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Normal forms of 3 degree of freedom Hamiltonian systems at equilibrium in the resonant case

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Abstract

An algorithm is given to compute a Stanley decomposition for the normal form of a three degree of freedom Hamiltonian at equilibrium in the semisimple resonant case. This algorithm is then applied to compute Stanley decompositions of the normal form of the first and second order resonances.

1 Introduction

We consider Hamiltonians at equilibrium with quadratic term

$$H^0 = \sum_{j=1}^{3} m_j x_j y_j,$$

where $x_j = q_j + ip_j$ and $y_j = q_j - ip_j$, and the q_j, p_j are the real canonical coordinates. We assume $m_j \in \mathbb{N}$, although it is straightforward to apply the results in the more general case $m_j \in \mathbb{Z}$. The signs are important in the nonsemisimple case, and of course, in the stability considerations. With these quadratic terms we speak of the semisimple resonant case. For the asymptotic analysis of such resonances, see [SV85]. Most bibliographic references are taken form the second edition of [SV85], in preparation. We now pose the problem to find the description of a general element

$$H \in k[[x_1, y_1, x_2, y_2, x_3, y_3]]$$

such that $\{H^0, H\} = 0$ (see [Mur03, Section 4.5]). Since the flow of H^0 defines a compact Lie group (S^1) action on $T^*\mathbb{R}^3$, we know beforehand that H can be written as a function of a finite number of invariants of the flow of H^0, that is, as

$$H = \sum_{k=1}^{q} F_k(\alpha_1, \cdots, \alpha_{p_k})\beta_k,$$

where $\{H^0, \alpha_\iota\} = \{H^0, \beta_\iota\} = 0$ for all relevant ι. The $\alpha_\iota, \beta_\iota$ are monomials in the $x_1, \cdots, y_3$ variables and are to be determined explicitly. The F_k are completely arbitrary polynomials or formal power series. If it follows from

$$\sum_{k=1}^{q} F_k(\alpha_1, \cdots, \alpha_{p_k})\beta_k = 0$$

that all the F_k are identically zero, we say that we have obtained a **Stanley decomposition** of the normal form. While the existence of the Stanley decomposition follows from the Hilbert finiteness theorem, it

is general not unique: both $F(x)$ and $c + G(x)x$ are Stanley decompositions of general functions in one variable x. Notice that the number of primary variables α_ι is in principle variable, contrary to the case of Hironaka decompositions.
One can define the minimum number q in the Stanley decomposition as the **Stanley dimension**. In general one can only obtain upper estimates on this dimension by a smart choice of decomposition.
We show that if $M = m_1 + m_2 + m_3$, the Stanley dimension of the ring of invariants of H^0 is bounded by $1 + 2M$.
We do this by giving an algorithm to compute a Stanley decomposition, and we illustrate this by giving the explicit formulae for the genuine zeroth, first and second order resonances, that is, those resonances which have more than one generator of degree ≤ 4, not counting complex conjugates and $x_j y_j$'s. These resonances are the most important ones from the point of view of the asymptotic approximation of the solutions.

2 The kernel of $\mathrm{ad}(H^0)$

First of all, we see immediately that the elements $\tau_j = x_j y_j$ all commute with H^0. We let $\mathcal{I} = k[[\tau_1, \tau_2, \tau_3]]$. In principle, we work with real Hamiltonians as they are given by a physical problem, but it is easier to work with complex coordinates, so we take the coefficients to be complex too. In practice, one can forget the reality condition and work over $\mathbb{C}$. In the end, the complex dimension will be the same as the real one, after applying the reality condition.
Any monomial in $\ker \mathrm{ad}(H^0)$ is an element of one of the spaces $\mathcal{I}[[y_1^{n_1} x_2^{n_2} x_3^{n_3}]]$, $\mathcal{I}[[x_1^{n_1} y_2^{n_2} x_3^{n_3}]]$, $\mathcal{I}[[x_1^{n_1} x_2^{n_2} y_3^{n_3}]]$, where $\mathbf{n} = (n_1, n_2, n_3)$ is a solution of $n_1 m_1 = n_2 m_2 + n_3 m_3$, $n_2 m_2 = n_1 m_1 + n_3 m_3$, $n_3 m_3 = n_1 m_1 + n_2 m_2$, respectively, and all the $n_j \geq 0$.
In the equation $n_1 m_1 = n_2 m_2 + n_3 m_3$ one cannot have a nontrivial solution of $n_1 = 0$, but if $n_1 > 0$, one can either have $n_2 = 0$ or $n_3 = 0$, but not both. We allow in the sequel n_2 to be zero, that is, we require $n_1 > 0$, $n_2 \geq 0$ and $n_3 > 0$.
We formulate this in general as follows. Consider the three equations

$$n_i m_i = n_{i^+} m_{i^+} + n_{i^{++}} m_{i^{++}}.$$

where the increment in the indices is in $\mathbb{Z}/3 = (1, 2, 3)$ (that is, $2^{++} \equiv 1$, etc.), where we allow n_{i^+} to be zero, but n_i and $n_{i^{++}}$ are strictly positive.
We now solve for given $\mathbf{m}$ the equation $n_1 m_1 = n_2 m_2 + n_3 m_3$, and then apply a cyclic permutation to the indices of $\mathbf{m}$.
Suppose that $\gcd(m_2, m_3) = g_1 > 1$. In that case, assuming $\mathbf{m}$ is primitive, we may conclude that $g_1 | n_1$. Let $n_1 = g_1 \bar{n}_1$, $m_j = g_1 \bar{m}_j$, $j = 2, 3$. Then

$$\bar{n}_1 m_1 = n_2 \bar{m}_2 + n_3 \bar{m}_3, \quad \gcd(\bar{m}_2, \bar{m}_3) = 1.$$

By cyclic permutation, we assume now that $\gcd(\bar{m}_i, \bar{m}_j) = 1$, and we call $\bar{\mathbf{m}}$ the **reduced** resonance. Observe that the Stanley dimension of the ring of invariants is the same for a resonance and its reduction. Obviously, keeping track of the divisions by the gcd's, one can reconstruct the solution of the original resonance problem from the reduced one. Observe that in terms of the coordinates, the division is equivalent to taking a root, and this is not a symplectic transformation.
Dropping the bars, we again consider $n_1 m_1 = n_2 m_2 + n_3 m_3$, but now we have $\gcd(m_2, m_3) = 1$.
If $m_1 = 1$, we are immediately done, since the solution is simply $n_1 = n_2 m_2 + n_3 m_3$, with arbitrary integers $n_2 \geq 0, n_3 > 0$.
So we assume $m_1 > 1$ and we calculate $\mathrm{mod} m_1$, keeping track of the positivity of our coefficients. Let $m_j = \bar{m}_j + k_j m_1$, $j = 2, 3$, with $0 < m_j < m_1$ since $\gcd(m_j, m_1) = 1$. Let $\tilde{m}_3 = m_1 - m_3$, so again

$0 < \tilde{m}_3 < m_1$. For $q = 0, \cdots, m_1 - 1$ let

$$\begin{aligned} n_2 &= q\tilde{m}_3 + l_2 m_1 \\ n_3 &= q\bar{m}_2 + l_3 m_1 \end{aligned}$$

with the condition that if $q = 0$, then $l_3 > 0$. Then

$$\begin{aligned} n_1 m_1 &= (q\tilde{m}_3 + l_2 m_1)m_2 + (q\bar{m}_2 + l_3 m_1)m_3 \\ &= q\tilde{m}_3 m_2 + q\bar{m}_2 m_3 + m_1(l_2 m_2 + l_3 m_3) \\ &= q\tilde{m}_3(\bar{m}_2 + k_2 m_1) + q\bar{m}_2(\bar{m}_3 + k_3 m_1) + m_1(l_2 m_2 + l_3 m_3) \\ &= q\tilde{m}_3\bar{m}_2 + q\bar{m}_2\bar{m}_3 + m_1(q\tilde{m}_3 k_2 + q\bar{m}_2 k_3 + l_2 m_2 + l_3 m_3) \\ &= m_1(q(k_2\tilde{m}_3 + (1 + k_3)\bar{m}_2) + l_2 m_2 + l_3 m_3) \end{aligned}$$

or

$$n_1 = q(k_2\tilde{m}_3 + (1 + k_3)\bar{m}_2) + l_2 m_2 + l_3 m_3, \quad q = 0, \cdots, m_1 - 1.$$

This is the general solution of the equation $n_1 = n_2 m_2 + n_3 m_3$.
The solution is not necessarily giving us an irreducible monomial: it could be the product of several monomials in $\ker \mathrm{ad}(H^0)$. To analyze this we put

$$q\bar{m}_2 = \psi_2^q m_1 + \phi_2^q, 0 \le \phi_2^q < m_1, \psi_2^q \ge 0$$

and

$$q\tilde{m}_3 = \psi_3^q m_1 + \phi_3^q, 0 \le \phi_3^q < m_1, \psi_3^q \ge 0.$$

We now write $y_1^{n_1} x_2^{n_2} x_3^{n_3}$ as $\langle n_1, n_2, n_3 \rangle$. Then

$$\begin{aligned} &\langle n_1, n_2, n_3 \rangle = \\ &= \langle q(k_2\tilde{m}_3 + (1 + k_3)\bar{m}_2) + l_2 m_2 + l_3 m_3, q\tilde{m}_3 + l_2 m_1, q\bar{m}_2 + l_3 m_1 \rangle \\ &= \langle m_2, m_1, 0 \rangle^{l_2} \langle m_3, 0, m_1 \rangle^{l_3} \langle q(k_2\tilde{m}_3 + (1 + k_3)\bar{m}_2), \psi_3^q m_1 + \phi_3^q, \psi_2^q m_1 + \phi_2^q \rangle. \end{aligned}$$

Let $\phi_1^q = q(k_2\tilde{m}_3 + (1 + k_3)\bar{m}_2) - \psi_2^q m_3$. Then

$$\begin{aligned} \phi_1^q &= q(k_2\tilde{m}_3 + (1 + k_3)\bar{m}_2) - \psi_2^q m_3 \\ &= k_2 q\tilde{m}_3 + (1 + k_3)(\psi_2^q m_1 + \phi_2^q) - \psi_2^q(\bar{m}_3 + k_3 m_1) \\ &= k_2 q\tilde{m}_3 + (1 + k_3)\phi_2^q + \psi_2^q m_1 - \psi_2^q \bar{m}_3 \\ &= k_2 q\tilde{m}_3 + (1 + k_3)\phi_2^q + \psi_2^q \tilde{m}_3 \ge 0. \end{aligned}$$

We now write $\phi_1^q = \tilde{\psi}_3^q m_2 + \chi_1^q$, and we let $\hat{\psi}_3^q = \min(\tilde{\psi}_3^q, \psi_3^q)$. We have

$$\begin{aligned} &\langle n_1, n_2, n_3 \rangle = \\ &= \langle m_2, m_1, 0 \rangle^{l_2 + \hat{\psi}_3} \langle m_3, 0, m_1 \rangle^{l_3 + \psi_2^q} \langle (\tilde{\psi}_3^q - \hat{\psi}_3^q)m_2 + \chi_1^q, (\psi_3^q - \hat{\psi}_3^q)m_1 + \phi_3^q, \phi_2^q \rangle. \end{aligned}$$

We define

$$\begin{aligned} \alpha_\iota &= \langle m_{\iota^+}, m_\iota, 0 \rangle \\ \beta_\iota^0 &= \langle m_{\iota^{++}}, 0, m_\iota \rangle \\ \beta_\iota^q &= \langle (\tilde{\psi}_{\iota^{++}}^q - \hat{\psi}_{\iota^{++}}^q)m_{\iota^+} + \chi_\iota^q, (\psi_{\iota^{++}}^q - \hat{\psi}_{\iota^{++}}^q)m_\iota + \phi_{\iota^{++}}^q, \phi_{\iota^+}^q \rangle. \end{aligned}$$

ι	α	β^0	
1	y_1x_2	y_1x_3	$\frac{t^2}{(1-t^2)^5}$
2	y_2x_3	x_1y_2	$\frac{t^2}{(1-t^2)^5}$
3	x_1y_3	x_2y_3	$\frac{t^2}{(1-t^2)^5}$

$$H_{1:1:1}(t) = \frac{1+4t^2+t^4}{(1-t^2)^5}$$

Table 1: The 1 : 1 : 1-resonance ([FHPY02])

Thus

$$\langle n_1, n_2, n_3 \rangle = \alpha_1^{l_2'}(\beta_1^0)^{l_3'}\beta_1^q, \quad l_2', l_3' \in \mathbb{N}, q = 0, \cdots, m_1 - 1,$$

or, in other words, $\langle n_1, n_2, n_3 \rangle \in \mathcal{I}[[\alpha_1, \beta_1^0]]\beta_1^q$. This means that $\mathcal{I}[[\alpha_1, \beta_1^0]]\beta_1^q$ is the solution space of the resonance problem. Notice that by construction these spaces have only 0 intersection.
Let $\mathcal{K}$ be defined as $\bigoplus_{\iota \in \mathbb{Z}/3} \mathcal{K}_\iota$, where

$$\mathcal{K}_\iota = \bigoplus_{q=0}^{m_\iota - 1} \mathcal{I}[[\alpha_\iota, \beta_\iota^0]]\beta_\iota^q.$$

Then we have

Theorem 1 *Let $\bar{\mathcal{K}}$ denote the space of complex conjugates (that is, x_j and y_j interchanged) of the elements of $\mathcal{K}$. Then $\mathcal{I} \oplus \mathcal{K} \oplus \bar{\mathcal{K}}$ is a Stanley decomposition of the $m_1 : m_2 : m_3$-resonance.*

Corollary 1 *In each $\mathcal{K}_\iota$ there are m_ι direct summands. Therefore there are $M = m_1 + m_2 + m_3$ direct summands in $\mathcal{K}$. This enables us to estimate the Stanley dimension from above by $1 + 2M$.*

Remark 1 *The number of generators need not be minimal. In particular the β^q's can be generated by one or more elements. We conjecture that the $\beta^q, q = 1, \cdots, m_\iota - 1$, are generated as polynomials by at most two invariants. Furthermore, the β_ι^q's, are for $q > 0$ not algebraically independent of α_ι and β_ι^0. The relations among them constitute what we will call here the defining curve. Since the Stanley decomposition is the ring freely generated by the invariants divided out by the ideal of the defining curve, this gives us a description of the normal form that is independent of the choices made in writing down the Stanley decomposition.*

Remark 2 *The generating functions of the following resonances have been computed by A. Fekken [Fek86]. They are the Poincaré-Hilbert series of the Stanley decomposition and can be computed by computing the Molien series of the group action given by the flow of H^0, that is, by computing circle integrals (or residues).*

The 15 tables contain all the information to compute the Stanley decomposition for the lower order resonances.

3 Nonsemisimple normal form of three degrees of freedom Hamiltonians

We consider Hamiltonians at equilibrium with quadratic term

$$H^0 = \sum_{j=1}^{3} m_j x_j y_j + x_2 y_3,$$

ι	α	β^0	
1	$y_1^2x_2$	$y_1^2x_3$	$\frac{t^3}{(1-t^2)^3(1-t^3)^2}$
2	y_2x_3	$x_1^2y_2$	$\frac{t^3}{(1-t^2)^4(1-t^3)}$
3	$x_1^2y_3$	x_2y_3	$\frac{t^2}{(1-t^2)^4(1-t^3)}$

$$P_{1:2:2}(t) = \frac{t^6 + 2\,t^4 + 2\,t^3 + 2\,t^2 + 1}{(1-t^3)^2\,(1-t^2)^3}$$

Table 2: The 1 : 2 : 2-resonance ([MMV81]). This is derived from the 1 : 1 : 1-resonance by squaring x_1 and y_1.

ι	α	β^0	
1	$y_1^3x_2$	$y_1^3x_3$	$\frac{t^4}{(1-t^2)^3(1-t^4)^2}$
2	y_2x_3	$x_1^3y_2$	$\frac{t^4}{(1-t^2)^4(1-t^4)}$
3	$x_1^3y_3$	x_2y_3	$\frac{t^2}{(1-t^2)^4(1-t^4)}$

$$P_{1:3:3}(t) = \frac{t^8 + 2\,t^6 + 4\,t^4 + 2\,t^2 + 1}{(1-t^4)^2\,(1-t^2)^3}$$

Table 3: The 1 : 3 : 3-resonance.This is derived from the 1 : 1 : 1-resonance by raising x_1 and y_1 to the third power.

ι	α	β^0	β^1	
1	y_1x_2	$y_1^2x_3$		$\frac{t^3}{(1-t^2)^4(1-t^3)}$
2	$y_2^2x_3$	x_1y_2		$\frac{t^2}{(1-t^2)^4(1-t^3)}$
3	$x_1^2y_3$	$x_2^2y_3$	$x_1x_2y_3$	$\frac{2t^3}{(1-t^2)^3(1-t^3)^2}$

$$P_{1:1:2}(t) = \frac{t^6 + 2\,t^4 + 4\,t^3 + 2\,t^2 + 1}{(1-t^3)^2\,(1-t^2)^3}$$

Table 4: The 1 : 1 : 2-resonance ([vdAS79, vdA83]). The defining curve is $((\beta_3^1)^2 - \alpha_3\beta_3^0)$.

ι	α	β^0	β^1	
1	$y_1^2x_2$	$y_1^4x_3$		$\frac{t^5}{(1-t^2)^3(1-t^3)(1-t^5)}$
2	$y_2^2x_3$	$x_1^2y_2$		$\frac{t^3}{(1-t^2)^3(1-t^3)^2}$
3	$x_1^4y_3$	$x_2^2y_3$	$x_1^2x_2y_3$	$\frac{t^3+t^4}{(1-t^2)^3(1-t^3)(1-t^5)}$

$$P_{1:2:4}(t) = \frac{t^9 + t^7 + 2t^6 + 3t^5 + 3t^4 + 2t^3 + t^2 + 1}{(1-t^2)^2(1-t^3)^2(1-t^5)}$$

Table 5: The 1 : 2 : 4-resonance ([vdA83]). This is derived from the 1 : 1 : 2-resonance by squaring x_1 and y_1.

ι	α	β^0	β^1
1	$y_1^3x_2$	$y_1^6x_3$	
2	$y_2^2x_3$	$x_1^3y_2$	
3	$x_1^6y_3$	$x_2^2y_3$	$x_1^3x_2y_3$

$$P_{1:3:6}(t) = \frac{1+t^2+t^3+2t^4+3t^5+2t^6+3t^7+2t^8+t^9+t^{10}+t^{12}}{(1-t^2)^2(1-t^3)(1-t^4)(1-t^7)}.$$

Table 6: The 1 : 3 : 6-resonance. This is derived from the 1 : 1 : 2-resonance by raising x_1 and y_1 to the third power.

ι	α	β^0	β^1	β^2
1	y_1x_2	$y_1^3x_3$		
2	$y_2^3x_3$	$x_1^3y_2$		
3	$x_1^3y_3$	$x_2^3y_3$	$x_1^2x_2y_3$	$x_1x_2^2x_3$

$$P_{1:1:3}(t) = \frac{1+2t^2+8t^4+2t^6+t^8}{(1-t^2)^3(1-t^4)^2}.$$

Table 7: The 1 : 1 : 3-resonance The defining curve is $(\beta_3^1\beta_3^2-\alpha_3\beta_3^0, \beta_3^1\beta_3^1-\alpha_3\beta_3^2, \beta_3^2\beta_3^2-\beta_3^0\beta_3^1)$.

ι	α	β^0	β^1	β^2
1	$y_1^2x_2$	$y_1^6x_3$		
2	$y_2^3x_3$	$x_1^6y_2$		
3	$x_1^6y_3$	$x_2^3y_3$	$x_1^4x_2y_3$	$x_1^2x_2^2y_3$

$$P_{1:2:6}(t) = \frac{1+t^2+t^3+2t^4+3t^5+4t^6+3t^7+2t^8+t^9+t^{10}+t^{12}}{(1-t^2)^2(1-t^3)(1-t^4)(1-t^7)}.$$

Table 8: The 1 : 2 : 6-resonance ([VdADW94]). This is derived from the 1 : 1 : 3-resonance by squaring x_1 and y_1.

ι	α	β^0	β^1	β^2
1	$y_1^3x_2$	$y_1^9x_3$		
2	$y_2^3x_3$	$x_1^9y_2$		
3	$x_1^9y_3$	$x_2^3y_3$	$x_1^6x_2y_3$	$x_1^3x_2^2y_3$

$$P_{1:3:9}(t) = \frac{1+t^2+3t^4+5t^6+6t^8+5t^{10}+3t^{12}+t^{14}+t^{16}}{(1-t^2)^2(1-t^4)^2(1-t^{10})}.$$

Table 9: The 1 : 3 : 9-resonance. This is derived from the 1 : 1 : 3-resonance by raising x_1 and y_1 to the third power.

ι	α	β^0	β^1	β^2
1	$y_1^2x_2$	$y_1^3x_3$		
2	$y_2^3x_3^2$	$x_1^2y_2$	$x_1y_2^2x_3$	
3	$x_1^3y_3$	$x_2^3y_3^2$	$x_1x_2y_3$	$x_1^2x_2^2y_3^2$

$$P_{1:2:3}(t) = \frac{1+t^2+3t^3+4t^4+4t^6+3t^7+t^8+t^{10}}{(1-t^2)^2(1-t^3)(1-t^4)(1-t^5)}.$$

Table 10: The 1 : 2 : 3-resonance. The defining curve is $(\beta_2^1\beta_2^1-\alpha_2\beta_2^0,(\beta_3^1)^3-\alpha_3\beta_3^0)$.

ι	α	β^0	β^1	β^2
1	$y_1^2x_2$	$y_1^3x_3^2$		
2	$y_2^3x_3^4$	$x_1^2y_2$	$x_1y_2^2x_3^2$	
3	$x_1^3y_3^2$	$x_2^3y_3^4$	$x_1x_2y_3^2$	$x_1^2x_2^2y_3^4$

$$P_{2:4:3}(t) = \frac{1+t^2+t^3+3t^4+4t^5+3t^6+3t^7+4t^8+3t^9+t^{10}+t^{11}+t^{13}}{(1-t^2)^2(1-t^3)(1-t^5)(1-t^7)}.$$

Table 11: The 2 : 4 : 3-resonance ([vdA83, Kum75]). This is derived from the 1 : 2 : 3-resonance by squaring x_3 and y_3.

ι	α	β^0	β^1	β^2	β^3	β^4
1	$y_1^2x_2$	$y_1^5x_3$				
2	$y_2^5x_3^2$	$x_1^2y_2$	$x_1y_2^3x_3$			
3	$x_1^5y_3$	$x_2^5y_3^2$	$x_1^3x_2y_3$	$x_1x_2^2y_3$	$x_1^4x_2^3y_3^2$	$x_1^2x_2^4y_3^2$

$$P_{1:2:5}(t) = \frac{1+t^2+t^3+3t^4+5t^5+4t^6+4t^7+4t^8+5t^9+3t^{10}+t^{11}+t^{12}+t^{14}}{(1-t^2)^2(1-t^3)(1-t^6)(1-t^7)}.$$

Table 12: The 1 : 2 : 5-resonance ([VdADW94, HW96, Hal99]). The defining curve is $((\beta_2^1)^2-\alpha_2\beta_2^0,\beta_3^3-\beta_3^1\beta_3^2,\beta_3^4-(\beta_3^2)^2,(\beta_3^2)^3-\beta_3^0\beta_3^1,(\beta_3^1)^2-\alpha_3\beta_3^2,\beta_3^1(\beta_3^2)^2-\alpha_3\beta_3^0)$.

ι	α	β^0	β^1	β^2	β^3
1	$y_1^3x_2$	$y_1^4x_3$			
2	$y_2^4x_3^3$	$x_1^3y_2$	$x_1y_2^3x_3^2$	$x_1^2y_2^2x_3$	
3	$x_1^4y_3$	$x_2^4y_3^3$	$x_1x_2y_3$	$x_1^2x_2^2y_3^2$	$x_1^3x_2^3y_3^3$

$$P_{1:3:4}(t) = \frac{1+t^2+2t^3+2t^4+5t^5+6t^6+4t^7+6t^8+5t^9+2t^{10}+2t^{11}+t^{12}+t^{14}}{(1-t^2)^2(1-t^4)(1-t^5)(1-t^7)}.$$

Table 13: The 1 : 3 : 4-resonance. The defining curve is $((\beta_2^2)^2-\beta_2^0\beta_2^1,(\beta_2^1)^2-\alpha_2\beta_2^2,\beta_2^1\beta_2^2-\alpha_2\beta_2^0,(\beta_3^1)^4-\alpha_3\beta_3^0)$.

ι	α	β^0	β^1	β^2	β^3	β^4
1	$y_1^3x_2$	$y_1^5x_3$				
2	$y_2^5x_3^3$	$x_1^2y_2$	$x_1^2y_2^4x_3^2$	$x_1y_2^2x_3$		
3	$x_1^5y_3$	$x_2^5y_3^3$	$x_1^2x_2y_3$	$x_1^4x_2^2y_3^2$	$x_1x_2^3y_3^2$	$x_1^3x_2^4y_3^3$

$$P_{1:3:5}(t) = \frac{1+t^2+6t^4+9t^6+12t^8+9t^{10}+6t^{12}+t^{14}+t^{16}}{(1-t^2)^2(1-t^4)(1-t^6)(1-t^8)}.$$

Table 14: The 1 : 3 : 5-resonance. The defining curve is $(\beta_2^1-(\beta_2^2)^2, (\beta_2^2)^3-\alpha_2\beta_2^0), \beta_3^4-\beta_3^1\beta_3^3, (\beta_3^1)^3-\alpha_3\beta_3^3, (\beta_3^3)^2-\beta_3^0\beta_3^1, (\beta_3^1)^2\beta_3^3-\alpha_3\beta_3^0)$.

ι	α	β^0	β^1	β^2	β^3	β^4	β^5	β^6
1	$y_1^3x_2$	$y_1^7x_3$						
2	$y_2^7x_3^3$	$x_1^3y_2$	$x_1y_2^5x_3^2$	$x_1^2y_2^3x_3$				
3	$x_1^7y_3$	$x_2^7y_3^3$	$x_1^4x_2y_3$	$x_1x_2^2y_3$	$x_1^5x_2^3y_3^2$	$x_1^2x_2^4y_3^2$	$x_1^6x_2^5y_3^3$	$x_1^3x_2^6y_3^3$

$$P_{1:3:7}(t) = \frac{1+t^2+4t^4+8t^6+11t^8+12t^{10}+11t^{12}+8t^{14}+4t^{16}+t^{18}+t^{20}}{(1-t^2)^2(1-t^4)(1-t^8)(1-t^{10})}.$$

Table 15: The 1 : 3 : 7-resonance ([VH92]). The defining curve is $((\beta_2^1)^2-\alpha_2\beta_2^2, (\beta_2^2)^2-\beta_2^0\beta_2^1, \beta_2^1\beta_2^2-\alpha_2\beta_2^0, \beta_3^3-\beta_3^1\beta_3^2, \beta_3^4-\beta_3^2\beta_3^2, \beta_3^5-\beta_3^1\beta_3^2\beta_3^2, \beta_3^6-\beta_3^2\beta_3^2\beta_3^2, \beta_3^1)^2-\alpha_3\beta_3^2, (\beta_3^2)^4-\beta_3^0\beta_3^1, \beta_3^1(\beta_3^2)^3-\alpha_3\beta_3^0)$.

where $x_j = q_j + ip_j$ and $y_j = q_j - ip_j$, $m_2 = m_3$, $gcd(m_1, m_2) = 1$, and the q_j, p_j are the real canonical coordinates. We assume $m_j \in \mathbb{N}$, although it is straightforward to apply the results in the more general case $m_j \in \mathbb{Z}$. The signs are important in the nonsemisimple case. With these quadratic terms we speak of the nonsemisimple resonant case. Most bibliographic references are taken form the second edition of [SV85], in preparation. We now pose the problem to find the description of a general element

$$H \in k[[x_1, y_1, x_2, y_2, x_3, y_3]]$$

such that $\{H_s^0, H\} = 0$, where $H_s^0 = \sum_{j=1}^3 m_j x_j y_j$ and $\{H_m^0, H\} = 0$, with $H_m^0 = x_3y_2$. In principle, the normal form theory is more difficult in the nonsemisimple case than in the semisimple case, but in practice it turns out to be much easier. This is caused by the fact that only two integers play a role, m_1 and m_2. We list the basic monomials in $\ker \mathrm{ad}(H_2^0)$: $\tau = x_1y_2, n = x_2y_3, m = y_2x_3, h = x_2y_2 - x_3y_3, e = x_2y_2 + x_3y_3$ and $x_1^{m_2}y_2^{k_2}y_3^{k_3}, y_1^{m_2}x_2^{k_2}x_3^{k_3}$, with $k_2 + k_3 = m_1$. So

$$H^0 = \tau + e + n.$$

We see that

$$\begin{aligned}
\{n, x_1^{m_2}y_2^{k_2}y_3^{k_3}\} &= k_2x_1^{m_2}y_2^{k_2-1}y_3^{k_3+1} \\
\{m, x_1^{m_2}y_2^{k_2}y_3^{k_3}\} &= k_3x_1^{m_2}y_2^{k_2+1}y_3^{k_3-1} \\
\{h, x_1^{m_2}y_2^{k_2}y_3^{k_3}\} &= (k_2-k_3)x_1^{m_2}y_2^{k_2}y_3^{k_3} \\
\{n, y_1^{m_2}x_2^{k_2}x_3^{k_3}\} &= -k_3y_1^{m_2}x_2^{k_2+1}x_3^{k_3-1} \\
\{m, y_1^{m_2}x_2^{k_2}x_3^{k_3}\} &= -k_2y_1^{m_2}x_2^{k_2-1}x_3^{k_3+1} \\
\{h, y_1^{m_2}x_2^{k_2}x_3^{k_3}\} &= -(k_2-k_3)y_1^{m_2}x_2^{k_2}x_3^{k_3}
\end{aligned}$$

In other words, the $x_1^{m_2}y_2^{k_2}y_3^{k_3}$ and $y_1^{m_2}x_2^{k_2}x_3^{k_3}$ are m_1+1-dimensional representation spaces of an $\mathfrak{sl}_2$ spanned by n, m and h. For the nonsemisimple normalform we need the elements in $\ker \mathrm{ad}(m)$, that is, τ, $\alpha = x_1^{m_2}y_2^{m_1}$, $\beta = y_1^{m_2}x_3^{m_1}$ and m itself. Observe that $\alpha\beta = \tau^{m_2}m^{m_1}$. The Stanley decomposition of the normal form space $\ker \mathrm{ad}(m) \cap \ker \mathrm{ad}(\tau + e)$ is given by

$$\mathbb{R}[\tau, m, \alpha] \oplus \mathbb{R}[\tau, m, \beta]\beta.$$

The generating function is (the $\mathrm{ad}(h)$-eigenvalues for τ, m and α, β are $0, 2$ and m_1, respectively)

$$P(u,t) = \frac{1 + u^{m_1}t^{m_1+m_2}}{(1-t^2)(1-u^2t^2)(1-u^{m_1}t^{m_1+m_2})}$$

The corresponding generating for the semisimple case is

$$\frac{\partial u P(u,t)}{\partial u}|_{u=1} = \frac{2m_1 t^{m_1+m_2}}{(1-t^2)^2(1-t^{m_1+m_2})^2} + \frac{(1+t^{m_1+m_2})(1+t^2)}{(1-t^2)^3(1-t^{m_1+m_2})}$$

Some of these have been computed (as integrals over S^1) in [Fek86].

4 Structure of the semisimple normal form

The idea is now to see the terms that span the Stanley decomposition as a module as orbits of elements in $\ker \mathrm{ad}(\tau + e)$. This may explain the great regularity that is found in all the normal form decompositions that have been computed so far.

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