FOURTH ORDER CONSERVATIVE TWIST SYSTEMS: SIMPLE CLOSED CHARACTERISTICS

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ABSTRACT. On the energy manifolds of fourth order conservative systems closed characteristics can be found in many cases via analogues of Twist-maps. The 'Twist property' implies the existence of a generating function which leads to second order recurrence relations. We study these recurrence relations to find simple closed characteristics and we give conditions when fourth order systems satisfy the 'Twist property'.

CONTENTS

1. INTRODUCTION

Various mathematical models for problems in nonlinear elasticity, nonlinear optics, solid mechanics, etc. are derived from *second order* Lagrangian principles, i.e. the differential equations are obtained as the Euler-Lagrange equations of a

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Lagrangian L that depends on a state variable u, and its first and second order derivatives. The Euler-Lagrange differential equations are fourth order and are of conservative nature. In scalar models the Lagrangian action is defined by $J[u] = \int L(u, u', u'')dt$. A second order Lagrangian systems is, under suitable assumptions on the u''-dependence of L, equivalent to a Hamiltonian system on $\mathbb{R}^4$. Trajectories of the Lagrangian system and thus Hamiltonian system lie on three dimensional sets $M_E \stackrel{\text{def}}{=} \{H = E\}$, where H is the Hamiltonian (conserved quantity). The sets M_E are smooth manifolds for all regular E values ($\nabla H|_{M_E} \neq 0$) and the energy manifolds M_E are non-compact for all $E \in \mathbb{R}$. It turns out that for Hamiltonian systems that come from second order Lagrangians, one can find a natural two dimensional section $\{u' = 0\} \cap M_E$ which bounded trajectories have to intersect finitely or infinitely many times (possibly only in the limit). This section will be denoted by Σ_E and $\Sigma_E \simeq N_E \times \mathbb{R}$, where N_E is a one dimensional set defined by:

$$N_E = \{(u, u'') \mid \frac{\partial L}{\partial u''}u'' - L(u, 0, u'') = E\} \tag{1}$$

(see Section 1.1 for more details). The Hamiltonian flow induces a return map to the section Σ_E, and closed trajectories — closed characteristics — correspond to fixed points of iterates of this map. In many situations the return map is an analogue of a monotone area-preserving Twist map (see e.g. [6, 19, 22]). The theory developed in this paper will be centered around this property. Lagrangian systems that allow such Twist maps will be referred to as *Twist systems*. More precise definitions and analysis will be given in the forthcoming sections. This paper will be concerned with the basic properties of Twist systems and the study of simple closed characteristics. These are periodic trajectories that, when represented in the (u, u')-plane (configuration plane of a Lagrangian system), are simple closed curves. In [31] we will investigate more elaborate types of characteristics via a Morse type theory. One of the main results of this paper is the following.

Theorem 1. *Let E be a regular value. If N_E has a compact connected component N_E^c, then the Lagrangian Twist system has at least one simple closed characteristic at energy level E with $u(t) \in \pi^u N_E^c$ for all $t \in \mathbb{R}$, where $\pi^u N_E^c$ is the projection of N_E^c onto the u-coordinate.*

A precise statement of this result will be presented in Section 3.1 together with information about the location and the Morse index of the trajectory (Theorem 11). The results in this paper are proved for Twist systems. We conjecture that Theorem 1 is still true even without such a requirement (some growth conditions on L may be needed).

For singular energy levels a similar theorem can be proved (Theorem 13). The bottom line is that under the same compactness assumptions there exists a simple closed characteristic in the broader sense of the word, i.e. depending on possible singularities a closed characteristics is either a regular simple closed trajectory, a simple homoclinic loop, or a simple heteroclinic loop. We also explain how singularities can lead to multiplicity of closed characteristics. This issue is addressed in full in [31]. In Chapter 4 we give some more background information on Twist maps and the relation to Aubry-LeDaeron-Mather theory [6, 19]. We also briefly discuss the analogues of KAM-tori/circles for second order Lagrangian systems, and the issue of integrability versus non-integrability. Throughout the Chapters

1-4 we will also give specific examples of physical systems such as the eFK and Swift-Hohenberg equations ($u'''' - \alpha u'' + F'(u) = 0$ with $\alpha \in \mathbb{R}$). The theory developed in this paper also applies to systems on $M = S^1$ by simply assuming L to be periodic in u (see Section 4.3 for more details).

1.1. **Second-order Lagrangians.** Let $L : \mathbb{R}^3 \to \mathbb{R}$ be a C^2-function of the variables u, v, w. For any smooth function $u : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$, define the functional $J[u] = \int_I L(u, u', u'')dt$, which is called is the *(Lagrangian) action* of u. The function L may be regarded as a function on the 2-jet of $\mathbb{R}$, and is generally referred to as the *Lagrangian* function[1]. The pair (L, dt) is called a second-order Lagrangian system on $\mathbb{R}$. The action J of the Lagrangian system is said to be stationary at a function u if $\delta J[u] = 0$ with respect to variations $\delta u \in C_c^\infty(I, \mathbb{R})$[2], i.e.

$$\begin{aligned}\delta J[u] &= \delta \int_I L(u, u', u'')dt = \int_I \left[\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial v}\delta u' + \frac{\partial L}{\partial w}\delta u''\right] dt \\ &= \int_I \left[\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial v} + \frac{d^2}{dt^2}\frac{\partial L}{\partial w}\right]\delta u dt = 0.\end{aligned}$$

A stationary function u thus satisfies the differential equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u''} = 0,$$

which is called the *Euler-Lagrange equation* of the Lagrangian system (L, dt). The Lagrangian action J is invariant under the $\mathbb{R}$-action $t \mapsto t + c$, which by Noether's Theorem yields the conservation law

$$\Big(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\Big)u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u'') = \text{ constant} \tag{2}$$

(see for instance [18]). This conservation law is called the *Hamiltonian*, and if the Lagrangian is strictly convex in the w-variable then the Lagrangian system (L, dt) is equivalent to a Hamiltonian system on $\mathbb{R}^4$ with the standard symplectic structure. Therefore we assume:

(H) $\partial_w^2 L(u, v, w) \geq \delta > 0$ *for all* (u, v, w).

The correspondence between a Lagrangian system (L, dt) on $\mathbb{R}$ and a Hamiltonian system (H, ω) on $\mathbb{R}^4$ can be explained as follows. Let $x = (p_u, p_v, u, v)$ be symplectic coordinates for $\mathbb{R}^4$ with the symplectic form given by $\omega = dp_u \wedge du + dp_v \wedge dv$. Define the Hamiltonian $H(x) = p_u v + L^*(u, v, p_v)$, where $L^*(u, v, p_v) = \max_{w \in \mathbb{R}}\{p_v w - L(u, v, w)\}$ is the Legendre transform of L. Since L is strictly convex in w we have that L^* is strictly convex in p_v. Moreover $\partial_{p_v} L^* = \big(\partial_w L\big)^{-1}(p_v) = w$, hence $H(x) = p_u v + p_v\big(\partial_w L\big)^{-1}(p_v) - L(u, v, \big(\partial_w L\big)^{-1}(p_v))$. For any function $x : I \to \mathbb{R}$ the Hamiltonian action is defined by $\mathcal{A}[x] = \int_I \big[p_u u' + p_v v' - H(x)\big]dt$. A function x is stationary for $\mathcal{A}$ if and only if the u-coordinate is stationary for J. In particular, the Euler-Lagrange equations for $\mathcal{A}$ are of the form $x' = X_H(x)$, where $X_H = \mathcal{J}\nabla H$ and $\mathcal{J}$ is defined by $\omega(x, \mathcal{J}y) = \langle x, y\rangle$ ($\langle x, y\rangle$ is the standard inner product in $\mathbb{R}^4$). X_H is called the Hamiltonian vector field associated

[1] In the case of a general smooth 1-dimensional manifold M we define L as a smooth function on the 2-jet of M. The action is then defined by considering functions $(u, u', u'') : I \to J^2 M$.

[2] If M is a arbitrary 1-dimensional manifold a different notion of variation is used.

to H. The correspondence between u and its derivatives and x is given by: $v = u'$, $p_u = \partial_{u'}L - p_v'$, and $p_v = \partial_{u''}L$ (see e.g. [4] for more details on this correspondence). The state space $\mathbb{R}^4$ of the Hamiltonian system (H, ω) is often referred to as the *phase space* and $J^1 = \mathbb{R}^2$ is called the *configuration space*[3].

If the Hamiltonian is sufficiently smooth then the Hamiltonian system $x' = X_H(x)$ generates a local flow on $\mathbb{R}^4$. If we assume strict convexity of L in the w-variable then H is of class C^1. Under Hypothesis (H)[4] the Hamiltonian $H(x)$ is a C^2-function, which in return generates a local C^1-flow ϕ_H^t on $\mathbb{R}^4$ via the equation $x' = X_H(x)$.

Stationary functions of J satisfy relation (2), which is equivalent to $H(x) = E \in \mathbb{R}$. For the associated Hamiltonian system (H, ω) this means that the stationary motions lie on the 3-dimensional sets $M_E = \{x \in \mathbb{R}^4 \mid H(x) = E\}$. If $\nabla H \neq 0$ on M_E then E is called a *regular value* and M_E is smooth manifold (noncompact) without boundary. The vector field X_H restricted to M_E is non-singular when E is a regular value. Indeed, the singular points of the vector field X_H, i.e. points x_* such that $X_H(x_*) = 0$, are exactly the critical points of the Hamiltonian, and thus only occur at singular energy levels. Singular points are of the form $x_* = (p_u, p_v, u, 0)$ and are given by: $\partial_u L(u, 0, 0) = 0$, $p_u = \partial_v L(u, 0, 0)$ and $p_v = \partial_w L(u, 0, 0)$. Equivalently, for a Lagrangian system an energy level E is said to be regular if and only if $\frac{\partial L}{\partial u}(u, 0, 0) \neq 0$ for all points $u \in \mathbb{R}$ that satisfy the relation $-L(u, 0, 0) = E$.

A bounded characteristic of the Lagrangian system (L, dt) is a function $u \in C_b^2(\mathbb{R}, \mathbb{R})$ for which $\delta \int_I L(u, u', u'')dt = 0$ with respect to variations $\delta u \in C_c^2(I, \mathbb{R})$ for any compact interval $I \subset \mathbb{R}$. Since the Lagrangian is a C^2-function of the variables (u, v, w) it follows from the Euler-Lagrange equations that $u \in C_b^3(\mathbb{R}, \mathbb{R})$, $\frac{\partial L}{\partial w}(\cdot) \in C_b^2(\mathbb{R}, \mathbb{R})$, and $\left(\frac{d}{dt}\frac{\partial L}{\partial w} - \frac{\partial L}{\partial v}\right)(\cdot) \in C_b^2(\mathbb{R}, \mathbb{R})$ (regularity of extremals of L). This is equivalent to having a function $x \in C_b^2(\mathbb{R}, \mathbb{R}^4)$ which is stationary for $\mathcal{A}[x]$; a bounded characteristic for the associated Hamiltonian system (H, ω).

The question now arises, given an 'energy-value' E, do there exist closed and/or bounded characteristics (see Sections 1.2 and 1.3 for a definition) on M_E, and how many, and how are these questions related to geometric and topological properties of M_E.

1.2. **Cross-sections and area-preserving maps.** From (H) it follows that bounded solutions of the Euler-Lagrange equations only have isolated extrema (well-posedness of the initial value problem for $x' = X_H(x)$). Consequently, a bounded characteristic has either finitely, or infinitely many isolated local extrema. For the associated Hamiltonian system this means that a bounded trajectory always intersects the section $\Sigma_E = \{v = 0\} \cap M_E = \{(p_u, p_v, u, 0) \mid p_u \in \mathbb{R},\ p_v = \partial_w L(u, 0, w),\ (u, w) \in N_E\}$, where N_E is defined by (1)[5]. In the case that there are only finitely many intersections x must be asymptotic, as $t \to \pm\infty$, to singular points of X_H, and thus critical points of H. If E is a regular value this possibility is excluded. A bounded solution u is therefore a concatenation of monotone laps between extrema — an increasing lap followed by a decreasing lap and vice versa

[3] In the general case the phase space is TJ^1M and the configuration space is J^1M.

[4] In order to study stationary points of $\mathcal{A}$ additional regularity for H is not required. One does usually need proper growth conditions on H.

[5] It is sometimes convenient to define N_E in terms of coordinates (u, p_v) by simply using the formula $p_v = \partial_w L$.

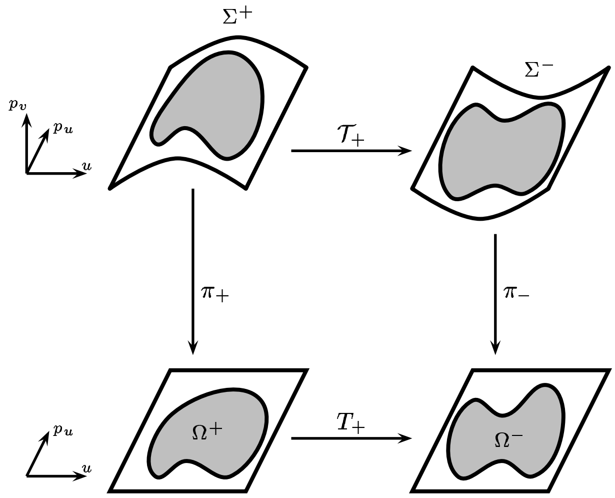

FIGURE 1. The map $\mathcal{T}_+$, which is induced by the flow, and its projection T_+.

— at least if we assume that u does not have critical inflection points, i.e. Σ_E is not intersected in a point where $w = 0$. In this context it is important to note that if E is a regular value then critical inflection points can only occur at the boundary of

$$\pi^u N_E \stackrel{\text{def}}{=} \{u|(u,w) \in N_E \text{ for some } w \in \mathbb{R}\} = \{u|L(u,0,0) + E \geq 0\}.$$

The last equality follows from the definition of N_E and the fact that $\partial_w(\partial_w Lw - L) = \partial_w^2 Lw$ in combination with hypothesis (H). We will be interested in bounded characteristics that avoid critical inflection points.

Recalling that $w = \partial_{p_v} L^*$ we define $N_E^+ = \{(u,p_v) \in N_E \mid \partial_{p_v} L^*(u,0,p_v) > 0\}$, $N_E^- = \{(u,p_v) \in N_E \mid \partial_{p_v} L^*(u,0,p_v) < 0\}$, and $N_E^0 = \{(u,p_v) \in N_E \mid \partial_{p_v} L^*(u,0,p_v) = 0\}$. It follows from Hypothesis (H) that N_E^+ and N_E^- are smooth graphs over the u-axis and $\pi^u N_E^+ = \pi^u N_E^-$. The sets $\Sigma_E^\pm \simeq N_E^\pm \times \mathbb{R}$ are smooth surfaces over the (p_u,u)-plane. Therefore the projections $\pi_\pm : \Sigma_E^\pm \to N_E^\pm$ are invertible. For a given bounded trajectory $x(t)$ we therefore only need to know the (p_u,u)-coordinates of the intersections of $x(t)$ with Σ_E. Thus bounded characteristics can be identified with sequences of points (p_{u_i},u_i) in the (p_u,u)-plane.

In the following we fix the energy level E and drop the subscript in the notation. The vector field X_H is tranverse to the section $\Sigma^+ \cup \Sigma^-$ (non-transverse at Σ^0). It therefore makes sense to consider the Poincare return maps, i.e. maps from Σ^+ to Σ^- and from Σ^- to Σ^+. Follow the flow φ_H^t starting at Σ^+ until it intersects Σ^-. It may happen that φ_H^t does not intersect Σ^- at all. For the points in Σ^+ for which the flow does intersect Σ^- we have defined a map $\mathcal{T}_+$ from Σ^+ to Σ^-[6]. The same can be done for the map $\mathcal{T}_-$ mapping from Σ^- to Σ^+. Since $\Sigma^\pm$ are graphs over the (p_u,u)-plane the above defined maps induce maps $T_\pm = \pi_\mp \mathcal{T}_\pm \pi_\pm^{-1}$ between open regions $\Omega^\pm \subset \pi_\pm\Sigma^\pm$, i.e. $T_\pm : \Omega^\pm \to \Omega^\mp$ (see also Figure 1). For any point

[6] In ODE theory the study of this map is often called a *shooting* method.

$(p_u, u) \in \Omega^{\pm}$, $T_{\pm}$ is a local C^1-diffeomorphism (since there are no critical inflection points in $N^{\pm}$).

Since bounded characteristics consist of increasing laps followed by decreasing laps we seeks fixed points of iterates of the composition map $T = T_- \circ T_+$ (or $T = T_+ \circ T_-$. Fixed points are contained in the set

$$\Omega^* = \bigcap_{n\in\mathbb{Z}} \bigl(T_- \circ T_+\bigr)^n(\Omega^+) \subset \mathbb{R}^2.$$

The maps $T_{\pm}$ are area-preserving maps with respect to the area $\alpha = dp_u \wedge du$. This means that for any region $U \subset \Omega^{\pm}$ it holds that $\int_U \alpha = \int_{T_{\pm} U} T_{\pm}^* \alpha$ (locally area-preserving). This was proved in [16] for the eFK-equation. We will give a different proof of this fact here. Let $(p_u, u) \in U \subset \Omega^+$, and recall that $\omega = dp_u \wedge du + dp_v \wedge dv$. Now $\mathcal{T}_+$ maps $\pi_+^{-1} U \subset \Sigma^+$ to $\mathcal{T}_+ \pi_+^{-1} U \subset \Sigma^-$. Since $\mathcal{T}_+$ preserves ω, and because $\Sigma^{\pm} \subset \{v = 0\}$ it follows that the 2-form $\alpha = dp_u \wedge du$ is preserved, and thus T_+, as a map from Ω^+ to Ω^-, is area-preserving. This implies that

$$p_{u_2} du_2 - p_{u_1} du_1 = dS_*(p_{u_1}, u_1), \tag{3}$$

where $(p_{u_1}, u_1) \in U$ and $(p_{u_2}, u_2) = T_+(p_{u_1}, u_1) \in T_+ U$, and S_* is a C^1-function of (p_{u_1}, u_1).

The map T_+ is a (local) *Twist map* if $u_2 = u_2(p_{u_1}, u_1)$ is strictly increasing in $-p_{u_1}$. It then follows from (3) that there exists a C^1-function $S_E(u_1, u_2) = S_*(p_{u_1}(u_1, u_2), u_1)$ such that $\partial_1 S_E = -p_{u_1}$ and $\partial_2 S_E = p_{u_2}$. This function is called the *generating function* of the Twist map. A similar construction can be carried out for T_-.

The function S_E can be used to formulate a variational principle for the u_i-variables. In the next chapter we will make a connection with the variational principle for the Lagrangian action

$$J_E[u] = \int_0^{\tau} \bigl(L(u, u', u'') + E\bigr) dt,$$

where the integration over $[0, \tau]$ is between two consecutive extrema of $u(t)$. In relation to this connection we note the following (which does not depend on T_+ being a Twist map or not).

Lemma 2. *Let $S_*(p_{u_1}, u_1) = J_E[u]$, where $u(t)$ is the trajectory starting at $\pi_+^{-1}(p_{u_1}, u_1) \in \Sigma^+$, and $\tau = \tau(p_{u_1}, u_1)$ is the first intersection time at Σ^-. Then S_* satisfies Equation* (3).

Proof. Define the Hamiltonian action $\mathcal{A}_E[x] = \int_0^T \{p_u u' + p_v v' - H(x) + E\} dt$, and let $(p_{u_1}, u_1) \in \Omega^+$. Consider the trajectory $\{\varphi_H^t(\pi_+^{-1}(p_{u_1}, u_1))\}_{t=0}^{t=\tau(p_{u_1}, u_1)}$, where $\tau(p_{u_1}, u_1)$ is the first intersection time at Σ^-. These trajectories vary smoothly with $(p_{u_1}, u_1) \in \Omega^+$. We now consider variations with respect to $(p_{u_1}, u_1) \in \Omega^+$. Using the fact that (p_u, p_v, u, v) obeys the Hamilton equations and $v(\tau(p_{u_1}, u_1)) = 0$, we obtain

$$\begin{aligned} \delta \mathcal{A}_E[x] &= p_u \delta u|_0^{\tau} + p_v \delta v|_0^{\tau} + \bigl[p_u u' + p_v v' - H(x) + E\bigr]_{\tau} \delta\tau \\ &= p_u \delta u(\tau) - p_u \delta u(0) + p_v \bigl[\delta v(\tau) + v'(\tau)\delta\tau\bigr] \\ &= p_{u_2} \delta u_2 - p_{u_1} \delta u_1, \end{aligned}$$

where $(p_{u_2}, u_2) = T_+(p_{u_1}, u_1)$. It may be clear that $\mathcal{A}_E[x] = J_E[u]$, which proves the lemma. ∎

If T_+ is a Twist map then for J_E this implies that there exists a local continuous family $u(t; u_1, u_2)$ of extremals (and $\tau(u_1, u_2)$ varies continuously). Conversely, we will show in the next chapter that the continuity conditions on the family of extremals $u(t; u_1, u_2)$ imply the Twist property.

We remark that instead of studying the maps $T_\pm$ one can study a related area-preserving map which is well defined when $T_\pm$ are Twist maps. From $T_\pm$ we construct the map $\tilde{T}$

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \tilde{T} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad u_{n-1}, u_n, u_{n+1} \in \pi^u N_E.$$

For this map we can use the generating function $S_E(u_1, u_2)$ to retrieve the the maps $T_\pm$. We refer to [3, 6] for more details.

1.3. **Closed characteristics.** A special class of bounded characteristics are closed characteristics. These are functions u that are stationary for $J[u]$ and are τ-periodic for some period τ. If we seek closed characteristics at a given energy level E we can invoke the following variational principle:

$$\text{Extremize}\big\{ J_E[u] \mid u \in \Omega_{\text{per}} \big\}, \tag{4}$$

where $J_E[u] = \int_0^\tau (L(u, u', u'') + E)dt$ and $\Omega_{\text{per}} = \cup_{\tau>0} C^2(S^1, \tau)$. It may be clear that τ is also a parameter in this problem. Indeed, Problem (4) is equivalent to

$$\text{Extremize}\big\{ J_E[v, \tau] \mid (v, \tau) \in C^2(S^1, 1) \times \mathbb{R}^+ \big\}, \tag{5}$$

where $J_E[v, \tau] = \int_0^1 (L(v, \frac{v'}{\tau}, \frac{v''}{\tau^2}) + E)\tau ds$. This equivalent variational characterization is convenient for technical purposes. Notice that the variations in τ guarantee that any extremal of (4) has energy $H(x) = E$. The variational problem of finding closed characteristics for a given energy value E can also be formulated in terms of unparametrized closed curves in the configuration plane.

The Morse index of a closed characteristic u is defined as the number of negative eigenvalues of the linearized operator $d^2 J_E[u] \in L(T_u \Omega_{\text{per}})$. The nullity is the dimension of the kernel of $d^2 J_E[u]$. The large Morse index is defined as the sum of the Morse index and the nullity.

2. Twist systems and Monotone recurrences

2.1. **Generating functions.** In this section we will introduce a class of Lagrangian systems which satisfy a variant of the Twist property. Such systems can be studied via generating functions. We start with systems for which the generating function is of class C^2. In Section 2.2 we will give a number of examples of such systems. In Section 2.3 we explain how the theory also works with C^1-generating functions which allows a weaker variant of the Twist property (see Condition (T′) in Section 2.3).

For a regular energy value E the set $\pi^u N_E$ is a union of closed intervals. Connected components of $\pi^u N_E$ are denoted by I_E and will be referred to as *interval components*. Since E is regular it holds that $L(u, 0, 0) + E > 0$ for $u \in \text{int}(I_E)$, and $L(u, 0, 0) + E = 0$ for $u \in \partial I_E$[7]. In terms of N_E this means that connected components of N_E are copies of $\mathbb{R}$ and/or S^1. Let $\Delta = \{(u_1, u_2) \in I_E \times I_E \mid u_1 = u_2\}$,

[7]The relation $L(u, 0, w) + E = 0$ automatically implies that $w = 0$ due to Hypothesis (H).

then for any pair $(u_1, u_2) \in I_E \times I_E \backslash \Delta$ we define

$$S_E(u_1, u_2) = \inf_{\substack{u \in X_\tau \\ \tau \in \mathbb{R}^+}} \int_0^\tau \Big(L(u, u', u'') + E \Big) dt, \tag{6}$$

where $X_\tau = X_\tau(u_1, u_2) = \{u \in C^2([0, \tau]) \mid u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = u'(\tau) = 0$ and $u'|_{(0,\tau)} > 0$ if $u_1 < u_2$, and $u'|_{(0,\tau)} < 0$ if $u_1 > u_2\}$. We remark that the notation S_E is slightly suggestive since it is not a priori clear that this definition of S_E is equivalent to the one in Section 1.2 (however, compare Lemma 2). If there is no ambiguity about the choice of E we simply write $S(u_1, u_2)$. At this point it is not clear whether S is defined on all of $I_E \times I_E \backslash \Delta$.

The u-laps from u_1 to u_2 that minimize $\int (L + E)$ are the analogues of broken geodesics. Our goal now is to formulate a variational problem in terms of the u_i-coordinates of bounded characteristics replacing the 'full' variational problem for $J_E[u]$. This will be a direct analogue of the method of broken geodesics.

As in (5) there is an equivalent formulation of the variational problem above. In view of this we consider the pair (v, τ), with $v(s) = u(t)$ and $s = t/\tau$. For $(u_1, u_2) \in \Delta$ we define $v(s) = u_1$ for $s \in [0, 1]$ and $\tau = 0$ (and $S(u_1, u_1) = 0$) A Lagrangian system (L, dt) is said to satisfy the *Twist property* on an interval component I_E if (with E a regular energy value):

(T) $\inf\{J_E[u] \mid u \in X_\tau(u_1, u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimizer* $u(t; u_1, u_2)$ *for all* $(u_1, u_2) \in I_E \times I_E \setminus \Delta$, *and* $v(s)$ *and* τ *are* C^1*-smooth function of* (u_1, u_2).

To be precise, by C^1-smoothness we mean that $(u_1, u_2) \to (v, \tau)$ is a C^1-function from $\mathrm{int}(I_E \times I_E \setminus \Delta)$ to $C^2([0, 1]) \times \mathbb{R}^+$ and a C^0-function on $I_E \times I_E$. The results presented in this paper will apply whenever the Twist property is satisfied on an interval component I_E[8].

If E is a singular energy level with non-degenerate critical points then we have the same formulation of the Twist property with the following exceptions. Firstly, C^1-smoothness is only required for all $(u_1, u_2) \in \mathrm{int}(I_E \times I_E \backslash \Delta)$ such that u_1 nor u_2 is a critical point[9]. Secondly, when an equilibrium point $u_* \in I_E$ is a saddle-focus or a center then $\tau(u_1, u_2)$ is not continuous at (u_*, u_*)[10]. In the case that u_1 and/or u_2 is an equilibrium point of real saddle type then τ can be ∞[11]. See Section 3.2 and Appendix A for more information on singular energy levels and equilibrium points.

Definition 3. *A Lagrangian system (L, dt) is called a Twist system on an interval component I_E if both Hypotheses* (H) *and* (T) *are satisfied.*

Using Hypothesis (T) we can derive the following regularity properties for S.

Lemma 4. *Let E be a regular value. If (L, dt) is a Twist system on an interval component I_E, then the function $S_E(U_1, U_2)$ is of class $C^2(\mathrm{int}(I_E \times I_E \setminus \Delta)) \cap C^1(I_E \times I_E \setminus \Delta) \cap C^0(I_E \times I_E)$.*

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[8] We note that most of the results in this paper also hold for slightly weaker conditions. For example, when we do not require the family of solutions/extrema to be minimizers of $J_E(u_1, u_2)$ then we obtain the same results, the information on the index excluded. For the case where the family is continuous but not C^1 we refer to Section 2.3.

[9] Singular energy levels connected components of $\pi^u N_E$ can have internal critical points. This will be discussed in Section 3.3.

[10] It still follows that $J_E[u(t; u_1, u_2)] \to 0$ as $(u_1, u_2) \to (u_*, u_*)$.

[11] We then consider u on either $[0, \infty)$, $(-\infty, 0]$ or $\mathbb{R}$ (whichever is appropriate) and require that $u(t; u_1, u_2)$ converges on compact sets as u_1 and/or u_2 tends to the critical point.

Proof. Due to the smoothness assumption in (T) and the regularity of solutions of the Euler-Lagrange equations (see Section 1.1), we have that $u(t;u_1,u_2)$ varies smoothly with (u_1,u_2) with values in C^2. It is easily seen that $S_E(u_1,u_2) = J_E[u(t;u_1,u_2)]$ is a C^1-function. Lemma 2 and Equation (3) show that $\partial_1 S(u_1,u_2) = -p_{u_1}$ and $\partial_2 S(u_1,u_2) = p_{u_2}$. It follows from the smoothness assumption in (T) and the fact that all solutions obey (2) that p_{u_1} and p_{u_2} are C^1-functions of (u_1,u_2), hence S_E is a C^2-function. ■

If S is considered on $I_E^1 \times I_E^2$, where I_E^i, $i=1,2$ are different connected components of $\pi^u N_E$ one does not expect S_E to be defined on all of $I_E^1 \times I_E^2$. The next lemma reveals some other important properties of the generating function S. For the remainder of this section we assume that E is a regular value and we consider interval components I_E on which (L,dt) is a Twist system.

Lemma 5. *Let E be a regular value. We have:*

(1) $\partial_1 S(u_1,u_2) = -p_{u_1}$ and $\partial_2 S(u_1,u_2) = p_{u_2}$ for any $(u_1,u_2) \in I_E \times I_E \backslash \Delta$,
(2) $\partial_1\partial_2 S(u_1,u_2) > 0$ for all $(u_1,u_2) \in \mathrm{int}(I_E \times I_E\backslash\Delta)$, and
(3) $\partial_{n_\pm} S|_{\mathrm{int}(\Delta)} = +\infty$, where $n_\pm = (\mp 1, \pm 1)$[12].

Proof. Part (1) has been dealt with in the proof of Lemma 3. For part (2) of this lemma we argue as follows: $\partial_1\partial_2 S(u_1,u_2) = \frac{\partial p_{u_2}}{\partial u_1} = -\frac{\partial p_{u_1}}{\partial u_2}$. Because of the uniqueness of the initial value problem for $x' = X_H(x)$ it easily follows that $-p_{u_1}$ is a strictly increasing function of u_2 (u_1 fixed). In exactly the same way p_{u_2} is a strictly increasing function of u_1 (u_2 fixed). Therefore $\partial_1\partial_2 S(u_1,u_2) \geq 0$. On the other hand using the smooth dependence on initial data for $x' = X_H(x)$ and the smoothness of $\tau(u_1,u_2)$, it follows that both $u_2 = u_2(u_1,p_{u_1})$ and $u_1 = u_1(u_2,p_{u_2})$ are smooth functions. This implies that $\frac{\partial p_{u_1}}{\partial u_2} \neq 0$ and $\frac{\partial p_{u_2}}{\partial u_1} \neq 0$, and thus $\partial_1\partial_2 S(u_1,u_2) > 0$.

As for for part (3) we only consider the derivative in the direction n_+ (the other case is similar). We have that $u'''(0), u'''(\tau) \to -\infty$ as $u_1 \to u_2$. For p_u it holds that $p_u = \partial_v L(u,0,w) - \partial^2_{vw} L(u,0,w)u'' - \partial^2_w L(u,0,w)u'''$ and thus $p_{u_i} \to \infty$, $i=1,2$. ■

The question of finding bounded characteristics for (L,dt) can now best be formulated in terms of S. Extremizing the action J_E over a space of 'broken geodesics' now corresponds to finding critical points of the formal sum $\sum_{n\in\mathbb{Z}} S(u_n,u_{n+1})$. Formally we seek 'critical points' (bounded sequences) of the infinite sum

$$W(\cdots,u_{-1},u_0,u_1,\cdots) = \sum_{i\in\mathbb{Z}} S(u_i,u_{i+1}).$$

Since this sum is usually not well-defined for bounded sequences $(u_i)_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, we say that a sequence is a *critical sequence*, or critical point of W, if:

$$\partial_2 S(u_{i-1},u_i) + \partial_1 S(u_i,u_{i+1}) = 0, \quad \forall i \in \mathbb{Z}. \tag{7}$$

Such equations are called *second order recurrences* and are closely related to Aubry-LeDaeron-Mather theory for Twist diffeomorphisms (see e.g. [6, 19] and Section 4.3). If (7) is satisfied for all $i \in Z$ then u-laps can be glued to a C^3-function for which all derivatives up to order three match. Indeed, Equation (7) means that the

[12] This should be read as follows: when we approach a point $(\tilde{u},\tilde{u}) \in \mathrm{int}(\Delta)$ from within the region $\{u_2 > u_1\}$ then $\partial_{n_+} S \to \infty$ as $(u_1,u_2) \to (\tilde{u},\tilde{u})$.

third derivatives match. Since every u-lap satisfies the Euler-Lagrange equations we thus get a C_b^3-function u that is stationary for $J[u]$. Of course if we seek periodic sequences, i.e. sequences $(u_i)_{i\in Z}$ with $u_{i+2n} = u_i$, where $n > 0$ is called the period, we may look for critical points of the restricted action $W_{2n} = \sum_{i=1}^{2n} S(u_i, u_{i+1})$ defined on I_E^{2n}.[13] This corresponds to finding closed characteristics for (L, dt). The period can be linked to various topological properties of u and x (in the Hamiltonian system (H, ω)) such as knotting and linking of closed characteristics. Moreover, periodic sequences as critical points of W_{2n} have a certain Morse index, which is exactly the Morse index of a closed characteristic u as critical point of $J|_{\Omega_{\text{per}}}$.

Lemma 6. *Let E be a regular value. Let $\mathbf{u} = (u_i)_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z}), u_i \in \text{int}(I_E)$ be a periodic sequence with period n, which is a stationary point of W_{2n} with index $\mu(\mathbf{u}) \le 2n$. Then the associated closed characteristic u for (L, dt) is stationary for $J[u]$ and the Morse index of u is also $\mu(\mathbf{u})$ and vice versa.*

Proof. Let $\mathbf{u}$ be stationary for W_{2n}, i.e. $dW_{2n}(\mathbf{u}) = 0$. Concatenating the u-laps between the consecutive extrema u_i yields a τ-periodic C^3-function u that satisfies the Euler-Lagrange equations of (L, dt). It may be clear that the function u is an extremal of (4). The statement concerning the Morse index $\mu(\mathbf{u}) = \mu(u)$ can be proved as follows. The assumption that $u_i \in \text{int}(I_E)$ implies that $u'' \ne 0$ at extrema of $u(t)$. The linear operator $d^2J_E[u]$ induces the orthogonal decomposition $T_u\Omega_{\text{per}} = L \oplus T_u^0\Omega_{\text{per}}$, where $\dim(L) = 2n$ and the functions φ in $T_u^0\Omega_{\text{per}}$ have the property that φ and φ' are zero at the 'breakpoints' T_i. It follows now that $d^2J_E[u]|_L$ and $d^2W_{2n}[\mathbf{u}]$ are conjugated which proves the lemma (for more details see e.g. [20]: case of broken geodesics). ■

For points $u_1 \in \partial I_E$ additional information about S can be obtained. Denote the left boundary points by u^- and right boundary points by u^+. If I_E is compact then $\partial I_E = \{u^-, u^+\}$.

Lemma 7. *Let E be a regular value. Let $u^- \in \partial I_E$ (assuming that there exists a left boundary point) then $\partial_1 S(u^-, \tilde{u}) > -\partial_v L(u^-, 0, 0)$ and $\partial_2 S(\tilde{u}, u^-) > -\partial_v L(u^-, 0, 0)$ for $\tilde{u} > u^-$. Similarly, if $u^+ \in \partial I_E$ then $\partial_1 S(u^+, \tilde{u}) < -\partial_v L(u^+, 0, 0)$ and $\partial_2 S(\tilde{u}, u^+) < -\partial_v L(u^+, 0, 0)$ for all $\tilde{u} < u^+$.*

Proof. Let us prove the above inequalities for $\partial_1 S$ as the case for $\partial_2 S$ leads to an analogous argument. We start with the left boundary point u^-. We seek an increasing lap from u^- to $\tilde{u}$. At $u_1 = u^-$ it holds that $-L(u^-, 0, 0) = E$, $u_1'' = 0$ and $\partial_u L(u^-, 0, 0) > 0$, which implies that $u_1'''(0) > 0$. By contradiction, suppose that $u'(0) = u''(0) = u'''(0) = 0$. On the one hand we have $p_v' = \partial_v L - p_u$ and on the other hand $p_v' = \partial_{uw}^2 L u' + \partial_{vw}^2 u'' + \partial_w^2 L u'''$. From the former we see that $p_v''(0) = -\partial_u L(0) < 0$, so that

$$\lim_{\varepsilon\to 0} \frac{(\partial_{uw}^2 L u' + \partial_{vw}^2 u'' + \partial_w^2 L) u'''(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\partial_w^2 L u'''(\varepsilon)}{\varepsilon} = -\partial_u L(0) < 0.$$

We conclude (using condition (H)) that $u'''(t) < 0$ in a right neighborhood of 0, which contradicts the fact that we are dealing with an increasing lap.

[13]The function W_{2n} is continuous on I_E^{2n} and is of class C^2 on the set $\{(u_1, ..., u_{2n}) \in \text{int}(I_E^{2n}) \mid u_i \ne u_{i+1}, \ \forall\, i = 1, .., 2n\}$ with $u_{2n+1} \equiv u_1$.

It now follows that $p_{u_1} = \partial_v L(u^-,0,0) - \partial_w^2 L(u^-,0,0)u_1'''(0) < \partial_v L(u^-,0,0)$. Therefore $\partial_1 S(u^-,\tilde{u}) = -p_{u^-} > -\partial_v L(u^-,0,0)$. For the right boundary point u^+ we obtain $\partial_1 S(u^+,\tilde{u}) = -p_{u^+} < -\partial_v L(u^+,0,0)$, since u is a decreasing lap. ■

2.2. **Examples of Twist systems.** An example of a class of Lagrangians for which we can verify the Twist property in various cases is given by $L(u,u',u'') = \frac{1}{2}u''^2 + K(u,u')$. Most of the fourth order equations coming from physical models are derived from Lagrangians of this form. We could tag such systems as *fourth order mechanical systems* based on the analogy with second order mechanical systems given by Lagrangians of the form $L(u,u') = \frac{1}{2}u'^2 + K(u)$ (integrable system). The Lagrangian L clearly satisfies Hypothesis (H) and (L, dt) is thus equivalent to the Hamiltonian system (H,ω) with ω the standard symplectic form on $\mathbb{R}^4$ (see previous chapter) and $H(x) = p_u v + \frac{1}{2}p_v^2 - K(u,v)$. For a regular energy value E the set $\pi^u N_E$ is given by $\pi^u N_E = \{u \mid K(u,0) + E \geq 0\}$. If E is regular it holds that $K(u,0) + E > 0$ for $u \in \mathrm{int}(I_E)$, and $K(u,0) + E = 0$ for $u \in \partial I_E$.

Lemma 8. *Let I_E a connected component of $\pi^u N_E$ (E not necessarily regular*[14]*). Assume that*

(a) $\frac{\partial K}{\partial v}v - K(u,v) - E \leq 0$ for all $u \in I_E$ and $v \in \mathbb{R}$,

(b) $\frac{\partial^2 K}{\partial v^2}|v|^2 - \frac{5}{2}\left\{\frac{\partial K}{\partial v}v - K(u,v) - E\right\} \geq 0$ for all $u \in I_E$ and $v \in \mathbb{R}$.

Then for any pair $(u_1,u_2) \in I_E \times I_E \backslash \Delta$ Problem (6) has a unique minimizer $(u,\tau) \in X_\tau \times \mathbb{R}^+$ (in fact the only critical point), and the minimizer $u(t;u_1,u_2)$ depends C^1-smooth on (u_1,u_2) for $(u_1,u_2) \in \mathrm{int}(I_E \times I_E \backslash \Delta)$ [15].

For the proof of this lemma we refer to Appendix B.

At this point we are not able to prove that the Twist property holds for more general systems under some mild growth conditions on K without assuming (a) and (b). However numerical experiments (see also Section 4.1) for various Lagrangians suggest that Lemma 8 is still valid, although we do not have a proof of this fact. Milder conditions on K sometimes only allow the existence of a continuous family $u(t;u_1,u_2)$. We come to this case in Section 2.3. The conditions given in Lemma 8 already allow for a large variety of Lagrangians that occur in various physical models. We will give a few examples of such systems now.

2.2.1. *The eFK/Swift-Hohenberg system.* The eFK/Swift-Hohenberg Lagrangian is given by $L(u,u',u'') = \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u)$, where $\alpha \in \mathbb{R}$ and F is a smooth potential function. The Hamiltonian in this case is $H(x) = p_u v + \frac{1}{2}p_v^2 - \frac{\alpha}{2}v^2 - F(u)$. Connected components of $\pi^u N_E$ are sets of the from $\{u \mid F(u) + E \geq 0\}$.

In the case that $\alpha > 0$ this L is referred to as the eFK-Lagrangian (see e.g. [13, 14, 15]), and in the case $\alpha \leq 0$ it is usually referred to as the Swift-Hohenberg Lagrangian [21, 28, 29]. For example $F(u) = \frac{1}{4}(u^2-1)^2$ is the classical eFK/Swift-Hohenberg potential [25, 26], $F(u) = \frac{1}{3}u^3 - \frac{1}{2}u^2$ gives the water-wave model [9], $F(u) = -\frac{1}{4}(u^2-1)^2$ is the potential of the nonlinear optics model [1].

If $\alpha \leq 0$ then the conditions (a) and (b) are satisfied for any interval component I_E. The Swift-Hohenberg systems is therefore a Twist system for all interval component. For $\alpha > 0$ this is not immediately clear (conditions (a) and (b) are not

[14] If E is a singular energy level then we require the critical points to be non-degenerate.

[15] If E is a singular energy level then C^1-regularity holds for all $(u_1,u_2) \in \mathrm{int}(I_E \times I_E \backslash \Delta)$ such that u_1 nor u_2 is a critical point.

satisfied)[16]. More details on eFK/Swift-Hohenberg systems are given in Section 3.4.

2.2.2. *The suspension-bridge model.* The suspension bridge model is a special case of the Swift-Hohenberg equation, i.e. $L(u,u',u'') = \frac{1}{2}u''^2 - \frac{c^2}{2}u'^2 + F(u)$, with $F(u) = e^u - u - 1$ (see [27]). Clearly, the suspension bridge model is a Twist system for all $c \in \mathbb{R}$. For more details see Section 3.4. This model is particularly intriguing due to the specific form of the potential function F. The growth of F for $u \to \infty$ is basically different from the growth for $u \to -\infty$ which has far reaching consequences for the set of closed characteristics.

2.2.3. *The fifth order KdV equation.* Consider $L(u,u',u'') = \frac{1}{2}u''^2 + K(u,u')$, $K(u,u') = \frac{1}{2}(\alpha + 2\mu u)u'^2 + F(u)$, with $F(u) = \frac{\kappa}{3}u^3 + \frac{\sigma}{2}u^2$, which describes a fifth order Korteweg-deVries equation (see e.g. [23]). In order for the theory to be applicable the above conditions on K imply that $\alpha + 2\mu u \le 0$ for $u \in I_E$. The case $\mu = 0$ is the Swift-Hohenberg equation again. Let us assume for example that $\kappa, \sigma > 0$, then one can find compact intervals I_E for values $0 \le E \le \frac{\sigma^3}{6\kappa^2}$. These intervals are contained in $[-\frac{3\sigma}{2\kappa}, 0]$. For $\mu > 0$ the condition becomes $u < -\frac{\alpha}{2\mu}$, which is satisfied for all $u \in I_E$ if $\alpha < 0$ for instance. For $\mu < 0$ the condition becomes $u > -\frac{\alpha}{2\mu}$, which is satisfied for all $u \in I_E$ if $\alpha < 2\frac{\sigma\mu}{\kappa}$. Many more combination can be found by also varying the signs of κ and σ.

2.3. **The C^0-Twist property.** As we already pointed out before, the theory developed in this paper can be adjusted to work for C^1-generating functions. We will point out the difficulties and how the theory has to be adjusted at the end of this section. First we start with a weaker version of the Twist property that ensures the existence of C^1-generating functions.

(T′) $\inf\{J_E[u] \mid u \in X_\tau(u_1,u_2),\ \tau \in \mathbb{R}^+\}$ *has a minimizer* $u(t;u_1,u_2)$ *for all* $u_1, u_2 \in I_E$*, and* $u(t;u_1,u_2)$ *is a continuous function of* (u_1,u_2)*.*

Condition (T′) is often easier to verify than the stronger Condition (T). Let I_E be an interval component and (L, dt) is a twist system on I_E with respect to Condition (T′). Then $p_{u_2}(u_1,u_2)$ is strictly increasing in u_1 and $-p_{u_2}(u_1,u_2)$ is strictly increasing in u_2, and both continuous in (u_1,u_2). The maps $T_\pm$ as described in Section 1.2 are therefore monotone (C^1-) Twist maps, which have a C^1-generating function $S_E(u_1,u_2) = J_E[u(t;u_1,u_2)]$.

Lemma 9. *Let I_E be an interval component. If (L, dt) is a Twist system with respect to Condition (T′), then S_E is a C^1-generating function on $I_E \times I_E \backslash \Delta$.*

Property (2) of Lemma 5 is now replaced by the property that $\partial_1 S$ and $\partial_2 S$ are increasing functions of u_1 and u_2 respectively. The difficulties in working with C^1-generating functions are the definition of the Morse index and the gradient flow of $W = \sum_i S(u_i, u_{i+1})$. In Section 3 we use the gradient flow of W to find other critical points besides minima and maxima. A way to deal with this problem is to approximate S by C^2-functions. A C^1-Morse/Conley index can then be defined (see for instance [7, 8]). An analogue of Lemma 6 can also be proved now. Other

[16] J. Kwapisz proves that the (C^0) Twist property (T′) (see Section 2.3) is satisfied for the eFK-Lagrangian ($\alpha > 0$) on interval components I_E for which $F(u) + E$ has at most one internal critical point (a maximum), see [17].

properties of S that we use in this paper, such as construction of isolating neighborhoods, do not need the C^2-regularity. For that reason we will continue with C^2-function keeping in mind that all result immediately carry over to the C^1-case.

3. Existence

3.1. **Simple closed characteristics for compact sections N_E.** The properties of S listed in the Section 2.1 can be used to derive an existence result for simple closed characteristics. Before stating the theorem we need to introduce some additional notation: $I_E \times I_E \backslash \Delta = D_E^+ \cup D_E^-$, where $D_E^+ = \{(u_1, u_2) \in I_E \times I_E \backslash \Delta \mid u_2 > u_1\}$, and D_E^- is defined analogously. The function $W_2(u_1, u_2) = S(u_1, u_2) + S(u_2, u_1)$ is a C^2-function on $\mathrm{int}(I_E \times I_E \backslash \Delta)$. Since $W_2(u_1, u_2) = W_2(u_2, u_1)$ we can restrict our analysis to D_E^+.

Throughout this section we again assume that E is regular and (L, dt) is a Twist system on I_E.

Lemma 10. *Assume that $\pi^u N_E$ contains a compact interval component I_E. Then W_2 has at least one maximum on D_E^+*[17]*.*

Proof. We have that $W_2|_\Delta = 0$ and W_2 is strictly positive near $\mathrm{int}(\Delta)$ by Lemma 5 part (3). Since the set $\overline{D_E^+}$ is compact, W_2 must attain a maximum on set $\overline{D_E^+}$. It follows that $\max_{(u_1,u_2)\in\overline{D_E^+}} W_2(u_1, u_2) > 0$.

Writing $I_E = [u^-, u^+]$ we denote by $n_1 = (1, 0)^T$ the inward pointing normal on the left boundary $B_1 = \{(u^-, u_2) \mid u_2 \in I_E\}$ and by $n_2 = (0, -1)^T$ the inward pointing normal on $B_2 = \{(u_1, u^+) \mid u_1 \in I_E\}$. Using Lemma 7 we can now compute $\frac{\partial W_2}{\partial n_1}$ and $\frac{\partial W_2}{\partial n_2}$. For example let $u_1 = u^-$, then $\frac{\partial W_2}{\partial n_1} = \partial_1 S(u^-, u_2) + \partial_2 S(u_2, u^-) > -\partial_v L(u^-, 0, 0) + \partial_v L(u^-, 0, 0) = 0$. Similarly, using Lemma 7, we derive that $\frac{\partial W_2}{\partial n_2}|_{B_2} > 0$. Since both $\frac{\partial W_2}{\partial n_1}|_{B_1} > 0$, $\frac{\partial W_2}{\partial n_2}|_{B_2} > 0$, and $W_2|_\Delta = 0$, the maximum is attained in $\mathrm{int}(D_E^+)$ (see also Figure 2). ■

If we study W_{2n}, $n > 1$ we do not necessarily find new closed characteristics for (L, dt). i.e. critical points of W_{2n} of higher index may be the same closed characteristic traversed more than once. In the next sections we will describe some mechanisms that yield more geometrically distinct closed characteristics.

The above lemma can be slightly rephrased for Lagrangian systems (see Lemma 6). We do not have information about the nullity of $d^2 J_E(u_1, u_2)$, so that the large Morse index[18] of the solutions may be greater than 2, but the Morse index is certainly smaller than or equal to 2.

Theorem 11. *Assume that $\pi^u N_E$ contains a compact interval component I_E. Then (L, dt) contains at least one simple closed characteristic $u(t) \in \mathrm{int}(I_E)$ with large Morse index greater than or equal to 2 and Morse index less than or equal to 2.*

Theorem 11 states that the associated Hamiltonian system (H, ω) has at least one closed characteristic on M_E. The above Theorem is reminiscent of first order Lagrangian systems: $L(u, u')$ with Euler-Lagrange equation $\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} = 0$. Such

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[17] From straightforward Morse theory for W_2 on D_E^+ we obtain in addition that $b_0 \geq 0$, $b_1 - b_0 \geq 0$ and $b_2 - b_1 + b_0 = 1$, where b_i is the number of critical points of index i.

[18] The large Morse index is defined as the sum of the Morse index and the nullity.

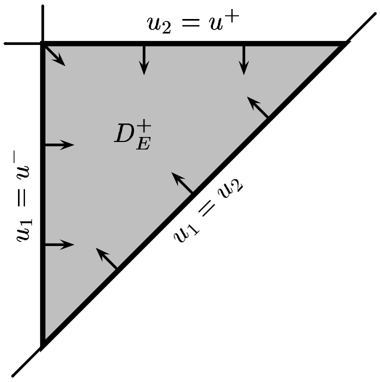

FIGURE 2. A picture of D_E^+. The arrows denote the direction of the gradient ∇W_2 schematically (of course the gradient is not perpendicular to the boundary everywhere). Clearly the maximum of W_2 is attained in the interior of D_E^+.

systems may be labeled as mechanical systems if $\partial_{u'} L > 0$. On the compact components of $\{(u, u') \mid \frac{\partial L}{\partial u'} u' - L(u, u') = E\}$ closed characteristics exist (integrable system).

If L is invariant with respect to the symmetry $t \mapsto -t$ it holds that $L(u, v, w) = L(u, -v, w)$ for all $(u, v, w) \in \mathbb{R}^3$. A consequence of this symmetry is that $S(u_1, u_2) = S(u_2, u_1)$ which implies that we can study just S (instead of W_2) to find simple closed characteristics in this case. Besides, this symmetry of L carries over to the simple closed characteristic: $u(t)$ is symmetric with respect to its extrema. Some Lagrangian systems are also invariant under the symmetry $u \mapsto -u$ which yields the relation $L(u, v, w) = L(-u, -v, -w)$. If $0 \in \pi^u N_E$ then there is at least one closed characteristic on the anti-diagonal $u_1 = -u_2$. If the global maximum of W_2 is not on the anti-diagonal $u_1 = -u_2$ then there are at least 2 more closed characteristics (by symmetry).

If we consider non-compact interval components I_E there is no topological restriction that forces the existence of closed characteristics, and there need not exist any. In order to deal with this case (in forthcoming sections) more information about L needed: asymptotic behavior (see Section 3.4).

3.2. **Singular energy levels.** If E is a singular energy level then there exist points $u \in \pi^u N_E$ for which $\partial_u L(u, 0, 0) = 0$ and $L(u, 0, 0) + E = 0$. For a singular value E the connected components of N_E are either smooth manifolds ($\mathbb{R}$ or S^1), or they are characterized as: $N_E^c \simeq (\mathbb{R}\vee)S^1 \vee \cdots \vee S^1(\vee\,\mathbb{R})$. The points in $\mathbb{R}^2$ on which N_E fails to be a manifold lie on the u-axis, and are exactly the points u for which $\partial_u L(u, 0, 0) = 0$ and $L(u, 0, 0) + E = 0$. The set of such points — critical points — is denoted by $\mathcal{C}(I_E)$. As before $\pi^u N_E$ is a union of closed intervals. An *interval component* I_E is defined as a subset of $\pi^u N_E$ such that $L(u, 0, 0) + E > 0$ for all $u \in \mathrm{int}(I_E)$ and $L(u, 0, 0) + E = 0$ for $u \in \partial I_E$. Since E is singular two interval components I_E^1 and I_E^2 may have non-empty intersection, i.e. $I_E^1 \cap I_E^2 = \{\text{one point}\} \subset \mathcal{C}(I_E)$. Concatenations of interval components are discussed in Section 3.3. If we consider interval components with critical points geometric properties come into play. We assume that (L, dt) is a Twist system for the interval components that we consider.

Let I_E be an interval component for which $u^- \in \partial I_E$ is a critical point. In order to prove the analogue of Lemma 7 we need to know whether $u'''(0)$ is zero or not. This is determined by $\tau(u^-, u_2)$, i.e. if $\tau(u^-, u_2) < \infty$ then $u'''(0) > 0$ (assuming $u_2 > u^-$), and if $\tau(u^-, u_2) = \infty$ then $u'''(-\infty) = 0$ (in the case that $\tau = \infty$ we consider u on $[-\tau, 0]$ using translation invariance). These two cases can be distinguished by studying the singularity at u^-. We can compute the spectrum of u^- which we will denote by $\sigma(u^-)$. We assume that we are dealing with *non-degenerate* singular points, i.e. $0 \notin \sigma(u^-)$. It is shown in Appendix A that there are three possible behaviors for $\sigma(u^-)$: $\sigma(u^-) \subset \mathbb{R}$, $\sigma(u^-) \subset i\mathbb{R}$, or $\sigma(u^-) \subset \mathbb{C}\backslash\{\mathbb{R} \cup i\mathbb{R}\}$. In the latter case there is one eigenvalue in each quadrant. For all three cases it holds that $\partial_u^2 L(u, 0, 0) > 0$ and the three possible behaviors are categorized as *real saddle*, *center* and *saddle-focus* respectively. If $\sigma(u^-) \subset \mathbb{C}\backslash\mathbb{R}$ (center or saddle-focus), then $\tau(u^-, u_2) < \infty$ for all $u_2 \in I_E$. It is immediately clear that Theorem 11 is still valid in that case. Also, if both u^+ and u^- are critical points and have their spectrum in $\mathbb{C}\backslash\mathbb{R}$, Theorem 11 remains true.

Theorem 12. *Let E be a singular value and assume that $\sigma(\mathcal{C}(I_E)) \subset \mathbb{C}\backslash\{\mathbb{R}\}$. Then (L, dt) contains at least one simple closed characteristic $u(t) \in \mathrm{int}(I_E)$ with large Morse index greater than or equal to* 2 *and Morse index less than or equal to* 2.

The way to attack the problem of finding closed characteristics at singular energy levels in general is to again consider the function $W_2(u_1, u_2) = S(u_1, u_2) + S(u_2, u_1)$. Since $W_2|_\Delta \equiv 0$ and strictly positive near Δ, W_2 attains its global maximum in $I_E \times I_E\backslash\Delta$. As was already pointed out before, the maximum is attained in the interior of $I_E \times I_E\backslash\Delta$ if there are no critical points of $L(u, 0, 0)$ in ∂I_E, or if critical points of $L(u, 0, 0)$ have complex spectrum (Theorem 12). Thus in order for W_2 to attain its global maximum on the boundary, the interval component ∂I_E needs to contain at least one critical point of $L(u, 0, 0)$ with real spectrum.

The next question is: suppose W_2 attains its maximum at $\partial(I_E \times I_E\backslash\Delta)$, does this maximum correspond to a simple closed trajectory for (L, dt)? Again from the previous we know that at a point $(u_1, u_2) \in \partial(I_E \times I_E\backslash\Delta)$ it holds that $\partial_1 W_2 \geq 0$ if $u_1 = u^-$ and $\partial_2 W_2 \leq 0$ if $u_2 = u^+$. A boundary maximum for which $u_1 = u^-$ and $u_2 = u^+$ is called a co-dimension 2 point, and the remaining boundary points are called co-dimension 1 points. It is clear that at a co-dimension 1 point, for example at $u_1 = u^-$, it holds that $\partial_2 W_2 = 0$. Since this point is a maximum, and because $\partial_1 W_2 \geq 0$ it follows that the maximum is in fact a critical point. The same holds for a co-dimension 1 point at $u_2 = u^+$. Such points correspond to solutions $u(t)$ for which $u'''(-\infty) = u'''(\infty) = 0$, and $u(-\infty) = u(\infty) = u^-$, and $u(t)$ is thus a homoclinic orbit. By the same reasoning co-dimension 2 points are also critical points. Such a point corresponds to a heteroclinic loop (two heteroclinic connections that form a loop).

Summarizing, we can introduce the notion of closed characteristic in the broad sense of the word: a simple closed periodic orbit, a simple homoclinic loop, or a simple heteroclinic loop (they all form a simple closed loop in the configuration plane). If we use this definition we obtain the following theorem.

Theorem 13. *Assume that $\pi^u N_E$ has a compact interval component I_E then (L, dt) has at least one simple closed characteristic in the broad sense.*

It is clear from the previous that a necessary condition for (L, dt) to have a simple homoclinic loop to u^- is that u^- is a critical point of $L(u, 0, 0)$ that has real spectrum (real saddle). The same holds for u^+. A necessary condition to find a simple heteroclinic loop between u^- and u^+ is that both u^- and u^+ are real saddles. Unfortunately, these conditions need not be sufficient[19].

One way to guarantee the existence of a simple homoclinic loop to u^- is that $\tau(u^-, u_2) = \tau(u_2, u^-) = \infty$ for all $u_2 \in I_E$[20], and $u^+ \notin \mathcal{C}(I_E)$, or u^+ has complex spectrum. In that case $\partial_1 S(u^-, u_2) = -\partial_v L(u^-, 0, 0) = 0$ for all $u_2 \in I_E$. In terms of W_2 this yields that $\partial_1 W_2(u^-, u_2) = 0$ for all $u_2 \in I_E$. We can now restrict W_2 to the line-segment $\{u_1 = u^-\} \times I_E$. Define $W_1(u) = W_2|_{\{u_1=u^-\}\times I_E} = S(u^-, u) + S(u, u^-)$. It easily follows that (compare Lemma 7) $W_1(u^-) = 0$, $W_1(u^- + \varepsilon) > 0$ for $\epsilon > 0$ sufficiently small[21] and $W_1'(u^+) < 0$, and thus W_1 has at least one global maximum u_* on (u^-, u^+). The point u_* corresponds to a homoclinic orbit to $u = u^-$.

Regarding the Morse index of this point/orbit we note the following. If u_* is a (local) maximum of W_2 on D_E^+ then the large Morse index is again equal to 2. The corresponding homoclinic orbit has large Morse index greater than or equal to 2 and Morse index less than or equal to 2. However, restricted to the class of functions that are homoclinic to u^- it has large Morse greater than or equal to 1 and Morse index less than or equal to 1 (mountain-pass critical point)[22].

3.3. **Concatenation of interval components.** Up to this point we have only considered single interval components I_E. When E is a singular value then two interval components I_E^1 and I_E^2 may have a common boundary point. This boundary point is then necessarily a critical point. The concatenation of the interval components I_E^i, $i = 1, 2$, will be denoted by $I_E^\#$, and the critical point in $I_E^1 \cap I_E^2$ is denoted by u_*. If (L, dt) is a Twist system on both interval components I_E^1 and I_E^2 it does does not necessarily mean that (L, dt) is a Twist system on the concatenated interval $I_E^\#$. One can easily give examples where (L, dt) fails to satisfy the Twist property on $I_E^\#$.[23] However if (L, dt) is Twist system on $I_E^\#$ more solutions can be found. In order to study this case we will use the gradient flow of W_2:

$$\begin{aligned} u_1' &= \partial_2 S(u_2, u_1) + \partial_1 S(u_1, u_2), && \text{(8a)}\\ u_2' &= \partial_2 S(u_1, u_2) + \partial_1 S(u_2, u_1), \quad u_i \in \operatorname{int}(I_E),\ \forall i = 1, 2. && \text{(8b)} \end{aligned}$$

[19] For the eFK Lagrangian with $F(u) = \frac{1}{4}(u^2 - 1)^2$ it has been shown that the simple closed characteristic found in Theorem 13 corresponds to a heteroclinic loop if and only if the equilibrium points are real saddles [25, 30, 16].

[20] It follows from Lemma 5 part (2) that it in fact suffices that $\tau(u^-, u_2) = \tau(u_2, u^-) = \infty$.

[21] It follows from the linearization around u^- that $p_{u_2} < 0$ for $u^- < u_2 < u^- + \varepsilon$ when ε is small enough.

[22] For the eFK Lagrangian with $F_a(u) = \int_1^u (s^2 - 1)(s - a)ds$, $0 \le a < 1$ and $\alpha \ge 2\sqrt{2(1-a)}$ the Twist property is satisfied on the interval component $I_0 = [u^-, 1]$ ($\tau(u^-, 1) = \infty$). Therefore there exists a homoclinic loop is this case. The existence of such solutions for this problem was first proved in [24] by means of a different method. If the case $a = 0$ is considered one obtains a heteroclinic loop (see e.g. [14]).

[23] For example consider the eFK Lagrangian with $F(u) = 1/4(u^2 - 1)^2$. Take $E = 0$, then $\pi^u N_0 = \mathbb{R}$ is the concatenation of three intervals. If $\alpha > 2\sqrt{2}$ then (L, dt) is not a Twist system on $I_0^\# = \mathbb{R}$. However for $\alpha \le 0$ the Twist property is satisfied on $\mathbb{R}$, and numerical experiments indicate the same for $2\sqrt{2} > \alpha > 0$. This is related to the behavior of the singularities $u = \pm 1$ (see Section 4.1).

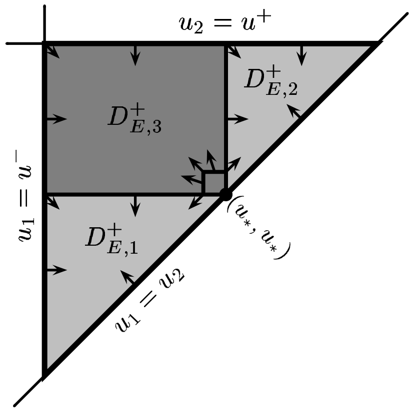

FIGURE 3. The triangle D^+_E when a connected component of $\pi^u N_E$ consists of two compact interval components. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has maximum in $D^+_{E,1}$ and $D^+_{E,2}$ and a saddle point in $D^+_{E,3}$.

As before we can restrict our analysis to D^+_E. Define $D^+_{E,1} = \{(u_1, u_2) \in I^1_E \times I^1_E \mid u_2 > u_1\}$, $D^+_{E,2} = \{(u_1, u_2) \in I^2_E \times I^2_E \mid u_2 > u_1\}$, and $D^+_{E,3} = I^1_E \times I^2_E \backslash (u_*, u_*)$. On the domains $D^+_{E,1}$ and $D^+_{E,2}$ one can again apply Theorem 13 which yields the existence of maxima on each of these components. Note that this is independent of the type of u_* (spectrum $\sigma(u_*)$). The following theorem will crucially use the fact that u_* is a critical point for which $\sigma(u_*) \subset \mathbb{C}\backslash\{\mathbb{R} \cup i\mathbb{R}\}$, i.e. a saddle-focus.

Lemma 14. *Let $I^\#_E$ be a concatenation of two compact interval components I^1_E and I^2_E and assume that the critical point $u_* \in I^1_E \cap I^2_E$ is a saddle-focus. Then W_2 has at least one maximum on each of the components $D^+_{E,i}$, $i = 1, 2$ and W_2 has a saddle point (critical points with large Morse index equal to 1) on the component $D^+_{E,3}$.*

Proof. The existence of at least one maximum on each of the components $D^+_{E,i}$, $i = 1, 2$ follows directly from Theorem 14. As for the existence of saddle points we argue as follows (see also Figure 3). Applying Lemma 7 we obtain that $\partial_1 W_2|_{\partial I^1_E \times I^2_E} > 0$ and $\partial_2 W_2|_{I^1_E \times \partial I^2_E} > 0$. In order to successfully apply Conley's Morse theory we need to choose an appropriate subset of $D_{E,3}$ which will serve as an isolating neighborhood. Near $(u_1, u_2) = (u_*, u_*)$ we can find a small solution of the Euler-Lagrangian equation by perturbing from a linear solution. Consider the unique solution $u(t)$ for which $u(0) = u_1 = u_* - \delta$ and $u(\tau) = u_* + \delta$. Since u_* is critical point of saddle-focus type follows that $u'''(0) < 0$ and $u'''(\tau) < 0$ for δ sufficiently small[24]. Straightforward calculation shows that $\partial_1\partial_2 W_2 = \partial_1\partial_2 S(u_1, u_2) + \partial_1\partial_2 S(u_2, u_1) > 0$. These two facts combined show that $\partial_1 W_2(u_* - \delta, u_2) < 0$ for all $u_2 \le u_* + \delta$, and $\partial_2 W_2(u_1, u_* + \delta) > 0$ for all $u_1 \ge u_* - \delta$. Define $N_\delta = D^+_{E,3} \backslash \{(u_1, u_2) \mid u_* - \delta < u_1 \le u_*, u_* + \delta > u_2 \ge u_*\}$.

[24] This follows for example from an explicit calculation of the solution for the linearized problem.

The set N_δ is a closed subset of $D^+_{E,3}$ and is isolating with respect to the gradient flow of W_2[25]. The next step is to compute the Conley index of the maximal invariant set $\mathrm{Inv}(N_\delta) \subset N_\delta$. It suffices here to compute the homological index (see [10]) of $\mathrm{Inv}(N_\delta)$. In order to do so we need to find an index pair for $\mathrm{Inv}(N_\delta)$. Let $\partial I^1_E = \{a_1^-, a_1^+\}$, $\partial I^2_E = \{a_2^-, a_2^+\}$. Let $N_\delta^- = \{u_1 = u_*, u_* + \delta \leq u_2 \leq a_2^+\} \cup \{a_1^- \leq u_1 \leq u_* - \delta, u_2 = u_*\}$, then (N_δ, N_δ^-) is an index pair for $\mathrm{Inv}(N_\delta)$, and $CH_*(\mathrm{Inv}(N_\delta)) = H_*(N_\delta, N_\delta^-)$. Consequently $CH_1(\mathrm{Inv}(N_\delta)) \simeq \mathbb{Z}$ and $CH_k(\mathrm{Inv}(N_\delta)) = 0$ for $k \neq 1$. The fact that the homological Conley index is non trivial for $k = 1$ and because (8) is a gradient flow we conclude that there exists at least one critical point of W_2 in N_δ with large Morse index equal to 1. ∎

With regard to the relative position of the extrema of W_2 we note the following. Let (b_i, c_i) be the minimum in $D^+_{E,i}$ for $i = 1, 2$. Since $\nabla W_2(b_i, c_i) = 0$ it follows from Lemma 5 part (2) that $\partial_1 W_2(b_1, u_2) > 0$ for all $u_2 > c_1$ and $\partial_2 W_2(u_1, c_2) < 0$ for all $u_1 < b_2$. Therefore, we may as well take use $\tilde{D}^+_{E,3} = \{b_1 \leq u_1 \leq u_*,\ u_* \leq u_2 \leq c_2\} \setminus (u_*, u_*)$ instead of $D^+_{E,3}$. We then obtain a saddle point $(b_3, c_3) \in \tilde{D}^+_{E,3}$ with $b_1 < b_3 < b_2$ and $c_1 < c_3 < c_2$.

In terms of closed characteristics for a Lagrangian systems the above lemma yields

Theorem 15. *Let $\pi^u N_E$ contain a concatenation $I^\#_E$ of two compact intervals I^1_E and I^2_E, and assume that (L, dt) is a Twist system on $I^\#_E$. If $u_* \in I^1_E \cap I^2_E$ is of saddle-focus type, then there exist at least 3 geometrically distinct closed characteristics.*

An analogue of the above theorem can also be proved for concatenations of more than two interval components. We leave this to the interested reader.

3.4. **Non-compact interval components.** As already indicated in the previous sections the theory developed in this paper is applicable to various model equations that we know from physics such as the eFK/Swift-Hohenberg type equations, 5th order KdV equations, suspension bridge model, etc. (see Section 2.2). In this section we will take a closer look at the class of eFK/Swift-Hohenberg type equations. This family of equations is given by a Lagrangian of the form: $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u)$, where F is the potential which is an arbitrary C^2-function of u. We have already proved that such Lagrangian systems are always Twist systems if $\alpha \leq 0$ (and we believe the same to be true also for $\alpha > 0$ (Twist property on interval components)). The results obtained in this paper prove that for any energy level E for which the set $\{u \mid F(u) + E \geq 0\}$ contains a compact interval component I_E, there exist a simple closed characteristic $u(t) \in \mathrm{int}(I_E)$. Let us by means of example consider a double equal-well potential F (like $\frac{1}{4}(u^2 - 1)^2$) with $\min_u F(u) = 0$. In this case the set $\{u \mid F(u) + E \geq 0\}$ always contains non-compact interval components. Without further geometric knowledge of the energy manifold M_E a general topological result proving existence of closed characteristics does not seem likely. Therefore we will consider a specific example here. Consider the energy level $E = 0$, then $I_0 = \mathbb{R}$, and I_0 is a concatenation of three interval components. The Lagrangian system with $\alpha \leq 0$ is a Twist system on I_0

[25] The flow is not well-defined on the boundary of D_{E_3}, but we can choose a slightly smaller isolating neighborhood inside $D_{E,3}$ with the same Conley index (alternatively we can use the Morse index for C^1-functions (see also Section 2.3)).

and therefore S is well-defined on $\mathbb{R}^2$. One way to deal with this non-compact case is to compactify the system (see [12]). This however requires detailed information about the asymptotic behavior of F. There is a weaker assumption that one can use in order to be able to restrict the analysis of W_2 to a compact subset of D_E^+. This boils down to a geometric property for S:

(D) *There exists a pair* $(u_1^*, u_2^*) \in D_E^+$ *(with* $|u_1^*|$ *and* $|u_2^*|$ *large) such that* $u'''_{a,b}(0) < 0$ *and* $u'''_{a,b}(\tau) < 0$ *for the unique minimizers* $u_a = u(t; u_1^*, u_2^*)$ *and* $u_b = u(t; u_2^*, u_1^*)$ *of* (6)[26].

If (L, dt) satisfies Hypothesis (D) on a (non-compact) interval component I_E, then the system is said to be *dissipative* on $I_E^* = [u_1^*, u_2^*]$.

Lemma 16. *If a Lagrangian system is dissipative on* I_E^*. *Then it holds that* $\partial_1 W_2(u_1^*, u_2) < 0$ *for all* $u_2 \in (u_1^*, u_2^*]$ *and* $\partial_2 W_2(u_1, u_2^*) > 0$ *for all* $u_1 \in [u_1^*, u_2^*)$.

Proof. It follows from (D) that $\partial_1 W_2(u_1^*, u_2^*) < 0$. Lemma 5 part (2) implies that $\partial_1 W_2(u_1^*, u_2)$ is increasing as a function of u_2. It easily follows that $\partial_1 W_2(u_1^*, u_2) < 0$ for all $u_2 \leq u_2^*$. The other case is proved in exactly the same way. ∎

For many nonlinearities $F(u)$ it can be proved that eFK/Swift-Hohenberg system is dissipative on some interval $I_E^* = [u_1^*, u_2^*]$ with $u_1^* < -1$ and $u_2^* > +1$ [27]. Notice that S need not have any critical points, for example for $E \gg 0$ (see [12]). For $E = 0$ there are two equilibrium points which will force S to have critical points.

Lemma 17. *If the Swift-Hohenberg Lagrangian satisfies* (D) *on* I_0^* *(with* $\pm 1 \in I_0^*$*) then it has at least two geometrically distinct simple closed characteristics (large and small amplitude). Moreover, if* $u = \pm 1$ *are both saddle-foci then there exist two more geometrically distinct simple closed characteristics.*

Proof. We consider the function W_2 on $I_E^* \times I_E^*$ and as before we define $D_E^+ = I_E^* \times I_E^* \cap \{u_2 > u_1\}$ (see also Figure 4). Define $A_1 = \{u_1 > -1, u_1 < u_2 < 1\}$ and $A_2 = D_E^+ \cap \{u_1 < -1, u_2 > 1\}$. As in the proof of Lemma 10 we have that $\partial_1 W_2(\pm 1, u_2) > 0$ and $\partial_2 W_2(u_1, \pm 1) < 0$. We now see from Lemma 16 that the gradient of W_2 points outwards on ∂A_2 and inwards on ∂A_1. Hence, on A_1 the function W_2 attains a maximum and on A_2 the function W_2 attains a minimum (index 2 and index 0 points), which proves the first part of the lemma.

As for the second part we argue as in the proof of Lemma 12. Since $u = \pm 1$ are saddle-foci one finds index 1 saddle points in both $A_3 = D_E^+ \cap \{-1 < u_1 < 1, u_2 > 1\}$ and $A_4 = D_E^+ \cap \{u_1 < -1, -1 < u_2 < 1, \}$. ∎

Concerning the relative position of the extrema of W_2 we argue in the same way as at the end of Section 3.3. Denoting by (b_i, c_i) the extremum in A_i (for $i = 1, 2, 3, 4$) we find that $b_2 < b_4 < b_1 < b_3$ and $c_4 < c_1 < c_3 < c_2$.

The result proved above have already been found in [21, 28] for the special case $F(u) = \frac{1}{4}(u^2 - 1)^2$ without information about the index of the solutions.

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[26] Notice that $u_a(t) = u_2^* - u_b(\tau - t)$ if $L(u, v, w)$ is symmetric in v.

[27] For example, when $F(u) \sim |u|^n$ as $|u| \to \infty$ for some $n > 2$ then this follows from a scaling argument. After scaling the Euler-Lagrange equation tends to $u'''' = -|u|u^{n-2}$. For this equation it is easy to see that $u(0) = u_1 < 0$, $u'''(0) = 0$ implies that $u(\tau) = u_2 > 0$ and $u'''(\tau) < 0$. A perturbation argument then shows that D is satisfied for the original equation for some (u_1^*, u_2^*) with $-u_1^*$ and u_2^* large.

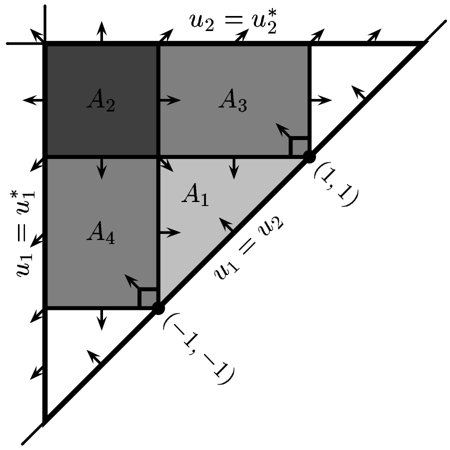

FIGURE 4. The triangle $D_E^+ = I_E^* \times I_E^* \cap \{u_2 > u_1\}$ for the case of a double-well potential. The arrows denote (schematically) the direction of the gradient ∇W_2. Clearly W_2 has at least one maximum and one minimum. Additionally, when the equilibrium points are saddle-foci then W_2 has two saddle points.

Many more examples can be considered with non-compact interval components. A rather tricky system is the suspension bridge model (see Section 2.2.2). The Lagrangian is given by $L(u, u', u'') = \frac{1}{2}u''^2 - \frac{c^2}{2}u'^2 + F(u)$, where $F(u) = e^u - u - 1$. This nonlinearity is especially hard to deal with when trying to compactify D_E^+. In this context it is interesting to note that there is no a priori L^∞ bound on the set of bounded solutions (see [27]) as opposed to nonlinearities with super-quadratic growth. From the analysis in [27] it follows that there exists a point $(u_1^*, u_2*) \in D_E^+$ such that $\partial_1 S(u_1^*, u_2^*) > 0$, $\partial_2 S(u_1^*, u_2^*) > 0$, and $\partial_1 S(u_1^*, u_2) > 0$ for all $u_2 > 0$. This is a different dissipativity condition. Upon examining S (for $E = 0$) on $I_E^* \times I_E^*$ we find at least one index 1 simple closed characteristic for the suspension bridge problem (this was already proved in [27], without information on the Morse index). In order for the argument the work the equilibrium point 0 has to be a saddle-focus. Moreover, for the dissipativity condition to be satisfied the coefficient in front of the second term in the Lagrangian has to be strictly positive. In [27] more complicated closed characteristics are also found. This will be subjected to a future study.

4. CONCLUDING REMARKS

4.1. **Numerical evidence for the Twist property.** In Lemma 8 we prove the Twist property for a class of Lagrangians including the well known Swift-Hohenberg Lagrangian. Numerical evidence suggests that the Twist property holds for a large class of other Lagrangians as well. As an example we depict in Figure 5 solutions of the eFK equation (i.e., the eFK Lagrangian with $F(u) = \frac{1}{4}(u^2 - 1)^2$) in the energy level $E = 0$ for the two different cases where the equilibrium points are real saddles and saddle-foci. While the Twist property certainly is not satisfied on the whole of $I_E = \mathbb{R}$ (it *is* satisfied on $[-1, 1]$) for the real saddle case, we conjecture

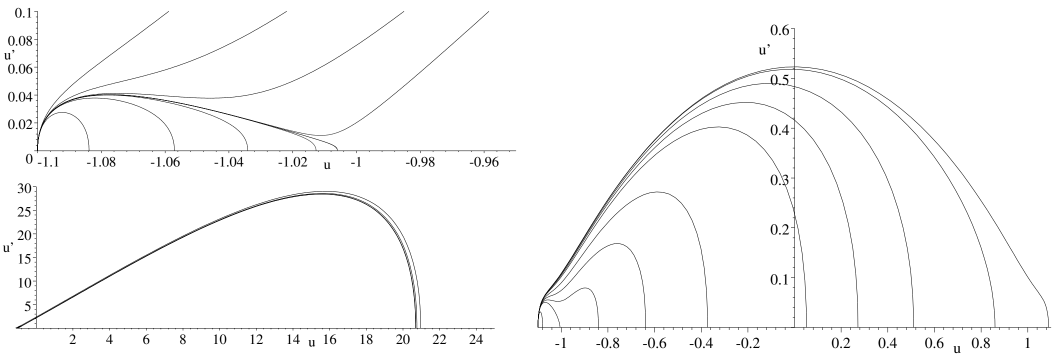

FIGURE 5. For fixed $u_1 = -1.1$ characteristics in the energy level $E = 0$ of the eFK Lagrangian are shown (in the (u, u')-plane). On the left the equilibrium points $u = \pm 1$ are real saddles ($\alpha = -5$). Notice the different scales needed to obtain an overall picture of the situation. The Twist property is only satisfied for $u_2 \in (u_1, -1)$. On the right the equilibrium points are saddle-foci ($\alpha = -1$). In this case the Twist property seemingly holds for all $u_2 > u_1$.

that the Twist property holds as long as the equilibrium points are saddle-foci. We also performed numerical calculations on the 5th order KdV equation (see Section 2.2.3) and it seems that the same is true for this system. It is of course impossible to make statements about the rich class of second order Lagrangians as a whole, but the Twist property seems to hold for a large subclass.

4.2. **Local behavior at equilibrium points.** In Section 3.2 we indicated that the critical points u_* with $\partial_u^2 L(u_*, 0, 0) > 0$ can be categorized into three classes: $\sigma(u_*) = \{\pm\lambda_1, \pm\lambda_2\}$ (real saddle), $\sigma(u_*) = \{\pm a \pm bi\}$ (saddle-focus), and $\sigma(u_*) = \{\pm ai, \pm bi\}$ (center). A fourth possibility does not occur for critical points on the boundary of interval components, namely $\sigma(u_*) = \{\pm\lambda, \pm ai\}$ (saddle-center). Such points do however occur as critical points. For these saddle-centers it holds that $\partial_u^2 L(u_*, 0, 0) < 0$. Since these points never occur on interval components one may ask how such point fit in.

Consider a compact interval component I_E, then $L(u, 0, 0) + E > 0$ for all $u \in \mathrm{int}(I_E)$ and $\partial_u L|_{\partial I_E} \geq 0$ (if $\partial_u L = 0$ at a boundary point then necessarily $\partial_u^2 L > 0$). There exists a point $u_* \in \mathrm{int}(I_E)$ such that $\partial_u L(u_*, 0, 0) = 0$ and $\partial_u^2 L(u_*, 0, 0) < 0$. As a matter of fact there may be many minima and maxima. Now let E increase until the next extremum is reached. If the extremum is a minimum then I_E splits into two components, and if this extremum is a maximum then I_E simply shrinks to the point u_* Conversely, if u_* is a saddle-center equilibrium point at energy level E_*, then there exists an $\epsilon > 0$ such that $\pi^u N_{E_* - \epsilon}$ contain a compact interval component $I_{E_* - \epsilon}$ which shrinks to u_* as $\epsilon \to 0$.

The local theory for saddle-centers reveals the existence of a family of closed characteristics on $I_{E_* - \epsilon}$ parametrized by ϵ (Lyapunov Center Theorem). Our theory not only provides the existence of closed characteristics for $E_* - \varepsilon < E < E_*$ but also guarantees the existence of closed characteristics *for all $E < E_*$ as long as* *the interval component I_E remains compact*. We should emphasize again the resemblance with the classical mechanical system $\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} = 0$.

4.3. **Aubry-LeDaeron-Mather theory.** The theory of Aubry-LeDaeron-Mather [6, 19] is concerned with finding periodic and quasi-periodic points of area-preserving Twist maps of the annulus $S^1 \times \mathbb{R}$ (this can also be translated in terms of the generalized Frenkel-Kantorovich model). The variational approach via generating functions has a lot of similarity with the techniques used in this paper, i.e. we also use a variational argument based on generating functions. There are however a lot of differences as well. For instance in ALM theory the generating function is defined on $S^1 \times \mathbb{R}$. In our case the generating function S is defined on $I_E \times I_E$ which is topologically much simpler. Therefore the topology by itself does not yield existence results, but the information on the boundary allows us to find critical points. We prove specific properties for S that are related to Lagrangian systems in order to prove general statements about periodic points of the associated Twist maps. We will explain that if $M = S^1$ our theory has more similarities with ALM theory. The analogues of the multiplicity results obtained by ALM theory will be addressed in [31].

Consider the case that $M = S^1$. It is now appropriate to consider L on the universal covering of $J^2 S^1$ which means that $L(u, u', u'') = L(u + 1, u', u'')$ for all $u \in \mathbb{R}$. If $E > -\min_{u\in[0,1)} L(u, 0, 0)$ and (L, dt) is a Twist system on $I_E = S^1$, then $T_\pm$ are area-preserving Twist maps of the annulus $S^1 \times \mathbb{R}$. The maps $T_\pm$ have the additional property that $u_{n+1} > u_n$ for T_+ and $u_{n+1} < u_n$ for T_-. It also holds that $S(u_1 + 1, u_2 + 1) = S(u_1, u_2)$. The function S is continuous on $\mathbb{R}^2$ and smooth on $\mathbb{R}^2 \setminus \{u_1 = u_2\}$ (S is zero on the diagonal). Notice that this case is very close to ALM theory.

Multiplicity results that have been proved for Lagrangian systems on $M = \mathbb{R}$ (and $M = S^1$) can be found in [14, 15]. There the existence of homoclinic and heteroclinic ground states is proved without using generating functions[28]. The study of homoclinic and heteroclinic solutions, especially higher index patterns, will be subject of future study.

4.4. **KAM theory.** For the Lagrangian systems that we study in this paper we may wonder whether such systems can be completely integrable. A Lagrangian system (L, dt) is said to completely integrable if the associated Hamiltonian system (H, ω) is completely integrable. Many of the interesting examples that we consider such as the eFK/Swift-Hohenberg system with $\alpha \le 0$ are far from integrable. An example of an integrable system is given by the Lagrangian $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{1}{4}u^4$ (see [12] for a proof). Integrability can also be addressed at the level of the Twist maps in the Lagrangian systems. Without going into too much detail let us look at a specific example. Consider again the eFK/Swift-Hohenberg family defined by the $L(u, u', u'') = \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + \frac{1}{4}(u^2 - 1)^2$, $\alpha \le 0$. Now let $E > 0$

[28] In [14] action integrals of the form $J[u] = \int_{\mathbb{R}} \frac{1}{2}u''^2 + \frac{\alpha}{2}u'^2 + F(u)dt$ are considered, where $\alpha < 0$ and with (i) $F \in C^2(\mathbb{R})$, (ii) $F > 0$ for $u \ne \pm 1$, and $F(\pm 1) = 0$, and (iii) $F(u) > -C_0 + C_1 u^2$ for some $C_0, C_1 > 0$.

Theorem 18. *Assume the above hypotheses on L and let $u = \pm 1$ be saddle-focus equilibria. Then there exists an infinity of local minimizers for J consisting of homoclinic and heteroclinic connections between $u = \pm 1$.*

This has profound consequences for the set of periodic minimizers (not necessarily simple) [15]. A similar result can be proved with more equilibrium points, or for $u \in S^1$.

and consider the area-preserving map T on $\Omega_E^* = \mathbb{R}^2$ as defined in Section 4.3. It follows from the compactification results in [12] that $\mathbb{R}^2 \backslash B_r(0)$ contains only invariant curves for the map T for $r > 0$ sufficiently large. Inside the ball $B_r(0)$ the map T can be chaotic (depending on the character of the equilibrium points). The invariant curves in $\mathbb{R}^2 \backslash B_r(0)$ can be interpreted as the analogues of KAM tori/circles. To get a feel for integrability of the map T on Ω_E^* for which I_E is compact we can look at the quadratic Lagrangian $L(u, u', u'') = \frac{1}{2}u''^2 - \frac{1}{2}u^2$. We will leave this to the interested reader.

The question of integrability versus non-integrability for second-order Lagrangian systems may be fairly complex. The results in [14, 15] and those proved in Section 3.3 seem to suggest that equilibrium points of saddle-focus and center type in combination with geometric and topological conditions on the system create regions of non-integrability. With the techniques presented in this paper and the methods in [31] we are trying to understand some of the dynamics of the system in this case. These questions will be subject of further study.

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Appendix A. Classification of equilibrium points

The equilibrium solutions of the Euler-Lagrange equation

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\frac{\partial L}{\partial u'} + \frac{d^2}{dt^2}\frac{\partial L}{\partial u''} = 0, \tag{9}$$

are given by the relation $\frac{\partial L}{\partial u}(u_*, 0, 0) = 0$. The sign of $\frac{\partial^2 L}{\partial u^2}(u_*, 0, 0)$ divides the behaviors of the equilibrium points onto two groups. We will not consider the case $\frac{\partial^2 L}{\partial u^2}(u_*, 0, 0) = 0$ which requires information on higher order derivatives. Equilibrium points for which $\frac{\partial^2 L}{\partial u^2}(u_*, 0, 0) \neq 0$ are usually called *non-degenerate*. In order to study the local structure of singular points we need to consider the second variation of $J[u]$ around a equilibrium solution $u(t) \equiv u_*$. This yields the following linear differential equation for the variations φ:

$$\frac{\partial^2 L}{\partial u^2}\varphi + \Big(2\frac{\partial^2 L}{\partial u \partial u''} - \frac{\partial^2 L}{\partial u'^2}\Big)\varphi'' + \frac{\partial^2 L}{\partial u''^2}\varphi'''' = 0, \tag{10}$$

where all partial derivatives of L are evaluated at $(u, u', u'') = (u_*, 0, 0)$. The characteristic equation is given by $\partial_u^2 L + \big(2\partial^2_{uu''}L - \partial^2_{u'}L\big)\lambda^2 + \big(\partial^2_{u''}L\big)\lambda^4 = 0$. For non-degenerate equilibrium solutions the following classification holds:

Lemma 19. *Let $u(t) \equiv u_*$ be an equilibrium solution.*

(i) *If $\partial_u^2 L < 0$, then $\sigma(u_*) = \{\pm\lambda, \pm ai\}$ (saddle-center).*
(ii) *If $\partial_u^2 L > 0$, then $\sigma(u_*) = \{\pm\lambda_1, \pm\lambda_2\}$, $\sigma(u_*) = \{\pm ai, \pm bi\}$, or $\sigma(u_*) = \{\pm a \pm bi\}$ (real saddle, center, and saddle-focus respectively) depending on $\partial^2_{uu''}L$ and $\partial^2_{u'}L$.*

Proof. From the characteristic equation we derive

$$\lambda^2_{\pm} = \frac{-\big(2\partial^2_{uu''}L - \partial^2_{u'}L\big) \pm \sqrt{D}}{2\partial^2_{u''}L},$$

where $D = \big(2\partial^2_{uu''}L - \partial^2_{u'}L\big)^2 - 4\big(\partial^2_{u''}L\big)\big(\partial^2_u L\big)$. Clearly if $\partial^2_u L < 0$, then $\sqrt{D} > |2\partial^2_{uu''}L - \partial^2_{u'}L|$ and thus $\lambda^2_- < 0$ and $\lambda^2_+ > 0$. This forces the spectrum to be $\{\pm\lambda, \pm ai\}$.

If $\partial^2_u L > 0$, then $\sqrt{|D|} < |2\partial^2_{uu''}L - \partial^2_{u'}L|$ and there are 3 possibilities:

1. $D > 0$, then $\sqrt{D} < |2\partial^2_{uu''}L - \partial^2_{u'}L|$ and $\lambda^2_\pm$ are both positive or negative. This depends on $\partial^2_{uu''}L$ and $\partial^2_{u'}L$. If both eigenvalues are negative the spectrum is given by $\{\pm ai, \pm bi\}$, and if both eigenvalues are positive the spectrum is $\{\pm\lambda_1, \pm\lambda_2\}$.
2. $D = 0$, then the same possibilities as for (1.) hold with the additional property that the eigenvalues all have multiplicity two.
3. $D < 0$, then $\lambda^2_\pm \in \mathbb{C}\backslash\mathbb{R}$ and there for the spectrum is $\{\pm a \pm bi\}$. ■

As indicated before we do not study the case $\partial^2_u L = 0$. In order to analyze degenerate equilibrium solutions a normal form analysis is required. An example of such type of analysis for a non-linear saddle-focus ($\lambda^2_\pm = 0$) can be found in [12]. The results proved in [12] for non-linear saddle-foci would suffice for the purposes of this paper.

Appendix B. The proof of Lemma 8

Stationary functions of the action functional $J_E[u]$, with $L(u, u', u'') = \frac{1}{2}u''^2 + K(u, u')$, satisfy the equation

$$u'''' - \frac{d}{dt}\frac{\partial K}{\partial u'} + \frac{\partial K}{\partial u} = 0. \tag{11}$$

Solutions of (11) satisfy the Hamiltonian relation $-u'u''' + \frac{1}{2}u''^2 + \frac{\partial K}{\partial u'}u' - K(u, u') - E = 0$. For an increasing lap from u_1 to u_2 the derivative u' can be represented as a function of u. Set $z(u) = u'\sqrt{u'}$ (see for example [5, 25] were similar substitutions are used). Using the Hamiltonian relation we find that z satisfies the equation $z'' = g(u, z)$, $z > 0$, $z(u_1) = z(u_2) = 0$, where $g(u, z) = \frac{3}{2}\frac{\frac{\partial K}{\partial u'}u' - K(u,u') - E}{z^{5/3}}$. The same holds for decreasing laps ($z < 0$). If $\frac{\partial K}{\partial u'}u' - K(u, u') - E \le 0$, and $\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\big(\frac{\partial K}{\partial v}v - K(u, v) - E\big) \ge 0$, for all $u \in I_E$, and $z \ge 0$ (condition (a) and (b) in Lemma 8), then $g(u, z) \le 0$, and $\frac{\partial g}{\partial z}(u, z) \ge 0$ respectively.

It then follows from results in [11] that the boundary value problem for the z-equation has a unique strictly concave positive solution. Consequently the u-laps from u_1 to u_2 are unique, and we thus obtain a family $u(t; u_1, u_2)$. These functions are global minimizers of J_E[29] (follows from $\frac{\partial g}{\partial z} \ge 0$). It follows from the smoothness of the initial value problem of (11) that these functions depend continuously on $\lambda = (u_1, u_2) \in \Lambda \stackrel{\text{def}}{=} I_E \times I_E\backslash\Delta$, and that the time $\tau(u_1, u_2)$ it takes for u to (monotonically) go from u_1 to u_2 depends continuously on u_1 and u_2 as well[30] and $\tau(u_1, u_2) < \infty$ for all $(u_1, u_2) \in \Lambda$[31].

The remainder of this proof will be concerned with showing that $u(t; \lambda)$ varies smoothly with respect to λ for all $\lambda \in \mathrm{int}(\Lambda)$ that are away from possible equilibrium points. Rescale the u-variable as $s = \frac{u - u_1}{u_2 - u_1}$ and set $y(s) = z(u)$. From the z-equation we obtain the following equation for y:

$$y'' = \tilde{g}(s, y; \lambda),\ y(0) = y(1) = 0,\ y > 0 \text{ on } (0, 1).$$

[29] In z-variables we have $J_E = \int_{u_1}^{u_2}\Big(\frac{2}{9}z'^2 + \frac{K(u, z^{2/3}) + E}{z^{2/3}}\Big)\,du$.

[30] It follows from $g \le 0$ and the analysis in Appendix A that equilibrium points (that are non-degenerate by assumption) can only be of saddle-focus or center type.

[31] Away from equilibrium points this is obvious. At equilibrium points this follows either by taking limits and using the uniqueness or from the local analysis performed in [26, Lemma 5.8].

Moreover $\tilde{g} \leq 0$ and $\frac{\partial \tilde{g}}{\partial y} \geq 0$, and we can write $\tilde{g}(s,u;\lambda) = \frac{h(s,y;\lambda)}{y^{5/3}}$ with $h(s,y;\lambda)$ a continuous function.

In order to obtain smooth dependence on the parameter λ we first consider the following equation: $y_\epsilon'' = \tilde{g}(s,y_\epsilon;\lambda)$, $y_\epsilon(0) = y_\epsilon(1) = \epsilon$ and $y_\epsilon > \epsilon$ on $(0,1)$. It follows from the maximum principle that $0 < y_\epsilon - y_0 \leq \epsilon$. For the y_ϵ-problem it is not difficult to show that $y_\epsilon(\cdot;\lambda)$ depends smoothly on λ. To prove this we consider the map $F(y_\epsilon,\lambda) = y_\epsilon'' - g(s,y_\epsilon;\lambda)$, where F maps from $X_\epsilon \times \Lambda$ (with $X_\epsilon = \epsilon + H_0^1(0,1)$) to $H^{-1}(0,1)$, and $F \in C^1(X_\epsilon \times \Lambda, H^{-1})$. From the Implicit Function Theorem we derive that

$$\frac{d}{d\lambda} y_\epsilon(\cdot;\lambda) = -\big(F_y(y_\epsilon,\lambda)\big)^{-1} F_\lambda(y_\epsilon,\lambda) \in C(\Lambda, X_\epsilon).$$

Our goal now is to derive a similar expression for $\frac{d}{d\lambda} y_0(\cdot;\lambda)$. We cannot apply the Implicit Function Theorem to y_0 directly because of the singularity of $\tilde{g}$ at $y=0$.

We define $\Phi_\epsilon(\lambda) \stackrel{\text{def}}{=} F_y(y_\epsilon(\cdot,\lambda),\lambda) = \frac{d^2}{ds^2} - \frac{\partial g}{\partial y}(s,y_\epsilon;\lambda) = \frac{d^2}{ds^2} - \frac{k(s,y_\epsilon;\lambda)}{y_\epsilon^{8/3}}$, where k is a continuous function. From the asymptotic behavior of y_0 at $s = 0,1$ (i.e. $y_0(s) = O(s^{3/4})$ as $s \downarrow 0$ and $y_0(s) = O((1-s)^{3/4})$ as $s \uparrow 1$) we conclude from Hardy's inequality that $\Phi_0(\lambda) \in B(H_0^1, H^{-1})$ for all $\lambda \in \Lambda$.

It holds that $\Phi_\epsilon(\lambda) \longrightarrow \Phi_0(\lambda)$ in $B(H_0^1,H^{-1})$ as $\epsilon \to 0$ and the same holds for the inverses in $B(H^{-1},H_0^1)$. Namely, we obtain that (writing $k_\epsilon = k(\cdot,y_\epsilon;\lambda)$)

$$\|\Phi_\epsilon(\lambda) - \Phi_0(\lambda)\| \leq C\|k_\epsilon\Big(\frac{y_0}{y_\epsilon}\Big)^{\frac{8}{3}} - k_0\|_{L^\infty}.$$

From the L^∞-convergence of y_ϵ to y_0 we then conclude that $\Phi_\epsilon(\lambda) \to \Phi_0(\lambda)$ as $\epsilon \to 0$. In order to obtain the above inequality we again used Hardy's inequality in combination with the asymptotic behavior of y_0 at $s = 0, 1$[32].

We now assert that $F_\lambda(y_\epsilon,\lambda) \longrightarrow F_\lambda(y_0,\lambda)$ in H^{-1} as $\epsilon \to 0$. We find that

$$\|F_\lambda(y_\epsilon,\lambda) - F_\lambda(y_0,\lambda)\|_{H^{-1}} \leq C\|\frac{\partial h_\epsilon}{\partial\lambda}\Big(\frac{y}{y_\epsilon}\Big)^{\frac{5}{3}} - \frac{\partial h_0}{\partial \lambda}\|_{L^\infty}.$$

As before, due to the L^∞-convergence of y_ϵ to y_0 the assertion follows.

We conclude that $\frac{d}{d\lambda} y_\epsilon(\cdot;\lambda)$ converges to $\big(F_y(y_0,\lambda)\big)^{-1} F_\lambda(y_0,\lambda) \stackrel{\text{def}}{=} \zeta_\lambda$. The next step is to consider the difference quotient $D_h y(\cdot;\lambda) = \frac{y(\cdot;\lambda+h) - y(\cdot;\lambda)}{h}$. We have that $D_h y_\epsilon \to D_h y_0$ in L^∞ as $\epsilon \to 0$, and $D_h y_\epsilon \to \frac{d}{d\lambda} y_\epsilon(\cdot;\lambda)$ as $h \to 0$ for $\epsilon > 0$. Combining these facts we obtain $\|\zeta_\lambda - D_h y_0\|_{L^\infty} \leq \|\zeta_\lambda - \frac{d}{d\lambda} y_\epsilon(\cdot;\lambda)\|_{L^\infty} + \|\frac{d}{d\lambda} y_\epsilon(\cdot;\lambda) - D_h y_\epsilon\|_{L^\infty} + \|D_h y_\epsilon - D_h y_0\|_{L^\infty} \to 0$ as $\epsilon, h \to 0$. This gives

$$\frac{d}{d\lambda} y_0(\cdot;\lambda) = -\big(F_y(y_0,\lambda)\big)^{-1} F_\lambda(y_0,\lambda) \in H_0^1(0,1),\ \forall \lambda \in \mathrm{int}(\Lambda).$$

It remains to be proved that $\frac{d}{d\lambda} y(\cdot;\lambda)$ depends continuously on λ.

It follows from an estimate similar to the ones above that $\frac{d}{d\lambda} y_0(\cdot;\lambda)$ depends continuously on λ for all $\lambda \in \mathrm{int}(\Lambda)$ that are away from equilibrium points. It follows from the differential equation that $y_0'(s;\lambda)$ and $y_0''(s;\lambda)$ are C^1-functions of λ for all $s \in (0,1)$, i.e. $y_0(\cdot;\lambda)$ is continuously differentiable as a C^2-function on any compact subset of $(0,1)$. This implies that $u(\cdot;\lambda)$ is continuously differentiable as a C^3-function (at least away from its extrema). Finally, a simple application of

[32] If k has a zero at $s=0$ or $s=1$ the asymptotic behavior of y_0 will be different (i.e. $y_0 = O(s)$ near $s=0$). In this case a slightly different inequality holds which proves the same statement.

the Implicit Function Theorem then shows that $\tau(\lambda)$ is continuously differentiable for all $\lambda \in \mathrm{int}(\Lambda)$ that are away from equilibrium points. ■