1. Show that any finite rank operator $K : X \to X$ can be written as in (4.21) with not only the x_j but also the x'_j linearly independent. If you get stuck, go on with the exercises below. In what follows we assume that K is already in this form.

2. Show that every solution of $x - Kx = y$ is of the form

$$x = \sum_k \xi_k x_k + y.$$

3. We write ξ for the n-vector with components $\xi_1, \dots, \xi_n$ and $\zeta(x)$ for the n-vector with components $\zeta_1(x) = x'_1(x), \dots, \zeta_n(x) = x'_n(x)$. Determine a matrix C such that x is a solution if and only if ξ is a solution of a matrix equation $\xi - C\xi = \zeta(y)$. What is therefore the relation between $N(I - C)$ and $N(I - K)$?

4. Express the adjoint operator $K' : X' \to X'$ by a formula similar to that of K above.

5. Show that every solution of the equation $x' - K'x' = y'$ is of the form

$$x' = \sum_k \xi'_k x'_k + y'.$$

6. Writing ξ' for the n-vector with components $\xi'_1, \dots, \xi'_n$ and $\zeta'(x)$ for the n-vector with components $\zeta'_1(x) = x'(x_1), \dots, \zeta'_n(x) = x'(x_n)$, show that x' is a solution of $x' - K'x' = y'$ if and only if ξ' is a solution of $\xi' - C'\xi' = \zeta'(y')$, where C' is the transposed matrix of C. What is te relation between $N((I - C)')$ and $N((I - K)')$?

7. Formulate (3.12), (3.14), the statement in Exercise 3.8 4a, and (3.25) for $A = I - C$.

8. Show that $R(I - K) =^\circ N((I - K)')$.

9. Show that $R((I - K)') = N(I - K)^\circ$.

10. Show that $N(I - K)$ and $N((I - K)')$ have the same finite dimension.

11. Let $X = L^2(0, 1)$ and $v_1, \dots, v_n, w_1, \dots, w_n$ be in X. Discuss how you would formulate and solve the integral equation

$$x(t) = y(t) + \sum_{k=1}^{n} v_i(t) \int_0^1 w_i(s) x(s) ds$$

as an equation of the form $x = Kx + y$ in X. What are the conditions for the solution to exist for every y in X? What are the conditions on y for solutions to exist if the above condition fails?