Journal of Difference Equations and Applications
Vol. 00, No. 00, June 2006, 1–16
Journal of Difference Equations and Applications ISSN 1023-6198 print/ ISSN 1563-5120 online

Journal of Difference Equations and Applications
Vol. 00, No. 00, June 2006, 2–16

Integrable systems in n-dimensional conformal geometry

Jan A. Sanders
Faculty of Sciences
Vrije Universiteit
1081 HV Amsterdam
The Netherlands
&
Jing Ping Wang
IMSAS
University of Kent
Canterbury CT2 7NF
UK

()

In this paper we show that if we write down the structure equations for the flow of the *parallel* frame of a curve embedded in a flat n-dimensional conformal manifold, this leads to two compatible Hamiltonian operators. The corresponding integrable scalar-vector equation is

$$D_t\psi = \psi_{xxx} + \frac{3}{2}(\psi^2 - (\Psi, \Psi))_x$$
$$D_t\Psi = \Psi_{xxx} + 3(\psi\Psi)_x,$$

where (Ψ, Ψ) is the standard Euclidean inner product of the vector Ψ with itself. These results are similar to those we obtained in the Riemannian case, implying that the method employed is well suited for the analysis of the connection between geometry and integrability.

1 Introduction

We study the connnection between the motion of a curve and the theory of integrable equations.

Recently we showed in [1] that if one writes down the structure equations for the evolution of the connection of a curve embedded in an n-dimensional Riemannian manifold with constant curvature, this leads to a symplectic, a

The authors acknowledge the support of the London Mathematical Society (Ref No. 2442)

ISSN 1563-5120 print; ISSN 1023-6198 online © 2006 Taylor & Francis Ltd.
DOI: 10.1080/10236190xxxxxxxxxxxxx

Hamiltonian and a hereditary operator. This gives us a natural connection among finite dimensional geometry, infinite dimensional geometry and integrable systems. These results were first obtained by using the *natural* or *parallel* frame and then we show how this can be gauged by a generalized Hasimoto transformation [2] to the (usual) *Frenêt* frame. It is interesting to notice that if one chooses the curvature of the manifold to be zero, as usual in the context of integrable systems, one loses information unless one works in the natural frame.

Poisson geometry is very important in the study of integrable systems, cf. [3, 4]. Once one has two compatible Hamiltonian operators, one can construct a hereditary operator, from which a hierarchy of integrable equations can be computed. They are all Hamiltonian with respect to these two compatible Hamiltonian operators, that is, biHamiltonian as defined in [6]. For the history of this type of problems we refer to [1], where we listed out some of the references.

The goal of the present paper is to generalize this analysis to conformal geometry. Although the Riemannian and conformal geometries are quite similar, it turns out that the more complicated looking conformal geometry has local Hamiltonian operators, which makes the proof of their compatibility rather trivial compared to the analogues proof in the Riemannian case. The whole construction can be explicitly computed, but that the initial choice of gauge (or moving frame) is crucial. One can of course remark that the results could have been obtained in any gauge equivalent frame. This is true, but if one were to carry out this programme in one's favourite frame, one would quickly see that the proofs that the operators found are indeed compatible Hamiltonian operators are quite complicated, the degree of difficulty depending very much on the choice of gauge. We should stress that our approach is, in principle, immediately applicable to any Cartan geometry.

We avoid the usual approach with Poisson brackets and reduction, as in [7] since we want to emphisize that this formal approach leads to interesting mathematics, that is, the appearance of biHamiltonian structures straight out of the structure equations in different geometries.

The paper is organized as follows. In section 2 we make some historical remark about the concepts of connection and curvature. In section 3, we compute the structure equations in the conformal case. In section 4, we choose a connection and obtain a Hamiltonian pair, leading to the integrable evolution equations, where we also compare our equation with the classification result in [12]. Finally, in section 5, we comment on future developments.

Acknowledgment The authors thank Gloria Marí Beffa for numerous discussions, which both influenced the questions that we asked ourselves as well as some the methods we used. The definition of the Ψ at the end of section 4 was suggested by Sasha Mikhailov and simplifies the compatibility analysis of

the Hamiltonian operators tremendously.

2 Connections, some history

An important role in this paper is played by connections. What are connections? A good introduction to the subject is [8]. Here we give a rather condensed account of the long story behind connections. The first occurrence of a connection is in Riemann's inaugural lecture, where he in so many words sketched how to define curvature and how to choose coordinates to put the metric terms in normal form. The normal form in first order is defined by the vanishing of the linear terms in the Taylor expansion of the metric at some arbitrary point, and the transformation doing this is a near identity transformation with certain quadratic terms. The coefficients of these quadratic terms are constants with three indices, Γ^k_{ij}, the **Christoffel symbols**, and these constants could be used to define a structure on the constant vectorfields $\frac{\partial}{\partial x_i}$, as in

$$\mathsf{d}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Gamma^k_{ij} \frac{\partial}{\partial x_k}.$$

This structure defines covariant differentiation, and it is seen that if one uses this structure to define a new Lie bracket on the vectorfields, by putting $[X, Y] = \mathsf{d}_X Y - \mathsf{d}_Y X$, the metric is automatically flat to first order, that is, induced Christoffel symbols will be zero. It was then found that these coefficients, which still depend on the arbitrary point, transform not as a tensor, but as something more complicated, and it was readily seen that if one took these transformation rules as definition, then the following proof that the corresponding curvature terms, that is, the terms of quadratic and higher order after the normalizing transformation, transforma as a tensor, became much easier, although it was still a rather long calculation. One proves that the covariant derivative of a tensor is again a tensor.

It then became customary to call anything that transformed like the Christoffel symbols a **connection**, and the specific case where the connection is induced by a metric, a **Levi-Civita connection**. By defining a dual basis on the tangent space, one can also give a dual formulation of the connection as a one-form taking its values in a Lie algebra, that is, the tangent space of transformations at the identity which leave the geometric structure (metric, symplectic form, etc.) invariant. If one compares the computational effort of setting up a moving Frenêt frame and the structure equations just using the metric (as it is usually done when one derives the evolution equation for the generalized curvatures of a curve imbedded in a Riemannian manifold) with the Cartan formulation in terms of connection, the difference is striking: not only can one not see what one is doing, but there is no need to see it, since everything goes right by construction, and the entities one writes down are automatically differential invariants. However, if one sees this for the first time, it may pay off to compare these two methods, just to get a feeling for what exactly goes on. Then one will see that even if one cannot see what one is doing in the abstract approach, the originators knew exactly what they were doing since they knew how to do it with the classical methods and could just follow the analogy.

3 Moving Frame Method: Conformal Geometry

For a general description of the moving frame method we refer to [1]. The actual calculations will be performed in a model that is slightly different from the general framework given there, but corresponds to what is usually done in the literature for this specific geometry. For the formulation of the Cartan formulation of the Möbius model of conformal geometry we refer to [9, p. 272]. We just give the notation here. Let

$$\Sigma_{n+1,1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & I_n & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then define the Lorentz group $L_{n+1,1}(\mathbb{R})$ by

$$L_{n+1,1}(\mathbb{R}) = \{g \in GL(n+2,\mathbb{R}) | g^t \Sigma_{n+1,1} g = \Sigma_{n+1,1}\},$$

and the Möbius group $\text{Möb}(n,\mathbb{R}) = L_{n+1,1}(\mathbb{R})/\{\pm I\}$. Let

$$H = \{h \in \text{Möb}(n,\mathbb{R}) | h[e_0] = [e_0]\}.$$

Here $[e_0]$ denotes the class of e_0 in $P(\mathbb{R}^{n+2})$, the projective space of lines in $\mathbb{R}^{n+2}$, where $\mathbb{R}^{n+2}$ has the basis $e_0, \cdots, e_{n+1}$. The groups $\text{Möb}(n,\mathbb{R})$ and H have Lie algebras, which we denote by $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Their elements are of the form

$$\begin{pmatrix} -\epsilon & v^t & 0 \\ \theta & \xi & v \\ 0 & \theta^t & \epsilon \end{pmatrix} \in \mathfrak{g}, \quad \begin{pmatrix} -\epsilon & v^t & 0 \\ 0 & \xi & v \\ 0 & 0 & \epsilon \end{pmatrix} \in \mathfrak{h}, \quad \xi \in \mathfrak{o}_n, v, \theta \in \mathbb{R}^n, \epsilon \in \mathbb{R}.$$

We use exactly the same notation for one-forms taking their values in $\mathfrak{g}, \mathfrak{h}$, respectively and for their evaluation at the general element D_t, to be introduced next. If this sounds confusing, one should realize that the various identifications of one-forms and coefficients of frames are completely intuitive, and so it does not pay to be very pedantic about notation here.

Given a curve, we know its tangent vector D_x. We choose a point p in the manifold, through which the curve goes, and an open neighborhood $p \in U$. The whole question now is to determine possible evolutions of the tangent vector at p, which we denote by the evolutionary vectorfield D_t (So, by definition, at time $t = 0$, $D_t = D_x$ and $[D_x, D_t] = 0$). We write, with ξ a 1-form taking its values in $\mathfrak{o}_n$, θ and v 1-forms taking their values in $\mathbb{R}^n$, and ϵ an $\mathbb{R}$-valued one-form, the Cartan gauge matrix $\omega = \omega_U : T_p(U) \to \mathfrak{g}$, which allows us to differentiate the vectors in the moving frame of our choice, as

$$\omega(D_t) = \begin{pmatrix} -\epsilon & v^t & 0 \\ \theta & \xi & v \\ 0 & \theta^t & \epsilon \end{pmatrix}, \quad \omega(D_x) = \begin{pmatrix} -\bar{\epsilon} & \bar{v}^t & 0 \\ \bar{\theta} & \bar{\xi} & \bar{v} \\ 0 & \bar{\theta}^t & \bar{\epsilon} \end{pmatrix}, \tag{1}$$

and we make a specific choice for these components in the next section. With D_x given, the form $\omega(D_x)$ is determined by the the choice of frame, that is by the choice of a certain gauge transformation (that is, an element in H) which maps an arbitrary frame to the frame of choice. We assume that the curvature of the Cartan gauge is zero. In [9] it is described how one can obtain from this local construction a principal bundle with Cartan connection.

A moving frame is a differentiable choice of a basis of the tangent space and is denoted by e and the individual basisvectors by $\mathsf{e}_i, i = 1, \cdots, n$. It can therefore be considered as a group element, since its matrix is invertible. The Cartan gauge matrix ω is used to define differentation of the vectors in the moving frame, that is, the (Koszul) connection d, by putting

$$\mathsf{de} = \omega \mathsf{e}.$$

This defines a one-form which is evaluated as follows.

$$\mathsf{de}(D_t) = \omega(D_t)\mathsf{e}.$$

Acting on functions, the d is the ordinary de Rham d, and one has the Leibniz rule for the connection d:

$$\mathsf{d}f\mathsf{e} = df\mathsf{e} + f\mathsf{de}.$$

This rule is often used to define a connection. The structure equation satisfies

$$0 = \mathsf{d}\omega - \omega \wedge \omega$$

and this leads to

$$
\begin{aligned}
0 &= \mathsf{d}\omega(D_x, D_t) - \omega(D_x)\omega(D_t) + \omega(D_t)\omega(D_x) = \\
&= D_x\omega(D_t) - D_t\omega(D_x) - \omega([D_x, D_t]) - \omega(D_x)\omega(D_t) + \omega(D_t)\omega(D_x) = \\
&= D_x\omega(D_t) - D_t\omega(D_x) - \omega(D_x)\omega(D_t) + \omega(D_t)\omega(D_x).
\end{aligned}
$$

We write this out in coordinates as follows:

$$
\begin{aligned}
0 = D_x &\begin{pmatrix} -\epsilon & v^t & 0 \\ \theta & \xi & v \\ 0 & \theta^t & \epsilon \end{pmatrix} - D_t \begin{pmatrix} -\bar{\epsilon} & \bar{v}^t & 0 \\ \bar{\theta} & \bar{\xi} & \bar{v} \\ 0 & \bar{\theta}^t & \bar{\epsilon} \end{pmatrix} \\
- &\begin{pmatrix} -\bar{\epsilon} & \bar{v}^t & 0 \\ \bar{\theta} & \bar{\xi} & \bar{v} \\ 0 & \bar{\theta}^t & \bar{\epsilon} \end{pmatrix} \begin{pmatrix} -\epsilon & v^t & 0 \\ \theta & \xi & v \\ 0 & \theta^t & \epsilon \end{pmatrix} + \begin{pmatrix} -\epsilon & v^t & 0 \\ \theta & \xi & v \\ 0 & \theta^t & \epsilon \end{pmatrix} \begin{pmatrix} -\bar{\epsilon} & \bar{v}^t & 0 \\ \bar{\theta} & \bar{\xi} & \bar{v} \\ 0 & \bar{\theta}^t & \bar{\epsilon} \end{pmatrix} \\
= D_x &\begin{pmatrix} -\epsilon & v^t & 0 \\ \theta & \xi & v \\ 0 & \theta^t & \epsilon \end{pmatrix} - D_t \begin{pmatrix} -\bar{\epsilon} & \bar{v}^t & 0 \\ \bar{\theta} & \bar{\xi} & \bar{v} \\ 0 & \bar{\theta}^t & \bar{\epsilon} \end{pmatrix} \\
- &\begin{pmatrix} \bar{v}^t\theta - v^t\bar{\theta} & -\bar{\epsilon}v^t + \bar{v}^t\xi + \epsilon\bar{v}^t - v^t\bar{\xi} & 0 \\ \bar{\epsilon}\theta - \bar{\theta}\epsilon + \bar{\xi}\theta - \xi\bar{\theta} & \bar{\theta}v^t + \bar{v}\theta^t + \bar{\xi}\xi - \xi\bar{\xi} - \theta\bar{v}^t - v\bar{\theta}^t & \bar{\xi}v + \epsilon\bar{v} - \bar{\epsilon}v - \xi\bar{v} \\ 0 & \bar{\epsilon}\theta^t + \bar{\theta}^t\xi - \theta^t\bar{\xi} - \epsilon\bar{\theta}^t & \bar{\theta}^t v - \theta^t\bar{v} \end{pmatrix}
\end{aligned}
$$

This leads to the following equations.

$$0 = -D_x\epsilon + D_t\bar{\epsilon} - \bar{v}^t\theta + v^t\bar{\theta} \tag{2}$$

$$0 = D_x v - D_t\bar{v} + \bar{\epsilon}v - \bar{\xi}v - \epsilon\bar{v} + \xi\bar{v} \tag{3}$$

$$0 = D_x\theta - D_t\bar{\theta} - \bar{\epsilon}\theta - \bar{\xi}\theta + \bar{\theta}\epsilon + \xi\bar{\theta} \quad \text{(Torsion)} \tag{4}$$

$$0 = D_x\xi - D_t\bar{\xi} - \bar{\theta}v^t - \bar{v}\theta^t - \bar{\xi}\xi + \xi\bar{\xi} + \theta\bar{v}^t + v\bar{\theta}^t. \tag{5}$$

4 Hamiltonian Pair in Conformal Geometry

We now choose $1 \leq m \leq n$ and fix the gauge such that formula (1) reads

$$\omega(D_x) = \begin{pmatrix} 0 & \sigma & k_2 & \cdots & k_m & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & -u_{m+1} & \cdots & -u_n & \sigma \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & k_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & k_m \\ 0 & u_{m+1} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & u_n & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This choice in any geometry determines the n-dimensional vectorspace (with coordinates $\sigma, k_2, \cdots, k_m, u_{m+1}, \cdots, u_n$) on which we will be working and in which we try and find integrable evolution equations of the coordinates. With any two different choices, one can simply compare them by comparing the complications that arise in proving that the operators one finds are indeed compatible Hamiltonian operators. It is optimal to put all generalized curvatures of the embedded curve in the highest row possible of the connection matrix. This choice however, does not give us the reduction to the Riemannian case. The lower one starts, the more nonlocal terms will appear in the course of the computation, which complicates the compatibility proofs for the Hamiltonian operators to be found enormously.

For $m = 1$, this frame is the natural generalization of the Riemannian case. This leads to a scalar-vector equation with the vmKdV as a subsystem. However, this is not going to generalize the result in [10], where for $n = 2$ a complexly coupled KdV system was obtained. But taking $m = n$ we generalize this result. In general, there are basically three cases to consider: $m = 1$; $1 < m < n$ and $m = n$. The third case is the simplest, and the second the most complicated. The reader may want to take m to be either 1 or n in checking the results of this paper, since this will make the computation a bit more transparent.

Under our choice of $\omega(D_x)$, formula (2), (3), (4) and (5) lead to

$$0 = -D_x\epsilon - \sigma\theta_1 - \sum_{j=2}^{m} k_j\theta_j + \underline{v_1}, \tag{6}$$

$$0 = D_x v_1 - \underline{D_t \sigma} + \sum_{j=m+1}^{n} u_j v_j - \epsilon\sigma + \sum_{j=2}^{m} \xi_{1j} k_j \tag{7}$$

$$0 = D_x v_i - \underline{D_t k_i} - \epsilon k_i + \xi_{i1}\sigma + \sum_{j=2}^{m} \xi_{ij} k_j, \quad 1 < i \leq m \tag{8}$$

$$0 = D_x \underline{v_i} - u_i v_1 + \xi_{i1}\sigma + \sum_{j=2}^{m} \xi_{ij} k_j, \quad m < i \leq n \tag{9}$$

$$0 = D_x \theta_1 + \sum_{j=m+1}^{n} u_j \theta_j + \underline{\epsilon} \tag{10}$$

$$0 = D_x \theta_i + \underline{\xi_{i1}} \quad 1 < i \leq m \tag{11}$$

$$0 = D_x \underline{\theta_i} - u_i \theta_1 + \xi_{i1}, \quad m < i \leq n \tag{12}$$

$$0 = D_x \xi_{1i} - \underline{v_i} - \sigma\theta_i - \sum_{j=m+1}^{n} u_j \xi_{ij} + \theta_1 k_i, \quad 1 < i \leq m, \tag{13}$$

$$0 = D_x \xi_{1i} + \underline{D_t u_i} - v_i - \sigma\theta_i - \sum_{j=m+1}^{n} u_j \xi_{ij}, \quad m < i \leq n, \tag{14}$$

$$0 = D_x \underline{\xi_{ij}} - k_i \theta_j + k_j \theta_i, \quad 1 < i < j \leq m, \tag{15}$$

$$0 = D_x \underline{\xi_{ij}} - k_i \theta_j + u_j \xi_{1i}, \quad 1 < i \leq m < j \leq n, \tag{16}$$

$$0 = D_x \underline{\xi_{ij}} - u_i \xi_{1j} - u_j \xi_{i1}, \quad m < i < j \leq n, \tag{17}$$

where we underline the expression we want to solve from each equation. We start with equations (10), (11) and (12). Let $\mathbf{u} = (u_{m+1}, \cdots, u_n)$ and $\langle \mathbf{p}, \mathbf{q} \rangle = \sum_{i=m+1}^{n} p_i q_i$. We find:

$$\begin{aligned} D_x \theta_1 &= -\langle \mathbf{u}, \theta \rangle - \epsilon \\ D_x \theta_i &= -\xi_{i1}, \quad 1 < i \leq m, \\ \xi_{1j} &= D_x \theta_j - u_j \theta_1, \quad m < j \leq n. \end{aligned}$$

Let $\top$ be the index lowering (or raising, depending on the context) operator, carrying $(\mu_1, \mu_2, \cdots, \mu_n)^t$ to $(-\mu_1, \mu_2, \cdots, \mu_n)$, as induced by the Killing form. The Killing form is the trace of the product of two matrices representing elements in the (conformal) Lie algebra and gives us an indefinite product that is invariant under the (conformal) group action. Then we have found an operator $\mathfrak{I}$, defined by

$$\mathfrak{I} = \top \cdot \begin{pmatrix} D_x^{-1} & 0 & D_x^{-1}\mathbf{u}^\top \\ 0 & -D_x^{-1} & 0 \\ \mathbf{u}D_x^{-1} & 0 & D_x + \mathbf{u}D_x^{-1}\mathbf{u}^\top \end{pmatrix} \tag{18}$$

where the matrix maps

$$\begin{pmatrix} \epsilon \\ \xi_{12} \\ \vdots \\ \xi_{1m} \\ \theta_{m+1} \\ \vdots \\ \theta_n \end{pmatrix} \longmapsto \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \\ \xi_{m+1,1} \\ \vdots \\ \xi_{n,1} \end{pmatrix}. \tag{19}$$

Here the choice of the elements is determined by our choice of gauge, that is, by $\omega(D_x)$, and by the taking a dual basis with repect to the Killing form. This operator has a right inverse

$$\mathfrak{I}^{-1} = \begin{pmatrix} D_x + \mathbf{u}^\top D_x^{-1}\mathbf{u} & 0 & -\mathbf{u}^\top D_x^{-1} \\ 0 & -D_x & 0 \\ -D_x^{-1}\mathbf{u} & 0 & D_x^{-1} \end{pmatrix} \top, \tag{20}$$

We now introduce the substitution to simplify this operator. Let us return to the torsion equations and put

$$\bar{\epsilon} = \epsilon + \sum_{j=m+1}^{n} u_j\theta_j \tag{21}$$

$$D_x\bar{\xi}_{1j} = \xi_{1j} + u_j\theta_1 \tag{22}$$

$$\bar{\theta}_j = D_x\theta_j \tag{23}$$

Then the torsion equations become

$$D_x\theta_1 = -\bar{\epsilon} \tag{24}$$

$$D_x\theta_i = -\xi_{i1}, \quad i = 2, \cdots, m \tag{25}$$

$$D_x\bar{\xi}_{1j} = \bar{\theta}_j, \quad j = m+1, \cdots, n \tag{26}$$

Using formula (6), we obtain

$$
\begin{aligned}
v_1 &= D_x\epsilon + \sigma\theta_1 + \sum_{i=2}^{m} k_i\theta_i \\
&= D_x(\bar{\epsilon} - \sum_{j=m+1}^{n} u_j\theta_j) + \sigma\theta_1 + \sum_{i_2=2}^{m} k_{i_2}\theta_{i_2} \\
&= -(D_x^2 - \sigma)\theta_1 - \sum_{j_2=m+1}^{n} D_x u_{j_2}\bar{\xi}_{1j_2} + \sum_{i_2=2}^{m} k_{i_2}\theta_{i_2}
\end{aligned}
$$

We now eliminate ξ_{ij} using equation (15), (16) and (17):

$$
\begin{aligned}
\xi_{i_1i_2} &= D_x^{-1}(k_{i_1}\theta_{i_2} - k_{i_2}\theta_{i_1}), \quad 1 < i_1 < i_2 \leq m; \\
\xi_{i_1j_1} &= D_x^{-1}(k_{i_1}\theta_{j_1} - u_{j_1}\xi_{1i_1}) = D_x^{-1}(k_{i_1}\bar{\xi}_{1j_1} - u_{j_1}D_x\theta_{i_1}), \quad 1 < i_1 \leq m < j_1 \leq n; \\
\xi_{j_1j_2} &= D_x^{-1}(u_{j_1}\xi_{1j_2} - u_{j_2}\xi_{1j_1}) = D_x^{-1}(u_{j_1}D_x\bar{\xi}_{1j_2} - u_{j_2}D_x\bar{\xi}_{1j_1}), \quad m < j_1 < j_2 \leq n.
\end{aligned}
$$

It follows from (9) and (13) that

$$
\begin{aligned}
v_{i_1} &= D_x\xi_{1i_1} - \sigma\theta_{i_1} - \sum_{j_2=m+1}^{n} u_{j_2}\xi_{i_1j_2} + k_{i_1}\theta_1 \\
&= D_x^2\theta_{i_1} - \sigma\theta_{i_1} - \sum_{j_2=m+1}^{n} u_{j_2}D_x^{-1}(k_{i_1}\bar{\xi}_{1j_2} - u_{j_2}D_x\theta_{i_1}) + k_{i_1}\theta_1, \quad i_1 = 2, \cdots, m.
\end{aligned}
$$

and

$$
\begin{aligned}
v_{j_1} &= D_x^{-1}(u_{j_1}v_1 - \sigma\xi_{j_1 1} + \sum_{i_2=2}^{m} k_{i_2}\xi_{i_2j_1}) \\
&= D_x^{-1}(u_{j_1}(-D_x^2\theta_1 - \sum_{j_2=m+1}^{n} D_x u_{j_2}\bar{\xi}_{1j_2} + \sum_{i_2=2}^{m} k_{i_2}\theta_{i_2}) + \sigma D_x\bar{\xi}_{1j_1} \\
&\quad + \sum_{i=2}^{m} k_i D_x^{-1}(k_i\bar{\xi}_{1j_1} - u_{j_1}D_x\theta_i)), \quad j_1 = m+1, \cdots, n
\end{aligned}
$$

We can now derive the evolution equations for $\sigma, k_{i=2,\cdots,m}, u_{i=m+1,\cdots,n}$ using (7), (8) and (14), respectively.

First for σ, we have

$$
\begin{aligned}
D_t\sigma &= D_x v_1 + \sum_{j_2=m+1}^{n} u_{j_2} v_{j_2} - \epsilon\sigma + \sum_{i_2=2}^{m} k_{i_2}\xi_{1i_2} \\
&= -(D_x^3 - D_x\sigma - \sigma D_x)\theta_1 - \sum_{j_2=m+1}^{n} (D_x^2+\sigma) u_{j_2}\bar{\xi}_{1j_2} + \sum_{i_2=2}^{m} (D_x k_{i_2} + k_{i_2} D_x)\theta_{i_2} \\
&\quad + \sum_{j_1=m+1}^{n} u_{j_1} D_x^{-1}(u_{j_1}(-D_x^2\theta_1 - \sum_{j_2=m+1}^{n} D_x u_{j_2}\bar{\xi}_{1j_2} + \sum_{i_2=2}^{m} k_{i_2}\theta_{i_2}) + \sigma D_x \bar{\xi}_{1j_1} \\
&\quad + \sum_{i=2}^{m} k_i D_x^{-1}(k_i \bar{\xi}_{1j_1} - u_{j_1} D_x\theta_i)),
\end{aligned}
$$

For $k_{i_1},\ 1 < i_1 \le m$, we have

$$
\begin{aligned}
D_t k_{i_1} &= D_x v_{i_1} - \epsilon k_{i_1} + \xi_{i_1 1}\sigma + \sum_{j_1=2}^{m} \xi_{i_1 j_1} k_{j_1} \\
&= D_x^3\theta_{i_1} - D_x\sigma\theta_{i_1} - \sum_{j_2=m+1}^{n} D_x u_{j_2} D_x^{-1}(k_{i_1}\bar{\xi}_{1j_2} - u_{j_2} D_x\theta_{i_1}) + D_x k_{i_1}\theta_1 \\
&\quad + k_{i_1} D_x\theta_1 + k_{i_1}\sum_{j_2=m+1}^{n} u_{j_2}\bar{\xi}_{1j_2} + \sigma D_x\theta_{i_1} + \sum_{j_1=2}^{m} k_{j_1} D_x^{-1}(k_{i_1}\bar{\xi}_{1j_1} - u_{j_1} D_x\theta_{i_1}),
\end{aligned}
$$

For $u_{j_1},\ m < j_1 \le n$, we have

$$
\begin{aligned}
D_t u_{j_1} &= -D_x\xi_{1j_1} + v_{j_1} + \sigma\theta_{j_1} + \sum_{j_2=m+1}^{n} u_{j_2}\xi_{j_1 j_2} \\
&= -D_x^2\bar{\xi}_{1j_1} + D_x u_{j_1}\theta_1 + \sigma\bar{\xi}_{1j_1} + \sum_{j_2=m+1}^{n} u_{j_2} D_x^{-1}(u_{j_1} D_x\bar{\xi}_{1j_2} - u_{j_2} D_x\bar{\xi}_{1j_1}) \\
&\quad + D_x^{-1}(u_{j_1}(-D_x^2\theta_1 - \sum_{j_2=m+1}^{n} D_x u_{j_2}\bar{\xi}_{1j_2} + \sum_{i_2=2}^{m} k_{i_2}\theta_{i_2}) + \sigma D_x\bar{\xi}_{1j_1} \\
&\quad + \sum_{i_2=2}^{m} k_{i_2} D_x^{-1}(k_{i_2}\bar{\xi}_{1j_1} - u_{j_1} D_x\theta_{i_2})).
\end{aligned}
$$

These equations implicitly define an operator $\mathfrak{H}$, that is,

$$D_t\mu = \mathfrak{H}\nu, \tag{27}$$

with $\mu = (\sigma, k_2, \cdots, k_m, u_{m+1}, \cdots, u_n)^t$, $\nu = (-\theta_1, \cdots, \theta_m, \bar{\xi}_{m+1,1}, \cdots, \bar{\xi}_{n1})$.

We make the following specialization: Let $\hat{\epsilon} = \sigma_x$, $\hat{\xi}_{1i} = k_{i,x}$, $i = 2, \cdots, m$ and $\hat{\theta}_j = -u_{j,x}$, $j = m+1, \cdots, n$. This choice is determined by the vector in formula (19), reflecting the translation symmetry of the equations (and hereditary operator, cf. [4, 5]). Define $\psi = -\sigma + \frac{1}{2}\sum_{l=m+1}^{n} u_l^2$ and take all integration constants as zero. It follows that

$$\begin{aligned}
\hat{\theta}_1 &= -\sigma + \tfrac{1}{2}\sum_{l=m+1}^{n} u_l^2 = \psi \\
\hat{\theta}_i &= k_i, \quad 1 < i \le m \\
\hat{\bar{\xi}}_{1j} &= u_{j,x}, \quad m < j \le n \\
\hat{v}_i &= \begin{cases} \sigma_{xx} + \psi\sigma + \sum_{j=2}^{m} k_j^2, & i = 1 \\ k_{i,xx} + 3\psi k_i + \sigma k_i, & 1 < i \le m \\ \sigma_x u_i - \sigma u_{i,x}, & m < i \le n \end{cases} \\
\hat{\xi}_{ij} &= \begin{cases} 0, & 1 < i < j \le m \\ -k_i u_j, & i \le m < j \le n \\ u_j u_{i,x} - u_i u_{j,x}, & m < i < j \le n \end{cases}
\end{aligned}$$

Finally we obtain the evolution equations:

$$\begin{cases} D_t\sigma = \sigma_{xxx} + 3\psi\sigma_x + 3\sum_{i=2}^{m} k_i k_{i,x} \\ D_t k_i = k_{i,xxx} + 3\psi k_{i,x} + 3\psi_x k_i, & 1 < i \le m. \\ D_t u_j = u_{j,xxx} + 3\psi u_{j,x}, & m < j \le n. \end{cases} \tag{28}$$

or, in vector notation, with $\psi = -\sigma + \frac{1}{2}\langle \mathbf{u}, \mathbf{u}\rangle$,

$$\begin{cases} D_t\sigma = \sigma_{xxx} + 3\psi\sigma_x + 3(\mathbf{k}, \mathbf{k}_x), \\ D_t\mathbf{k} = \mathbf{k}_{xxx} + 3(\psi\mathbf{k})_x, \\ D_t\mathbf{u} = \mathbf{u}_{xxx} + 3\psi\mathbf{u}_x. \end{cases} \tag{29}$$

Remark 1 System (29) is invariant under the translation: $\sigma \to \sigma + \lambda$ and

$x \to x + 3\lambda t$, which leads to a Lax pair of the system, that is,

$$L = D_x - \lambda \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} - \omega(D_x).$$

System (29) is equivalent to

$$\begin{cases} D_t\psi = \psi_{xxx} + \frac{3}{2}(\psi^2 - (\mathbf{k},\mathbf{k}) - \langle \mathbf{u}_x, \mathbf{u}_x\rangle)_x, \\ D_t\mathbf{k} = \mathbf{k}_{xxx} + 3(\psi\mathbf{k})_x, \\ D_t\mathbf{u} = \mathbf{u}_{xxx} + 3\psi\mathbf{u}_x. \end{cases} \tag{30}$$

We observe that the equation is **invariant** under the interchange $\mathbf{k} \leftrightarrow \mathbf{u}_x$, if we view ψ as an independent variable. Let $\Psi = \begin{pmatrix} \mathbf{k} \\ -\mathbf{u}_x \end{pmatrix}$. System (30) can be rewritten as

$$\begin{cases} D_t\psi = \psi_{xxx} + \frac{3}{2}(\psi^2 - (\Psi,\Psi))_x \\ D_t\Psi = \Psi_{xxx} + 3(\psi\Psi)_x. \end{cases} \tag{31}$$

Under the above transformation, the corresponding operators (20) and (27) transform into

$$\mathfrak{H}_1 = \begin{pmatrix} -D_x & 0 \\ 0 & D_x \end{pmatrix} \top$$

and

$$\mathfrak{H}_2 = \begin{pmatrix} -D_x^3 - \psi D_x - D_x\psi & -D_x\Psi^\top - \Psi^\top D_x \\ -D_x\Psi - \Psi D_x & D_x^3 + \psi D_x + D_x\psi - \sum_{i<j} J_{ij}\Psi D_x^{-1}(J_{ij}\Psi)^\top \end{pmatrix} \top,$$

where the J_{ij} are anti-symmetric matrices with nonzero entry of (i,j) being 1 if $i<j$, that is, $(J_{ij})_{kl} = \delta^i_k\delta^l_j - \delta^i_l\delta^j_k$. Using the techniques developed in [4] and explictly employed in [11] and [1], one can easily prove:

LEMMA 4.1 *The operators $\mathfrak{H}_1$ and $\mathfrak{H}_2$ form a Hamiltonian pair.*

System (31) is a biHamiltonian system. It can be written as

$$D_t \begin{pmatrix} \psi \\ \Psi \end{pmatrix} = \mathfrak{H}_1\delta H_2 = \mathfrak{H}_2\delta H_1,$$

where $H_1 = \frac{(\Psi,\Psi)-\psi^2}{2}$ and $H_2 = \frac{\psi_x^2-(\Psi_x,\Psi_x)}{2} + \frac{3}{2}\psi(\Psi,\Psi) - \frac{\psi^3}{2}$.

Since Hamiltonian operators are still Hamiltonian under Miura transformation, it follows that

Theorem 4.2 *The flow of the parallel connection of a curve embedded in a flat n-dimensional conformal manifold naturally leads to two compatible Hamiltonian operators (20) and (27).*

Similarly, systems (29) and (30) are both biHamiltonian. Recently, a list of third order scalar-vector systems with fifth order symmetries was given in [12]. System (31) and the systems obtained from (29) and (30) by taking $m = 1$ or $m = n$ are listed there.

By choosing $m = n = 2$ we can now identify the operator found in [10] with

$$-\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathfrak{H}_2 \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

by taking $\psi = I_2$, $\Psi = I_1$, where we use the notation of the cited paper.

One might ask whether the Riemannian case can be seen as a reduction of the conformal case. Although the equations can be obtained by reduction, this is not the case for the Hamiltonian operators. In Riemannian geometry, we can not write the inverse of the operator $\mathfrak{I}$ in the same form as itself, see [1].

5 Conclusion and Discussion

The procedure that we followed in this paper once again shows how easy it is to obtain the biHamiltonian structure straight from the original geometry. By taking the natural frame one can avoid the *guessing game* (in finding the Hamiltonian pair) which has so far prevented substantial progress in the Poisson bracket formulation of the evolution of embeddings in spaces with given geometry, since the only thing one could do was stare at the operator until one saw the left and right actions which would make it manifestly Hamiltonian. This does not imply that in the next geometry we immediately know what to do, but we can now make a more educated guess! Anyone who thinks we are just replacing one guessing game by another, is invited to do the same analysis in an unnatural frame, just to see what a guessing game looks like!

We should also mention here that the usual Poisson reduction scheme [13,14] carries over to this situation without any (formal) difficulties. There might be analytical problems, but these have nothing to do with integrability as we define it by the existence of infinitely many generalized symmetries, without specifying a function space. However, there is no simple way to compare our results with those in [7], since yet another type of *natural frame* is used in that paper, which is gauge equivalent to our frame. The transformation is highly nonlocal, therefore, even for low dimensional cases, to show equivalence is tricky. Nevertheless, there is no doubt that the connection we employ is geometrical in the sense that the entries in $\omega(D_x)$ are invariants for the generic curves in conformal geometry.

References

[1] Jan A. Sanders and Jing Ping Wang. Integrable systems in n-dimensional Riemannian geometry. *Moscow Mathematical Journal*, 3(4):1369–1393, 2003.

[2] Jing Ping Wang. Generalized Hasimoto transformation and vector Sine-Gordon equation. In S. Abenda, G. Gaeta, and S. Walcher, editors, *Symmetry and Perturbation Theory, SPT 2002*. World Scientific, 2003.

[3] Irene Dorfman. *Dirac structures and integrability of nonlinear evolution equations*. John Wiley & Sons Ltd., Chichester, 1993.

[4] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[5] Jan A. Sanders and Jing Ping Wang. Integrable systems and their recursion operators. In *Proceedings of the Third World Congress of Nonlinear Analysts, Part 8 (Catania, 2000)*, volume 47, pages 5213–5240, 2001.

[6] F. Magri. A simple model of integrable Hamiltonian equation. *Journal of Mathematical Physics*, 19(5):1156–1162, 1978.

[7] G. Marí Beffa. Poisson brackets associated to the conformal geometry of curves. Technical report, University of Wisconsin, 2003. To appear in the Transactions of the American Mathematical Society.

[8] Michael Spivak. *A comprehensive introduction to differential geometry. Vol. II.* Publish or Perish Inc., Wilmington, Del., second edition, 1979.

[9] R. W. Sharpe. *Differential geometry*. Springer-Verlag, New York, 1997. Cartan's generalization of Klein's Erlangen program, With a foreword by S. S. Chern.

[10] G. Marí Beffa. Corrigenda: "Conformal analogue of the Adler-Gel' fand-Dikii bracket in two dimensions". *J. Phys. A*, 33(47):8601, 2000.

[11] G. Marí Beffa, J. A. Sanders, and Jing Ping Wang. Integrable systems in three-dimensional Riemannian geometry. *J. Nonlinear Sci.*, 12(2):143–167, 2002.

[12] T. Tsuchida and T. Wolf. Classification of integrable coupled system with one scalar and one vector unknown (in japanese). In *Proceedings of the conference "New Developments in the Research of Integrable Systems — Continuous, Discrete, Ultra-discrete"*, pages 68–90, 2003.

[13] Richard S. Palais. The symmetries of solitons. *Bull. Amer. Math. Soc. (N.S.)*, 34(4):339–403, 1997.

[14] Chuu-Lian Terng and Gudlaugur Thorbergsson. Completely integrable curve flows on adjoint orbits. *Results Math.*, 40(1-4):286–309, 2001. Dedicated to Shiing-Shen Chern on his 90th birthday.