A Simple Proof of the Optimality of a Threshold Policy in a Two-Server Queueing System

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Abstract

Lin & Kumar [2] introduced a control model with a single queue and two heterogeneous servers. They showed, using policy iteration, that the slower server should only be used if the queue length is above a certain level, i.e., the optimal policy is of threshold type. In this note we give a simple iterative proof of this result.

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1. Introduction

Consider a queue with Poisson arrivals (rate λ) and two exponential servers, with service rates μ_1 and μ_2 (with $\mu_1 > \mu_2$). The customers in the queue have to be assigned non-preemptively to the servers, i.e., after each event (which can either be an arrival or a service completion) a controller decides if it sends a customer in the queue to a free server (if available).

Lin & Kumar [2] studied this model and showed that the fast server should always be used (as long as there are customers in the queue), and that the slower server should only be used if the number of customers in queue exceeds a certain level. Note that a customer sent to the slower server cannot be sent back to the queue; if that were the case then the model would reduce to a simpler service rate control model.

In this note we give a simple iterative proof of the threshold optimality.

Walrand [4] gives yet another proof of this result, Viniotis & Ephremides [3] consider several generalizations like Erlang servers.

2. The Result

We will show, using value iteration, that the optimal policy is of threshold type. This proof is considerably simpler than the proofs in [2, 3] (where policy iteration is used) and [4] (where it is shown that any policy which is not a threshold policy can be improved).

The fact that the faster server should always be used is already proven in [2] using value iteration. Therefore we assume that the state space is of the form (x, i) where x is the total number of customers in the queue and at the first server, and $i \in \{0, 1\}$ denotes the number of customers at the second server. In [2], $c(x,i) = x+i$ is taken as direct costs. Here we prove the optimality of a threshold policy for a larger class of cost functions, including $c(x,i) = x+i$.

Assume, by scaling, that $\lambda + \mu_1 + \mu_2 = 1$. We will study the continuous time model by looking just at the jump times. If one of the servers idles, a fictitious transition is added to make the times between each two jump times equally (exponentially) distributed. We write $V^n(x,i)$ for the expected costs over the next n jumps, in the embedded discrete time model, starting in state (x,i). To simplify the notation and the analysis, we use W^n below as an intermediate step (we can interpret W^n as the costs to go just after an event, but before a decision).

We can write V^n as follows:

$$V^{n+1}(x,i) = c(x,i) + \lambda W^n(x+1,i) + \mu_1 W^n((x-1)^+,i) + \mu_2 W^n(x,0),\ i = 0,1,$$

$$W^n(x,0) = \min\{V^n(x,0), V^n(x-1,1)\} \text{ if } x > 0,$$

$$W^n(0,i) = V^n(0,i),\ W^n(x,1) = V^n(x,1),$$

$$V^0(x,i) = c(x,i).$$

Note that we also allow a customer to move from server 1 to 2, as we took $W^n(1,0) = \min\{V^n(1,0), V^n(0,1)\}$. This however will not occur; $V^n(1,0) \leq V^n(0,1)$ is one of the inequalities needed to show that server 1 should always be used (in [2], it is inequality ii) of lemma 1, with $x = (1,0,0)$).

Remark 1. We can consider discounted costs by multiplying the transition probabilities by a discount factor $\alpha \in [0,1)$, or, equivalently, by taking $\lambda + \mu_1 + \mu_2 < 1$. In the following analysis $\lambda + \mu_1 + \mu_2 = 1$ is not used, thus the results hold also for discounted costs.

To prove the optimality of a threshold policy, it is sufficient to show that

$$V^n(x+1,0) - V^n(x,1) \leq V^n(x+2,0) - V^n(x+1,1). \tag{2.1}$$

Indeed, if $V^n(x+1,0) - V^n(x,1) \leq 0$, then it is optimal not to use server 2, and vice versa. Thus if $V^n(x+1,0) - V^n(x,1)$ is increasing in x, a threshold policy is optimal.

Note however that the threshold depends on n. But, as $n \to \infty$, the optimal policy converges to the average optimal policy, which is consequently also of threshold type. Adding a discount factor to the value function would yield that also the discounted optimal policy is of threshold type.

Inequality (2.1), together with several other inequalities, will be shown to hold in the next section, given that c satisfies the same inequalities.

3. Proof of the Result

We define a class of functions $\mathcal{F}$ as follows: $f \in \mathcal{F}$ if for all $x \in \mathbb{N}_0$

$$f(x+1,0) + f(x+1,1) \leq f(x+2,0) + f(x,1), \tag{3.1}$$

$$f(x+1,0) + f(x,1) \leq f(x,0) + f(x+1,1), \tag{3.2}$$

and

$$f(x,i) \leq f(x+1,i),\ i = 0,1. \tag{3.3}$$

Theorem. *If $c \in \mathcal{F}$, then also $W^n \in \mathcal{F}$ and $V^n \in \mathcal{F}$.*

The first inequality is equivalent to (2.1). It is easily seen that $c(x,i) = g(x+i)$, with g increasing and convex, satisfies the inequalities. However, g should also satisfy the inequalities which are used to prove that idleness of the first server is never optimal. They are given in lemma 1 in [2], and, if g is increasing and convex, they are satisfied as well. (In [2] just $c(x,i) = x+i$ is considered, but lemma 1 can easily be generalized to handle other cost functions as well.)

Remark 2. Inequality (3.2) is known as *supermodularity*. It can be seen that recent results on sub and supermodularity for more general models (as in Glasserman & Yao [1]) cannot be used to prove our result.

Proof of the theorem. Before proving the inequalities by induction, we will first derive two other helpful inequalities. Summing (3.1) and (3.2) yields

$$2f(x+1,0) \leq f(x,0) + f(x+2,0), \tag{3.4}$$

i.e., $f(x,0)$ is convex in x. If we sum (3.1) and (3.2), with x replaced by $x+1$ in (3.2), we find that also $f(x,1)$ is convex in x:

$$2f(x+1,1) \leq f(x,1) + f(x+2,1). \tag{3.5}$$

Now assume that $V^n \in \mathcal{F}$. (Note that $V^0 \in \mathcal{F}$ by assumption.) We will show that both $W^n \in \mathcal{F}$ and $V^{n+1} \in \mathcal{F}$, starting with W^n. Denote keeping server 2 idle with action 0, and sending a customer to server 2 with action 1.

Denote with $a(x,i)$ the optimal action in (x,i) at stage n. We start with (3.1). If $a(x+2,0) = 1$, we have

$$W^n(x+1,0) + W^n(x+1,1) \leq V^n(x,1) + V^n(x+1,1) = W^n(x,1) + W^n(x+2,0),$$

the inequality following from the suboptimality of action 1 in $(x+1,0)$. For $a(x+2,0) = 0$, we have

$$W^n(x+1,0) + W^n(x+1,1) \leq V^n(x+1,0) + V^n(x+1,1) \leq V^n(x+2,0) + V^n(x,1).$$

The first inequality follows from the suboptimality of action 0 in $(x+1,0)$, the second holds by induction. As $a(x+2,0) = 0$, the r.h.s. is equal to $W^n(x+2,0) + W^n(x,1)$.

Inequality (3.2) with $a(x,0) = 0$ can be shown similarly. If $a(x,0) = 1$ (and of course $x > 0$), we get (3.5), by taking action 1 in $(x+1,0)$.

By taking $a(x+1,0)$ in $(x,0)$ if $x > 0$, also (3.3) follows easily for $f = W^n$.

Now consider V^{n+1}. For each of the four terms separately we show that they satisfy the inequalities. By assumption, $c \in \mathcal{F}$. By induction, also the arrival terms, i.e., the terms with coefficient λ, satisfy the corresponding inequalities. Concerning the μ_1-terms, the only non-trivial cases arise for $x = 0$. Inequality (3.1) then reduces to (3.3), (3.2) becomes an equality, and (3.3) also becomes an equality.

When looking at the μ_2-terms, (3.1) reduces to (3.4), (3.2) becomes an equality, (3.3) with $i = 0$ remains the same, and (3.3) with $i = 1$ reduces to (3.3) with $i = 0$. □

Remark 3. We assumed that the arrivals are Poisson, but without losing the optimality result, we can assume the arrival process to be more general, for example a Markov arrival process (MAP). Note that the optimal policy in that case also depends on the state of the arrival process.

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