nm} |x_m - z_m|\, \|z^i_n\| d\tau + \int_0^t e^{-\lambda_j\tau} H^n_j \|x^i_n - z^i_n\| d\tau.$$

Plugging in the bounds on $|x_m - z_m|$ and $\|z^i_n\|$ from Proposition 3.12, we obtain the desired result. $\square$

Theorem 4.10. *Let $\{\mu_k\}_{k=1}^{N_\mu}$ and let the tensor $\widehat{K} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ be as defined in Remark 4.8. When K is the output of Algorithm A.5 taken with input $\widehat{K}$ and some $N_{bootstrap} \geq 1$, then K satisfies Condition 4.5.*

The proof of Theorem 4.10 follows from the argument outlined in Appendix A, where Conditions A.1 and A.2 correspond to Proposition 4.9 and Condition 4.5 respectively.

Theorem 4.11. *Let $\bar{P} \in (\mathbb{R}^{m_s})^{\otimes 3}$ and assume $K \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{N_\mu}$ satisfies Condition 4.5. Define the tensor $\tilde{P} \in \mathbb{R}^{m_u} \otimes (\mathbb{R}^{m_s})^{\otimes 2}$ as*

$$\tilde{P}^{il}_{j'} := \sum_{0 \leq k_1, k_2 \leq m_s} (\lambda_{j'} - \gamma_{k_1} - \gamma_{k_2})^{-1} S^{nm}_{j'} G^l_{m,k_1} G^i_{n,k_2} + \sum_{1 \leq k \leq N_\mu} (\lambda_{j'} - \mu_k)^{-1} H^n_{j'} K^{il}_{n,k}. \tag{39}$$

Then for all $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ we have $\operatorname{Lip}(\partial_i \Psi[\alpha])^l_{j'} \leq \tilde{P}^{il}_{j'}$. If $\tilde{P}^{il}_{j'} \leq \bar{P}^{il}_{j'}$ then $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ is well defined.

Proof. Let $\eta, \zeta \in B_s(\rho)$ and define $x(t) = x(t, \eta, \alpha)$ and $z(t) = x(t, \zeta, \alpha)$. Define $x^i_j(t) = \frac{\partial}{\partial \xi_i} x_j(t, \eta, \alpha)$ and likewise for $z^i_j(t)$. From Definition 2.8 we have

$$\Psi[\alpha](\eta) - \Psi[\alpha](\zeta) = -\int_0^\infty e^{-\Lambda_u t}\big(\mathcal{N}_u(x(t), \alpha(x(t))) - \mathcal{N}_u(z(t), \alpha(z(t)))\big)dt.$$

By using Proposition 4.6 we obtain

$$\left\|\Psi[\alpha]^i_{j'}(\eta) - \Psi[\alpha]^i_{j'}(\zeta)\right\| \leq \int_0^\infty e^{-\lambda_{j'} t}\left(S^{nm}_{j'} |x_m - z_m| \, \|z^i_n\| + H^n_{j'} \|x^i_n - z^i_n\|\right) dt.$$

Plugging in the bounds on $|x_m - z_m|$ and $\|z^i_n\|$ from Proposition 3.12, as well as the bounds on $|x^i_n - z^i_n|$ from Proposition 4.9, we infer that

$$\begin{aligned} \left\|\Psi[\alpha]^i_{j'}(\eta) - \Psi[\alpha]^i_{j'}(\zeta)\right\| \leq & \int_0^\infty e^{-\lambda_{j'} t} \sum_{0 \leq k_1, k_2 \leq m_s} e^{(\gamma_{k_1} + \gamma_{k_2})t} S^{nm}_{j'} G^l_{m,k_1} G^i_{n,k_2} |\xi_l - \zeta_l| dt \\ & + \int_0^\infty e^{-\lambda_{j'} t} \sum_{1 \leq k \leq N_\mu} e^{\mu_k t} H^n_{j'} K^{il}_{n,k} |\eta_l - \zeta_l| dt \\ = & \tilde{P}^{il}_{j'} |\eta_l - \zeta_l|. \end{aligned}$$

Hence, we have obtained the desired bound $\operatorname{Lip}(\partial_i \Psi[\alpha])^l_{j'} \leq \tilde{P}^{il}_{j'}$. □

5 Contraction Mapping

Remark 5.1. *Throughout this section we will suppose that all of the assumptions on the positive vector $\rho \in \mathbb{R}^{m_s}$, the positive tensor $P \in \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s}$, and the tensor $G \in (\mathbb{R}^{m_s})^{\otimes 2} \otimes \mathbb{R}^{m_s+1}$ made in Remark 4.1 are satisfied. Additionally, we fix a tensor $K \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{N_\mu}$ satisfying Condition 4.5, and a positive tensor $\bar{P} \in (\mathbb{R}^{m_s})^{\otimes 3}$. We assume that all the hypotheses of Theorem 4.4 and Theorem 4.11 are satisfied, so that both $\Psi : \mathcal{B}^{0,1}_{\rho,P} \to \mathcal{B}^{0,1}_{\rho,P}$ and $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ are well defined maps.*

5.1 Bounding the Difference Between Two Projected Systems

We wish to show that the Lyapunov-Perron operator is a contraction mapping when considered with an appropriate norm. That is, for two maps $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$ the distance between $\Psi[\alpha]$ and $\Psi[\beta]$ is smaller than the distance between α and β. The norm we will use is somewhat weaker than the notion used to define $\mathcal{B}^{0,1}_{\rho,P}$ in Definition 2.5.

Definition 5.2. *For $\alpha \in \mathcal{E} := \{\alpha \in Lip(B_s(\rho), X_u) : \alpha(0) = 0\}$ define the following semi-norms:*

$$\|\alpha\|^i_{i'\mathcal{E}} := \sup_{\xi \in B_s(r); \xi_i \neq 0} \frac{|\alpha_{i'}(\xi) - \alpha_{i'}(\xi - \xi_i)|}{|\xi_i|}.$$

where $i \in I$ and $i' \in I'$. Taken together these semi-norms define a norm:

$$\|\alpha\|_{\mathcal{E}} := \sum_{i \in I, i' \in I'} \|\alpha\|^i_{i'\mathcal{E}}.$$

Note that $\|\alpha\|^i_{i'\mathcal{E}} \leq Lip(\alpha)^i_{i'}$ and $|\alpha(\xi)| \leq \sum_{i' \in I'} \|\alpha\|^i_{i'\mathcal{E}} |\xi_i| \leq \|\alpha\|_{\mathcal{E}} |\xi| \big(\max_{i \in I} p_i\big)$. With this norm both $\mathcal{B}^{0,1}_{\rho,P}$ and $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ are complete metric spaces (cf. [15, Chapter 4]).

Before we can show that Ψ is a contraction, we first need to derive estimates on $x(t, \xi, \alpha) - x(t, \xi, \beta)$, the difference between two solutions to the projected system in (16) using different maps $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$. Classically, this results in an estimate of the form $|x(t, \xi, \alpha) - x(t, \xi, \beta)| \leq k e^{\gamma t} |\xi| \|\alpha - \beta\|_{\mathcal{E}}$, for some constants k, γ. This estimate can be notably tightened, as at time zero $|x(0, \xi, \alpha) - x(0, \xi, \beta)| = |\xi - \xi| = 0$. To that end, we are interested in providing a bound on $|x(t, \xi, \alpha) - x(t, \xi, \beta)|$ using a tensor F as described below.

Condition 5.3. *Fix some $\gamma_{-1} > \gamma_0$ and define $\{\mu_k\}_{k=1}^{m_s+2} = \{\gamma_k\}_{k=-1}^{m_s}$. A tensor $F \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ is said to satisfy Condition 5.3 if:*

$$|x_m(t, \xi, \alpha) - x_m(t, \xi, \beta)| \leq \sum_{-1 \leq k \leq m_s} e^{\gamma_k t} F^{ni'}_{mi,k} \|\alpha - \beta\|^i_{i'\mathcal{E}} |\xi_n|.$$

for all $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$ and $\xi \in B_s(\rho)$ and $m \in I$.

We may obtain such a tensor F by applying our bootstrapping method, as we did in Sections 3 and 4, which is presented in a general setting in Appendix A. However, in this section we will encounter a slight resonance problem with γ_0, and we augment $\{\gamma_k\}_{k=1}^{m_s}$, defining

$$\gamma_{-1} := \gamma_0/2.$$

In this manner we obtain an indexed set $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_k\}_{k=-1}^{m_s}$. The exact choice of γ_{-1} is somewhat arbitrary; it should satisfy $\lambda_{1'} > \gamma_{-1} > \gamma_0$, and $(\gamma_{-1} - \gamma_0)^{-1}$ should not be too large. We augment the tensor G fixed in Remark 4.1 by defining $G^n_{i,-1} = 0$ for all $i, n \in I$. To overcome the resonance problem we will use the map $\mathcal{Q}_0$ (following the notation convention from Appendix A) defined as

$$\mathcal{Q}_0(G)^n_{i,k} = \begin{cases} G^n_{i,0} & \text{if } k = -1 \\ 0 & \text{if } k = 0 \\ G^n_{i,k} & \text{if } 1 \leq k \leq m_s \end{cases} \qquad \text{for } i, n \in I. \tag{40}$$

In Proposition 5.4 and Remark 5.5 below, we identify an initial tensor $\widehat{F}$ which satisfies Condition 5.3.

Proposition 5.4. *Fix maps $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$ and fix some $\gamma_{-1} > \gamma_0$ and define $\mathcal{Q}_0$ as in (40). Define the tensor $\widetilde{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ as*

$$\widetilde{F}^{ni'}_{ji,k} := \begin{cases} C_s (\gamma_k - \gamma_0)^{-1} p_j C^{i'}_j \, \mathcal{Q}_0(G)^n_{i,k} & \text{if } k \neq 0, \\ 0 & \text{if } k = 0. \end{cases}$$

Then we have

$$|x(t, \xi, \alpha) - x(t, \xi, \beta)| \leq \sum_{-1 \leq k \leq m_s, j \in I} \left(e^{\gamma_k t} - e^{\gamma_0 t}\right) \widetilde{F}^{ni'}_{ji,k} \|\alpha - \beta\|^i_{i'\mathcal{E}} |\xi_n|,$$

for all $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$, and $\xi \in B_s(\rho)$.

Proof. Fix an initial condition $\xi \in B_s(\rho)$ and define $x(t) := x(t,\xi,\alpha)$ and $y(t) := x(t,\xi,\beta)$. We compute $|x(t) - y(t)|$ by variation of constants:

$$x(t) - y(t) = \int_0^t e^{(\Lambda_s + L_s^s)(t-\tau)} \Big(L_s^u \alpha(x(\tau)) + \hat{\mathcal{N}}_s(x(\tau), \alpha(x(\tau))) - L_s^u \beta(y(\tau)) - \hat{\mathcal{N}}_s(y(\tau), \beta(y(\tau))) \Big)\, d\tau.$$

By applying the usual splitting $\alpha(x) - \beta(y) = [\alpha(x) - \alpha(y)] + [\alpha(y) - \beta(y)]$ and using the definition of $\hat{\mathcal{H}}$ we obtain

$$\begin{aligned}\Big| L_s^u \alpha(x) + \hat{\mathcal{N}}_s(x, \alpha(x)) - L_s^u \beta(y) - \hat{\mathcal{N}}_s(y, \beta(y)) \Big| &\le \hat{\mathcal{H}} |x - y| \\ &\quad + \Big| L_s^u \alpha(y) + \hat{\mathcal{N}}_s(y, \alpha(y)) - L_s^u \beta(y) - \hat{\mathcal{N}}_s(y, \beta(y)) \Big|.\end{aligned}$$

To shorten notation we set $E_{i'}^i := \|\alpha - \beta\|_{i'\mathcal{E}}^i$. Since $|\alpha_{i'}(y) - \beta_{i'}(y)| \le E_{i'}^i |y_i|$ we have

$$\Big| L_s^u \alpha(y) + \hat{\mathcal{N}}_s(y, \alpha(y)) - L_s^u \beta(y) - \hat{\mathcal{N}}_s(y, \beta(y)) \Big| \le \sum_{j \in I} p_j (\hat{C}_j^{i'} + D_j^{i'}) E_{i'}^i |y_i|.$$

Combining these estimates we obtain

$$e^{-\lambda_s t} |x(t) - y(t)| \le \int_0^t C_s e^{-\lambda_s \tau} \sum_{j \in I} p_j C_j^{i'} E_{i'}^i |y_i(\tau)| d\tau + \int_0^t C_s e^{-\lambda_s \tau} \hat{\mathcal{H}} |x(\tau) - y(\tau)| d\tau.$$

We would now like to substitute in our bound $|y_i(\tau)| \le \sum_{0 \le k \le m_s} e^{\gamma_k \tau} G_{i,k}^n |\xi_n|$ from Theorem 3.12 and apply Lemma 3.9. However, this integral inequality will encounter a resonance problem with γ_0, which we overcome by replacing G by $\mathcal{Q}_0(G)$. We then obtain

$$\begin{aligned} e^{-\lambda_s t} |x(t) - y(t)| \le \int_0^t C_s e^{-\lambda_s \tau} & \sum_{-1 \le k \le m_s; j \in I} p_j C_j^{i'} E_{i'}^i e^{\gamma_k \tau} \mathcal{Q}_0(G)_{i,k}^n |\xi_n| d\tau \\ & + \int_0^t C_s e^{-\lambda_s \tau} \hat{\mathcal{H}} |x(\tau) - y(\tau)| d\tau. \end{aligned}$$

Now we apply Lemma 3.9 and infer that

$$|x(t) - y(t)| \le C_s \sum_{-1 \le k \le m_s; j \in I} \frac{e^{\gamma_k t} - e^{\gamma_0 t}}{\gamma_k - \gamma_0} p_j C_j^{i'} \mathcal{Q}_0(G)_{i,k}^n E_{i'}^i |\xi_n|. \qquad \square$$

Remark 5.5. *For some fixed $\gamma_{-1} > \gamma_0$, define the tensor $\widetilde{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ as in Proposition 5.4. Define the tensor $\widehat{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ by*

$$\widehat{F}_{mi,k}^{ni'} := \begin{cases} p_m \sum_{j \in I} \widetilde{F}_{ji,k}^{ni'} & \text{if } k \neq 0, \\ -p_m \sum_{j \in I} \sum_{-1 \le k_1 \le m_s} \widetilde{F}_{ji,k_1}^{ni'} & \text{if } k = 0. \end{cases}$$

It follows that $\widehat{F}$ satisfies Condition 5.3.

To turn the initial norm estimate from Proposition 5.4 into more refined estimates on the components we use the following auxiliary proposition.

Proposition 5.6. *Fix $\alpha, \beta \in \mathcal{B}_{\rho,P}^{0,1}$ and fix an initial condition $\xi \in B_s$, and define*

$$\begin{aligned} u_i(t) &:= |x_i(t,\xi,\alpha) - x_i(t,\xi,\beta)| \\ E_{i'}^i &:= \|\alpha - \beta\|_{i'\mathcal{E}}^i \\ V_j(t) &:= \int_0^t e^{-\lambda_j \tau} \sum_{0 \le k \le m_s} e^{\gamma_k \tau} E_{i'}^i C_j^{i'} G_{i,k}^n |\xi_n| \, d\tau. \end{aligned}$$

Then we have

$$e^{-\lambda_j t} u_j(t) \le V_j(t) + \int_0^t e^{-\lambda_j \tau} H^i_j u_i(\tau) d\tau \,. \tag{41}$$

Proof. Let $x(t) := x(t, \xi, \alpha)$ and $y(t) := x(t, \xi, \beta)$. We compute $|x_j(t) - y_j(t)|$ for $j \in I$ by variation of constants:

$$x_j(t) - y_j(t) = \int_0^t e^{\Lambda_j (t-\tau)} \big(\mathcal{N}_j(x(\tau), \alpha(x(\tau))) - \mathcal{N}_j(y(\tau), \beta(y(\tau))) \big) d\tau.$$

By the triangle inequality we obtain

$$\begin{aligned} |\alpha_{i'}(x) - \beta_{i'}(y)| &\le |\alpha_{i'}(y) - \beta_{i'}(y)| + |\alpha_{i'}(x) - \alpha_{i'}(y)| \\ &\le \|\alpha - \beta\|^i_{i'\mathcal{E}} |y_i| + P^i_{i'} |x_i - y_i|, \end{aligned}$$

hence

$$|\mathcal{N}_j(x, \alpha(x)) - \mathcal{N}_j(y, \beta(y))| \le C^{i'}_j E^i_{i'} |y_i| + H^i_j |x_i - y_i|. \tag{42}$$

By applying our bounds from Theorem 3.12 we obtain

$$\begin{aligned} e^{-\lambda_j t} |x_j - y_j| &\le \int_0^t e^{-\lambda_j \tau} \left(C^{i'}_j E^i_{i'} |y_i| + H^i_j |x_i - y_i| \right) d\tau \\ &= \int_0^t e^{-\lambda_j \tau} C^{i'}_j E^i_{i'} |y_i| d\tau + \int_0^t e^{-\lambda_j \tau} H^i_j |u_i| d\tau \\ &\le \int_0^t e^{-\lambda_j \tau} \sum_{0 \le k \le m_s} C^{i'}_j E^i_{i'} e^{\gamma_k \tau} G^n_{i,k} |\xi_n| ds + \int_0^t e^{-\lambda_j \tau} H^i_j u_i(\tau) d\tau. \end{aligned}$$

With our initial definition of $V_j(t)$, the above inequality is of the form stated in (41).

□

Theorem 5.7. *Define $N_\lambda = m_s$ and $\{\mu_k\}_{k=1}^{N_\mu} = \{\gamma_k\}_{k=-1}^{m_s}$. Let $\widehat{F} \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ denote the tensor defined in Remark 5.5. When F is the output of Algorithm A.5 taken with input $\widehat{F}$ and some $N_{bootstrap} \ge 1$, then F satisfies Condition 5.3.*

Proof. By Proposition 5.4 the initial tensor F satisfies Condition 5.3. We note that Proposition 5.6 is a special case of Condition A.1 and Condition 5.3 is a special case of Condition A.2. Hence Proposition A.6 applies, yielding the result. □

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5.2 Contraction Mapping

We would like to provide conditions on ρ and P for which the limit $\|\Psi \circ \ldots \circ \Psi[\alpha] - \Psi \circ \ldots \circ \Psi[\beta]\|_{\mathcal{E}}$ tends to 0 for any choice of $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$. To that end, we define a tensor J below which takes $m_s \times m_u$ matrices to $m_s \times m_u$ matrices and provides a bound on $\|\Psi[\alpha] - \Psi[\beta]\|^i_{i'\mathcal{E}}$.

Definition 5.8. *Define the tensor $J \in (\mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u})^{\otimes 2}$ by*

$$J^{i'n}_{j'i} := \sum_{-1 \le k \le m_s} (\lambda_{j'} - \gamma_k)^{-1} \left(C^{i'}_{j'} G^n_{i,k} + H^m_{j'} F^{ni'}_{mi,k} \right). \tag{43}$$

Theorem 5.9. *If the tensor $F \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ satisfies Condition 5.3, then $\|\Psi[\alpha] - \Psi[\beta]\|^n_{j'\mathcal{E}} \leq J^{i'n}_{j'i} \|\alpha - \beta\|^i_{i'\mathcal{E}}$ for all $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$.*

Proof. Select charts $\alpha, \beta \in \mathcal{B}^{0,1}_{\rho,P}$. Pick some $\xi \in B_s(\rho)$, and define $x := x(t,\xi,\alpha)$, and $y := x(t,\xi,\beta)$. By the definition of the Lyapunov-Perron operator, we have

$$\Psi[\alpha](\xi) - \Psi[\beta](\xi) = -\int_0^\infty e^{-\Lambda_u t} \left[\mathcal{N}_u(x,\alpha(x)) - \mathcal{N}_u(y,\beta(y))\right] dt.$$

By using (42), as well as the estimates provided in Condition 3.4 and Condition 5.3, we obtain

$$\begin{aligned} |\Psi[\alpha]_{j'}(\xi) - \Psi[\beta]_{j'}(\xi)| &\leq \int_0^\infty e^{-\lambda_{j'} t} \left(C^{i'}_{j'} E^i_{i'} |y_i| + H^i_{j'} |x_i - y_i| \right) dt \\ &\leq \int_0^\infty e^{-\lambda_{j'} t} \sum_{-1 \leq k \leq m_s} e^{\gamma_k t} E^i_{i'} \left(C^{i'}_{j'} G^n_{i,k} + H^m_{j'} F^{ni'}_{mi,k} \right) |\xi_n| dt. \end{aligned}$$

By integrating we infer that

$$|\Psi[\alpha]_{j'}(\xi) - \Psi[\beta]_{j'}(\xi)| \leq E^i_{i'} J^{i'n}_{j'i} |\xi_n|,$$

where the coefficients $J^{i'n}_{j'i}$ are defined as in (43). It follows that $\|\Psi[\alpha] - \Psi[\beta]\|^n_{j'\mathcal{E}} \leq E^i_{i'} J^{i'n}_{j'i}$. ∎

Remark 5.10. *The tensor J is a linear operator which maps $m_s \times m_u$ matrices to $m_s \times m_u$ matrices. If we represent an $m_s \times m_u$ matrix E as an $m_s \cdot m_u$ dimensional vector $\tilde{E}$ with components $\tilde{E}_{(i'-1)m_s+i} = E^i_{i'}$, then the action of J can be represented as a $m_s m_u \times m_s m_u$ matrix $\tilde{J}$ with components $\tilde{J}^{(i'-1)m_s+i}_{(j'-1)m_s+n} \equiv J^{i'n}_{j'i}$.*

We are principally interested in whether the Lyapunov-Perron operator Ψ has a unique fixed point. By Theorem 5.9, this will be true if an iterative application of J to any $m_s \times m_u$ matrix E limits to zero, that is:

$$\lim_{k\to\infty} \underbrace{J \circ \cdots \circ J}_{k} \cdot E = 0.$$

Algebraically, this limits to zero if and only if the spectral radius of J, denoted by $\rho(J)$, is less than 1. Since J is finite dimensional, then $\rho(J)$ is equal to the norm of its eigenvalue with largest magnitude. This may be bounded as $\rho(J) \leq \|J^k\|^{1/k}$ for any positive integer $k \geq 1$ and any matrix norm $\|\cdot\|$.

In the theorem below we summarize and collect all of our major results thus far.

Theorem 5.11. *Take the assumptions made in Remarks 4.1 and 5.1. Suppose the tensor $F \in (\mathbb{R}^{m_s})^{\otimes 3} \otimes \mathbb{R}^{m_u} \otimes \mathbb{R}^{m_s+2}$ satisfies Condition 5.3 and define $J \in (\mathbb{R}^{m_s} \otimes \mathbb{R}^{m_u})^{\otimes 2}$ as in Definition 5.8. If the spectral radius of J is less than 1, then there exists a unique fixed point $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ for which $\Psi[\alpha] = \alpha$. Furthermore, the graph*

$$M_{\text{loc}} := \{(\mathrm{x}_s, \alpha(\mathrm{x}_s)) \in X_s \times X_u : \mathrm{x}_s \in B_s(\rho)\}$$

is an invariant manifold under the flow (3), and points in M_{loc} converge asymptotically to 0.

In addition, suppose that $\tilde{h}$ is an equilibrium solution to (2) satisfying $|\tilde{h}_\mathbf{i}| < \epsilon_\mathbf{i}$ for $\mathbf{i} \in \mathbf{I}$. Define $\tilde{\alpha}(\mathrm{x}_s) := \alpha(\mathrm{x}_s - \tilde{h}_s) + \tilde{h}_u$. The graph

$$\widetilde{M}_{loc} := \{(\mathrm{x}_s, \tilde{\alpha}(\mathrm{x}_s)) \in X_s \times X_u : \mathrm{x}_s \in B_s(r_s - \epsilon_s)\},$$

is an invariant manifold under the flow (2), and points in $\widetilde{M}_{loc}$ converge asymptotically to $\tilde{h}$. Moreover, we have the estimates

$$|\tilde{\alpha}_{i'}(\mathrm{x}_s)| \leq P^i_{i'}(|\mathrm{x}_i| + \epsilon_i) + \epsilon_{i'} \qquad \|\tilde{\alpha}^i_{i'}(\mathrm{x}_s)\| \leq P^i_{i'} \qquad \mathrm{Lip}(\partial_i \tilde{\alpha})^j_{i'} \leq \bar{P}^{ij}_{i'},$$

for all $\mathrm{x}_s \in B_s(r_s - \epsilon_s)$ and $i, j \in I$ and $i' \in I'$.

Proof. We infer from the assumptions made in Remarks 4.1 and 5.1, all of which can be verified *a posteriori*, that the map $\Psi : \mathcal{B}^{1,1}_{\rho,P,\bar{P}} \to \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ is a well defined endomorphism. Since the spectral radius of J is less than 1, there exists a unique fixed point $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ for which $\Psi[\alpha] = \alpha$, see Remark 5.10. As discussed in Section 2.3, the fixed point of the Lyapunov-Perron operator provides us with a chart for a local invariant manifold for the differential equation defined in (3). By construction $\alpha(0) = 0$, hence the origin is contained in the manifold. It follows from the proof of Proposition 3.13 that points in M_{loc} converge asymptotically to the origin.

As (3) is conjugate to (2) via the change of variables $\mathrm{x} \to \mathrm{x} + \tilde{h}$, it follows that $\tilde{\alpha}(\mathrm{x}_s)$ is a graph for a local invariant manifold (having a slightly smaller domain) for the differential equation defined in (2). Furthermore this manifold contains the equilibrium $\tilde{h}$, a point to which trajectories in $\widetilde{M}_{loc}$ are asymptotically attracted. The error estimates follow by virtue of $\alpha \in \mathcal{B}^{1,1}_{\rho,P,\bar{P}}$. ☐

As discussed at the end of in Section 2.3, the fixed point of the Lyapunov-Perron operator provides us with a chart for the local stable manifold provided we have captured all stable eigenvalues.

6 Application I: Linear Change of Variables

6.1 The Swift-Hohenberg Equation

We are interested in constructing a validated numerical approximation of the stable manifold of a hyperbolic fixed point, with the ultimate goal of applying Theorem 5.11. To do so, we must produce a change of coordinates bringing whatever differential equation we wish to study into the notational framework set up in Section 2. For an application, we study the Swift-Hohenberg equation given in (1) and repeated below:

$$u_t = -\beta_1 u_{xxxx} + \beta_2 u_{xx} + u - u^3,$$

for $\beta_1 > 0$ and $\beta_2 \in \mathbb{R}$ on a spatial domain $x \in [0, \pi]$ with Neumann boundary conditions

$$u_x(0) = u_x(\pi) = 0 \qquad \text{and} \qquad u_{xxx}(0) = u_{xxx}(\pi) = 0.$$

We wish to study the Swift-Hohenberg equation using a Fourier cosine series. Without yet introducing a norm, let us define the one-sided sequence space $Y = \mathbb{R}^{\mathbb{N}}$ with standard Schauder basis $\{e_k\}_{k=0}^{\infty}$. For a time varying sequence $a \in C(\mathbb{R}, Y)$, we define a Fourier cosine series

$$u(t, x) = a_0(t) + 2\sum_{k=1}^{\infty} a_k(t)\cos(kx).$$

By plugging this function into (1) we obtain the following sequence of differential equations that a must satisfy:

$$\dot{a}_k = (-\beta_1 k^4 - \beta_2 k^2 + 1)a_k - (a * a * a)_k, \tag{44}$$

where we define the discrete convolution $*$ for $a, b \in Y$ as

$$(a * b)_k = \sum_{\substack{k_1+k_2=k \\ k_1,k_2 \in \mathbb{Z}}} a_{|k_1|} b_{|k_2|}.$$

We will work with two norms defined on the space Y. One will be a "natural" norm on a space $\ell^1_\nu \cong Y$ $(\nu \geq 1)$ corresponding to functions with geometrically decaying Fourier coefficients. The second space $X \cong Y$ will have a norm which is well aligned with the linear dynamics about an equilibrium. We delay defining a norm on X until (49), and define the ℓ^1_ν norm as

$$|a|_{\ell^1_\nu} := \sum_{k=0}^{\infty} |a_k| \omega_k(\nu),$$

where

$$\omega_k(\nu) = \omega_k := \begin{cases} 1 & k = 0 \\ 2\nu^k & k \geq 1. \end{cases}$$

We may then rewrite (44) using a (densely defined) vector field $F \colon \ell^1_\nu \to \ell^1_\nu$ given by

$$F(a) := \mathfrak{L}a - a * a * a, \tag{45}$$

where $\mathfrak{L}$ is the diagonal linear operator

$$\mathfrak{L}(a)_k := (-\beta_1 k^4 - \beta_2 k^2 + 1)a_k, \qquad \text{for all } k \geq 0. \tag{46}$$

Remark 6.1. *To enter into the notational framework established in Section 2 we must provide a change of variables such that the conjugate differential equation to* (44) *is of the type given in* (2)*. At first blush* (45) *appears to be of the proper form, and indeed it is for the equilibrium* $0 \in \ell^1_\nu$*. That is to say, the equilibrium occurs at the zero in our vector space, we can naturally write* ℓ^1_ν *as the Cartesian product of the stable/unstable eigenspaces of* $\mathfrak{L}$*, and the derivative of the nonlinearity in* (45) *vanishes at the zero equilibrium.*

For nontrivial equilibria, however, it is essential that we construct a change of variables which transforms (45) *into the "normal form" described in Section 2. Only after we have performed this change of variables can we begin to bound all of the constants needed to satisfy the hypotheses of Theorem 5.11.*

Note that by design this "normal form" is not unique. This stems from our desire to obtain a computer assisted proof and account for numerical errors, which vary with the implementation used. The smaller our linear defect L is, the more our estimates improve. Moreover, if one employs a nonlinear change of coordinates, as we do in Section 7, this can reduce the size of our nonlinearity in certain directions and greatly improve our overall estimates.

For the remainder of Section 6.1 we develop notation and derive estimates that will be used both when constructing a linear or a nonlinear approximation of the stable manifold. To begin, fix some $N \in \mathbb{N}$ and define a Galerkin projection $\pi_N : \ell^1_\nu \to \mathbb{R}^{N+1} \subseteq \ell^1_\nu$ by

$$\pi_N(a) := (a_0, a_1 \dots a_{N-1}, a_N, 0, 0, 0, \dots). \tag{47}$$

We define the Galerkin projection of F by $F_N := \pi_N \circ F \circ \pi_N$. There is an extensive literature on finding equilibria to partial differential equations, and providing computer assisted proofs of their existence, local uniqueness, and the accuracy of the numerical approximation. Such techniques often rely on solving the finite dimensional problem $F_N(\bar{a}) = 0$ for the Galerkin projection of F, and then using an implicit function type argument to show that there is a point $\tilde{a} \in \ell^1_\nu$ close to $\bar{a}$ for which $F(\tilde{a}) = 0$. We use the techniques described in [31, 58] to obtain such a point $\bar{a} \in \ell^1_\nu$ and bounds $\epsilon = |\bar{a} - \tilde{a}|_{\ell^1_\nu}$.

Remark 6.2. *The Morse index of the stationary point* $\tilde{a}$ *is established rigorously using a straightforward implementation based on the ideas and techniques from [61, 59]. The Morse index is denoted by* n_u*.*

Ideally we would define our Banach space X as a Cartesian product of the eigenspaces of $DF(\tilde{a})$. Unfortunately the operator $DF(\tilde{a})$, let alone its eigenspaces, are not something we can compute exactly. What we can compute are approximate eigenspaces arising from the Galerkin projection. Let us fix some numerically constructed matrix $A_N^\dagger \in \mathrm{Mat}(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ constructed so that $A_N^\dagger \approx DF_N(\bar{a})$.

The Swift-Hohenberg PDE is a gradient system, hence $A_N^\dagger$ has real eigenvalues with $N+1$ linearly independent eigenvectors. Indeed, this is most easily established by working with the slightly adapted

$\widetilde{F}$ rather than F directly, where

$$\widetilde{F}(a)_k = \begin{cases} F(a)_0/2 & \text{for } k=0, \\ F(a)_k & \text{for } k \geq 1, \end{cases}$$

so that $D\widetilde{F}_N(\bar{a})$ is symmetric with respect to the standard inner product on $\mathbb{R}^{N+1}$. However, this is a minor technical point.

Furthermore, suppose that $A_N^\dagger$ has n_u unstable eigenvalues (i.e. it captures the correct Morse index, see Remark 6.2). We denote by $\{\mu_{k'}\}_{k'=1'}^{n_u'}$ approximations of the positive (unstable) eigenvalues of $A_N^\dagger$ and by $\{\mu_k\}_{k=1}^{n_f}$ with $n_f = N+1-n_u$ its (stable) negative eigenvalues. Without loss of generality, we suppose that these eigenvalues are ordered as follows:

$$\mu_{n_u'} \geq \cdots \geq \mu_{1'} > 0 > \mu_1 \geq \cdots \geq \mu_{n_f}.$$

For our application here in Section 6 we set $m_u = 1$ and $m_s = 2$, whereas in Section 7 we will choose $m_s = 3$. We then define X by defining its subspaces as

$$X_{1'} := \mathbb{R}^{n_u'} \qquad X_1 := \mathbb{R}^{n_f} \qquad X_2 := \{a \in \ell_\nu^1 : a_k = 0 \text{ for } k \leq N\}.$$

We define $X_u := X_{1'}$ and $X_s := X_1 \times X_2$ and $X = X_u \times X_s$. We will also will use the notational shorthand $X_f := X_1$ and $X_\infty := X_2$. It follows that the map π_N as defined in (47) can also be considered to be the projection $\pi_N : X \to X_N \subseteq X$ where we define $X_N := X_{1'} \times X_1 \cong \mathbb{R}^{N+1}$. We additionally define the map $\pi_\infty : X \to X_\infty$ by $\pi_\infty x := x - \pi_N x$. Fix a Schauder basis $\{\hat{e}_n\}_{n\in\mathbb{N}}$ for X such that

$$X_{1'} := \operatorname{span}\{\hat{e}_0, \ldots, \hat{e}_{n_u-1}\} \qquad X_1 := \operatorname{span}\{\hat{e}_{n_u}, \ldots, \hat{e}_N\} \qquad X_2 := \overline{\operatorname{span}\{\hat{e}_{N+1}, \hat{e}_{N+2}, \ldots\}}.$$

We may then express $\phi \in X$ by $\phi = \sum_{n=0}^\infty \phi_n \hat{e}_n$.

We define a change of variables from X to ℓ_ν^1 as follows. Fix $Q_u \in \operatorname{Mat}(\mathbb{R}^{n_u}, \mathbb{R}^{N+1})$ and $Q_f \in \operatorname{Mat}(\mathbb{R}^{n_f}, \mathbb{R}^{N+1})$ as numerical approximations of unstable/stable eigenvectors of $A_N^\dagger$. For $\phi = (\phi_u, \phi_f, \phi_\infty) \in X_u \times X_f \times X_\infty$, we define the linear map $Q : X \to \ell_\nu^1$ by

$$Q(\phi) = Q_u\phi_u + Q_f\phi_f + \phi_\infty. \tag{48}$$

We now define a norm on X. Let $\phi^N = \pi_N\phi$ and let Q^N be the $(N+1)\times(N+1)$ invertible matrix constructed from Q_u and Q_f so that the transformation $Q\colon X \to \ell_\nu^1$ may be expressed by

$$[Q\phi]_n = \begin{cases} [Q^N\phi^N]_n & 0 \leq n \leq N, \\ \phi_n & n > N+1, \end{cases}$$

for all $\phi \in X$. Denote the columns of Q by q_n, $n \in \mathbb{N}$. Note that $q_n = e_n$ when $n \geq N+1$ and that $q_n = Q_n^N$, the n-the column of Q^N, for $0 \leq n \leq N$. Define the norm on X by

$$\begin{aligned} |\phi|_X &:= \sum_{n=0}^N |\phi_n Q\hat{e}_n|_{\ell_\nu^1} \qquad (49) \\ &= \sum_{n=0}^N |\phi_n||q_n|_{\ell_\nu^1} + \sum_{n=N+1}^\infty |\phi_n|\omega_n \\ &= \sum_{n=0}^N |\phi_n||q_n|_{\ell_\nu^1} + |\phi_\infty|_{\ell_\nu^1}. \end{aligned}$$

Note that $|\phi|_X = \sum_{\mathbf{i}\in\mathbf{I}} |\phi_{\mathbf{i}}|$ for $\phi \in X$, so this norm and decomposition of X into subspaces satisfies the hypotheses of Proposition 2.7.

We now consider the induced norm for bounded linear operators in $\mathcal{L}(X,X)$, $\mathcal{L}(X,\ell^1_\nu)$ and $\mathcal{L}(\ell^1_\nu, X)$. Suppose that M^N is a $(N+1)\times(N+1)$ matrix and define the linear operator $M\colon X \to X$ by

$$[M\phi]_n = \begin{cases} [M^N\phi^N]_n & 0 \le n \le N, \\ 0 & n \ge N+1. \end{cases}$$

A standard calculation shows that

$$\|M\|_{\mathcal{L}(X,X)} = \sup_{|\phi|_X=1} |M\phi|_X \le \max_{0\le k\le N} \frac{|M^N_k|_X}{|q_k|_{\ell^1_\nu}}, \tag{50}$$

where M^N_k denotes the k-th column of M^N.

Suppose that Ω^N is a $(N+1)\times(N+1)$ matrix and define the linear operator $\Omega\colon X \to \ell^1_\nu$ by

$$[\Omega\phi]_n = \begin{cases} [\Omega^N\phi^N]_n & 0 \le n \le N, \\ \phi_n & n \ge N+1. \end{cases}$$

Another standard calculation shows that

$$\|\Omega\|_{\mathcal{L}(X,\ell^1_\nu)} = \sup_{|\phi|_X=1} |\Omega\phi|_{\ell^1_\nu} \le \max\left(\max_{0\le k\le N} \frac{|\Omega^N_k|_{\ell^1_\nu}}{|q_k|_{\ell^1_\nu}}, 1 \right), \tag{51}$$

where Ω^N_k denotes the k-th column of Ω^N. From this it follows that $\|Q\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$.

To compute the norm of $Q^{-1} : \ell^1_\nu \to X$ let us first define B^N to be the matrix inverse of Q^N. The action of Q^{-1} can be expressed as:

$$\left[Q^{-1}a\right]_n = \begin{cases} \left[B^N a^N\right]_n & 0 \le n \le N, \\ a_n & n \ge N+1. \end{cases}$$

Hence, we obtain the following bound:

$$\|Q^{-1}\|_{\mathcal{L}(\ell^1_\nu,X)} = \sup_{|a|_{\ell^1_\nu}=1} |Q^{-1}a|_X \le \max\left(\max_{0\le k\le N} \frac{|B^N_k|_X}{\omega_k}, 1 \right). \tag{52}$$

For any $\mathbf{i} \in \mathbf{I}$ we define projection maps $\pi_{\mathbf{i}} : X \to X_{\mathbf{i}}$. Again, π_∞ coincides with its usual definition. By our choice of norm on X, we have $\|\pi_{\mathbf{i}}\|_{\mathcal{L}(X,X_{\mathbf{i}})} = 1$. Hence in view of inequality (5) we set $p_u = p_s = p_{\mathbf{i}} = 1$.

Lastly, we define Λ as follows:

$$\Lambda_{1'} := \operatorname{diag}\{\mu_{n_u}, \dots, \mu_{1'}\}, \qquad \Lambda_1 := \operatorname{diag}\{\mu_1, \dots, \mu_{n_f}\}, \qquad \Lambda_2 := \mathfrak{L} \circ \pi_\infty.$$

We show that the norm on X is well aligned with the semigroup $e^{\Lambda t}$. Fix a point $\phi = (\phi_u, \phi_f, \phi_\infty) \in X$ and write $\phi_u = (\phi_0, \dots, \phi_{n_u-1})$ and $\phi_f = (\phi_{n_u}, \dots, \phi_N)$ and $\phi_\infty = (\phi_{N+1}, \phi_{N+2}, \dots)$. We then have for $t \in \mathbb{R}$ the following:

$$\begin{aligned} e^{\Lambda_{1'}t}\phi_u &= \sum_{1\le k\le n_u} e^{\mu_{k'}t}\phi_{k-1}\hat{e}_{k-1}, \\ e^{\Lambda_1 t}\phi_f &= \sum_{1\le k\le n_f} e^{\mu_k t}\phi_{k+n_u-1}\hat{e}_{k+n_u-1}, \\ e^{\Lambda_2 t}\phi_\infty &= \sum_{k=N+1}^{\infty} e^{(-\beta_1 k^4 - \beta_2 k^2 + 1)t}\phi_k\hat{e}_k. \end{aligned}$$

Let us define $\lambda_{1'}$, λ_1 and λ_2 as

$$\lambda_{1'} := \text{Re } \mu_{1'}, \qquad \lambda_1 := \text{Re } \mu_1, \qquad \lambda_2 := -\beta_1(N+1)^4 - \beta_2(N+1)^2 + 1. \tag{53}$$

It follows that $\lambda_{1'} \leq \text{Re } \mu_{k'}$ for $1' \leq k' \leq n'_u$, and $\lambda_1 \geq \text{Re } \mu_k$ for $1 \leq k \leq n_f$, and $\lambda_2 \geq (-\beta_1 k^4 - \beta_2 k^2 + 1)$ for $k \geq N+1$. We assume here that we have chosen N sufficiently large so that $-\beta_1 k^4 - \beta_2 k^2 + 1$ is negative and decreasing in k for $k \geq N+1$. Hence, we have the following bounds on the norm:

$$\begin{aligned}
|e^{\Lambda_{1'} t}\phi_u|_X &\leq \sum_{0 \leq k \leq n_u - 1} e^{\lambda_{1'} t} |Q\phi_k|_{\ell^1_\nu}, && \text{for } t \leq 0, \\
|e^{\Lambda_1 t}\phi_f|_X &\leq \sum_{n_u \leq k \leq N} e^{\lambda_1 t} |Q\phi_k|_{\ell^1_\nu}, && \text{for } t \geq 0, \\
|e^{\Lambda_2 t}\phi_\infty|_X &\leq \sum_{k=N+1}^{\infty} e^{\lambda_2 t} |Q\phi_k|_{\ell^1_\nu} && \text{for } t \geq 0.
\end{aligned}$$

We infer from the expression (49) for the norm on X that (6) and (7) are satisfied.

6.2 Bounds for the Linear Change of Coordinates

Remark 6.3. *In the previous subsection, we established notation and performed estimates for studying stable manifolds of equilibria to the Swift-Hohenberg equation. For computing either a linear or a nonlinear approximation to the stable manifold, such notation and estimates are largely identical. However, the subsequent estimates will vary depending on how we make our approximation. Nevertheless, they both follow the same general outline.*

1. *Define a change of variables $K : X \supseteq U \to \ell^1_\nu$ such that $K(0) = \bar{a}$.*
   For the equilibrium $\tilde{h} = K^{-1}(\tilde{a})$, obtain bounds $|\pi_{\mathbf{i}}\tilde{h}| \leq \epsilon_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$.

2. *Pull back the vector field from ℓ^1_ν to U, creating the conjugate differential equation*

$$\dot{\text{x}} = DK(\text{x})^{-1} F(K(\text{x})).$$

   Define $\tilde{\mathcal{N}} \in C^2_{loc}(U, X)$ as $\tilde{\mathcal{N}}(\text{x}) := DK(\text{x})^{-1}F(K(\text{x})) - \Lambda \text{x}$.

3. *Obtain constants $\tilde{C}^{\mathbf{ik}}_{\mathbf{j}}(r_s, r_u)$ which bound $\|\tilde{\mathcal{N}}^{\mathbf{ik}}_{\mathbf{j}}\|_{(r_s+\epsilon_s, r_u+\epsilon_u)}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$.*

4. *Obtain constants $\tilde{D}^{\mathbf{i}}_{\mathbf{j}}$ which bound $\|\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)\|$ for $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.*

5. *Obtain constants C_s, λ_s which satisfy equation (8) to bound $e^{(\Lambda_s + L^s_s)t}$.*

In the remainder of this section we follow the outline given in Remark 6.3 for a linear change of coordinates K. The results of such a calculation are then presented in Section 6.3. Later in Section 7 we follow the outline given in Remark 6.3 again, but for a nonlinear change of coordinates K.

6.2.1 Estimate 1 – Defining a Change of Variables

We define an affine change of coordinates $K : X \to \ell^1_\nu$ by:

$$K(\phi) := \bar{a} + Q\phi. \tag{54}$$

If $\epsilon = |\bar{a} - \tilde{a}|_{\ell^1_\nu}$ is our bound on the distance between our approximate solution and the true solution, then we may define $\epsilon_{\mathbf{i}} := \epsilon \|\pi_{\mathbf{i}} Q^{-1}\|_{\mathcal{L}(\ell^1_\nu, X_{\mathbf{i}})}$ for $\mathbf{i} \in \mathbf{I}$ as needed in Proposition 2.4. This step will be more involved for the nonlinear change of variables in Section 7.

6.2.2 Estimate 2 – Defining the Conjugate Differential Equation

Using the change of coordinates in (54), the Swift-Hohenberg equation in (45) is conjugate to the differential equation

$$\dot{\phi} = \Lambda\phi + \tilde{\mathcal{N}}(\phi) \qquad \text{with} \qquad \tilde{\mathcal{N}}(\phi) := DK(\phi)^{-1}F(K(\phi)) - \Lambda\phi. \tag{55}$$

We note that this form of $\tilde{\mathcal{N}}$ is not immediately easy to work with. We begin to expand $\tilde{\mathcal{N}}$ into an affine part and a purely nonlinear part. Define functions $E, R : X \to \ell^1_\nu$ as

$$E(\phi) := F(\bar{a}) + DF(\bar{a})Q\phi - Q\Lambda\phi, \qquad\qquad R(\phi) := -3\bar{a} * (Q\phi)^{*2} - (Q\phi)^{*3}.$$

A computation shows that $E + R = F \circ K - DK \cdot \Lambda$, where we note that $DK(\phi) = Q$ for all $\phi \in X$. Hence it follows that $\tilde{\mathcal{N}}(\phi) = Q^{-1}(E(\phi) + R(\phi))$.

6.2.3 Estimate 3 – Bounding $\tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}}$

All second derivatives of E are zero. Hence $\partial_{\mathbf{i}}\partial_{\mathbf{j}}\pi_{\mathbf{k}}\tilde{\mathcal{N}} = \tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}} = (Q^{-1}R)^{\mathbf{ij}}_{\mathbf{k}}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$. Below we perform a series of calculations to derive bounds on the derivatives of R. To alleviate our notation, for $\phi \in X$ we define

$$\mathbf{Q} := Q\phi = Q_f\phi_f + Q_u\phi_u + \phi_\infty. \tag{56}$$

Note that all terms in R have $\mathbf{Q} * \mathbf{Q}$ terms in them. For notational ease, we set

$$\mathbf{Q}^2 := \mathbf{Q} * \mathbf{Q} \qquad \text{and} \qquad \mathbf{Q}^3 := \mathbf{Q} * \mathbf{Q} * \mathbf{Q}.$$

Then we have that $R(\phi) = -3\bar{a} * \mathbf{Q}^2 - \mathbf{Q}^3$. First we calculate the derivatives of $\mathbf{Q}$:

$$\partial_f \mathbf{Q} \cdot h_f = Q_f h_f, \qquad\qquad \partial_u \mathbf{Q} \cdot h_u = Q_u h_u, \qquad\qquad \partial_\infty \mathbf{Q} \cdot h_\infty = h_\infty,$$

where $h_f \in X_f$, $h_u \in X_u$ and $h_\infty \in X_\infty$. Since $\|Q\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$, then $\|\partial_{\mathbf{i}}\mathbf{Q}\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$ for $\mathbf{i} \in \mathbf{I}$. As $\partial_{\mathbf{i}}\mathbf{Q}$ is a linear operator, the second derivatives $\partial_{\mathbf{ij}}\mathbf{Q}$ vanish for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.

The derivatives of $\mathbf{Q}^2$ and $\mathbf{Q}^3$ are given by

$$\partial_{\mathbf{ij}}\mathbf{Q}^2 = 2\partial_{\mathbf{i}}\mathbf{Q} * \partial_{\mathbf{j}}\mathbf{Q} \qquad \text{and} \qquad \partial_{\mathbf{ij}}\mathbf{Q}^3 = 6\mathbf{Q} * \partial_{\mathbf{i}}\mathbf{Q} * \partial_{\mathbf{j}}\mathbf{Q},$$

hence

$$\partial_{\mathbf{ij}}R = -6(\bar{a} + \mathbf{Q}) * \partial_i\mathbf{Q} * \partial_j\mathbf{Q}.$$

Recall that $\|\partial_{\mathbf{i}}\mathbf{Q}\|_{\mathcal{L}(X,\ell^1_\nu)} = 1$ for all $\mathbf{i} \in \mathbf{I}$. If we consider a point $\phi = (\phi_u, \phi_s) \in B_u(r_u) \times B_s(r_s)$ with $r_s = (r_f, r_\infty)$, then $|Q\phi| \leq r_u + r_f + r_\infty$. Hence, if we define

$$C^{\mathbf{ij}}_{\mathbf{k}} := 6\|\pi_{\mathbf{k}}Q^{-1}\|_{\mathcal{L}(\ell^1_\nu,X)}\left(|\bar{a}| + r_u + r_f + r_\infty + \epsilon_u + \epsilon_f + \epsilon_\infty\right), \tag{57}$$

then $\|\tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}}\|_{(r_s+\epsilon_s, r_u+\epsilon_u)} \leq C^{\mathbf{ij}}_{\mathbf{k}}$ for $\mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{I}$.

6.2.4 Estimate 4 – Bounding $\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)$

Note that since $\partial_{\mathbf{i}}R(0) = 0$ and $\partial_\phi DK(\phi)^{-1}E(\phi) = Q^{-1}DF(\bar{a})Q - \Lambda$, then we have

$$\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0) = \pi_{\mathbf{j}}\left(Q^{-1}DF(\bar{a})Q - \Lambda\right)\pi_{\mathbf{i}}.$$

First, we approximate $DF(\bar{a})$ by defining an operator $A^\dagger : \ell^1_\nu \to \ell^1_\nu$ for $v \in \ell^1_\nu$ by:

$$(A^\dagger v)_k := \begin{cases} (A^\dagger_N v)_k & k \leq N \\ (\mathfrak{L}v)_k & k > N. \end{cases}$$

We bound $\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0)$ by adding and subtracting $Q^{-1}A^{\dagger}Q$:

$$\left\|\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0)\right\|_{\mathcal{L}(X,X)} \leq \left\|\pi_{\mathbf{j}} Q^{-1}\left(DF(\bar{a}) - A^{\dagger}\right) Q\pi_{\mathbf{i}}\right\|_{\mathcal{L}(X,X)} + \left\|\pi_{\mathbf{j}}\left(Q^{-1}A^{\dagger}Q - \Lambda\right)\pi_{\mathbf{i}}\right\|_{\mathcal{L}(X,X)}. \tag{58}$$

To bound the right summand in (58), note that $\pi_{\mathbf{j}}\left(Q^{-1}A^{\dagger}Q - \Lambda\right)\pi_{\mathbf{i}}$ vanishes when either $\mathbf{i} = \infty$ or $\mathbf{j} = \infty$, hence the right-summand in (58) can be computed directly using (50).

To bound the left summand in (58) we split this into four cases, depending on whether $\mathbf{i}$ or $\mathbf{j}$ equals ∞. Each of these terms involves $(DF(\bar{a}) - A^{\dagger}) \in \mathcal{L}(\ell_\nu^1, \ell_\nu^1)$ which we calculate below:

$$\left(DF(\bar{a})h - A^{\dagger}h\right)_k = \begin{cases} -3(\bar{a} * \bar{a} * \pi_\infty h)_k + ((DF_N(\bar{a}) - A_N^{\dagger})\pi_N h)_k & 0 \leq k \leq N \\ -3(\bar{a} * \bar{a} * h)_k & k \geq N+1. \end{cases} \tag{59}$$

We now bound $\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0)$ according to the four cases mentioned above.

The case $\mathbf{i} = \infty$ and $\mathbf{j} = \infty$. Since ℓ_ν^1 is a Banach algebra and π_∞ projects onto the modes $k \geq N+1$, by using (59) we obtain

$$\left|\pi_\infty\left(DF(\bar{a}) - A^{\dagger}\right)h\right| \leq 3|\bar{a} * \bar{a}|_{\ell_\nu^1}|h|_{\ell_\nu^1}.$$

Hence $\left\|\pi_\infty\left(DF(\bar{a}) - A^{\dagger}\right)\right\|_{\mathcal{L}(\ell_\nu^1,\ell_\nu^1)} \leq 3|\bar{a} * \bar{a}|_{\ell_\nu^1}$, and by defining

$$\tilde{D}_\infty^{\mathbf{i}} := 3|\bar{a} * \bar{a}|_{\ell_\nu^1}, \tag{60}$$

it follows that $\|\tilde{\mathcal{N}}_\infty^{\mathbf{i}}(0)\|_{\mathcal{L}(X,X)} \leq \tilde{D}_\infty^{\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}$.

The case $\mathbf{i} \neq \infty$ and $\mathbf{j} \neq \infty$. The operator $\pi_{\mathbf{j}}\left(Q^{-1}DF(\bar{a})Q - \Lambda\right)\pi_{\mathbf{i}}$ can be represented by an $(N+1) \times (N+1)$ matrix whose norm we may explicitly bound. Let us define

$$\tilde{D}_{\mathbf{j}}^{\mathbf{i}} := \left\|\pi_{\mathbf{j}}\left(Q^{-1}DF(\bar{a})Q - \Lambda\right)\pi_{\mathbf{i}}\right\|_{\mathcal{L}(X,X)}, \tag{61}$$

then it follows that $\|\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0)\| \leq \tilde{D}_{\mathbf{j}}^{\mathbf{i}}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{I} - \{\infty\}$.

The case $\mathbf{i} = \infty$ and $\mathbf{j} \neq \infty$. It follows from (59) that

$$\pi_{\mathbf{j}}[DF(\bar{a}) - A^{\dagger}]_k = 0 \qquad \text{for } k > 3N,$$

where we recall that the subscript k denotes the k-th column. Since $Q\pi_\infty = \pi_\infty$ and by using the appropriate analogue of (50) for a matrix of a larger size, we thus set

$$\tilde{D}_{\mathbf{j}}^{\infty} := \max_{N+1 \leq k \leq 3N} \frac{|\pi_{\mathbf{j}} Q^{-1}[DF(\bar{a}) - A^{\dagger}]_k|_X}{\omega_k}. \tag{62}$$

It follows that $\|\tilde{\mathcal{N}}_{\mathbf{j}}^{\infty}(0)\| \leq \tilde{D}_{\mathbf{j}}^{\infty}$ for all $\mathbf{j} \in \mathbf{I} - \{\infty\}$.

The case $\mathbf{i} \neq \infty$ and $\mathbf{j} = \infty$. Note that since $\pi_\infty Q^{-1} = \pi_\infty$ and $\pi_\infty A^{\dagger}\pi_N = 0$, it follows that

$$\pi_{\mathbf{j}} Q^{-1}\left(DF(\bar{a}) - A^{\dagger}\right) Q\pi_{\mathbf{i}} = \pi_\infty DF(\bar{a}) Q\pi_{\mathbf{i}}.$$

Using the formula in (50) we thus set

$$\tilde{D}_\infty^{\mathbf{i}} := \max_{0 \leq k \leq N} \frac{|[\pi_\infty DF(\bar{a})Q\pi_{\mathbf{i}}]_k|_X}{|q_k|_{\ell_\nu^1}}. \tag{63}$$

It follows that $\|\tilde{\mathcal{N}}_{\mathbf{j}}^{\infty}(0)\| \leq \tilde{D}_{\mathbf{j}}^{\infty}$ for all $\mathbf{j} \in \mathbf{I} - \{\infty\}$. With our definitions of $\tilde{D}_{\mathbf{j}}^{\mathbf{i}}$ in Equations (60), (61), (62) and (63) we have thus obtained bounds on $\|\tilde{\mathcal{N}}_{\mathbf{j}}^{\mathbf{i}}(0)\|_{\mathcal{L}(X,X)}$ for all $\mathbf{i}, \mathbf{j} \in \mathbf{I}$.

6.2.5 Estimate 5 – Semigroup Bounds

To calculate C_s and λ_s needed in (8) we use Proposition B.1 and Remark B.3. First define $D^{\mathbf{i}}_{\mathbf{j}} := \tilde{D}^{\mathbf{i}}_{\mathbf{j}} + \tilde{C}^{\mathbf{i}l}_{\mathbf{j}}\epsilon_l + \tilde{C}^{\mathbf{i}l'}_{\mathbf{j}}\epsilon_{l'}$ for $\mathbf{i},\mathbf{j} \in \mathbf{I}$ as in Proposition 2.4, and then define the following constants:

$$\mu_1 := \lambda_1 \qquad \delta_a := D^f_f \qquad \delta_b := D^\infty_f$$

$$\mu_\infty := \lambda_2 = \lambda_\infty \qquad \delta_c := D^f_\infty \qquad \delta_d := D^\infty_\infty \qquad \varepsilon := \sum_{\tilde{\mu}_k \in \sigma(\Lambda_1)} \frac{|\mu_\infty|^{-1}}{1 - |\mu_\infty|^{-1}(\delta_d + |\tilde{\mu}_k|)}.$$

We note that $\|\Lambda_\infty^{-1}\| = |\mu_\infty|^{-1}$. When the inequalities

$$1 > |\mu_\infty|^{-1}\Big(\delta_d + \sup_{\tilde{\mu}_k \in \sigma(\Lambda_1)} |\tilde{\mu}_k|\Big), \qquad \mu_1 > \mu_\infty + \delta_d + \varepsilon\delta_b\delta_c(1 + \varepsilon^2\delta_b\delta_c), \tag{64}$$

are satisfied, then it follows from Proposition B.1 and Remark B.3 that

$$\|e^{(\Lambda_s + L^s_s)t}\| \leq C_s e^{\lambda_s t},$$

where

$$\begin{aligned} C_s &:= (1 + \varepsilon\delta_b)^2(1 + \varepsilon\delta_c)^2 \\ \lambda_s &:= \mu_1 + \delta_a C_s + \Delta \\ \Delta &:= \varepsilon\delta_b\delta_c \max\left\{1 + \varepsilon\delta_c(1 + \varepsilon\delta_b), \varepsilon\delta_b(2 + \varepsilon^2\delta_b\delta_c)\right\}. \end{aligned}$$

6.3 Numerical Results

For the theorem below, the following input are required: parameters values $\beta_1 < 0$, $\beta_2 \in \mathbb{R}$ in the PDE (1); computational parameter $N \in \mathbb{N}$ – the Galerkin projection; $\nu \geq 1$ – the degree of analyticity; a specific approximate equilibrium $\bar{a} \in \ell^1_\nu$; and a positive vector $\rho \in \mathbb{R}^2$ which determines the size of the domain $B_s(\rho)$ for charts $\alpha \in \mathcal{B}_{\rho,P,\bar{P}}$. The rest of the constants are then computationally determined such that the hypotheses of Theorem 5.11 are satisfied (if possible). Choosing different parameters allows us to prove a variety of theorems. In Theorem 6.4 we present one such theorem describing the equilibrium displayed in Figure 1, and where we take ρ_f, the radius in the finite dimensional subspace X_f of the domain $B_s(\rho) \subseteq X_f \times X_\infty$, as large as possible. In Section 7.6 we discuss the performance of our computer assisted proofs, comparing both the linear and nonlinear approximation of the stable manifold.

Theorem 6.4. *Fix parameter $\beta_1 = 0.05$ and $\beta_2 = -0.35$. For computational parameters $\nu = 1.001$ and $N = 30$ we compute an approximate equilibrium $\bar{a} \in \ell^1_\nu$ which is $\epsilon = 1.61 \times 10^{-14}$ close to a true equilibrium $\tilde{a} \in \ell^1_\nu$. Take $\rho = \left(2.2 \times 10^{-2}, 10^{-5}\right)$ and take:*

$$P = \begin{pmatrix}0.153, & 1.38 \times 10^{-5}\end{pmatrix} \qquad \bar{P} = \begin{pmatrix} 16.9 \times 10^{-0} & 1.37 \times 10^{-3} \\ 1.37 \times 10^{-3} & 2.14 \times 10^{-4} \end{pmatrix}. \tag{65}$$

There exists a unique map $\tilde{\alpha} \in C^{1,1}(B_s(\rho - \epsilon_s), X_u)$ where $\epsilon_{\mathbf{i}} \leq 4.97 \times 10^{-14}$ such that the stable manifold of $\tilde{a} \in \ell^1_\nu$ for the differential equation (1) is locally given by

$$\mathrm{x}_s \mapsto K\left(\mathrm{x}_s, \tilde{\alpha}(\mathrm{x}_s)\right)$$

for K given in (54) and $\tilde{\alpha}$ satisfying the following estimates:

$$|\tilde{\alpha}_{i'}(\xi)| \leq 3.36 \times 10^{-3} \qquad \|\tilde{\alpha}^i_{i'}(\xi)\| \leq P^i_{i'} \qquad \mathrm{Lip}(\partial_i\tilde{\alpha})^j_{i'} \leq \bar{P}^{ij}_{i'},$$

for all $\xi \in B_s(\rho - \epsilon_s)$ and $i, j \in I$, $i' \in I'$ and $\mathbf{i} \in \mathbf{I}$.

Proof. In script `main.m` we calculate all of the constants and verify all of the hypotheses in Theorem 5.11. In particular we have a contraction constant $\|J\| < 0.356$. The entire computation took about 4 seconds and was run on MATLAB 2019a with INTLAB on a i7-8750H processor. □

7 Application II: Nonlinear Change of Variables

Again we consider the problem of approximating stable manifolds of equilibria for the Swift-Hohenberg equation, this time though, using a nonlinear approximation of the stable manifold (cf. Section 2.4). We use the same notation and estimates established in Section 6.1 with some minor adjustments. In particular, we will use $m_u = 1$ and $m_s = 3$. Adopting some of the notation from Section 2.4, we set $n_u = m_{\text{unst}}$, $n_\theta := m_{\text{slow}}$, $n_f = m_{\text{fast}} + m_{\text{slow}}$, and $N = n_u + n_f - 1$, and define

$$X_{1'} := \mathbb{R}^{n_u} \qquad X_1 := \mathbb{R}^{n_\theta} \qquad X_2 := \mathbb{R}^{n_f - n_\theta} \qquad X_3 := \{a \in \ell^1_\nu : a_k = 0 \text{ for } k \leq N\}.$$

We define $X_u := X_{1'}$ and $X_s := X_1 \times X_2 \times X_3$ and $X = X_u \times X_s$. We will also use the notational shorthand $X_\theta := X_1$ (slow stable), $X_f := X_2$ (fast but finite stable) and $X_\infty := X_3$ (stable tail). The map π_N as defined in (47) can also be considered to be the projection $\pi_N : X \to X_N \subseteq X$ where we define $X_N := X_{1'} \times X_1 \times X_2 \cong \mathbb{R}^{N+1}$. We define the map $\pi_\infty : X \to X_\infty$ by $\pi_\infty x := x - \pi_N x$. We define Λ as

$$\Lambda_{1'} := \operatorname{diag}\{\mu_{n_u'}, \dots, \mu_{1'}\}, \quad \Lambda_1 := \operatorname{diag}\{\mu_1, \dots, \mu_{n_\theta}\}, \quad \Lambda_2 := \operatorname{diag}\{\mu_{n_\theta+1}, \dots, \mu_{n_f}\}, \quad \Lambda_3 := \mathfrak{L} \circ \pi_\infty,$$

with μ defined in Section 6.1, and $\mathfrak{L}$ defined in (46). We define constants $\lambda_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$ as follows:

$$\lambda_{1'} := \mu_{1'} \qquad \lambda_1 := \mu_1, \qquad \lambda_2 := \mu_{n_\theta+1}, \qquad \lambda_3 := -\beta_1(N+1)^4 - \beta_2(N+1)^2 + 1. \tag{66}$$

By the same argument as the one given at the end of Section 6.1, the inequalities (6) and (7) are satisfied. We now follow the scheme for stable manifold validation as outlined in Remark 6.3.

7.1 Estimate 1 – Defining a Change of Variables

Using the parameterization method and the good coordinates discussed in Section 2.4, we define a slow stable manifold and finite dimensional (un)stable bundles:

$$\begin{aligned} P &: [-1,1]^{n_\theta} \to X_N, \\ Q_f(\theta) &: [-1,1]^{n_\theta} \to \operatorname{Mat}(\mathbb{R}^{n_f - n_\theta}, X_N) \\ Q_u(\theta) &: [-1,1]^{n_\theta} \to \operatorname{Mat}(\mathbb{R}^{n_u}, X_N), \end{aligned}$$

which are chosen to solve (21)–(22) *approximately*. The error terms

$$E_\theta : [-1,1]^{n_\theta} \to \ell^1_\nu \qquad\qquad E_f : [-1,1]^{n_\theta} \to \mathcal{L}(X_f, \ell^1_\nu) \tag{67a}$$

$$E_u : [-1,1]^{n_\theta} \to \mathcal{L}(X_u, \ell^1_\nu) \qquad\qquad E_\infty : [-1,1]^{n_\theta} \to \mathcal{L}(X_\infty, \ell^1_\nu), \tag{67b}$$

are defined by

$$E_\theta(\theta) := F(P(\theta)) - DP(\theta)\Lambda_\theta\theta \tag{68a}$$

$$E_f(\theta) := DF(P(\theta))Q_f(\theta) - DQ_f(\theta)\Lambda_\theta\theta - Q_f(\theta)\Lambda_f \tag{68b}$$

$$E_u(\theta) := DF(P(\theta))Q_u(\theta) - DQ_u(\theta)\Lambda_\theta\theta - Q_u(\theta)\Lambda_u \tag{68c}$$

$$E_\infty(\theta) := DF(P(\theta))\pi_\infty - \Lambda_\infty. \tag{68d}$$

Define $U := B(r_s + \epsilon_s, r_u + \epsilon_u) \subseteq X_u \times [-1,1]^{n_\theta} \times X_f \times X_\infty$, and define a normal frame bundle $Q : [-1,1]^{n_\theta} \to \mathcal{L}(X/X_1, \ell^1_\nu)$ and a local diffeomorphism $K : U \subseteq X \to \ell^1_\nu$ as follows:

$$Q(\theta)\phi := Q_f(\theta)\phi_f + Q_u(\theta)\phi_u + \phi_\infty \tag{69}$$

$$K(\theta, \phi) := P(\theta) + Q(\theta)\phi. \tag{70}$$

We define the norm $|\cdot|_X$ as in (49) relative to the linear map $Q_0 : X \to \ell^1_\nu$ defined by

$$Q_0 \cdot (h_\theta, h_\phi) := DK(0,0) \cdot (h_\theta, h_\phi) = \partial_\theta P(0) h_\theta + Q(0) h_\phi, \tag{71}$$

where $h_\theta \in X_\theta$ and $h_\phi \in X_u \times X_f \times X_\infty$.

While we do not have an explicit expression for the inverse function K^{-1}, to bound the norm of $\tilde{h} = K^{-1}(\tilde{a})$ note that $K^{-1}(a) = Q_0^{-1}(a - \bar{a}) + \mathcal{O}(|a - \bar{a}|^2)$. If $\epsilon = |\bar{a} - \tilde{a}|_{\ell^1_\nu}$ is our bound on the distance between the approximate solution and the true solution in ℓ^1_ν, then we may apply standard techniques from rigorous numerics to bound $|\pi_{\mathbf{i}} \tilde{h}| \leq \epsilon_{\mathbf{i}}$ for $\mathbf{i} \in \mathbf{I}$ as needed in Proposition 2.4, in terms of ϵ, $\|\pi_{\mathbf{i}} Q_0^{-1}\|$, and the polynomial coefficients of $K(\theta, \phi)$.

7.2 Estimate 2 – Defining the Conjugate Differential Equation

Again as in (55), using the change of coordinates in (70), we see that the Swift-Hohenberg equation in (45) is conjugate to the differential equation

$$\dot{\mathrm{x}} = \Lambda \mathrm{x} + \tilde{\mathcal{N}}(\mathrm{x}), \qquad \tilde{\mathcal{N}}(\mathrm{x}) := DK(\mathrm{x})^{-1} F(K(\mathrm{x})) - \Lambda \mathrm{x}, \tag{72}$$

for $\mathrm{x} \in U$.

We perform a Taylor expansion of $F(K(\mathrm{x}))$ for $\mathrm{x} \in U$. To alleviate our notation, for $\mathrm{x} = (\theta, \phi)$ where $\theta \in [-1,1]^{n_\theta}$ and $\phi \in X_u \times X_f \times X_\infty$, let us define

$$\mathbf{P} := P(\theta) \qquad \mathbf{Q} := Q(\theta)\phi. \tag{73}$$

Starting from (45) we expand $F(K(\theta, \phi))$ as

$$\begin{aligned} F(K(\theta,\phi)) &= \mathfrak{L}[\mathbf{P} + \mathbf{Q}] - (\mathbf{P} + \mathbf{Q})^3 \\ &= \left(\mathfrak{L}\mathbf{P} - \mathbf{P}^3\right) + \left(\mathfrak{L}\mathbf{Q} - 3\mathbf{P}^2 * \mathbf{Q}\right) - 3\mathbf{P} * \mathbf{Q}^2 - \mathbf{Q}^3, \end{aligned}$$

where the powers denote convolution products. Note that for $a, h \in \ell^1_\nu$ the derivative of F is given by:

$$DF(a) \cdot h = \mathfrak{L}h - 3(a * a * h).$$

Hence, it follows that

$$F(\mathbf{P}) = \mathfrak{L}\mathbf{P} - \mathbf{P}^3, \qquad DF(\mathbf{P}) \cdot \mathbf{Q} = \mathfrak{L}\mathbf{Q} - 3(\mathbf{P}^2 * \mathbf{Q}).$$

If we now define a remainder term $R : U \subseteq X \to \ell^1_\nu$ by

$$\mathbf{R} = R(\theta,\phi) := -3P(\theta) * (Q(\theta)\phi) * (Q(\theta)\phi) - (Q(\theta)\phi) * (Q(\theta)\phi) * (Q(\theta)\phi) = -3\mathbf{P} * \mathbf{Q}^2 - \mathbf{Q}^3, \tag{74}$$

then $F(K(\theta, \phi))$ simplifies as

$$F(K(\theta,\phi)) = F(\mathbf{P}) + DF(\mathbf{P}) \cdot \mathbf{Q} + \mathbf{R}. \tag{75}$$

We use the (approximate) conjugacy relations in (68) to (approximately) linearize the non-remainder components in (75). We calculate

$$\begin{aligned} F(P(\theta)) + DF(P(\theta))\left[Q_f(\theta)\phi_f + Q_u(\theta)\phi_u + \phi_\infty\right] &= E_\theta(\theta) + DP(\theta)\Lambda_\theta\theta \\ &\quad + E_f(\theta)\phi_f + DQ_f(\theta)(\Lambda_\theta\theta, \phi_f) + Q_f(\theta)\Lambda_f\phi_f \\ &\quad + E_u(\theta)\phi_u + DQ_u(\theta)(\Lambda_\theta\theta, \phi_u) + Q_u(\theta)\Lambda_u\phi_u \\ &\quad + E_\infty(\theta)\phi_\infty + \Lambda_\infty\phi_\infty \\ &= E(\theta,\phi) + DK(\theta, \phi_f, \phi_u, \phi_\infty) \begin{pmatrix} \Lambda_\theta\theta \\ \Lambda_f\phi_f \\ \Lambda_u\phi_u \\ \Lambda_\infty\phi_\infty \end{pmatrix}, \end{aligned}$$

where $E : U \to \ell^1_\nu$ is defined by

$$E(\theta, \phi) := E_\theta(\theta) + E_f(\theta)\phi_f + E_u(\theta)\phi_u + E_\infty(\theta)\phi_\infty. \tag{76}$$

It then follows that for $\mathrm{x} \in U$ we have

$$\begin{aligned} DK(\mathrm{x})^{-1}F(K(\mathrm{x})) &= DK(\mathrm{x})^{-1}\big(E(\mathrm{x}) + DK(\mathrm{x})\Lambda \mathrm{x} + R(\mathrm{x})\big) \\ &= \Lambda \mathrm{x} + DK(\mathrm{x})^{-1}\left(E(\mathrm{x}) + R(\mathrm{x})\right). \end{aligned}$$

Thus, we have decomposed the differential equation into a diagonalized part and nonlinear error terms. Hence, it follows that

$$\tilde{\mathcal{N}}(\theta, \phi) = DK(\theta, \phi)^{-1}\left(E(\theta, \phi) + R(\theta, \phi)\right). \tag{77}$$

In our previous calculation for the linear change of coordinates, $DK(\theta, \phi)^{-1}$ was a constant linear operator Q^{-1}. In the current case it has nontrivial first and second derivatives.

7.3 Estimate 3 – Bounding $\tilde{\mathcal{N}}^{\mathbf{ij}}_{\mathbf{k}}$

In this subsection we compute bounds on the derivatives of the three components of (77) separately. Then we compute bounds on the derivatives of $\tilde{\mathcal{N}}$ using the product rule. In general each estimate is reduced down to a bound involving terms of the form $P(\theta)$ and $Q(\theta)$, for which we have *explicit* expressions, and their derivatives.

Throughout we will take points in the ball $(\theta, \phi) \in U = B(r_s + \epsilon_s, r_u + \epsilon_u)$, so we may assume that $|\phi_u| \leq r_u + \epsilon_u$, $|\phi_f| \leq r_f + \epsilon_f$, and $|\phi_\infty| \leq r_\infty + \epsilon_\infty$. Additionally, we choose some $\delta_\theta \in (0, 1]$ such that if $|\theta|_X \leq r_\theta + \epsilon_\theta$ then $(\theta)_k \leq \delta_\theta$ for all components $1 \leq k \leq n_\theta$, whereby $U = B(r_s + \epsilon_s, r_u + \epsilon_u) \subseteq X_u \times [-\delta_\theta, \delta_\theta]^{n_\theta} \times X_f \times X_\infty$.

7.3.1 Bounding the Derivatives of DK and its Inverse

Fix $h = (h_\theta, h_f, h_u, h_\infty) \in X_\theta \times X_f \times X_u \times X_\infty$. We compute

$$DK(\theta, \phi) \cdot h = \big(\partial_\theta P(\theta) + \partial_\theta Q_f(\theta)\phi_f + \partial_\theta Q_u(\theta)\phi_u\big)h_\theta + Q_f(\theta)h_f + Q_u(\theta)h_u + h_\infty. \tag{78}$$

We condense notation by defining the maps

$$\begin{aligned} A_0(\theta) \cdot h &:= \partial_\theta P(\theta)h_\theta + Q_f(\theta)h_f + Q_u(\theta)h_u + h_\infty, \\ A_1(\theta, \phi) \cdot h &:= \partial_\theta Q_f(\theta)\phi_f h_\theta + \partial_\theta Q_u(\theta)\phi_u h_\theta, \end{aligned}$$

so that $DK = A_0 + A_1$. The norm of A_1 can be controlled by taking $|\phi|$ small.

From now on we will assume $A_0(\theta)$ is invertible for all $\theta \in [-\delta_\theta, \delta_\theta]^{n_\theta}$ with inverse $B(\theta) := A_0(\theta)^{-1}$. Indeed, the action of the operator $A_0(\theta) : X_N \times X_\infty \to \ell^1_\nu \cong X_N \times X_\infty$ is invariant in both the subspaces X_N and X_∞. The action of the operator $A_0(\theta)$ in the finite dimensional component can be represented by a polynomial in θ with $(N+1) \times (N+1)$ matrix coefficients, and its action in the infinite dimensional component is precisely the identity map. Hence the operator $B(\theta) = A_0(\theta)^{-1}$ can be computed as an infinite power series in θ with Taylor coefficients defined recursively by power matching.

We now write the inverse $DK^{-1} : \ell^1_\nu \to X$ as

$$DK(\theta, \phi)^{-1} = B(\theta)\big(I + A_1(\theta, \phi)B(\theta)\big)^{-1}.$$

We will compute bounds on the derivatives of $DK(\theta, \phi)^{-1}$ using the product rule. We can explicitly compute finitely many terms in the power series of $B(\theta)$. To bound the Taylor remainder of $B(\theta)$

and its derivatives, we use a Neumann series argument similar to the one given below to bound $\big(I+A_1(\theta,\phi)B(\theta)\big)^{-1}$. Indeed, for ϕ sufficiently small we obtain from the Neumann series the bound

$$\begin{aligned}\|\big(I+A_1(\theta,\phi)B(\theta)\big)^{-1}\| &\le \frac{1}{1-\|A_1(\theta,\phi)B(\theta)\|_{\mathcal{L}(\ell^1_\nu,\ell^1_\nu)}}\\ &\le \big[1-(|\phi_f|+|\phi_u|)\|\partial_\theta Q(\theta)\|_{\mathcal{L}(X_\theta\otimes X,\ell^1_\nu)}\|B(\theta)\|_{\mathcal{L}(\ell^1_\nu,X)}\big]^{-1}.\end{aligned}$$

To bound the derivatives of $\big(I+A_1(\theta,\phi)B(\theta)\big)^{-1}$, we use that for any smooth path of invertible matrices it holds that

$$\frac{\partial Y^{-1}}{\partial t} = -Y^{-1}\frac{\partial Y}{\partial t}Y^{-1}.$$

By product rule we obtain for the second derivatives

$$\frac{\partial^2 Y^{-1}}{\partial t\partial s} = Y^{-1}\left(\frac{\partial Y}{\partial s}Y^{-1}\frac{\partial Y}{\partial t} - \frac{\partial^2 Y}{\partial t\partial s} + \frac{\partial Y}{\partial t}Y^{-1}\frac{\partial Y}{\partial s}\right)Y^{-1}.$$

Hence, to bound the derivatives of $\big(I+A_1(\theta,\phi)B(\theta)\big)^{-1}$, it suffices to have a bound on the inverse itself, and to have bounds on the derivatives of $I+A_1(\theta,\phi)B(\theta)$.

For fixed $(\theta,\phi)\in U$ and $\mathbf{i}\in\mathbf{I}$ we compute the nontrivial first derivatives $\partial_{\mathbf{i}}A_1(\theta,\phi): X\otimes X_{\mathbf{i}}\to\ell^1_\nu$:

$$\partial_\theta A_1(\theta,\phi) = \partial_{\theta\theta}Q_f(\theta)\phi_f + \partial_{\theta\theta}Q_u(\theta)\phi_u, \qquad \partial_\star A_1(\theta,\phi) = \partial_\theta Q_\star(\theta) \quad \text{for } \star\in\{f,u\}.$$

For fixed $(\theta,\phi)\in U$ and $\mathbf{i},\mathbf{j}\in\mathbf{I}$ we also compute the nontrivial second derivatives $\partial_{\mathbf{i}}\partial_{\mathbf{j}}A_1(\theta,\phi):$ $X\otimes X_{\mathbf{i}}\otimes X_{\mathbf{j}}\to\ell^1_\nu$, namely

$$\partial_{\theta\theta}A_1(\theta,\phi) = \partial_{\theta\theta\theta}Q_f(\theta)\phi_f + \partial_{\theta\theta\theta}Q_u(\theta)\phi_u, \qquad \partial_{\theta\star}A_1(\theta,\phi) = \partial_{\theta\theta}Q_\star(\theta) \quad \text{for } \star\in\{f,u\}.$$

We note that $\partial_\infty DK^{-1}=0$. Furthermore, note that $\pi_\infty DK^{-1}=\pi_\infty$, and so $\pi_\infty\partial_{\mathbf{i}}(DK^{-1})=0$ for all $\mathbf{i}\in\mathbf{I}$. Hence, to obtain all the bounds on DK^{-1} and its derivatives, it suffices to bound

$$\|\pi_\circ B(\theta)\|_{\mathcal{L}(\ell^1_\nu,X)} \qquad \left\|\pi_\circ\tfrac{\partial^k}{\partial\theta^k}B(\theta)\right\|_{\mathcal{L}\left(X_\theta^{\otimes k}\otimes\ell^1_\nu,X\right)} \qquad \left\|\pi_\circ\tfrac{\partial^k}{\partial\theta^k}Q(\theta)\right\|_{\mathcal{L}\left(X\otimes X_\theta^{\otimes k},\ell^1_\nu\right)}, \tag{79}$$

where $\pi_\circ\in\{\pi_N,\pi_\infty\}$ and $k=1,2,3$. Since we have either explicit expressions (we may take a supremum over $\theta\in[-\delta_\theta,\delta_\theta]^{n_\theta}$ using interval arithmetic) or explicit bounds for each of these, we conclude that we thus obtain explicit bounds on DK^{-1} and its derivatives. We note that bounds on $\pi_{\mathbf{k}}DK(\theta,\phi)^{-1}=\pi_{\mathbf{k}}B(\theta)\big(I+A_1(\theta,\phi)B(\theta)\big)^{-1}$ can be further improved by bounding $\|\pi_{\mathbf{k}}B(\theta)\|_{\mathcal{L}(\ell^1_\nu,X)}$ for $\mathbf{k}\in\mathbf{I}$, and likewise for the derivatives.

7.3.2 Bounding E

We we want to bound $E:U\to\ell^1_\nu$ defined in (76), and repeated below:

$$E(\theta,\phi) = E_\theta(\theta) + E_f(\theta)\phi_f + E_u(\theta)\phi_u + E_\infty(\theta)\phi_\infty,$$

see also (67) and (68). Note that bounds on E_θ are calculated in the $|\cdot|_{\ell^1_\nu}$ norm, whereas bound on E_f,E_u,E_∞ are calculated in the $\|\cdot\|_{\mathcal{L}(X,\ell^1_\nu)}$ norm.

We calculate

$$\partial_\theta E(\theta,\phi)\cdot h = \Big(\partial_\theta E_\theta(\theta) + \partial_\theta E_f(\theta)\phi_f + \partial_\theta E_u(\theta)\phi_u + \partial_\theta E_\infty(\theta)\phi_\infty\Big)\cdot h_\theta.$$

The other first derivatives of E are

$$\partial_\star E(\theta,\phi)\cdot h = E_\star(\theta)\cdot h_f, \qquad \text{for } \star\in\{f,u,\infty\}.$$

We next determine the nontrivial second derivatives of E:

$$\partial_{\theta\theta} E(\theta,\phi)\cdot(h^1,h^2) = (\partial_{\theta\theta}E_\theta + \partial_{\theta\theta}E_f\phi_f + \partial_{\theta\theta}E_u\phi_u + \partial_{\theta\theta}E_\infty\phi_\infty)\cdot(h^1_\theta,h^2_\theta),$$
$$\partial_{\theta\star} E(\theta,\phi)\cdot(h^1,h^2) = \partial_\theta E_\star(\theta)\cdot(h^1_\theta,h^2_\star), \qquad \text{for } \star\in\{f,u,\infty\}.$$

We recall that we have an explicit finite dimensional polynomial representation for the functions E_θ, E_f and E_u. For E_∞ and its derivatives we have

$$\begin{aligned} E_\infty(\theta)\cdot\phi_\infty &= -3P(\theta)*P(\theta)*\phi_\infty \\ \partial_\theta E_\infty(\theta)\cdot(\phi_\infty,h_\theta) &= -6\,(\partial_\theta P(\theta)h_\theta)*P(\theta)*\phi_\infty \\ \partial_{\theta\theta}E_\infty(\theta)\cdot(\phi_\infty,h^1_\theta,h^2_\theta) &= -6(\partial_{\theta\theta}P(\theta)\cdot(h^1_\theta,h^2_\theta))*P(\theta)*\phi_\infty - 6(\partial_\theta P(\theta)h^1_\theta)*(\partial_\theta P(\theta)h^2_\theta)*\phi_\infty. \end{aligned}$$

By using bounds on $|\phi|$, explicit expressions for the polynomials P, Q, and the expressions above, we can obtain bounds on E over all of $U\subseteq X$. In summary, to obtain all the bounds on E and its derivatives, we bound

$$\left\|\pi_\circ \tfrac{\partial^k}{\partial\theta^k}E_\theta(\theta)\right\|_{\mathcal{L}\left(X_\theta^{\otimes k},\ell^1_\nu\right)} \qquad \left\|\pi_\circ \tfrac{\partial^k}{\partial\theta^k}E_\star(\theta)\right\|_{\mathcal{L}\left(X\otimes X_\theta^{\otimes k},\ell^1_\nu\right)}, \tag{80}$$

where we take $\pi_\circ\in\{\pi_N,\pi_\infty\}$ and $\star\in\{u,f,\infty\}$ and $k=0,1,2$ and a supremum over $\theta\in[-\delta_\theta,\delta_\theta]^{n_\theta}$. Here $X^{\otimes k}$ is the k-fold tensor product of X, and $X^{\otimes 0}$ is the trivial vector space.

7.3.3 Bounding R

We first recall that, see (73) and (74),

$$\mathbf{P} := P(\theta), \qquad \mathbf{Q} := Q_f(\theta)\phi_f + Q_u(\theta)\phi_u + \phi_\infty, \qquad \mathbf{R} := -3\mathbf{P}*\mathbf{Q}^2 - \mathbf{Q}^3.$$

To calculate bounds on $R(\theta,\phi)=\mathbf{R}$ and its derivatives, we start by calculating the derivatives of $\mathbf{Q}$:

$$\partial_\theta\mathbf{Q}\cdot h = (\partial_\theta Q_f\phi_f + \partial_\theta Q_u\phi_u)\cdot h_\theta, \qquad \partial_\star\mathbf{Q}\cdot h = Q_\star\cdot h_\star \quad \text{for } \star\in\{f,u\}, \qquad \partial_\infty\mathbf{Q}\cdot h = h_\infty.$$

The only nonvanishing second derivatives of $\mathbf{Q}$ are given by

$$\partial_{\theta\theta}\mathbf{Q}\cdot(h^1,h^2) = (\partial_{\theta\theta}Q_f\phi_f + \partial_{\theta\theta}Q_u\phi_u)\cdot(h^1_\theta,h^2_\theta), \quad \partial_{\star\theta}\mathbf{Q}\cdot(h^1,h^2) = \partial_\theta Q_\star\cdot(h^1_\theta,h^2_\star) \quad \text{for } \star\in\{f,u\}.$$

The only nonvanishing derivatives of $\mathbf{P}$ are with respect to θ. From this, bounds on $\mathbf{Q}^2$, $\mathbf{Q}^3$, and $\mathbf{P}*\mathbf{Q}^2$ and their partial derivatives can all be estimated using the product rule.

Thus, using that $\mathbf{R} = -3\mathbf{P}*\mathbf{Q}^2 - \mathbf{Q}*\mathbf{Q}^2$, we have expressions for all of the first and second derivatives of R. Hence, to obtain all the bounds on R and its derivatives, it suffices to bound

$$\left\|\pi_\circ \tfrac{\partial^k}{\partial\theta^k}P(\theta)\right\|_{\mathcal{L}\left(X_\theta^{\otimes k},\ell^1_\nu\right)} \qquad \left\|\pi_\circ \tfrac{\partial^k}{\partial\theta^k}Q_\star(\theta)\right\|_{\mathcal{L}\left(X\otimes X_\theta^{\otimes k},\ell^1_\nu\right)}, \tag{81}$$

where we take $\pi_\circ\in\{\pi_N,\pi_\infty\}$ and $\star\in\{u,f\}$ and $k=0,1,2$ and a supremum over $\theta\in[-\delta_\theta,\delta_\theta]^{n_\theta}$. The rest of the bounds follow from the product rule (as detailed above), the Banach algebra property of ℓ^1_ν, and bounds on $|\phi|$ resulting from it being restricted to a ball $B(r_s+\epsilon_s,r_u+\epsilon_u)$.

7.3.4 Bounding $\tilde{\mathcal{N}}$

In the previous subsections, we have outlined how to obtain bounds on DK^{-1}, E and R and their derivatives, by computing the necessary bounds on the expressions in (79), (80) and (81) respectively. The derivatives of $\tilde{\mathcal{N}} = DK^{-1}(E+R)$ can be calculated using the product rule. Using the formulas for derivatives derived in Section 7.3, in the code we have implemented expression for the constants $\tilde{C}^{\mathbf{ik}}_{\mathbf{j}}$ bounding $\|\tilde{\mathcal{N}}^{\mathbf{ik}}_{\mathbf{j}}\|_{(r_s+\epsilon_s,r_u+\epsilon_u)}$ for $\mathbf{i},\mathbf{j},\mathbf{k}\in\mathbf{I}$ needed to apply Proposition 2.4.

7.4 Estimate 4 – Bounding $\tilde{\mathcal{N}}^{\mathbf{i}}_{\mathbf{j}}(0)$

We aim to compute a tensor $\tilde{D}$ bounding $\|\tilde{\mathcal{N}}(0)\|$, which is needed to apply Proposition 2.4. We infer from the computations in Section 7.3 that both $\mathbf{Q}^2(\theta,0)=0$ and $D\mathbf{Q}^2(\theta,0)=0$, that $D(\mathbf{Q}*\mathbf{Q}^2)=0$ and $D(\mathbf{P}*\mathbf{Q}^2)=0$ if $\phi=0$, and that thereby $DR(\theta,0)=0$. Since $R(\theta,0)=0$ as well, we infer that

$$\partial_{\mathbf{i}}\tilde{\mathcal{N}}(0) = DK(0)^{-1}\partial_{\mathbf{i}}E(0,0) + (\partial_{\mathbf{i}}DK(0)^{-1})E(0,0) \qquad \text{for } \mathbf{i}\in\mathbf{I}. \tag{82}$$

The first summand in (82) is essentially the same term we studied in Section 6.2.4. To see this, starting from (76) we compute the first derivatives of E at $(\theta,\phi)=(0,0)$:

$$\partial_\theta E(0,0)\cdot h = \partial_\theta E_\theta(0)\cdot h_\theta, \qquad \partial_\star E(0,0)\cdot h = E_\star(0)\cdot h_\star \quad \text{for } \star\in\{f,u,\infty\}.$$

We deduce from the definition of E in (68) and the substitution $P(0)=\bar{a}$ that

$$\begin{aligned} \partial_\theta E_\theta(0)\pi_\theta &= \left(DF(\bar{a})\partial_\theta P(0) - \partial_\theta P(0)\Lambda_\theta\right)\pi_\theta, \\ E_\star(0)\pi_\star &= \left(DF(\bar{a})Q_\star(0) - Q_\star\Lambda_\star\right)\pi_\star \qquad \text{for } \star\in\{f,u,\infty\}. \end{aligned}$$

Using Q_0 defined in (71) we obtain the simplification

$$\partial_{\mathbf{i}}E(0,0)h = \left(DF(\bar{a})Q_0 - Q_0\Lambda\right)\pi_{\mathbf{i}} \qquad \text{for } \mathbf{i}\in\mathbf{I}.$$

Finally, the first summand in (82) simplifies to

$$DK(0,0)^{-1}\partial_{\mathbf{i}}E(0,0) = \left(Q_0^{-1}DF(\bar{a})Q_0 - \Lambda\right)\pi_{\mathbf{i}} \qquad \text{for } \mathbf{i}\in\mathbf{I}.$$

We may then bound $\|\pi_{\mathbf{j}}\left(Q_0^{-1}DF(\bar{a})Q_0-\Lambda\right)\pi_{\mathbf{i}}\|_{\mathcal{L}(X,X)}$ in exactly the same manner as we did in Section 6.2.4, with the trivial addition that we have an additional projection map π_θ to consider.

To bound the second summand in (82) we first note that $E(0,0)=E_\theta(0)$, for which we have an explicit expression. From a calculation in the same vein as in Section 7.3.1, we obtain

$$\left(\partial_{\mathbf{i}}DK(0)^{-1}\right)E(0,0) = -Q_0^{-1}\left(\partial_{\mathbf{i}}DK(0)\right)Q_0^{-1}E_\theta(0).$$

We may further compute

$$\partial_\theta DK(0) = \partial_\theta A_0(0), \qquad \partial_\star DK(0) = \partial_\theta Q_\star(0) \quad \text{for } \star\in\{f,u,\infty\}.$$

The norm $|E_\theta(0)|_{\ell^1_\nu}$ is usually quite small, and it suffices to obtain a rough bound on the norm of $\partial_{\mathbf{i}}DK(0)^{-1}$. Thus, for $\mathbf{i},\mathbf{j}\in\mathbf{I}$ we use the following bounds on the various components of (82):

$$\tilde{D}^{\mathbf{i}}_{\mathbf{j}} := \|\pi_{\mathbf{j}}\left(Q_0^{-1}DF(\bar{a})Q_0-\Lambda\right)\pi_{\mathbf{i}}\|_{\mathcal{L}(X,X)} + \left\|\pi_{\mathbf{j}}Q_0^{-1}\right\|_{\mathcal{L}(\ell^1_\nu,X)}\|\partial_{\mathbf{i}}DK(0)\|_{\mathcal{L}(X_{\mathbf{i}}\otimes X,\ell^1_\nu)}\left|\pi_N Q_0^{-1}E_\theta(0)\right|_X.$$

We note that there is some additional cancellation, as $\pi_{\mathbf{j}}Q_0^{-1}\left(\partial_{\mathbf{i}}DK(0)\right)=0$ when $\mathbf{i}=\infty$ or $\mathbf{j}=\infty$.

7.5 Estimate 5 – Semigroup Bounds

To obtain the constants C_s and λ_s, we apply Theorem B.1 as we did in Section 6.2.5. The only difference is that X_s should be split into 2 subspaces to be able to apply Theorem B.1, whereas we have split X_s into 3 subspaces in Section 7. We argue as follows. First defining $D^{\mathbf{i}}_{\mathbf{j}} := \tilde{D}^{\mathbf{i}}_{\mathbf{j}} + \tilde{C}^{\mathbf{i}l}_{\mathbf{j}}\epsilon_l + \tilde{C}^{\mathbf{i}l'}_{\mathbf{j}}\epsilon_{l'}$ as in Proposition 2.4, we then define:

$$\begin{aligned} &\mu_1 := \lambda_1 && \delta_a := \max_{1\le i\le m_s-1}\sum_{1\le j\le m_s-1} D^i_j && \delta_b := \sum_{1\le j\le m_s-1} D^{m_s}_j, \\ &\mu_\infty := \lambda_3 = \lambda_\infty && \delta_c := \max_{1\le i\le m_s-1} D^i_{m_s} && \delta_d := D^{m_s}_{m_s}. \end{aligned}$$

The rest of the computation for C_s and λ_s follows exactly as described in Section 6.2.5.

7.6 Conclusion and Numerical Results

Our overall goal is to produce a large piece of the local stable manifold, while keeping our error bounds small. These goals are at odds with each other, as our error bounds generally increase with the size of our approximation. We recall that the parameter $\rho = (\rho_\theta, \rho_f, \rho_\infty)$ determines the size of the domain

$$B_s(\rho) = \{(\mathrm{x}_\theta, \mathrm{x}_f, \mathrm{x}_\infty) \in X_s : |\mathrm{x}_\theta| \leq \rho_\theta, |\mathrm{x}_f| \leq \rho_f, |\mathrm{x}_\infty| \leq \rho_\infty\} ,$$

for charts $\alpha \in \mathcal{B}_{\rho,P,\bar{P}}$, where X_s is decomposed in terms of the eigenspaces X_θ, X_f, and X_∞ of Λ_s corresponding to the slow stable eigenvalues, the fast-but-finite stable eigenvalues, and the remaining infinite stable eigenvalues respectively. This parameter ρ has a significant impact on nearly every aspect of our analysis.

For a given application it may be advantageous to choose certain components of $\rho = (\rho_\theta, \rho_f, \rho_\infty)$ to be large and others small. For example, we generically expect connecting orbits to have a larger projection into the slow-stable subspace X_θ and a smaller projection into the other stable subspaces. In Theorem 7.1 we present one such theorem where we take ρ_θ as large as possible. The parameters are the same as the ones used to produce Figure 1. This nonlinear approximation of the stable manifold produces significantly better error estimates than a linear approximation: the C^0 error bounds in Theorem 7.1 are of size 7.43×10^{-12}, whereas the approximate manifold in Theorem 6.4 has C^0 error bounds of 3.36×10^{-3}.

Theorem 7.1. *Fix parameter $\beta_1 = 0.05$ and $\beta_2 = -0.35$. For computational parameters $\nu = 1.001$ and $N = 30$. We compute an approximate equilibrium $\bar{a} \in \ell^1_\nu$ which is 1.61×10^{-14} close to a true equilibrium $\tilde{a} \in \ell^1_\nu$. Using the techniques discussed in Section 2.4, we compute a slow stable manifold and finite dimensional (un)stable bundles, represented by Taylor polynomials of degree 20. We take*

$$\rho = \begin{pmatrix} 3.18 \times 10^{-2} & 10^{-6} & 10^{-10} \end{pmatrix} ,$$

and we take

$$P = \begin{pmatrix} 9.43 \times 10^{-11} \\ 4.41 \times 10^{-6} \\ 3.31 \times 10^{-6} \end{pmatrix} \qquad \bar{P} = \begin{pmatrix} 1.30 \times 10^{-9} & 5.60 \times 10^{-5} & 1.04 \times 10^{-4} \\ 5.60 \times 10^{-5} & 2.72 \times 10^{-0} & 8.20 \times 10^{-4} \\ 1.04 \times 10^{-4} & 8.20 \times 10^{-4} & 1.41 \times 10^{-4} \end{pmatrix} .$$

There exists a unique map $\tilde{\alpha} \in C^{1,1}(B_s(\rho - \epsilon_s), X_u)$ where $\epsilon_{\mathbf{i}} \leq 4.51 \times 10^{-14}$ such that the stable manifold of $\tilde{a} \in \ell^1_\nu$ for the differential equation (1) is locally given by

$$\mathrm{x}_s \mapsto K\left(\mathrm{x}_s, \tilde{\alpha}(\mathrm{x}_s)\right) ,$$

for K given in (70) and $\tilde{\alpha}$ satisfying the following estimates:

$$|\tilde{\alpha}_{i'}(\xi)| \leq 7.43 \times 10^{-12} \qquad \|\tilde{\alpha}^i_{i'}(\xi)\| \leq P^i_{i'} \qquad \mathrm{Lip}(\partial_i \tilde{\alpha})^j_{i'} \leq \bar{P}^{ij}_{i'},$$

for all $\xi \in B_s(\rho - \epsilon_s)$ and $i, j \in I$, $i' \in I'$ and $\mathbf{i} \in \mathbf{I}$.

Proof. In script `main_NL.m` we calculate all of the constants and verify all of the hypotheses in Theorem 5.11. In particular we have a contraction constant $\|J\| < 5.86 \times 10^{-6}$. It takes approximately 11 seconds to construct the slow-stable manifold and normal bundles, 23 seconds to compute the bounds detailed in Section 7, and 12 seconds to compute all the bounds in Sections 3-5 needed to validate the stable manifold. These we run on MATLAB 2019a with INTLAB on a i7-8750H processor.

□

The nonlinear approximation Theorem 7.1 produces a larger validated part of the manifold in the direction of the slow stable eigenvector, where we would generically expect to find connecting orbits. We note that in Theorem 7.1 the gap between eigenvalues of $\Lambda_{1'}$, Λ_1 and Λ_2 is not very large:

$$\lambda_{1'} = 1.01, \qquad \lambda_1 = -1.41, \qquad \lambda_2 = -1.99, \qquad \lambda_3 = -4.58 \times 10^4.$$

We took the slow-stable eigenspace to be one dimensional. If a particular application required a stable manifold which was wider along the second slowest stable eigendirection, we could increase ρ_f at a cost of also increasing P, $\bar{P}$, etc. These error estimates could be improved somewhat by splitting X_f into two subspaces. Moreover, if we wanted to significantly increase the radius of our approximation along the second slowest stable eigendirection, we could use a higher dimensional slow stable manifold.

From the classical theory [15] we expect our derivative bound $P \geq \|D\alpha\|$ to be at least as large as the ratio between the derivative of the nonlinearity and the spectral gap, roughly

$$|P| \gtrsim \frac{\|D\mathcal{N}\|}{\lambda_u - \lambda_s} \gtrsim \frac{\|L\| + \|D^2\mathcal{N}\|\rho}{\lambda_u - \lambda_s}.$$

This bound should roughly increase linearly with ρ and be bounded below by $\|L\|$, the error from not perfectly splitting $X_u \times X_s$ into eigenspaces. This scaling can be observed in Figure 3, where we display how the error bounds in Theorem 6.4 and Theorem 7.1 change when varying ρ. Our nonlinear approximation is able to maintain small error bounds despite taking ρ_θ large, because our change of variables prepares our nonlinearity such that $\|\partial_\theta D\mathcal{N}\|$ is small. Note that one should be mindful in comparing the two graphs in Figure 3, as in Theorem 6.4 we split $X_s = X_f \times X_\infty$ with $\dim(X_f) = N$ and in Theorem 7.1 we split $X_s = X_\theta \times X_f \times X_\infty$ with $\dim(X_\theta) = 1$ and $\dim(X_f) = N - 1$.

Using the linear approximation, for a large range of ρ_f, the contraction constant, and the tensor P and the minimal choice of ρ_∞ all scale linearly with ρ_f. The C^0 error of the manifold, given by $|\tilde{\alpha}_{i'}| \leq P^i_{i'}(\rho_i + \epsilon_i) + \epsilon_{i'}$ in Theorem 5.11, is dominated by the error in validating the equilibrium until $\rho_f \approx 10^{-7}$, where it begins to scale quadratically with ρ_f. The $C^{1,1}$ error bounds on the norm of the components of $\bar{P}$ do not improve much for $\rho < 10^{-3}$, and increase quite rapidly for $\rho_f > 10^{-2}$.

For the nonlinear approximation, the error in validating the equilibrium dominates the C^0 bound until $\rho_\theta \approx 10^{-2}$, a point where P^θ_u also begins to increase marginally. The contraction constant scales similarly, although it begins to increase around $\rho_\theta \approx 10^{-3}$. The C^1 bounds in the X_f and X_∞ subspaces are bounded below by our accuracy in decomposing the eigenspaces of $DF(\bar{a})$, and increase linearly with ρ_θ. For the whole range of admissible ρ_θ, both ρ_f and ρ_∞ can be taken exceedingly small, and do not significantly contribute to the overall error estimates.

We do not expect to validate a global stable manifold with the Lyapunov-Perron approach; if ρ is too large, the various hypotheses of Theorem 5.11 may no longer be satisfied. For example, we may be unable to prove the image of Ψ is contained within $\mathcal{B}^{0,1}_{\rho,P}$ or $\mathcal{B}^{1,1}_{\rho,P,\bar{P}}$ as detailed in Theorems 4.2 or Theorem 4.4. Other causes for failure would be if $\|J\| > 1$ whereby Ψ is not a contraction mapping, or if we are unable to prove solutions $x(t, \xi, \alpha)$ are contained within $B_s(\rho)$ for all $t \geq 0$ as required by Proposition 3.13. When using a linear approximation, many of these hypotheses all simultaneously fail for larger values of ρ. In contrast, for the nonlinear approximation in Section 7, the dominant limiting factor is the condition $\gamma_0 = \lambda_s + C_s\hat{\mathcal{H}} < 0$ as required in Proposition 3.13.

Overall, the framework developed in Sections 2 - 5 allow us to leverage to great effect our estimates on our approximate stable manifold made in Sections 6-7. In future work, we aim to combine this with a rigorous integrator to prove connecting orbits. To develop the theory further, we aim to develop a constructive stable manifold theory for normally hyperbolic invariant manifolds, such as periodic orbits.

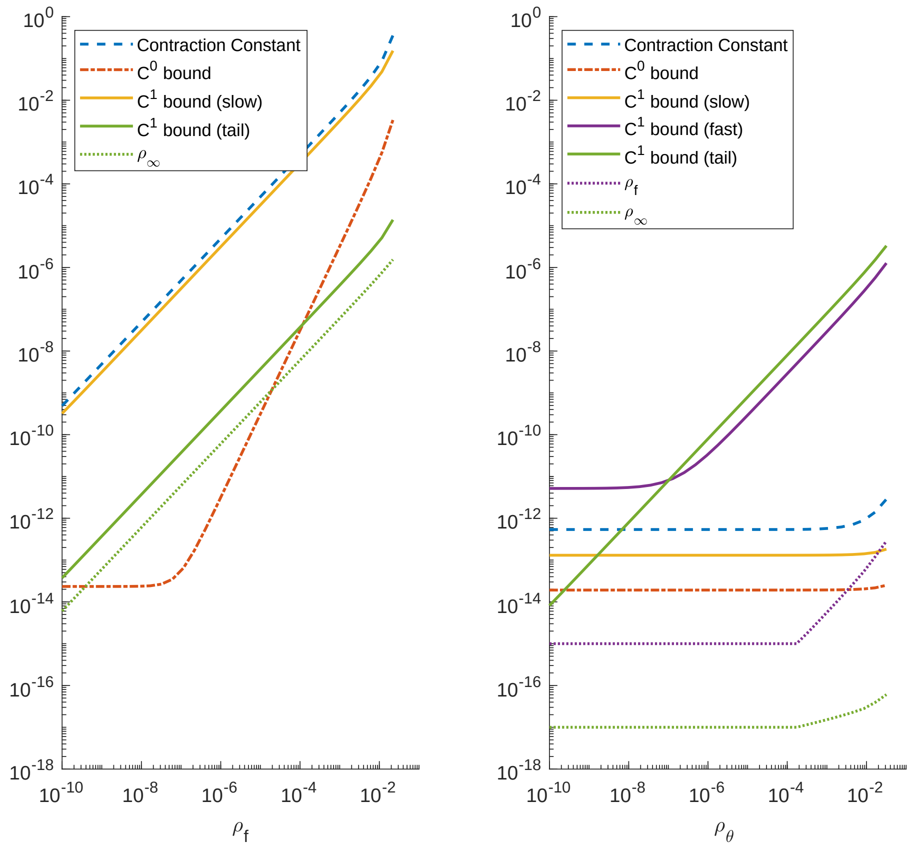

Figure 3: (Left) Using the estimates from Section 6, the bounds produced by a computer assisted proof for a range of radii $\rho_f \in [10^{-10}, 0.022]$, with ρ_∞ chosen to be as small as possible. (Right) Using the estimates from Section 7, the bounds produced for a range of radii $\rho_\theta \in [10^{-10}, 0.0318]$, with ρ_f and ρ_∞ chosen to be as small as possible. Note that the nonlinear approximation yields smaller C^0 error bounds (red dash-dotted lines).

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A General Strategy for Bootstrapping Gronwall's Inequality

We generalize the bootstrapping argument used in Section 3 so that it can be applied in Section 4 and Section 5. To unify the class of functions we wish to bound, and the set of assumptions we make on these functions, we define Condition A.1 below. In a slight abuse of notation, here we define $\mathcal{B}$ to be a tensor, distinct from its previous usage as a ball of functions in Definition 2.5.

Condition A.1. *Fix $\lambda_1, \ldots, \lambda_{N_\lambda} \in \mathbb{R}$, fix $H \in \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\lambda}$ and define $\gamma_k := \lambda_k + H_k^k$ for $1 \leq k \leq N_\lambda$. For $N_\mu \in \mathbb{N}$, fix some $\mu_k \in \mathbb{R}$ for $1 \leq k \leq N_\mu$. Assume that $\{\gamma_j\}_{j=1}^{N_\lambda} \subseteq \{\mu_k\}_{k=1}^{N_\mu}$, and suppose that both $\gamma_k > \gamma_{k+1}$ and $\mu_k > \mu_{k+1}$. Assume further that $\mu_1 > \gamma_1$.*

For $M \in \mathbb{N}$, and $N_i \in \mathbb{N}$ for $1 \leq i \leq M$ and basis elements $e_{n_i} \in \mathbb{R}^{N_i}$ where $1 \leq n_i \leq N_i$, we fix tensors

$$\mathcal{A} \in \Big(\bigotimes_{i=1}^{M} \mathbb{R}^{N_i}\Big) \otimes \mathbb{R}^{N_\lambda} \otimes \mathbb{R}^{N_\mu}, \qquad \mathcal{B} \in \Big(\bigotimes_{i=1}^{M} \mathbb{R}^{N_i}\Big) \otimes \mathbb{R}^{N_\lambda}$$

component-wise by:

$$\mathcal{A}_{j,k} := A_{j,k}^{n_1 \ldots n_M} \cdot e_{n_1} \otimes \cdots \otimes e_{n_M}, \qquad \mathcal{B}_j := B_j^{n_1 \ldots n_M} \cdot e_{n_1} \otimes \cdots \otimes e_{n_M}.$$

For this arrangement of constants, we say that a pair (u, ω) satisfies Condition A.1 on a time interval $[0, T]$ if the functions $u = (u_j)_{j=1}^{N_\lambda}$ and the positive tensor $\omega \in \bigotimes_{i=1}^{M} \mathbb{R}^{N_i}$ satisfy the inequalities

$$e^{-\lambda_j t} u_j(t) \leq \mathcal{B}_j \omega + \int_0^t e^{-\lambda_j \tau} \sum_{0 \leq k \leq N_\mu} e^{\mu_k \tau} \mathcal{A}_{j,k} \omega \, d\tau + \int_0^t e^{-\lambda_j \tau} H_j^i u_i(\tau) \, d\tau \qquad \textit{for all } t \in [0, T]. \tag{83}$$

In all cases where we consider constants satisfying Condition A.1, we take $N_\lambda = m_s$, and $\lambda_1, \ldots, \lambda_{N_\lambda}$ as in (6), and H_j^i as in Definition 2.6. Hence, the definition of γ_k here coincides with that given in Definition 3.3. For the other variables, we take them in the various sections according to the following table.

	Section 3	Section 4	Section 5
u_j	$\lvert x_j(t,\xi,\alpha) - x_j(t,\zeta,\alpha)\rvert$	$\lvert \partial_i x_j(t,\eta,\alpha) - \partial_i x_j(t,\zeta,\alpha)\rvert$	$\lvert x_j(t,\xi,\alpha) - x_j(t,\xi,\beta)\rvert$
ω	$\lvert \xi_n - \zeta_n \rvert$	$\lvert \eta_l - \zeta_l \rvert$	$\lvert \xi_{n_1}\rvert \otimes \lVert \alpha - \beta \rVert_{n_2', \mathcal{E}}^{n_3}$
$\mathcal{A}_{j,k}$	0	$S_j^{nm} G_{m,k_1}^{l} G_{n,k_2}^{i}$	$C_j^{n_2'} G_{n_3,k}^{n_1}$
$\mathcal{B}_j$	δ_j^n	0	0
$\{\mu_k\}$	$\{\gamma_k\}_{k=0}^{m_s}$	$\{\gamma_k\}_{k=0}^{m_s} \cup \{\gamma_{k_1} + \gamma_{k_2}\}_{k_1,k_2=0}^{m_s}$	$\{\gamma_k\}_{k=-1}^{m_s}$

We note that for $\mathcal{A}_{j,k}$ in Section 4 we use a double index (k_1, k_2) to index over the elements of $\{\mu_k\}$. For a system given as in Condition A.1 we are interested in finding a tensor $\mathcal{G}$ satisfying Condition A.2 below.

Condition A.2. *Given μ as in Assumption A.1 and a pair (u,ω) of functions $u=(u_j)_{j=1}^{N_\lambda}$ on $[0,T]$ and a positive tensor $\omega\in\bigotimes_{i=1}^M\mathbb{R}^{N_i}$, we say that the tensor $\mathcal{G}\in\left(\bigotimes_{i=1}^M\mathbb{R}^{N_i}\right)\otimes\mathbb{R}^{N_\lambda}\otimes\mathbb{R}^{N_\mu}$ with components*

$$\mathcal{G}_{j,k}:=G_{j,k}^{n_1\dots n_M}e_{n_1}\otimes\cdots\otimes e_{n_M},$$

satisfies Condition A.2 if $u_j(t)\le\sum_{k=1}^{N_\mu}e^{\mu_k t}\mathcal{G}_{j,k}\omega$ for all $t\in[0,T]$.

From these two conditions, we can bootstrap our bounds on a tensor $\mathcal{G}$.

Proposition A.3. *Assume the pair (u,ω) satisfies Condition A.1 on $[0,T]$ and assume $\mathcal{G}$ satisfies Condition A.2. Fix $1\le j\le N_\lambda$. If $\mathcal{A}_{j,k}=0$ and $\mathcal{G}_{i,k}=0$ whenever $\mu_k=\gamma_j$, then we have:*

$$u_j(t)\le e^{\gamma_j t}\mathcal{B}_j\omega+\sum_{\substack{1\le k\le N_\mu\\ \mu_k\ne\gamma_j}}\frac{e^{\mu_k t}-e^{\gamma_j t}}{\mu_k-\gamma_j}\Big(\mathcal{A}_{j,k}+\sum_{\substack{1\le i\le N_\lambda\\ i\ne j}}H_j^i\mathcal{G}_{i,k}\Big)\omega\qquad\text{for all } t\in[0,T].\tag{84}$$

In other words, define a map $\mathcal{T}_{j,k}:\left(\bigotimes_{i=1}^M\mathbb{R}^{N_i}\right)\otimes\mathbb{R}^{N_\lambda}\otimes\mathbb{R}^{N_\mu}\to\bigotimes_{i=1}^M\mathbb{R}^{N_i}$ by:

$$\mathcal{T}_{j,k}(\mathcal{A},\mathcal{B},\mathcal{G}):=\begin{cases}(\mu_k-\gamma_j)^{-1}\Big(\mathcal{A}_{j,k}+\sum_{\substack{1\le i\le N_\lambda\\ i\ne j}}H_j^i\mathcal{G}_{i,k}\Big) & \text{if } \mu_k\ne\gamma_j\\[2ex] \mathcal{B}_j-\sum_{\substack{0\le m\le N_\mu\\ \mu_m\ne\gamma_j}}(\mu_m-\gamma_j)^{-1}\Big(\mathcal{A}_{j,m}+\sum_{\substack{1\le i\le N_\lambda\\ i\ne j}}H_j^i\mathcal{G}_{i,m}\Big) & \text{if } \mu_k=\gamma_j.\end{cases}\tag{85}$$

Then $\mathcal{G}$ also satisfies Condition A.2 if we replace $\mathcal{G}_{j,k}$ by $\mathcal{T}_{j,k}(\mathcal{A},\mathcal{B},\mathcal{G})$ for all k.

Proof of Proposition A.3. Splitting $H_j^iu_i=\sum_{i\ne j}H_j^iu_i+H_j^ju_j$, we write (83) as

$$e^{-\lambda_j t}u_j(t)\le\mathcal{B}_j\omega+\int_0^t e^{-\lambda_j\tau}v(\tau,\omega)d\tau+\int_0^t e^{-\lambda_j\tau}H_j^ju_j(\tau)d\tau.$$

where

$$v(\tau,\omega)=\sum_{\substack{1\le k\le N_\mu\\ \mu_k\ne\gamma_j}}e^{\mu_k\tau}\mathcal{A}_{j,k}\omega+\sum_{\substack{1\le i\le N_\lambda\\ i\ne j}}H_j^iu_i(\tau).$$

By plugging in the bound assumed in Condition A.2, we obtain

$$v(\tau,\omega)\le\sum_{\substack{1\le k\le N_\mu\\ \mu_k\ne\gamma_j}}e^{\mu_k\tau}\Big(\mathcal{A}_{j,k}\omega+\sum_{\substack{1\le i\le N_\lambda\\ i\ne j}}H_j^i\mathcal{G}_{i,k}\omega\Big).$$

By applying Lemma 3.9 we obtain (84). □

In order to obtain tensors satisfying the requirement that $\mathcal{A}_{j,k},\mathcal{G}_{i,k}=0$ whenever $\mu_k=\gamma_j$, we define an operator $\mathcal{Q}_j$ as below.

Proposition A.4. *Fix $1\le j\le N_\lambda$ and define a map $\mathcal{Q}_j:\left(\bigotimes_{i=1}^M\mathbb{R}_+^{N_i}\right)\otimes\mathbb{R}^{N_\lambda}\otimes\mathbb{R}^{N_\mu}\to\left(\bigotimes_{i=1}^M\mathbb{R}_+^{N_i}\right)\otimes\mathbb{R}^{N_\lambda}\otimes\mathbb{R}^{N_\mu}$ by*

$$\mathcal{Q}_j(\mathcal{G})_{i,k}^{n_1\dots n_M}:=\begin{cases}0 & \text{if } \mu_k=\gamma_j\\ G_{i,k}^{n_1\dots n_M}+G_{i,(k+1)}^{n_1\dots n_M} & \text{if } \mu_{k+1}=\gamma_j,\text{ and } G_{i,(k+1)}^{n_1\dots n_M}>0\\ G_{i,k}^{n_1\dots n_M}+G_{i,(k-1)}^{n_1\dots n_M} & \text{if } \mu_{k-1}=\gamma_j,\text{ and } G_{i,(k-1)}^{n_1\dots n_M}<0\\ G_{i,k}^{n_1\dots n_M} & \text{otherwise.}\end{cases}$$

Then $\mathcal{Q}_j(\mathcal{G})_{i,k}=0$ whenever $\mu_k=\gamma_j$. Furthermore, if $\mathcal{G}$ satisfies Condition A.2 then $\mathcal{Q}_j(\mathcal{G})$ satisfies Condition A.2.

We are able to generalize Algorithm 3.11 as follows.

Algorithm A.5. *Take as input all the constants in Condition A.1, an input tensor $\widehat{\mathcal{G}}$ satisfying Condition A.2, and a computational parameter $N_{bootstrap}$. The algorithm outputs a tensor $\mathcal{G}$.*

$\mathcal{G} \leftarrow \widehat{\mathcal{G}}$
for $1 \leq i \leq N_{bootstrap}$ ***do***
  for $1 \leq j \leq m_s$ ***do***
    $\mathcal{G}_{j,k} \leftarrow \mathcal{T}_{j,k}(\mathcal{Q}_j(\mathcal{A}), \mathcal{B}, \mathcal{Q}_j(\mathcal{G}))$
  end for
end for
return $\mathcal{G}$

Proposition A.6. *If the input tensor $\widehat{\mathcal{G}}$ to Algorithm A.5 satisfies Condition A.2, then the output tensor $\mathcal{G}$ satisfies Condition A.2.*

The proof of Proposition A.4 follows from the assumption that $\mu_k > \mu_{k+1}$. The proof of Proposition A.6 follows from an induction argument which uses Proposition A.3 for the inductive step. Both proofs are left to the reader.

B Semigroup Estimates for Fast-Slow Systems

In equation (8) we require constants C_s, λ_s satisfying

$$|e^{(\Lambda_s + L_s^s)t} \mathrm{x}_s| \leq C_s e^{\lambda_s t} |\mathrm{x}_s|, \qquad t \geq 0, \mathrm{x}_s \in X_s. \tag{86}$$

Our assumption that $\lambda_s < 0$, and moreover that $\gamma_0 = \lambda_s + C_s \hat{\mathcal{H}} < 0$, is essential. In Proposition 3.13 this is used to prove that solutions $x(t, \xi, \alpha)$ stay inside the ball $B_s(\rho)$ for all $t \geq 0$. While our method of bootstrapping Gronwall's inequality greatly mitigates the effect of these constants C_s, λ_s on our final estimates, for the Lyapunov-Perron operator to be well defined it is essential that we prove $\gamma_0 < 0$.

There are two types of estimates which we will apply to obtain pairs (C_s, λ_s) satisfying (86). First, for linear operators $A, B \in \mathcal{L}(X, X)$ with $|e^{At}\mathrm{x}| \leq k e^{\lambda t} |\mathrm{x}|$ for all $\mathrm{x} \in X$ and $t \geq 0$, and $\|B\| < \infty$, we have (the proof is analogous to the one of Proposition 3.2)

$$|e^{(A+B)t} \mathrm{x}| \leq k e^{(\lambda + k\|B\|)t} |\mathrm{x}|, \qquad \text{for all } t \geq 0, \mathrm{x} \in X. \tag{87}$$

This estimate by itself is not enough, as the largest eigenvalue of Λ_s is often small in comparison with $\|L_s^s\|$. For example, in Section 6 we showed that $|e^{\Lambda_i t} \mathrm{x}_i| \leq e^{\lambda_i t} |\mathrm{x}_i|$ and $\|L_j^i\| \leq D_j^i$ with values

$$\lambda_1 = -1.41, \qquad \lambda_2 = -4.58 \times 10^4, \qquad D_s^s = \begin{pmatrix} 4 \times 10^{-10} & 1.6 \\ 1.6 & 5.7 \end{pmatrix}.$$

Since $\lambda_1 + \|L_s^s\| > 0$, just an estimate of the type in (87) with A the diagonal part of D_s^s and B the off-diagonal part will not suffice. We further note that our estimates for D_s^s do not improve with a larger Galerkin projection dimension. Hence we want to change basis to diagonalize $\Lambda_s + L_s^s$, at least approximately, and then take advantage of the identity $e^{PJP^{-1}t} = Pe^{Jt}P^{-1}$ in our estimates. To motivate our construction, we first consider a 2×2 matrix

$$M = \begin{pmatrix} \lambda_1 & \delta_b \\ \delta_c & \lambda_\infty \end{pmatrix}.$$

If λ_∞ is much larger in absolute value than the other matrix entries, then the eigenvalues of M are approximately given by λ_1 and λ_∞. In particular, if $|\delta_b \delta_c| < |\lambda_1 \lambda_\infty|$ and $\lambda_1, \lambda_\infty < 0$, then all of the eigenvalues of M have negative real part. Below in Theorem B.1 we prove an analogous theorem where we replace λ_1 by a finite dimensional matrix, and λ_∞ by an infinite dimensional linear operator. This is the second type of estimate that we use to find pairs (C_s, λ_s) satisfying (86).

Theorem B.1. *Consider Banach spaces $\mathbb{C}^N$ and X_∞ with arbitrary norms, and their product $\mathbb{C}^N \times X_\infty$ with norm $|(x_N, x_\infty)| = (|x_N|^p + |x_\infty|^p)^{1/p}$ for any $1 \le p \le \infty$.*

Consider the linear operators $M, \Lambda, L : \mathbb{C}^N \times X_\infty \to \mathbb{C}^N \times X_\infty$ given by

$$M = \Lambda + L, \qquad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_\infty \end{pmatrix}, \qquad L = \begin{pmatrix} L_1^1 & L_1^\infty \\ L_\infty^1 & L_\infty^\infty \end{pmatrix}. \tag{88}$$

We require Λ to be densely defined and L to be bounded. Suppose that Λ_1 is diagonal *and that Λ_∞ has a bounded inverse.*

Fix constants $\mu_1, \mu_\infty, C_1, C_\infty \in \mathbb{R}$ such that for all $t \ge 0$ we have

$$\|e^{\Lambda_1 t}\| \le C_1 e^{\mu_1 t}, \qquad \|e^{\Lambda_\infty t}\| \le C_\infty e^{\mu_\infty t}.$$

Fix constants $\delta_1, \delta_b, \delta_c, \delta_d, \varepsilon > 0$ such that

$$\|L_1^1\| \le \delta_a, \qquad \|L_1^\infty\| \le \delta_b, \qquad \|L_\infty^1\| \le \delta_c, \qquad \|L_\infty^\infty\| \le \delta_d,$$

and set

$$\varepsilon := \sum_{\lambda \in \sigma(\Lambda_1)} \frac{\|\Lambda_\infty^{-1}\|}{1 - \|\Lambda_\infty^{-1}\|(\delta_d + |\lambda|)}.$$

Assume that the inequalities

$$\|\Lambda_\infty^{-1}\| \left(\delta_d + \sup_{\lambda_k \in \sigma(\Lambda_1)} |\lambda_k| \right) < 1, \qquad \mu_\infty + C_\infty \left(\delta_d + \varepsilon \delta_b \delta_c (1 + \varepsilon^2 \delta_b \delta_c) \right) < \mu_1, \tag{89}$$

are satisfied. Then we have

$$\|e^{Mt}\| \le C_s e^{\lambda_s t},$$

where

$$\begin{aligned} C_s &:= (1 + \varepsilon \delta_b)^2 (1 + \varepsilon \delta_c)^2 \max\{C_1, C_\infty\} \\ \lambda_s &:= \mu_1 + C_s \delta_a + \Delta \max\{C_1, C_\infty\} \\ \Delta &:= \varepsilon \delta_b \delta_c \left(1 + \varepsilon(2\delta_b + \delta_c) + \varepsilon^2 \delta_b \delta_c (1 + \varepsilon \delta_b) \right). \end{aligned}$$

First we prove a lemma for general Banach spaces which allows us to approximately diagonalize our matrix. When $|\cdot|$ denotes the norm on a Banach space, then by $|\cdot|_*$ we denote the norm on its dual.

Lemma B.2. *For a Banach space X_∞ consider the linear operator $M_1 : \mathbb{C}^N \times X_\infty \to \mathbb{C}^N \times X_\infty$ defined as*

$$M_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Suppose that $\sigma(A) \cap \sigma(D) = \emptyset$ and that A has distinct eigenvalues $\lambda_1, \dots, \lambda_N$ with eigenvectors $v_1, \dots, v_N$, and dual eigenvectors $u_1, \dots, u_N$ (the corresponding eigenvectors of A^*). Normalize the vectors so that $u_i^* v_j = \delta_{ij}$, the Kronecker delta.

We define $W_b : X_\infty \to \mathbb{C}^N$ and $W_c : \mathbb{C}^N \to X_\infty$ as a sum of products between vectors in their codomains, and dual vectors acting on their domains:

$$W_b := \sum_{k=1}^{N} v_k \left[(D^* - \lambda_k^* I_\infty)^{-1} B^* u_k^*\right], \qquad W_c := \sum_{k=1}^{N} - \left[(D - \lambda_k I_\infty)^{-1} C v_k\right] u_k^*,$$

where $D^* : X_\infty^* \to X_\infty^*$ and $B^* : (\mathbb{C}^N)^* \to X_\infty^*$ are the dual transformations. Define invertible operators $P_b, P_c : \mathbb{C}^N \times X_\infty \to \mathbb{C}^N \times X_\infty$ by

$$P_b = \begin{pmatrix} I_N & W_b \\ 0 & I_\infty \end{pmatrix} \qquad P_c = \begin{pmatrix} I_N & 0 \\ W_c & I_\infty \end{pmatrix}.$$

Then

$$(P_c P_b)^{-1} M_1 (P_c P_b) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + E,$$

where

$$E = \begin{pmatrix} (I_N + W_b W_c) B W_c & B W_c W_b + W_b W_c B (I + W_c W_b) \\ -W_c B W_c & -W_c B (I_\infty + W_c W_b) \end{pmatrix}.$$

Proof. First we show that

$$P_b^{-1} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} P_b = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad P_c^{-1} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} P_c = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \tag{90}$$

We begin with the second equality in (90), and calculate

$$P_c^{-1} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} P_c = \begin{pmatrix} A & 0 \\ -W_c A + C + D W_c & D \end{pmatrix}.$$

We compute the action of $-W_c A + C + D W_c$ on an eigenvector v_k of A as follows:

$$(-W_c A + C + D W_c) v_k = C v_k + (D - \lambda_k I_\infty) W_c v_k.$$

To see that the right hand side is equal to zero, we calculate, using $u_i^* v_j = \delta_{ij}$,

$$W_c v_k = -\left(D - \lambda_k I_\infty\right)^{-1} C v_k.$$

Since the eigenvectors $v_1 \dots v_N$ span $\mathbb{C}^N$, then $-W_c A + C + D W_c = 0$, yielding the desired equality.

The argument is analogous for the first identity in (90). Again we begin by calculating

$$P_b^{-1} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} P_b = \begin{pmatrix} A & A W_b + B - W_b D \\ 0 & D \end{pmatrix}.$$

Hence, we would like to show the map $(A W_b + B - W_b D) : X_\infty \to \mathbb{C}^N$ is the zero map, which we do by arguing that $u_k^*(A W_b + B - W_b D) = 0$ for all k. The latter follows from a calculation similar to the one performed above.

Finally, we calculate $(P_c P_b)^{-1} M_1 P_c P_b$ as follows:

$$\begin{aligned}
(P_c P_b)^{-1} M_1 (P_c P_b) &= P_b^{-1} \left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + P_c^{-1} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} P_c \right) P_b \\
&= P_b^{-1} \left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} + \begin{pmatrix} B W_c & 0 \\ -W_c B W_c & -W_c B \end{pmatrix} \right) P_b \\
&= \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} (I_N + W_b W_c) B W_c & B W_c W_b + W_b W_c B (I + W_c W_b) \\ -W_c B W_c & -W_c B (I_\infty + W_c W_b) \end{pmatrix}.
\end{aligned}$$ □

Proof of Theorem B.1. Let $M = M_1 + M_2$, where

$$M_1 := \begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \Lambda_1 & L_1^\infty \\ L_\infty^1 & \Lambda_\infty + L_\infty^\infty \end{pmatrix}, \qquad M_2 := \begin{pmatrix} L_1^1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We will apply Lemma B.2 to the matrix M_1. Since we have assumed that Λ_1 is diagonal we may take $u_k = v_k = e_k$, the standard basis vectors in $\mathbb{C}^N$. We begin by proving $\|W_b\| \leq \varepsilon\delta_b$ and $\|W_c\| \leq \varepsilon\delta_c$. We first calculate

$$(D - \lambda_k I_\infty)^{-1} = (\Lambda_\infty + L_\infty^\infty - \lambda_k I_\infty)^{-1} = (I_\infty + \Lambda_\infty^{-1}(L_\infty^\infty - \lambda_k I_\infty))^{-1}\Lambda_\infty^{-1}.$$

By our hypothesis, we are allowed to apply the Neumann series and we obtain

$$\|(D - \lambda_k I_\infty)^{-1}\| \leq \frac{\|\Lambda_\infty^{-1}\|}{1 - \|\Lambda_\infty^{-1}\|(\delta_d + |\lambda_k|)}. \tag{91}$$

We note that the same estimate holds for the dual operator $(D^* - \lambda_k^* I_\infty)^{-1}$.

We now show that $\|W_b\| \leq \varepsilon\delta_b$. Namely, by using that $\|u_k^*\|_{(\mathbb{C}^N)^*} = \|v_k\|_{\mathbb{C}^N} = 1$ we find that

$$\begin{aligned}
\|W_b\| &= \sup_{x \in X_\infty, \|x\|=1} \left\| \sum_{\lambda_k \in \sigma(\Lambda_1)} v_k \left[(D^* - \lambda_k^* I_\infty)^{-1} B^* u_k^T\right] x \right\|_{\mathbb{C}^N} \\
&\leq \sup_{x \in X_\infty, \|x\|=1} \sum_{\lambda_k \in \sigma(\Lambda_1)} \left| \left[(D^* - \lambda_k^* I_\infty)^{-1} B^* u_k^T\right] x \right| \\
&\leq \sum_{\lambda_k \in \sigma(\Lambda_1)} \left\| (D^* - \lambda_k^* I_\infty)^{-1} B^* \right\|_{\mathcal{L}((\mathbb{C}^N)^*, X_\infty^*)} \\
&\leq \|B^*\| \sum_{\lambda_k \in \sigma(\Lambda_1)} \frac{\|\Lambda_\infty^{-1}\|}{1 - \|\Lambda_\infty^{-1}\|(\delta_d + |\lambda_k|)}.
\end{aligned}$$

Hence, by plugging in $\|B^*\| = \|L_1^\infty\|$ we obtain $\|W_b\| \leq \varepsilon\delta_b$. The proof of the estimate $\|W_c\| \leq \varepsilon\delta_c$ is analogous. Next, we note that

$$\|P_b\|, \|P_b^{-1}\| \leq 1 + \varepsilon\delta_b \qquad \|P_c\|, \|P_c^{-1}\| \leq 1 + \varepsilon\delta_c.$$

By Lemma B.2 we have

$$(P_c P_b)^{-1}(M_1 + M_2)(P_c P_b) = M_3 + M_4 + (P_b P_b)^{-1} M_2 (P_c P_b), \tag{92}$$

where

$$\begin{aligned}
M_3 &:= \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_\infty + L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b) \end{pmatrix}, \\
M_4 &:= \begin{pmatrix} (I_N + W_b W_c) L_1^\infty W_c & L_1^\infty W_c W_b + W_b W_c L_1^\infty (Id + W_c W_b) \\ -W_c L_1^\infty W_c & 0 \end{pmatrix}.
\end{aligned}$$

For $(x_N, x_\infty) \in \mathbb{C}^N \times X_\infty$ we see that

$$e^{M_3 t}(x_N, x_\infty) = \left(e^{\Lambda_1 t} x_N, e^{(\Lambda_\infty + L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b))t} x_\infty\right).$$

We also have $\|L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b)\| \leq \delta_d + \varepsilon\delta_b\delta_c(1 + \varepsilon_b\varepsilon_c)$. By applying the estimate (87) we obtain, for all $t \geq 0$,

$$\begin{aligned}
\|e^{\Lambda_1 t} x_N\| &\leq C_1 e^{\mu_1 t} \|x_N\|, \\
\|e^{(\Lambda_\infty + L_\infty^\infty - W_c L_1^\infty (I_\infty + W_c W_b))t} x_\infty\| &\leq C_\infty e^{(\mu_\infty + C_\infty[\delta_d + \varepsilon\delta_b\delta_c(1 + \varepsilon_b\varepsilon_c)])t} \|x_\infty\|.
\end{aligned}$$

From our assumption in (89) that $\mu_1 > \mu_\infty + C_\infty[\delta_d + \varepsilon\delta_b\delta_c(1+\varepsilon^2\delta_b\delta_c)]$, we obtain, for any p-norm, $1 \le p \le \infty$, on the product $\mathbb{C}^N \times X_\infty$,

$$\|e^{M_3 t}(x_N, x_\infty)\| \le \max\{C_1, C_\infty\} e^{\mu_1 t} \|(x_N, x_\infty)\|.$$

We may estimate the norm of the components of M_4 as

$$\begin{aligned}
\|(I_N + W_b W_c) L_1^\infty W_c\| &\le \varepsilon\delta_b\delta_c(1+\varepsilon^2\delta_b\delta_c), \\
\|-W_c L_1^\infty W_c\| &\le \varepsilon^2\delta_b\delta_c^2, \\
\|L_1^\infty W_c W_b + W_b W_c L_1^\infty (Id + W_c W_b)\| &\le \varepsilon^2\delta_b^2\delta_c(2+\varepsilon^2\delta_b\delta_c).
\end{aligned}$$

We then obtain the bound

$$\|M_4\| \le \Delta := \varepsilon\delta_b\delta_c\left(1 + \varepsilon(2\delta_b + \delta_c) + \varepsilon^2\delta_b\delta_c(1+\varepsilon\delta_b)\right)$$

by summing the component bounds.

We now perform the final estimate. By using (92) we obtain

$$e^{Mt} = (P_c P_b)\exp\left\{\left[M_3 + M_4 + (P_c P_b)^{-1} M_2 (P_c P_b)\right] t\right\}(P_c P_b)^{-1}.$$

By then applying (87) to the sum of M_3 and the bounded operator $M_4 + (P_c P_b)^{-1} M_2 (P_c P_b)$ we obtain, with $C_{1,\infty} := \max\{C_1, C_\infty\}$,

$$\|e^{Mt}\| \le \|P_c P_b\| \cdot \|(P_c P_b)^{-1}\| C_{1,\infty} \exp\left\{\mu_1 + C_{1,\infty}\left\|M_4 + (P_c P_b)^{-1} M_2 (P_c P_b)\right\| t\right\}.$$

Defining $C_s = \max\{C_1, C_\infty\}(1+\varepsilon\delta_b)^2(1+\varepsilon\delta_c)^2$ and plugging in our bounds, we finally infer

$$\|e^{Mt}\| \le C_s e^{(\mu_1 + C_s\delta_a + \Delta\max\{C_1, C_\infty\})t}. \qquad \square$$

Remark B.3. *If we use the* $p = 1$ *norm for the product space* $\mathbb{C}^N \times X_\infty$ *then our bound for* Δ *can be sharpened to*

$$\|M_4\| \le \varepsilon\delta_b\delta_c \max\left\{1 + \varepsilon\delta_c(1+\varepsilon\delta_b), \varepsilon\delta_b(2+\varepsilon^2\delta_b\delta_c)\right\}.$$