Hodge decomposition and conservation laws

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Abstract

We give a relatively simple algorithm to do Hodge decomposition on the variational bicomplex based on the imbedding of the total differentiation operator in a Heisenberg algebra. The method is illustrated with several examples taken from the literature. It has been implemented by the authors in the computer algebra system Form [Ver91].

1 Introduction

In the calculus of variations, and especially in the inverse problem, i.e. the determination of the Lagrangian from the differential equations, the so called variational bicomplex plays a prominent role. (Cf [AP95] for recent developments). In this paper we give a method to do Hodge decompostion in this bicomplex in a way that is both simple and algorithmic, based on the use of the Heisenberg algebra. This is based on the following observation, made in [SR94]. Let

$$D = D_x = \partial_x + \sum_{i=1}^{m} \sum_{j=0}^{\infty} u_{j+1}^{i} \partial_{u_j}$$

be the total differentiation operator in one variable. Then one can find operators

$$E = \sum_{i=1}^{m} \sum_{j=0}^{\infty} u_j^i \partial_{u_j^i}.$$

and

$$F = \sum_{i=1}^{m} \sum_{j=0}^{\infty} j u_{j-1}^{i} \partial_{u_j^i},$$

such that

$$[E, D] = 0, \qquad [E, F] = 0, \qquad [F, D] = E,$$

where the brackets are the ordinary commutators of operators. These are then used to formulate an algorithm to compute a splitting of the space of functions into $\mathrm{Ker}\, E \oplus \mathrm{Im}\, E$ and $\mathrm{Im}\, E = kerF \oplus \mathrm{Im}\, D$. This splitting is in the case of one dependent and one independent variable equivalent to partial integration, and algorithmically less efficient. However, the same algorithm holds when there are more dependent and independent variables and this makes it interesting. What makes the algorithm terminate is the fact that F is locally nilpotent on polynomials, i.e. given a polynomial p in a number of variables and their derivatives, there is always a k such that $F^k p = 0$. This will prevent things from working in general, but one can slightly remove the requirement of polynomial functions, by admitting general C^∞ functions with arguments in $KerF$ as their coefficient ring, since they do not hurt the nilpotency.

Probably the same things could be done using partial integration, once one understands how it is done, but it is not clear how to do this in an elegant way. Of course, this would be interesting, since it could increase the efficiency of the whole approach. To see what is at stake here, consider the fifth derivative u_5. Clearly $u_5 = Du_4$ and any algorithm ought to see this immediately. Not so for the Heisenberg algebra approach. Here one first has to compute F on u_5 repeatedly, obtaining $5u_4, 20u_3, 60u_2, 120u_1, 120u, 0$. Then one has to go back with D, scaling all the while in order to finally get the same result. So a practical implementation would be to first use partial integration, and then use the Heisenberg approach on the remaining element in the complement of $\mathrm{Im}\, D$ (in this case 0). The problem with partial integration can be seen from an expression like $u_x u_{yy}$. What to do here? The Heisenberg algorithm gives $u_x u_{yy} = \frac{1}{2}(u_x u_{yy} - u u_{xyy}) + \frac{1}{2} D_x(u u_{yy})$, where the first expression is in $KerF_x$. So this at least does something well defined and useful when one tries to do Hodge decomposition as shown in this paper.

2 The basic algorithm

First we formulate a slight generalization of Algorithm 1 and 2 in [SR94], to which paper we refer for the exact formulation and the proofs. Instead of polynomials in the dependent variables and their derivatives with constant coefficients, we now allow the coefficients to be C^∞-functions on Ker F. Clearly this does not affect the nilpotency of F, which is the main ingredient in the algorithm. The operator μ^E is defined by $\mu^E = \sum_\sigma \mu^\sigma \pi_\sigma$, where the σ's are eigenvalues of E and the π_σ projections on the eigenspaces. In our context this means that the dependent variables are scaled with μ. Let $\mathcal{T}(\mathbb{R})$ be a tensor space of the space of variables u_i, $i = 0, \cdots, \infty$ and coefficients in $C^\infty(U, \mathbb{R})$. The algorithm applies to elements like

$$c(x) exp(uw^2) w_x u_{xx}^3 \, du \wedge du_{xxx} \wedge dw_{xx}$$

with $c \in C^\infty(U, \mathbb{R})$, $exp(uw^2) \in C^\infty(\text{Ker } F)$ and $w_x u_{xx}^3 \, du \wedge du_{xxx} \wedge dw_{xx} \in \mathcal{T}(\mathbb{R})$.

Algorithm 1 *Let* $\alpha \in \mathrm{Im}\, E|_{C^\infty(\text{Ker } F) \otimes \mathcal{T}(\mathbb{R})}$.

1. *Put* $\alpha^{(0)} = \alpha$*, and let* $m = 0$.
2. **while** $\alpha^{(m)} \neq 0$ **do**
   (a) *Put* $\alpha^{(m+1)} = F\alpha^{(m)}$.
   (b) *Let* $m = m + 1$.

   od
3. *Put* $\beta^{(m)} = 0$.
4. **while** $m \neq 0$ **do**
   (a) *Put* $\gamma^{(m-1)} = \int\int \beta^{(m)} \mu^{-1} d\mu d\lambda$.
   (b) *Put* $\beta^{(m-1)} = D\gamma^{(m-1)} + \mu^E(\alpha^{(m-1)} - D\gamma^{(m-1)} \mid_{\lambda=1,\mu=1})$.
   (c) *Let* $m = m - 1$.

   od
5. *Put* $\int \alpha = \beta^{(0)} \mid_{\lambda=0,\mu=1}$ *and* $\partial\alpha = \gamma^{(0)} \mid_{\lambda=1,\mu=1}$.

Then $\alpha = \int \alpha + D\partial\alpha \in \operatorname{Ker} F \oplus \operatorname{Im} D$.

Example 1 *We compute an expression to be used later on in section 4. In this computation we have two independent variables* x *and* y*. We adapt our notation to this case by employing a subindex* (x) *or* (y) *attached to the functions* α, β *and* γ*. Let* $\alpha_{(x)} = u_x u_{yy}$*. Then* $\alpha^{(0)}_{(x)} = u_x u_{yy}$*. We have* $\alpha^{(1)}_{(x)} = F_x \alpha^{(0)}_{(x)} = u u_{yy} \in \operatorname{Ker} F_x$*, i.e.* $\alpha^{(2)}_{(x)} = 0$*. Then* $\beta^{(2)}_{(x)} = 0$ *and* $\gamma^{(1)}_{(x)} = 0$*. Then*

$$
\begin{aligned}
\beta^{(1)}_{(x)} &= D_x\gamma^{(1)}_{(x)} + \mu^E(\alpha^{(1)}_{(x)} - D_x\gamma^{(1)}_{(x)} \mid_{\lambda=1,\mu=1}) \\
&= \mu^E u u_{yy} = \mu^2 u u_{yy} \\
\gamma^{(0)}_{(x)} &= \int\int \mu u u_{yy} d\mu d\lambda \\
&= \frac{1}{2}\mu^2\lambda u u_{yy} \\
\beta^{(0)}_{(x)} &= D_x\gamma^{(0)}_{(x)} + \mu^E(\alpha^{(0)}_{(x)} - D_x\gamma^{(0)}_{(x)} \mid_{\lambda=1,\mu=1}) \\
&= \frac{1}{2}\mu^2\lambda D_x(u u_{yy}) + \mu^E\left(u_x u_{yy} - \frac{1}{2}\mu^2\lambda D_x(u u_{yy}) \mid_{\lambda=1,\mu=1}\right) \\
&= \frac{1}{2}\mu^2\lambda\left(u_x u_{yy} + u u_{xyy}\right) + \frac{1}{2}\mu^2\left(u_x u_{yy} - u u_{xyy}\right)
\end{aligned}
$$

Therefore $\int_{(x)} \alpha_{(x)} = \frac{1}{2}\left(u_x u_{yy} - u u_{xyy}\right)$ *and* $\partial_{(x)}\alpha_{(x)} = \frac{1}{2} u u_{yy}$*. The compuation in the* y *variable is completely analogous, but we give it here anyway. Let* $\alpha^{(0)}_{(y)} =$

$u_x u_{yy}$. *We have* $\alpha^{(1)}_{(y)} = F_y \alpha^{(0)}_{(y)} = 2u_x u_y$ *and* $\alpha^{(2)}_{(y)} = F_y \alpha^{(1)}_{(y)} = 2uu_x$. *Clearly* $\alpha^{(3)}_{(y)} = 0$.

Then $\beta^{(3)}_{(y)} = 0$ *and* $\gamma^{(2)}_{(y)} = 0$. *Then*

$$\begin{aligned}
\beta^{(2)}_{(y)} &= D_y \gamma^{(2)}_{(y)} + \mu^E (\alpha^{(2)}_{(y)} - D_y \gamma^{(2)}_{(y)} \mid_{\lambda=1,\mu=1}) \\
&= \mu^E 2uu_x = 2\mu^2 uu_x \\
\gamma^{(1)}_{(y)} &= 2 \int\int \mu uu_x d\mu d\lambda \\
&= \mu^2 \lambda uu_x \\
\beta^{(1)}_{(y)} &= D_y \gamma^{(1)}_{(y)} + \mu^E (\alpha^{(1)}_{(y)} - D_y \gamma^{(1)}_{(y)} \mid_{\lambda=1,\mu=1}) \\
&= \mu^2 \lambda D_y(uu_x) + \mu^E \left(2u_x u_y - \mu^2 \lambda D_y uu_x \mid_{\lambda=1,\mu=1} \right) \\
&= \mu^2 \lambda \left(u_y u_x + uu_{xy} \right) + \mu^2 \left(u_x u_y - uu_{xy} \right) \\
\gamma^{(0)}_{(y)} &= \int\int (\mu\lambda \left(u_y u_x + uu_{xy} \right) + \mu \left(u_x u_y - uu_{xy} \right)) d\mu d\lambda \\
&= \frac{1}{4} \mu^2 \lambda^2 \left(u_y u_x + uu_{xy} \right) + \frac{1}{2} \mu^2 \lambda \left(u_x u_y - uu_{xy} \right) \\
\beta^{(0)}_{(y)} &= D_y \gamma^{(0)}_{(y)} + \mu^E (\alpha^{(0)}_{(y)} - D_y \gamma^{(0)}_{(y)} \mid_{\lambda=1,\mu=1}) \\
&= \frac{1}{4} \mu^2 \lambda^2 \left(u_{yy} u_x + 2u_y u_{xy} + uu_{xyy} \right) + \frac{1}{2} \mu^2 \lambda \left(u_x u_{yy} - uu_{xyy} \right) \\
&\quad + \frac{1}{4} \mu^2 \left(u_x u_{yy} - 2u_y u_{xy} + uu_{xyy} \right)
\end{aligned}$$

Therefore $\int_{(y)} \alpha_{(y)} = \frac{1}{4} \left(u_x u_{yy} - 2u_y u_{xy} + uu_{xyy} \right)$ *and* $\partial_{(y)} \alpha_{(y)} = \frac{3}{4} u_x u_y - \frac{1}{4} uu_{xy}$.

Example 2 *Let* $v = Q(x;u)(f(x)u_{xx} + g(x)u_x^2)$. *Then*

$$\begin{aligned}
\int v &= \frac{1}{u}(uu_{xx} - u_x^2)(Q(x;u)f(x) - \int Q(x;\mu u)(f(x) + \mu g(x)u) \, d\mu \mid_{\mu=1}) \\
&\quad - u \int log(\mu)((Q(x;\mu u)f(x))_{xx} + \mu(Q(x;\mu u)g(x))_{xx} u) \, d\mu \mid_{\mu=1}
\end{aligned}$$

and

$$\begin{aligned}
\partial v &= u_x \int Q(x;\mu u)(f(x) + \mu g(x)u) \, d\mu \mid_{\mu=1} \\
&\quad + u \int log(\mu)((Q(x;\mu u)f(x))_x + \mu(Q(x;\mu u)g(x))_x u) \, d\mu \mid_{\mu=1} .
\end{aligned}$$

3 The variational complex

The results presented here can be found in [Dic91]. There they involve the construction of the 'Tulczyjev's operator', denoted by D, which is "not very simple". Cf. also [Dor93]. We give a simple algorithm which has been implemented in the *FORM* [Ver91] language. The construction depends only on the unique decomposition of $V = C_k \oplus \operatorname{Im} D_k$. Any procedure which gives a unique splitting will do, the algorithm does not depend on the choice of the complement C_k to the image of D_k. So one can also carry this out using partial integration and some ordering on the dependent variables and their derivatives. Let $U \subset \mathbb{R}^n$. We consider differential forms of the type

$$\omega = \sum_I f_I dx_I$$

where $f_I \in \mathcal{T}(\mathbb{R}^n)$, and $\mathcal{T}(\mathbb{R}^n)$ is a tensor space of the space of variables u^i_J, with $J = \{j_1, \cdots, j_n\}$ and coefficients in $C^\infty(U, \mathbb{R})$. The total differential operators D_k are defined as shifts on the variables: $D_k u^i_J = u^i_{J+1_k}$. Here $J \pm 1_k$ is short for $\{j_1, \cdots, j_k \pm 1, \cdots, j_n\}$. In [SR94] it is shown how expressions in $f \in \mathcal{T}(\mathbb{R}^n)$ can be uniquely decomposed in $f = f^0_k + D_k f^1_k$ with $f^0_k \in \operatorname{Ker} F_k$, where $F_k u^i_J = j_k u^i_{J-1_k}$ (Cf Algorithm 1). We denote in the sequel $f^0_k = \int_k f$ and $f^1_k = \partial_k f$. We see that $[\partial_k, D_k] = \int_k$.

We define (as usual) an operator $d_h : \bigwedge^p \longrightarrow \bigwedge^{p+1}$ by

$$d_h\omega = \sum_{k \in \complement I} D_k f_I dx_k dx_I,$$

where $\complement I$ is the complement of I. We now dualize by defining $\delta_h : \bigwedge^p \longrightarrow \bigwedge^{p-1}$ by

$$\delta_h \omega = \sum_{k \in I} (-1)^{\#k+1} \partial_k f_I dx_{I-1_k}.$$

Notice that $\delta_h^2 = 0$. Here $\#k$ denotes the position of k in I, to take account of the antisymmetry of the differential forms. For each f_I denote by Γ_{f_I} the set of all indices $i \in I$ such that $\partial_i f_I \neq 0$, and let $\#(f_I)$ denote $\#\Gamma_{f_I}$, i.e. the cardinality of Γ_{f_I}. E.g. take $\omega = uu_y dx \wedge dy$. Then $I = \{1, 2\}$, $\complement I = \emptyset$, $f_{\{1,2\}} = uu_y$ and $\Gamma_{f_{\{1,2\}}} = \{2\}, \#(f_{\{1,2\}}) = 1$.

We now compute $\Delta_h = \delta_h d_h + d_h \delta_h : \bigwedge^p \longrightarrow \bigwedge^p$.

$$\begin{aligned}
&(\delta_h d_h + d_h \delta_h) f_I dx_I = \\
&= \delta_h \sum_{k \in \complement I} D_k f_I dx_k dx_I + d_h \sum_{m \in I} (-1)^{\#m+1} \partial_m f_I dx_{I-1_m} \\
&= \sum_{m \in I} (-1)^{\#m} \partial_m \sum_{k \in \complement I} D_k f_I dx_k dx_{I-1_m} + \sum_{k \in \complement I} \partial_k D_k f_I dx_I \\
&\quad + \sum_{k \in \complement I} D_k \sum_{m \in I} (-1)^{\#m+1} \partial_m f_I dx_k dx_{I-1_m} + \sum_{k \in I} D_k \partial_k f_I dx_I \\
&= \sum_{k \in \complement I} \partial_k D_k f_I dx_I + \sum_{k \in I} D_k \partial_k f_I dx_I \\
&= \sum_{k \in \complement I} f_I dx_I + \sum_{k \in I} f_I dx_I - \sum_{k \in I} \int_k f_I dx_I
\end{aligned}$$

$$= \quad (n-p+\#(f_I))f_I dx_I - \sum_{k\in\Gamma_(f_I)} \int_k f_I dx_I$$

Since $\#(\int_k f_I) < \#(f_I)$, this allows us to express ω in terms of its components in Im δ_h and Im d_h plus terms with smaller $\#$-value. Applying $\delta_h d_h + d_h \delta_h$ to these terms will eventually lead to a decomposition of ω over Im δ_h and Im d_h, except in the case $|I| = n$ (i.e. $n-p=0$), where there are also *harmonic* terms, i.e. terms in Ker $\delta_h \bigcap$ Ker d_h. We remark that Ker $\Delta_h =$ Ker $\delta_h \bigcap$ Ker d_h, since both $(n-p+\#(f_I))f_I dx_I$ and $\sum_{k\in\Gamma_(f_I)} \int_k f_I dx_I$ have to vanish (since they have different $\#$-values). Thus one has $n-p=0$, implying $d_h\omega = 0$ and $\#(f_I) = 0$, implying $\delta_h = 0$.

Theorem 1 *Let $\omega \in \bigwedge^p$. Then for $p < n$, ω can be written as $\omega = d_h\alpha + \delta_h\beta$, while for $p = n$ we have $\omega = d_h\alpha + \gamma$, where $\alpha \in \bigwedge^{p-1}, \beta \in \bigwedge^{p+1}$ and $\gamma \in$* Ker $\Delta_h =$ Ker $d_h \bigcap$ Ker δ_h.

We now have to prove that this decomposition is unique. The same inductive reasoning leads us to the conclusion that the space of differential forms can be split into eigenspaces of Δ_h with positive integer eigenvalues. In particular we have a decomposition Ker $\Delta_h \oplus$ Im Δ_h. Since Im $d_h \bigcap \text{Im}\, \delta_h \subset$ Ker Δ_h, we have Im $\Delta_h =$ Im $d_h \oplus$ Im δ_h.

Theorem 2 $\bigwedge^p =$ Ker $\Delta_h \oplus$ Im Δ_h *and* Im $\Delta_h =$ Im $d_h \oplus$ Im δ_h.

Algorithm 2 *Let $\omega \in \bigwedge^p$ and define $\omega^{(i)} = \Delta_h^i \omega$ for $i = 0, \cdots, p$. First suppose $p < n$. Let*

$$f^{(p)}(x) = \prod_{i=n-p}^{n} (x-i) = \sum_{i=0}^{p+1} f_i^{(p)} x^i.$$

Then

$$0 = \prod_{i=n-p}^{n} (\Delta_h - i)\omega = \sum_{i=0}^{p+1} f_i^{(p)} \Delta^i \omega.$$

It follows that

$$\omega = -\Delta_h \sum_{i=0}^{p} \frac{f_{i+1}^{(p)}}{f_0^{(p)}} \omega^{(i)}.$$

If $p = n$, then let

$$f^{(n)}(x) = \prod_{i=1}^{n}(x-i) = \sum_{i=0}^{n} f_i^{(n)} x^i.$$

Then

$$0 = \prod_{i=0}^{n}(\Delta_h - i)\omega = \Delta_h \sum_{i=0}^{n} f_i^{(n)} \Delta_h^i \omega.$$

In other words,

$$\omega + \Delta_h \sum_{i=0}^{n-1} \frac{f_{i+1}^{(n)}}{f_0^{(n)}} \omega^{(i)} \in \operatorname{Ker} \Delta_h,$$

gives the desired decomposition over Ker $\Delta_h \oplus$ Im Δ_h.

4 A horizontal example

In this section we consider the following example in two variables (see next section for $\bigwedge$ notation):

$$\omega = u_x u_{yy}\, dx \wedge dy \in \bigwedge_0^2 (\mathcal{T}(\mathbb{R}^2)).$$

Clearly, $d\omega = 0$. Using Example 1 we see that

$$\delta_h \omega = \frac{1}{2} u u_{yy} dy - \frac{1}{4}(3u_x u_y - u u_{xy}) dx.$$

It follows that

$$d_h \delta_h \omega = \frac{1}{4}(5u_x u_{yy} + u u_{xyy} + 2u_y u_{xy})\, dx \wedge dy.$$

We now let $\omega^{(1)} = 2\omega - \Delta_h \omega = \frac{1}{4}(3u_x u_{yy} - u u_{xyy} - 2u_y u_{xy})\, dx \wedge dy$. Define $f_{11}^{(1)} = \frac{1}{4}(3u_x u_{yy} - u u_{xyy} - 2u_y u_{xy})$. We now repeat the algorithm. We find that $F_x f_{11}^{(1)} = \frac{1}{2}(u u_{yy} - u_y^2) \in \operatorname{Ker} F_x$. Thus $\partial_1 f_{11}^{(1)} = \frac{1}{4}(u u_{yy} - u_y^2)$. On the other hand $F_y f_{11}^{(1)} = u_x u_y - u u_{xy} \in \operatorname{Ker} F_y$ and $\partial_2 f_{11}^{(1)} = \frac{1}{2}(u_x u_y - u u_{xy})$. We have

$$\delta_h \omega^{(1)} = \frac{1}{4}(u u_{yy} - u_y^2) dy - \frac{1}{2}(u_x u_y - u u_{xy}) dx.$$

It follows that

$$\Delta_h \omega^{(1)} = d_h \delta_h \omega^{(1)} = \frac{1}{4}(3u_x u_{yy} - u u_{xyy} - 2u_y u_{xy})\, dx \wedge dy.$$

We define $\omega^{(2)} = \omega^{(1)} - \Delta_h \omega^{(1)}$ and see that it is equal to zero. It follows that $\gamma = 0$ in Theorem 1. It follows that $\omega = \frac{1}{2}\Delta_h(\omega + \omega^{(1)})$. We find that $\frac{1}{2}(\omega + \omega^{(1)}) = \frac{1}{8}(7u_x u_{yy} - u u_{xyy} - 2u_y u_{xy})\, dx \wedge dy$. Using our earlier computations we see that

$$\alpha = \delta_h \frac{1}{2}(\omega + \omega^{(1)}) = \frac{1}{8}((3u u_{yy} - u_y^2)\, dy - (5u_x u_y - 3u u_{xy})\, dx).$$

We check that

$$\begin{aligned} d_h \alpha &= \frac{1}{8} d_h((3u u_{yy} - u_y^2)\, dy - (5u_x u_y - 3u u_{xy})\, dx) \\ &= \frac{1}{8}((3u_x u_{yy} - 2u_y u_{xy} + 3u u_{xyy})\, dx \wedge dy \\ &\quad +(5u_{xy} u_y - 3u_y u_{xy} + 5u_x u_{yy} - 3u u_{xyy})\, dx \wedge dy) \\ &= u_x u_{yy}\, dx \wedge dy = \omega, \end{aligned}$$

showing the equation $\omega = 0$ is a conservation law.

5 Vertical forms

We now consider the case $\mathcal{T}(\mathbb{R}^n) = \mathcal{Y}^{pol} \otimes (\bigwedge_{.} \mathcal{Y}_\infty)$, where $\mathcal{Y}^{pol}$ is the space of polynomials in the dependent variables and their derivatives, without the constant term (so $u \in \mathcal{Y}^{pol}, 1 \notin \mathcal{Y}^{pol}$).

Definition 1 *One defines an operator* $d_v : \mathcal{Y}^{pol} \otimes (\bigwedge_q \mathcal{Y}_\infty) \to \mathcal{Y}^{pol} \otimes (\bigwedge_{q+1} \mathcal{Y}_\infty)$ *by* $d_v \sum_I f_I du^I = \sum_I \sum_{k \in I} D_k f_I du_k du^I$.

Definition 2 *One defines an operator* $\delta_v : \mathcal{Y}^{pol} \otimes (\bigwedge_q \mathcal{Y}_\infty) \to \mathcal{Y}^{pol} \otimes (\bigwedge_{q-1} \mathcal{Y}_\infty)$ *by* $\delta_v \sum_I f_I du^I = \sum_I \sum_{k \in I} (-1)^{\#k} u_k f_I du^{I-1_k}$.

We can now define (co)homology theory much as in the ordinary de Rham case. We define $\Delta_v = d_v \delta_v + \delta_v d_v$. As in the ordinary case, we find for homogeneous expressions that $\Delta_v f[u] = E f[u]$, where $E = \sum_{k=0}^{\infty} u_k \frac{\partial}{\partial u_k}$. This makes the algorithm clear.

What remains to be shown is that the decomposition is unique, i.e.

$$\mathrm{Im}\ d_v \bigcap \mathrm{Im}\ \delta_v = 0.$$

Since $\mathcal{Y}^{pol} \otimes (\bigwedge_q \mathcal{Y}_\infty) \bigcap \mathrm{Ker}\ \Delta_v = 0$ and $\mathrm{Im}\ d_v \bigcap \mathrm{Im}\ \delta_v \subset \mathrm{Ker}\ \Delta_v$ this follows immediately.

We now give the diagram describing all the maps so far: We denote by $\bigwedge_p^q \mathcal{Y}_\infty$ the space of forms $du_{i_1} \wedge \cdots \wedge du_{i_p} \otimes dx_{j_1} \wedge \cdots \wedge dx_{j_q}$, i.e. $\bigwedge_p \otimes \bigwedge^q$.

$$\begin{array}{ccccc}
\mathcal{Y}^{pol} \otimes (\bigwedge_{p-1}^{q-1} \mathcal{Y}_\infty) & \underset{\delta_h}{\overset{d_h}{\rightleftarrows}} & \mathcal{Y}^{pol} \otimes (\bigwedge_{p-1}^{q} \mathcal{Y}_\infty) & \underset{\delta_h}{\overset{d_h}{\rightleftarrows}} & \mathcal{Y}^{pol} \otimes (\bigwedge_{p-1}^{q+1} \mathcal{Y}_\infty) \\
d_v \downarrow \uparrow \delta_v & & d_v \downarrow \uparrow \delta_v & & d_v \downarrow \uparrow \delta_v \\
\mathcal{Y}^{pol} \otimes (\bigwedge_{p}^{q-1} \mathcal{Y}_\infty) & \underset{\delta_h}{\overset{d_h}{\rightleftarrows}} & \mathcal{Y}^{pol} \otimes (\bigwedge_{p}^{q} \mathcal{Y}_\infty) & \underset{\delta_h}{\overset{d_h}{\rightleftarrows}} & \mathcal{Y}^{pol} \otimes (\bigwedge_{p}^{q+1} \mathcal{Y}_\infty) \\
d_v \downarrow \uparrow \delta_v & & d_v \downarrow \uparrow \delta_v & & d_v \downarrow \uparrow \delta_v \\
\mathcal{Y}^{pol} \otimes (\bigwedge_{p+1}^{q-1} \mathcal{Y}_\infty) & \underset{\delta_h}{\overset{d_h}{\rightleftarrows}} & \mathcal{Y}^{pol} \otimes (\bigwedge_{p+1}^{q} \mathcal{Y}_\infty) & \underset{\delta_h}{\overset{d_h}{\rightleftarrows}} & \mathcal{Y}^{pol} \otimes (\bigwedge_{p+1}^{q+1} \mathcal{Y}_\infty)
\end{array}$$

6 A vertical example

Take $\omega = u^2 u_x^2\, du \wedge du_{xxx} \in \mathcal{Y}^{pol} \otimes \bigwedge_2^0 \mathcal{Y}_\infty$. Then

$$d_v \omega = -2u^2 u_x\, du \wedge du_x \wedge du_{xxx}$$

and

$$\delta_v\omega = u^3u_x^2du_{xxx} - u^2u_x^2u_{xxx}du.$$

It follows that

$$\begin{aligned}\delta_v d_v\omega &= -2u^3u_x\,du_x\wedge du_{xxx} + 2u^2u_x^2\,du\wedge du_{xxx}\\ &\quad -2u^2u_xu_{xxx}\,du\wedge du_x\end{aligned}$$

and

$$\begin{aligned}d_v\delta_v\omega &= 2u^3u_x\,du_x\wedge du_{xxx} + 3u^2u_x^2\,du\wedge du_{xxx}\\ &\quad +2u^2u_xu_{xxx}\,du\wedge du_x + u^2u_x^2\,du\wedge du_{xxx}.\end{aligned}$$

Adding the two expressions results in

$$\begin{aligned}\Delta_v\omega &= 6u^2u_x^2\,du\wedge du_{xxx}\\ &= 6\omega\end{aligned}$$

This leads to

$$\omega = \frac{1}{6}(\delta_v(-2u^2u_x\,du\wedge du_x\wedge du_{xxx}) + d_v(u^3u_x^2\,du_{xxx} - u^2u_x^2u_{xxx}\,du)),$$

the desired Hodge decomposition.

7 A mixed example

Let $\omega = (u_{xxxx} + 2uu_{xx} + u_x^2)du\ dx \in \mathcal{Y}^{pol}\otimes\bigwedge_1^1\mathcal{Y}_\infty$ (Cf [Olv93] example 5.93). We do the vertical splitting first. Applying the algorithm to each term of ω respectively, we obtain $\omega = (d_v(\frac{1}{2}uu_{xxxx} + \frac{2}{3}u^2u_{xx} + \frac{1}{3}uu_x^2) + \delta_v(\frac{1}{2}\,du_{xxxx}\wedge du + \frac{2}{3}u\,du_{xx}\wedge du + \frac{2}{3}u_x\,du_x\wedge du))\ dx$. We can use partial integration assuming the underlying space has no boundary or the forms vanish on the boundary. So

$$\omega \equiv d_v(\frac{1}{2}uu_{xxxx} + \frac{2}{3}u^2u_{xx} + \frac{1}{3}uu_x^2)\,dx.$$

Then we do the horizontal splitting to the form

$$(\frac{1}{2}uu_{xxxx} + \frac{2}{3}u^2u_{xx} + \frac{1}{3}uu_x^2)\,dx.$$

This leads to

$$\omega \equiv d_v(\frac{1}{16}uu_{xxxx} + \frac{1}{3}u^2u_{xx} - \frac{1}{3}uu_x^2 - \frac{1}{4}u_xu_{xxx} + \frac{3}{16}u_{xx}^2)\,dx.$$

In fact, $d_v\omega = 0$ is the condition that ω is the Euler-Lagrange expression for some variational problem. We notice the Lagrangian is unique under the meaning of equivalence.

It is obvious that the vertical splitting and the horizontal splitting are commutative. So we obtain the same result if we do the horizontal splitting first. But we can easily determine whether $d_v\omega = 0$ in this way. For this example

$$\begin{aligned}\omega &\equiv (\frac{1}{16}u_{xxxx}du + \frac{1}{16}udu_{xxxx} - \frac{1}{4}u_{xxx}du_x - \frac{1}{4}u_x du_{xxx} + \frac{3}{8}u_{xx}du_{xx} \\ &\quad -\frac{1}{3}u_x^2 du + \frac{2}{3}uu_{xx}du - \frac{2}{3}uu_x du_x + \frac{1}{3}u^2 du_{xx})\ dx.\end{aligned}$$

and then we obtain the same result by doing the vertical splitting.

8 A 3-dimensional mixed example

In this section we consider the wave equation in three variables (Cf [Olv93], p.255):

$$\omega = (u_{tt} - u_{xx} - u_{yy})\, du \wedge dx \wedge dy\ dt \in \mathcal{Y}^{pol} \otimes \bigwedge_1^3 \mathcal{Y}_\infty.$$

We do the vertical splitting first. $\omega = (\frac{1}{2}d_v(uu_{tt} - uu_{xx} - uu_{yy}) + \frac{1}{2}\delta_v\,(du_{tt} \wedge du - du_{xx} \wedge du - du_{yy} \wedge du))\, dx \wedge dy \wedge dt$. Using partial integration assuming the underlying space has no boundary or the forms vanish on the boundary, we obtain:

$$\omega \equiv \frac{1}{2}d_v(uu_{tt} - uu_{xx} - uu_{yy})\, dx \wedge dy \wedge dt.$$

Then we do the horizontal splitting to the form:

$$\frac{1}{2}(uu_{tt} - uu_{xx} - uu_{yy})\, dx \wedge dy \wedge dt.$$

It follows that

$$\omega \equiv \frac{1}{4}d_v(uu_{tt} - u_t^2 - uu_{xx} + u_x^2 - uu_{yy} + u_y^2)\, dx \wedge dy \wedge dt.$$

The same as the Section 7, we can do the horizontal splitting first.

$$\begin{aligned}\omega &\equiv (\frac{1}{4}u_{tt}du - \frac{1}{2}u_t du_t + \frac{1}{4}udu_{tt} - \frac{1}{4}u_{xx}du + \frac{1}{2}u_x du_x \\ &\quad -\frac{1}{4}udu_{xx} - \frac{1}{4}u_{yy}du + \frac{1}{2}u_y du_y - \frac{1}{4}udu_{yy})\, dx \wedge dy \wedge dt.\end{aligned}$$

9 Inverse problem: a simple example

We consider the equation (taken from [Sau89])

$$\ddot{q} = \dot{q}^2.$$

This leads to a differential form $\omega = (\dot{q}^2 - \ddot{q})\, dq \wedge dt \in \mathcal{Y}^{pol} \otimes \bigwedge_1^1 \mathcal{Y}_\infty$. First we compute $\int \omega$, i.e. the projection on Ker F of ω. We find that

$$\begin{aligned} &\int (\dot{q}^2 - \ddot{q})\, dq \wedge dt = \\ &= \frac{5}{9}(\dot{q}^2 - q\ddot{q})\, dq \wedge dt + (\frac{1}{9}q - \frac{1}{4})(\ddot{q}\, dq - 2\dot{q}\, d\dot{q} + q\, d\ddot{q}) \wedge dt. \end{aligned}$$

We find that $d_v \int \omega \neq 0$. Therefore we multiply ω with $f \in C^\infty(\text{Ker } F)$. To make things simple, we first try $f = f(q)$.

Indeed, $d_v f(q) \otimes (\dot{q}^2 - \ddot{q})\, dq \wedge dt = 2f(q) \otimes \dot{q}\, d\dot{q} \wedge dq \wedge dt - f(q) \otimes d\ddot{q} \wedge dq \wedge dt \equiv (2f(q) + \frac{\partial f(q)}{\partial q}) \otimes \dot{q}\, d\dot{q} \wedge dq \wedge dt$. We take $f(q) = e^{-2q}$ to make ω exact. Now $e^{-2q} \otimes (\dot{q}^2 - \ddot{q})\, dq \wedge dt \equiv e^{-2q} \otimes \dot{q}\,(d\dot{q} - \dot{q} dq) \wedge dt$ and $L = \delta_v e^{-2q} \otimes \dot{q}\,(d\dot{q} - \dot{q} dq) \wedge dt = \frac{1}{2} e^{-2q} \otimes \dot{q}^2\, dt$. Notice that the computation thus far relies on making the right choice of representative of the functional for ω. We now compute $\int L$ using Algorithm 1. This results in

$$\int L = e^{-2q} \frac{1 + 2q}{8q^2} \otimes (q\ddot{q} - \dot{q}^2).$$

Notice that this result is simpler than the one given in [Sau89]. The only reason seems to be that there the μ-integration in the homotopy formula is from 0 to 1, while in Algorithm 1 we substitute $\mu = 1$ after computing the primitive function. Furthermore,

$$d_v \int L = -e^{-2q} \frac{1 + 2q + 2q^2}{4q^3} \otimes (q\ddot{q} - \dot{q}^2) dq + e^{-2q} \frac{1 + 2q}{8q^2} \otimes (\ddot{q} dq - 2\dot{q} d\dot{q} + q d\ddot{q}).$$

10 The Khoklov-Zabolotskaya equation

The Khoklov-Zabolotskaya equation (Cf [Sha89]) is given by

$$0 = \frac{1}{2} \frac{\partial^2 (u^2)}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2}, \quad x_1, x_2 > 0$$

and describes the propagation of sound in nonlinear media without dispersion and absorption. We consider the form (with $h^{(1)} = h^{(1)}(x_1, x_2, x_3, x_4)$)

$$\omega = h^{(1)} (\frac{1}{2} \frac{\partial^2 (u^2)}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2})\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \in \mathcal{Y}^{pol} \otimes \bigwedge_0^4 \mathcal{Y}_\infty$$

and we see, using Algorithm 2, that in order for the harmonic component of ω to vanish we have to require that

$$\frac{\partial^2 h^{(1)}}{\partial x_1^2} = 0 \qquad (1)$$

$$\frac{\partial^2 h^{(1)}}{\partial x_1 \partial x_2} = \frac{\partial^2 h^{(1)}}{\partial x_3^2} + \frac{\partial^2 h^{(1)}}{\partial x_4^2} \qquad (2)$$

The conservation law is (Cf. [Sha89], p. 92)

$$\begin{aligned}\alpha &= (-\frac{1}{2}h^{(1)}u_{x_2} + \frac{1}{2}\frac{\partial h^{(1)}}{\partial x_2}u + h^{(1)}uu_{x_1} - \frac{1}{2}\frac{\partial h^{(1)}}{\partial x_1}u^2)\, dx_2 \wedge dx_3 \wedge dx_4 \\ &+(-h^{(1)}u_{x_4} + \frac{\partial h^{(1)}}{\partial x_4}u)\, dx_1 \wedge dx_2 \wedge dx_3 \\ &+(h^{(1)}u_{x_3} - \frac{\partial h^{(1)}}{\partial x_3}u)\, dx_1 \wedge dx_2 \wedge dx_4 \\ &+\frac{1}{2}(h^{(1)}u_{x_1} - \frac{\partial h^{(1)}}{\partial x_1}u)\, dx_1 \wedge dx_3 \wedge dx_4,\end{aligned}$$

i.e. $\omega = d_h\alpha$ under (1) and (2). Clearly the equation itself is a conservation law.

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