Monotonicity in Markov Reward and Decision Chains

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A well-known result

- Single queue, Poisson arrivals, a fast and a slow server
- Control: when to assign customer to slow server
- Result: threshold optimality (Lin & Kumar '84)
- Method: dp (K '95)

Another well-known result

- Two parallel (heterogeneous) queues, Poisson arrivals
- Control: where to assign an arriving customer
- Result: switching curve
- Method: dp (Hajek '84)

Yet another well-known result

- Two queues in tandem, Poisson arrivals
- Control: server speeds
- Result: monotone server speeds
- Method: dp (Weber & Stidham '87)

Method

- Formulate dp value function
- Write down equations needed for anticipated result
- try to propagate value functions
- add equations as needed

Example (Hajek model)

- Value function:

$$V_{n+1}(x) = |x| + \lambda \min_{i=\{1,2\}} \{V_n(x+e_i)\} + \sum_{i=1}^{2} \mu_i V_n((x-e_i)^+)$$

- monotonicity: if route to Q2 in x, then also in $x + e_1$
- route to Q2 in $x \Rightarrow V_n(x+e_2) - V_n(x+e_1) \leq 0$
- monotonicity if $V_n(x+e_1+e_2) - V_n(x+2e_1) \leq V_n(x+e_2) - V_n(x+e_1)$
- thus propagate $V_n(x+e_1+e_2) + V_n(x+e_1) \leq V_n(x+2e_1) + V_n(x+e_2)$

Example (Lin-Kumar model)

- Value function: $V_{n+1}(x) = |x| + \lambda V_n'(x + e_1) + \sum_{i=1}^{2} \mu_i V_n'((x - e_i)^+)$
  $V_n'(x) = \min\{V_n(x - e_1 + e_2), V_n(x)\}$ if $x_1 > 0$, $V_n(x)$ otherwise
- monotonicity: if route to server 2 in x, then also in $x + e_1$
- route to server 2 in $x \Rightarrow V_n(x + e_2) - V_n(x + e_1) \leq 0$
- monotonicity if $V_n(x+e_1+e_2) - V_n(x+2e_1) \leq V_n(x+e_2) - V_n(x+e_1)$
- thus propagate $V_n(x+e_1+e_2) + V_n(x+e_1) \leq V_n(x+2e_1) + V_n(x+e_2)$

Same equations

- "Conclusion": Lin-Kumar model $\equiv$ Hajek model $\equiv$ Weber-Stidham model

- Central role to set of equations

- For each set of equations a set of "operators" that propagate (Operators: things that happen in system such as arrivals, departures, environment changes,...)

- Dp equation = concatenation of operators

Example (Hajek model)

Value function:

$$V_{n+1}(x) = T_{\text{costs}}(T_{\text{cc}}(T_R, T_{D1}, T_{D2}))V_n(x)$$

with

$$T_{\text{costs}}f(x) = C(x) + f(x),$$
$$T_{\text{cc}}(f_1, \dots, f_m)(x) = \sum_i p_i f_i(x),$$
$$T_R f(x) = \min_i f(x + e_i),$$
$$T_{Di} f(x) = f((x - e_i)^+)$$

My contribution

- Identified many interesting operators
- Identified relevant (in)equalities
- Matched them
- Wrote an overview about it
  (Foundations and Trends on Stochastic Systems 1:1–73, 2006)

Operators

- (controlled) environment
- arrivals, admission control, routing
- single server, multiple servers, assignable server
- tandem server

Classes of inequalities

- First-order (e.g., $f(x + e_1) \leq f(x + e_2)$)
- Schur convexity ($x \prec y$ if x more balanced than y)
- Convexity (componentwise convex, sub/supermodular, sub/superconvex, multimodular)

Typical results

- First-order: optimality of μc rule (single server) and LEPT (multiple servers)
- Schur convexity: optimality of join the shortest queue
- Convexity in 1 dimension: monotonicity and optimality of threshold policy for concave service rates
- Convexity in multiple dimensions: monotonicity of control tandem model (W & S), convexity of value function of multi-server tandem system
- Convexity in two dimensions: results of Lin & Kumar, Hajek

Use of this all

- Characterization optimal policies
- Monotonicity in parameters (on-off vs. Poisson, convexity in arrival rate)

Some examples

- Assignment to parallel queues with controlled arrivals
- Comparison of arrival processes

Example: Assignment to parallel queues

- Two parallel queues, 1 server
- service rates μ_1, μ_2, holding costs c_1, c_2
- Inequalities ($\bar{\mu}_i = \mu - \mu_i$ with $\mu = \max_i\{\mu_i\}$):

$$\mu_1 f(x - e_1) + \bar{\mu}_1 f(x) \leq \mu_2 f(x - e_2) + \bar{\mu}_2 f(x)$$
$$f(x - e_1) \leq f(x), \quad f(x - e_2) \leq f(x)$$

Allowable cost functions

- Fill in $C(x) = c_1 x_1 + c_2 x_2$
- Conclusion: $\mu_1 c_1 \geq \mu_2 c_2$, $c_1, c_2 \geq 0 \Longrightarrow \mu c$ rule

Environment

- New feature: environment
- Add dimension 0, and operator T_{env}:
- $T_{env}(f_1,\dots,f_l)(x) = \sum_y \lambda(x_0,y)\sum_j q^j(x_0,y)f_j(x^*)$, $x_0^* = y$, $x_i^* = x_i$, $i > 0$
- Extension: controlled environment
- $T_{Cenv}(f_1,\dots,f_l)(x) = \min_a\{\sum_y \lambda(x_0,a,y)\sum_j q^j(x_0,a,y)f_j(x^*)\}$

Results

- Environment without control: μc rule
- Environment with control: μc rule and $\mu_1 \leq \mu_2$
- Counterexamples to $\mu_1 > \mu_2$
- Application: 2 "μc" nodes in tandem

Example: Comparison of arrival processes

- $T_A f(x) = f(x + e_1)$, $T'_A f(x) = 0.5 f(x) + 0.5 f(x + 2e_1)$
- $V_{n+1}(x) = T_{\text{costs}}(T_{\text{cc}}(T_A, T_{D1})V_n(x)$, $V'_{n+1}(x) = T_{\text{costs}}(T_{\text{cc}}(T'_A, T_{D1})V'_n(x)$
- All our operators: $f \leq f' \Longrightarrow Tf \leq Tf'$
- Result: $f \leq f'$, $f \in Cx \Longrightarrow T_A f \leq T'_A f'$
- Proof: $T_A f(x) = f(x + e_1) \leq 0.5 f(x) + 0.5 f(x + 2e_1) = T'_A f(x) \leq T'_A f'(x)$
- Conclusion: $V_n \leq V'_n$ if $V_0 \leq V'_0$

"Big" open problems

- > 2 servers in Lin-Kumar model $\equiv$ Hajek model
- Hysteresis
  - dim 1: queue length, dim 2: server speed
  - server costs, holding costs, switching costs
  - optimal switching rule $=$ hysteresis?

Open problem 1

- Hajek model (2 dim) with multi-server queues
- Superconvexity = $f(x+e_1+e_2)+f(x+e_1) \leq f(x+2e_1)+f(x+e_2)$
- also needed: Componentwise convexity & Supermodularity = $f(x+e_1)+f(x+e_2) \leq f(x)+f(x+e_1+e_2)$
- propagating SuperC through multi-server operator leads to Submodularity = contradiction
- no positive results (de Véricourt & Zhou '06)

Open (?) problem 2

- Lu & Serfozo '84 (again '84!!): hysteresis optimal
- Hipp & Holzbauer '88: counterex. to a condition in L&S
- Kitaev & Serfozo '99: "repairs" error without going into detail
- My opinion: "clean" proof needed