Math Meth Oper Res (2003) 58:221–236
DOI 10.1007/s001860300292

Mathematical Methods
of Operations Research

On the uniqueness of solutions to the Poisson equations for average cost Markov chains with unbounded cost functions

Sandjai Bhulai[1], **Flora M. Spieksma**[2]

[1] Vrije universiteit Amsterdam, Faculty of Sciences, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands (e-mail: sbhulai@cs.vu.nl)
[2] University of Leiden, Mathematical Institute, P.O. Box 9512, 2300 RA Leiden, The Netherlands (e-mail: spieksma@math.leidenuniv.nl)

Manuscript received: September 2002/Final version received: April 2003

Abstract. We consider the Poisson equations for denumerable Markov chains with unbounded cost functions. Solutions to the Poisson equations exist in the Banach space of bounded real-valued functions with respect to a weighted supremum norm such that the Markov chain is geometrically ergodic. Under minor additional assumptions the solution is also unique. We give a novel probabilistic proof of this fact using relations between ergodicity and recurrence. The expressions involved in the Poisson equations have many solutions in general. However, the solution that has a finite norm with respect to the weighted supremum norm is the unique solution to the Poisson equations. We illustrate how to determine this solution by considering three queueing examples: a multi-server queue, two independent single server queues, and a priority queue with dependence between the queues.

Key words: Geometric ergodicity, Poisson equations, Lyapunov functions, Multi-server queue, Priority queue

1 Introduction

Markov decision processes have been applied to a wide range of stochastic control problems. Particularly, inventory management, inspection-maintenance-replacement systems, and economic planning and consumption models were the earliest applications of this theory. Typically, these systems and models have a bounded cost function, possibly with an infinite state space. A great deal of literature is devoted to such models, see, e.g., Bertsekas [1, 2], Kumar and Varaiya [15], Puterman [19], Ross [20], and Tijms [23].

When Markov decision processes are applied to the control of queueing systems, unbounded cost functions with an infinite state space are common.

The algorithms and methods (e.g., value iteration and policy iteration) developed for bounded cost functions are valid in the Banach space of bounded real-valued measurable functions on the state space. Clearly, this does not hold anymore when working with unbounded cost functions. The remedy to this situation is to introduce a larger suitable Banach space.

Lippman [16] constructs a Banach space with a weighted norm such that the dynamic programming operator remains a contraction mapping with respect to the weighted norm. In the discounted cost case some minor additional assumptions are needed to ensure the existence of optimal stationary policies. Under additional conditions the same result for the average cost case is derived via limits of the discounted cost case. Sennott [21] relaxes the conditions of Lippman and derives conditions for the existence of deterministic average optimal policies.

Dekker [6] and Spieksma [22] study ergodicity and recurrence concepts and enforce stronger conditions for the existence of average optimal policies. However, the results are more practical since they allow explicit formulas and results on convergence of algorithms. Hernández-Lerma and Lasserre [12] also use the stronger conditions for the same reason (the more general setting is discussed in Hernández-Lerma and Lasserre [11]). They give a detailed overview of the theory of Markov decision processes with Borel state spaces and unbounded cost functions.

In this paper we consider the Poisson equations of average cost Markov chains with denumerable state spaces and unbounded cost functions. In contrast to the case of finite state spaces these equations may have more than one solution, even in the bounded cost case. Dekker [6] shows that the solution to the Poisson equations is unique (up to a constant) when we restrict to a Banach space given by a suitable weighted norm. We give an alternative probabilistic proof of Dekker's result using relations between ergodicity and recurrence. The main contribution of this paper is to show how to derive the unique solution to the Poisson equations in practical situations by construction of a suitable weighted norm.

The results are relevant to a variety of cases. Value iteration requires that the dynamic programming operator is a contraction mapping with respect to some Banach space. When cost functions are unbounded, one needs to construct this space by choosing an appropriate norm. In Section 5 we show how to construct such a norm based on results of Spieksma [22].

In policy iteration one starts with a fixed policy. The next step is to solve the Poisson equations induced by this policy in order to obtain the average cost and the value function for this policy. Next, using these expressions one can improve the policy and iterate the procedure. In this case, improving the policy using the right value function is particularly important. When the wrong value function is used convergence of the algorithm is not guaranteed and can be very slow (see, e.g., Chen and Meyn [4], and Meyn [17]).

Ott and Krishnan [18] introduce the idea of one-step policy improvement for deriving approximations to complex systems. The approach in this idea is that for specific policies the average cost and the value function of complex systems can be derived explicitly (see, e.g., Bhulai and Koole [3], Groenevelt, Koole and Nain [10], Koole and Nain [14]). Then, one step of policy iteration can result in a good approximation to the optimal policy of the system. However, using a value function with the wrong structure may give bad approximations.

The organization of this paper is as follows. We start with some preliminary results on ergodicity and recurrence in Section 2. In Section 3 we study the Poisson equations for Markov chains with a multichain structure. Then the unichain case is studied in Section 4. Finally, we apply the results to some queueing systems in Section 5.

2 Preliminaries

Consider a discrete-time uncontrolled Markov chain, given by the tuple (X, P, c). Here, $\{X_t\}_{t\in\mathbb{N}_0}$ is the X-valued uncontrolled Markov chain, P the matrix of transition probabilities, and c the cost function. We will assume that X is a denumerable set. Let $\mathbb{B}(X)$ denote the Banach space of bounded real-valued functions u on X endowed with the supremum norm, i.e.,

$$\|u\| = \sup_{x\in X} |u(x)|.$$

We want to study Markov chains with unbounded cost functions. However, these cost functions are not contained in the Banach space $\mathbb{B}(X)$. A remedy to this situation is to consider suitable larger Banach spaces instead of the space $\mathbb{B}(X)$. In order to construct such a space, consider a function $w : X \to [1, \infty)$, which we will refer to as a weight function. The w-norm is defined as

$$\|u\|_w = \left\|\frac{u}{w}\right\| = \sup_{x\in X} \frac{|u(x)|}{w(x)}.$$

The operator norm of a matrix A on $X \times X$ is given by

$$\|A\|_w = \sup_{i\in X} \frac{1}{w(i)} \sum_{j\in X} |A_{ij}| w(j).$$

A function u is said to be bounded if $\|u\| < \infty$, and w-bounded if $\|u\|_w < \infty$. Let $\mathbb{B}_w(X)$ denote the normed linear space of w-bounded functions on X. Note that for every function u which is bounded, $\|u\|_w \leq \|u\| < \infty$. Furthermore, it is easy to see that every Cauchy-sequence $\{u_n\}$ in w-norm has a w-limit. Hence, $\mathbb{B}_w(X)$ is a Banach space that contains $\mathbb{B}(X)$.

Consider the transition matrix P of the Markov chain. The 0^{th} iterate P^0 will be equal to the identity matrix. The number of closed classes in the Markov chain will be denoted by κ. A state that communicates with every state it leads to, is called essential; otherwise inessential. A set $B \subset X$ will be called a set of reference states if it contains exactly one state from each closed class and no other states. Since the stationary distribution of the Markov chain depends on the initial distribution of the chain, we will consider the stationary matrix P^* given by

$$P^*_{ij} = \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} P^t_{ij},$$

for $i, j \in X$. The Markov chain is said to be stable if P^* is a stochastic matrix. Define the taboo transition matrix for $i \in X$ and $M \subset X$ by

$$_{M}P_{ij} = \begin{cases} P_{ij}, & j \notin M, \\ 0, & j \in M. \end{cases}$$

We will write ${}_MP^n$ for the nth iterate of ${}_MP$, and set ${}_MP^0$ equal to the identity matrix. Let $F_{iM}^{(n)}$ denote the probability that the system, starting in state i, reaches set M for the first time after $n \geq 1$ steps. Then

$$F_{iM}^{(n)} = \sum_{m \in M} ({}_MP^{n-1} \cdot P)_{im}, \quad n \in \mathbb{N}.$$

The probability that set M is eventually reached is given by $F_{iM} = \sum_{n \in \mathbb{N}} F_{iM}^{(n)}$. For a stable Markov chain, one has $F_{iB} = 1$ for all sets B of reference states, and all classes are positive recurrent (see Theorems 6.3 to 7.4 of Chung [5]).

Let $M \subset X$ be finite, and let w be a weight function. The following properties of P will be used in the sequel (cf. Dekker and Hordijk [7,8], Dekker, Hordijk and Spieksma [9], Hordijk and Spieksma [13]). A Markov chain is called

- w-geometrically ergodic [w-GE] if there are constants $r > 0$ and $\beta < 1$ such that $\|P\|_w < \infty$ and $\|P^n - P^*\|_w \leq r\beta^n$ for $n \in \mathbb{N}_0$;
- w-geometrically recurrent with respect to M [w-GR(M)] if there exists an $\epsilon > 0$ such that $\|{}_MP\|_w \leq 1 - \epsilon$;
- w-weak geometrically recurrent with respect to M [w-WGR(M)] if there are constants $r > 0$ and $\beta < 1$ such that $\|{}_MP^n\|_w \leq r\beta^n$ for $n \in \mathbb{N}$;
- w-weak geometrically recurrent with respect to reference states [w-WGRRS(M)] if there is a set of reference states $B \subset M$ such that the Markov chain is w-WGR(B).

The relation between these recurrence and ergodicity concepts is summarized in the following lemma.

Lemma 1(Ch. 2, [22]). *Let $M \subset X$ be finite, and let w be a weight function.*

- *The condition that P is the transition matrix of an aperiodic Markov chain and that w-WGR(M) holds for some M is equivalent to w-GE with $\kappa < \infty$;*
- *w-GR(M) implies w-WGR(M);*
- *w-WGR(M) is equivalent to w-WGRRS(M).*

In the sequel w-geometric ergodicity will play an important role. However, it is not easy to check if a Markov chain is w-GE. Lemma 1 shows that w-GR(M) with the condition that P is the transition matrix of an aperiodic Markov chain implies w-GE. This condition is easier to check, since w-GR(M) is a property which only concerns the transition matrix P, instead of the power matrix P^n for $n \in \mathbb{N}$. We will use this fact in Section 5 to show that the queueing models under study are indeed geometrically ergodic.

If a Markov chain is geometrically ergodic, then the Markov chain is also geometrically recurrent (see Theorem 1.1 of [22]). However, this does not necessarily hold for the same weight function. When the weight function is bounded, then w-geometric ergodicity is equivalent to w-geometric recurrence.

3 The multichain case

Consider a discrete-time uncontrolled Markov chain, given by the tuple (X, P, c). Here $\{X_t\}_{t \in \mathbb{N}_0}$ is the X-valued uncontrolled Markov chain, P the matrix of transition probabilities, and c the cost function. Let $\mathbb{B}_w(X)$ be a normed vector space of w-bounded real-valued functions on X for some weight function w, such that $c \in \mathbb{B}_w(X)$. Furthermore, let κ denote the number of positive recurrent classes of the Markov chain. The Poisson equations for this system are given by

$$\varphi = P\varphi, \tag{1}$$

$$\varphi + v = c + Pv, \tag{2}$$

for some functions φ and v in $\mathbb{B}_w(X)$. By virtue of this definition we will call φ and v a solution to the Poisson equation when they have a finite norm with respect to w in the Banach space. The existence of solutions to the Poisson equations is guaranteed under the following assumptions.

Lemma 2. ([13], Prop 5.1; [22]). *Assume that the Markov chain is w-GE with $\kappa < \infty$. Then solutions (φ, v) to the Poisson equations exist in the space of w-bounded functions $\mathbb{B}_w(X)$.*

Note that by Lemma 1 the requirement that P is the matrix of an aperiodic Markov chain with w-GR(M) for some M, instead of w-GE and $\kappa < \infty$ also suffices. The ergodicity conditions are hard to verify in practice, whereas the recurrence conditions as a property of the one-step transition matrices are relatively easy to verify. We prefer to work with the ergodicity condition in the sequel, as this is the more general concept.

As a first step towards proving uniqueness of the solution to the Poisson equations, we study the average cost first. The average cost is uniquely determined by the transition matrix P and the cost function c. The following theorem summarizes this result.

Theorem 3. *Suppose that the Markov chain is w-GE, and that $\kappa < \infty$. Let the functions φ and v in $\mathbb{B}_w(X)$ be a solution to the Poisson equations. Then*

$$\varphi = \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} P^t \varphi = \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} P^t c = P^* c. \tag{3}$$

Proof. By iterating Equation (1) we find that $\varphi = P^t \varphi$ for $t \in \mathbb{N}_0$. Taking a finite sum over t from 0 to $n-1$ yields

$$n\varphi = \sum_{t=0}^{n-1} P^t \varphi, \quad n \in \mathbb{N}, \tag{4}$$

which implies the first equality. The second equality is obtained by writing Equation (2) as $v = (c - \varphi) + Pv$. By iteration of this expression we derive

$$v = \sum_{t=0}^{n-1} P^t (c - \varphi) + P^n v, \quad n \in \mathbb{N}. \tag{5}$$

Since the Markov chain is w-GE and $\kappa < \infty$, there exist non-negative constants r and $\beta < 1$ such that

$$\|P^n - P^*\|_w \leq r\beta^n, \quad n \in \mathbb{N}.$$

As a consequence there exists a constant C such that $\|P^n\|_w \leq C$ for all $n \in \mathbb{N}$. Hence,

$$\left\| \frac{1}{n} P^n v \right\|_w \leq \frac{1}{n} \|P^n\|_w \|v\|_w \to 0, \quad n \to \infty.$$

The last implication directly follows from w-geometric ergodicity. □

Note that in Theorem 3 the w-GE condition is only used for the second equality. The results of Theorem 3 are also stated in Corollary 7.5.6(a) and (b) of Hernández-Lerma and Lasserre [12]. They do not impose that the Markov chain should be geometrically ergodic, but merely assume that $P^n v/n \to 0$ as n grows to infinity. We have shown that this condition is fulfilled when the Markov chain is geometrically ergodic. Hence, this also holds under the more easily verifiable geometric recurrence condition.

Next, we move on to study the properties of the relative value function v. Since a Markov chain with a multichain structure can have classes with different state classifications, one can not expect the value function v to be uniquely determined. However, when the Markov chain is stable, then more can be said about the value function. The following result can be found in Dekker [6, Ch. 3, Lemma 2.2] in the setting of Blackwell optimality equations using Laurent series expansion techniques. We give an alternative probabilistic proof using the relations between ergodicity and recurrence.

Theorem 4. *Suppose that the Markov chain is stable, w-GE, and that $\kappa < \infty$. Then a solution to the Poisson equations is given by*

$$\varphi = P^* c, \quad v = \sum_{n=0}^{\infty} (P^n - P^*) c = Dc,$$

with D the deviation matrix. Furthermore, if both (φ, v) and (φ', v') are solutions to the Poisson Equations [with all functions in $\mathbb{B}_w(X)$]. Then $\varphi = \varphi'$ and

$$v - v' = \lim_{n \to \infty} \frac{1}{n} \sum_{t=0}^{n-1} P^t (v - v'). \tag{6}$$

Moreover, the functions v and v' only differ by a constant on each closed class.

Proof. From Theorem 3 it immediately follows that $\varphi = P^* c$ and $\varphi = \varphi'$. The fact that $v = Dc$ is a solution follows from direct insertion into Equation (2), and using w-boundedness. Equation (2) yields $v - v' = P(v - v')$. Consequently, the derivation of Expression (6) proceeds along the same lines as the derivation of Expression (3).

For the last statement, write Equation (2) as $v = (c - \varphi) + Pv$. Since the Markov chain is w-GE, it is also w-WGRRS(M) for some finite set $M \subset X$. Hence, there exist non-negative constants r and $\beta < 1$, and a set $B = \{b_1, \ldots, b_\kappa\} \subset M$ such that $\|{}_B P^n\|_w \leq r\beta^n$ for $n \in \mathbb{N}$. Let i be an essential state, and observe that

$$v_i = (c - \varphi)_i + (Pv)_i = (c - \varphi)_i + ({}_B Pv)_i + \sum_{k=1}^{\kappa} P_{ib_k} v_{b_k}.$$

By iteration of this expression we derive

$$v_i = \left[\sum_{t=0}^{n-1} {}_BP^t(c-\varphi)\right]_i + ({}_BP^n v)_i + \sum_{k=1}^{\kappa}\sum_{t=0}^{n-1} ({}_BP^t \cdot P)_{ib_k} v_{b_k}, \quad n \in \mathbb{N}. \tag{7}$$

Note that $({}_BP^t \cdot P)_{ib} = F_{ib}^{(t+1)}$; the probability that the system, starting in state i, enters state b for the first time after $t+1$ steps. A similar expression also holds for v'. Hence, by substituting $\varphi' = \varphi$ in the expression of v', we find

$$(v-v')_i = ({}_BP^n v)_i - ({}_BP^n v')_i + \sum_{k=1}^{\kappa}\sum_{t=0}^{n-1} F_{ib_k}^{(t+1)}(v_{b_k} - v'_{b_k}), \quad n \in \mathbb{N}.$$

The first two terms converge geometrically fast to zero as $n \to \infty$. Therefore, by taking the limit we find

$$(v-v')_i = \lim_{n\to\infty} \sum_{k=1}^{\kappa}\sum_{t=0}^{n-1} F_{ib_k}^{(t+1)}(v_{b_k} - v'_{b_k}) = \sum_{k=1}^{\kappa} F_{ib_k}(v_{b_k} - v'_{b_k}).$$

Since the Markov chain is stable, all closed classes $\mathcal{C}_{b_1}, \ldots, \mathcal{C}_{b_\kappa}$ of the Markov chain are positive recurrent. Thus, there is exactly one state $j = j(i) \in \{1, \ldots, \kappa\}$ such that $i \in \mathcal{C}_{b_j}$. Furthermore, for all $i \in \mathcal{C}_{b_j}$ we have $F_{ib_j} = 1$ and zero otherwise. Therefore,

$$(v-v')_i = v_{b_j} - v'_{b_j}, \quad i \in \mathcal{C}_{b_j}.$$

Thus, v and v' differ by only a constant on each closed class. □

4 The unichain case

In this section we will assume that the Markov chain has a unichain structure, thus $\kappa = 1$, i.e., there is only one recurrent class with a possibly empty set of transient states. In this case more can be said about the solution to the Poisson equations. We will therefore reformulate the theorems of the previous section, such that they convey more information and utilize the structure of the Markov chain.

Theorem 5. *Suppose that the Markov chain is stable and w-GE. Let the functions φ and v in $\mathbb{B}_w(X)$ be a solution to the Poisson equations. Then P^* has equal rows, say π, and φ is a constant given by*

$$\varphi = \sum_{i\in X} \pi_i c_i.$$

Proof Since the Markov chain is stable, P^* is a stochastic matrix. Due to the unichain structure the rows of the matrix P^* are identical (Theorem A.2 and Expression (A.6) of Puterman [19]). Therefore, the result follows by Theorem 3. □

Hernández-Lerma and Lasserre [12] work with Borel state spaces. In order to derive the equivalent result of Theorem 5, they have to work with the invariant probability measure, and would have to state that the result holds almost everywhere (in probability). Since this would not result in an "explicit" expression for φ, they have chosen to consider Markov processes for which the result holds for every $x \in X$. Note that in the case of denumerable state spaces, this is not necessary.

The specialized result of Theorem 4 is as follows.

Theorem 6. *Suppose that the Markov chain is stable and* w-GE. *Let both* (φ, v) *and* (φ', v') *be solutions to the Poisson Equations* [*with all functions in* $\mathbb{B}_w(X)$]. *Then* $\varphi = \varphi'$, *and* v, v' *only differ by a constant. Furthermore, for any positive recurrent state* m

$$v = \sum_{t=0}^{\infty} {}_mP^t(c - \varphi) + v_m.$$

Proof. Due to the unichain structure, there is only one closed class. The possibly empty set of transient states all reach states in the positive recurrent class with probability one. Therefore, the first two statements follow from Theorem 4. The last statement follows by taking the limit $n \to \infty$ in Expression (7). □

5 Examples

In this section we will study three queueing examples; a multi-server queue, two independent single server queues, and a priority queue with switching costs. In all three cases we will derive explicit solutions to the Poisson equations. Furthermore, we will construct (or show the existence of) suitable weight functions, such that we can obtain the unique solution that is finite with respect to the weighted norm.

The first example is an one-dimensional queueing problem. The example illustrates how a suitable weight function can be constructed. The construction is based on Chapter 3 of Spieksma [22]. The second example is a two-dimensional model. Since the queues are independent, one would expect to obtain a simple weight function. However, this is not the case; an explicit weight function is difficult to formulate. We will show that there exists a weight function with the right properties. In contrast to the second example, the third example concerns a queueing problem with dependence between the queues.

The multi-server queue. Consider a queueing system with one queue and s identical independent servers. The arrivals are determined by a Poisson process with parameter λ. The service times are exponentially distributed with parameter μ. Let state $x \in \mathbb{N}_0$ denote the number of customers in the system. The costs occurring in the system are made up of holding costs h per unit of time a customer is in the system. Assume that the stability condition $\rho = \lambda/s\mu < 1$ holds. Then the Markov chain satisfies the unichain condition, hence the Poisson equations for this system are given by

$$\varphi + \big(\lambda + \min\{x,s\}\mu\big)V(x) = hx + \lambda V(x+1) + \min\{x,s\}\mu V\big([x-1]^+\big), \quad x \in \mathbb{N}_0,$$

with $[x]^+ = \max\{x, 0\}$. Bhulai and Koole [3, Theorems 2.1 and 2.2] show that Equations (1) and (2) are satisfied by

$$V(x) = \frac{\varphi}{\lambda}\sum_{i=1}^{x} F(i) - \frac{h}{\lambda}\sum_{i=1}^{x}(i-1)F(i-1),$$

for $x = 0, \ldots, s$ with

$$F(x) = \sum_{k=0}^{x-1} \frac{\Gamma(x)}{\Gamma(x-k)}\left(\frac{\mu}{\lambda}\right)^k,$$

where $\Gamma(x) = (x-1)!$ when x is integer. For $x > s$ the solution is given by

$$V(x) = -\frac{x-s}{\mu(1-\rho)}\varphi + \left[\frac{(x-s)(x-s+1)}{2\mu(1-\rho)} + \frac{(x-s)\big(\rho + s(1-\rho)\big)}{\mu(1-\rho)^2}\right]h$$
$$+ V(s) + \frac{\left(\frac{1}{\rho}\right)^{x-s} - 1}{1-\rho}\left[\frac{1}{\mu(1-\rho)}\varphi + \Delta V(s) - \frac{\rho + s(1-\rho)}{\mu(1-\rho)^2}h\right],$$

with $\Delta V(s) = V(s) - V(s-1)$. Observe that the solution depends on φ. Thus, for every arbitrarily chosen value of φ, a function V exists such that Equations (1) and (2) are satisfied. However, among the pairs of solutions there will be one pair such that $V \in \mathbb{B}_w(\mathbb{N}_0)$ for a weight function w such that the Markov chain is w-GE; this will be the unique solution to the Poisson equations.

Bhulai and Koole [3] first study the multi-server queue with a finite buffer under holding, waiting, and rejection costs. In this case, the Poisson equations have a unique solution, since the state space is finite. Afterwards, they take the limit to infinity with respect to the buffer size to obtain expressions for the infinite buffer case. However, solving the Poisson equations for the finite buffer case is more difficult due to boundary effects, and therefore undesirable when only expressions for the infinite buffer case are needed.

We will now illustrate how the results of the previous sections can be applied to this model. In order to do that, we have to construct a suitable weight function w. Spieksma [22, Chapter 3] shows that a suitable weight function for this model is of the form $w(x) = (1+k)^x$ for $k < k^*$, where k^* has to be determined. In order to determine k^* explicitly, we adopt the same technique described in Chapter 3 of Spieksma [22]. Hence, we will first determine a weight function w such that the Markov chain is w-GR(M) for some finite set $M \subset X$.

First, observe that due to the stability condition $\rho = \lambda/s\mu < 1$, the average cost $\varphi < \infty$. Furthermore, it also makes the Markov chain stable. Assume without loss of generality that $\lambda + s\mu < 1$ (this can always be obtained through scaling). Then the multi-server queue has the following transition rate matrix.

$$\begin{array}{lll} P_{0,0} = 1-\lambda, & P_{0,1} = \lambda, & \\ P_{x,x-1} = x\mu, & P_{x,x} = 1-\lambda-x\mu, & P_{x,x+1} = \lambda, \quad x = 1,\ldots,s-1, \\ P_{x,x-1} = s\mu, & P_{x,x} = 1-\lambda-s\mu, & P_{x,x+1} = \lambda. \end{array}$$

Let $M = \{0, \ldots, s-1\}$, and assume that $w(x) = (1+k)^x$ for some k. Now consider

$$\sum_{y \notin M} \frac{P_{xy} w(y)}{w(x)} = \begin{cases} \lambda(1+k), & x = s-1, \\ (1-\lambda-s\mu) + \lambda(1+k), & x = s, \\ \frac{s\mu}{1+k} + (1-\lambda-s\mu) + \lambda(1+k), & x > s. \end{cases}$$

We need to choose k such that all expressions are strictly less than 1. The first expression immediately gives $k < (1-\lambda)/\lambda$. The second expression gives $k < s\mu/\lambda$. Solving the third expression gives $0 < k < (s\mu - \lambda)/\lambda$. Since $s\mu - \lambda < 1 - \lambda$ and $s\mu - \lambda < s\mu$, we finally obtain that $k^* = (s\mu - \lambda)/\lambda$. Observe that we have shown that for $w(x) = (1+k)^x$ with $0 < k < k^*$, there exists an $\epsilon > 0$ such that $\|_M P\|_w \leq 1 - \epsilon$. Hence, the Markov chain is w-GR(M), and therefore also w-GE.

It is clear that the cost function $c \in \mathbb{B}_w(\mathbb{N}_0)$ with $c(x) = hx$. Observe that $w(x) = (1+k)^x = z^x$ with $z < 1/\rho$. Since the exponential term $(1/\rho)^{(x-s)}$ appears in $V(x)$ for $x > s$, the expression $V(x)/w(x)$ will contain an exponential term with a base greater than zero. Therefore, $\|V\|_w = \infty$, unless φ is chosen such that the exponential term cancels. Hence, to meet the final requirement that $V \in \mathbb{B}_w(\mathbb{N}_0)$, i.e., $\|V\|_w < \infty$, we need that

$$\frac{1}{\mu(1-\rho)} \varphi + \Delta V(s) - \frac{\rho + s(1-\rho)}{\mu(1-\rho)^2} h = 0.$$

Since $\Delta V(s) = (\varphi/\lambda) \cdot F(s) - (h/\lambda) \cdot (s-1) F(s-1)$, we find that

$$\begin{aligned} \varphi &= \left[\frac{1}{1-\rho} + \frac{1}{\rho} F(s)\right]^{-1} \left[\frac{\rho + s(1-\rho)}{(1-\rho)^2} + \frac{s-1}{\rho} F(s-1)\right] h \\ &= \left[\frac{1}{1-\rho} + \frac{1}{\rho} F(s)\right]^{-1} \left[\frac{\rho}{(1-\rho)^2} + \frac{s\rho}{1-\rho} + sF(s)\right] h. \end{aligned}$$

The expression for φ is exactly the expression in Theorem 2.3 in Bhulai and Koole [3]. Observe that the choice of $M = \{0, \ldots, s-1\}$ permits us to look at large states only. In that way we do not have to deal with the transition probabilities for states in M, which have more complicated transition probabilities depending on the state itself.

Two independent single server queues. Consider a queueing system with two queues with their own dedicated server (two parallel $M/M/1$ queues). The arrivals to queue i are determined by a Poisson process with parameter λ_i for $i = 1, 2$. The service times at queue i are exponentially distributed with parameter μ_i for $i = 1, 2$. Let state $(x, y) \in \mathbb{N}_0^2$ denote that there are x customers at queue 1 and y customers at queue 2, respectively. The costs occurring in the system are made up of holding costs h_i per unit of time a customer is in queue i for $i = 1, 2$. Let $\rho_i = \lambda_i/\mu_i$ for $i = 1, 2$, and assume that the (strong) ergodicity condition $\rho_1 + \rho_2 < 1$ holds. Then the Markov chain satisfies the unichain condition, hence the Poisson equations for this system are given by

$$\begin{aligned} \varphi + (\lambda_1 + \lambda_2 + \mu_1 + \mu_2) V(x, y) = {} & h_1 x + h_2 y + \lambda_1 V(x+1, y) + \lambda_2 V(x, y+1) \\ & + \mu_1 V\big([x-1]^+, y\big) + \mu_2 V\big(x, [y-1]^+\big), \quad x, y \in \mathbb{N}_0. \end{aligned}$$

Note that the average cost φ is composed of φ_1 and φ_2, which are due to customers in the first and second queue respectively. Since the two queues are independent, the value function of this system is given by the sum of the value functions of the $M/M/1$ queueing systems with holding costs h_1 and h_2, respectively. Note that the these value functions can be easily derived from the previous example with $s = 1$. Hence, the value function $V = V_\varphi + V_{h_1} + V_{h_2}$ is given by

$$\begin{aligned}
V_\varphi(x,y) &= -\frac{x}{\mu_1(1-\rho_1)}\varphi_1 + \left[\left(\frac{1}{\rho_1}\right)^x - 1\right]\frac{1}{\mu_1(1-\rho_1)^2}\varphi_1 \\
&\quad - \frac{y}{\mu_2(1-\rho_2)}\varphi_2 + \left[\left(\frac{1}{\rho_2}\right)^y - 1\right]\frac{1}{\mu_2(1-\rho_2)^2}\varphi_2, \\
V_{h_1}(x,y) &= \frac{x(x+1)}{2\mu_1(1-\rho_1)}h_1 + \frac{\rho_1 x}{\mu_1(1-\rho_1)^2}h_1 - \left[\left(\frac{1}{\rho_1}\right)^x - 1\right]\frac{\rho_1}{\mu_1(1-\rho_1)^3}h_1, \\
V_{h_2}(x,y) &= \frac{y(y+1)}{2\mu_2(1-\rho_2)}h_2 + \frac{\rho_2 y}{\mu_2(1-\rho_2)^2}h_2 - \left[\left(\frac{1}{\rho_2}\right)^y - 1\right]\frac{\rho_2}{\mu_2(1-\rho_2)^3}h_2.
\end{aligned}$$

One could expect that the weight function has a similar structure, i.e., $w(x,y) = (1+k_1)^x(1+k_2)^y$ with $0 < k_i < k_i^* = (\mu_i - \lambda_i)/\lambda_i$ for $i = 1,2$. However, this turns out not to be true. In order to find a suitable weight function we apply the same technique as in the previous example.

First, observe that due to the ergodicity condition $\rho_i = \lambda_i/\mu_i < 1$ for $i = 1,2$, and the average cost $\varphi < \infty$. Furthermore, it makes the Markov chain stable. Assume without loss of generality that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 < 1$ (this can always be obtained through scaling). Then the system has the following transition rate matrix.

$$\begin{aligned}
&P_{(x,y)(x+1,y)} = \lambda_1, \qquad\qquad P_{(x,y)(x,y+1)} = \lambda_2, \\
&P_{(x,y)(x-1,y)} = \mu_1 \mathbb{1}(x>0), \quad P_{(x,y)(x,y-1)} = \mu_2 \mathbb{1}(y>0), \\
&P_{(x,y)(x,y)} = 1 - P_{(x,y)(x+1,y)} - P_{(x,y)(x-1,y)} - P_{(x,y)(x,y+1)} - P_{(x,y)(x,y-1)}.
\end{aligned}$$

Let $M = \{(0,0)\}$, and assume that $w(x,y) = (1+k_1)^x(1+k_2)^y$ for some k_1 and k_2. Now consider

$$\sum_{(x',y')\notin M} \frac{P_{(x,y)(x',y')}w(x',y')}{w(x,y)},$$

which is given by

$$\begin{cases}
\lambda_1(1+k_1)+\lambda_2(1+k_2), & (x,y)=(0,0), \\
\lambda_1(1+k_1)+\lambda_2(1+k_2)+(1-\lambda_1-\lambda_2-\mu_1), & (x,y)=(1,0), \\
\lambda_1(1+k_1)+\lambda_2(1+k_2)+(1-\lambda_1-\lambda_2-\mu_2), & (x,y)=(0,1), \\
\lambda_1(1+k_1)+\lambda_2(1+k_2)+\frac{\mu_1}{1+k_1}+(1-\lambda_1-\lambda_2-\mu_1), & x>1, y=0, \\
\lambda_1(1+k_1)+\lambda_2(1+k_2)+\frac{\mu_2}{1+k_2}+(1-\lambda_1-\lambda_2-\mu_2), & x=0, y>1, \\
\lambda_1(1+k_1)+\lambda_2(1+k_2)+\frac{\mu_1}{1+k_1}+\frac{\mu_2}{1+k_2}+(1-\lambda_1-\lambda_2-\mu_1-\mu_2), & x>0, y>0.
\end{cases}$$

We need to choose k_1 and k_2 such that all expressions are strictly less than 1. Observe that if the fourth and fifth expression are less than 1, then all others are also satisfied. Furthermore, $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 < 1$ implies $\lambda_1 + \lambda_2 + \max\{\mu_1, \mu_2\} < 1$. Hence we can restrict our attention to the system

$$f_1(k_1,k_2)=1+\lambda_1 k_1+\lambda_2 k_2-\frac{\mu_1 k_1}{1+k_1},\quad f_2(k_1,k_2)=1+\lambda_1 k_1+\lambda_2 k_2-\frac{\mu_2 k_2}{1+k_2},$$

with the assumptions $\lambda_1+\lambda_2+\max\{\mu_1,\mu_2\}<1$ and $\rho_1+\rho_2<1$.

Observe that $f_1(0,0)=f_1\big((\mu_1-\lambda_1)/\lambda_1,0\big)=1$. Thus, the curve $f_1(k_1,k_2)=1$ passes through $(0,0)$ and $\big((\mu_1-\lambda_1)/\lambda_1,0\big)$. Furthermore, it satisfies the equality $k_2=\mu_1/\lambda_2-\mu_1/\big(\lambda_2(1+k_1)\big)-\lambda_1/\lambda_2$. Note that this function has a maximum value at $k_1=\sqrt{1/\rho_1}-1$. Hence, this description determines the form of f_1; the curve $f_1(k_1,k_2)=1$ starts in $(0,0)$, and increases to an extreme point, and then decreases to the k_1-axis again. The curve f_2 has a similar form, but with the role of the k_1-axis interchanged with the k_2-axis.

The curves determine an area of points (k_1,k_2) such that f_1 and f_2 are strictly less than one, if the partial derivative to k_1 at $(0,0)$ of the curve $f_1(k_1,k_2)=1$ is greater than the partial derivative to k_2 of the curve $f_2(k_1,k_2)=1$ at $(0,0)$. These partial derivatives are given by $(\mu_1-\lambda_1)/\lambda_2$ and $\lambda_1/(\mu_2-\lambda_2)$, respectively. Since $\rho_1+\rho_2<1$, we have $\lambda_1\mu_2+\lambda_2\mu_1<\mu_1\mu_2$. Adding $\lambda_1\lambda_2$ to both sides gives $\lambda_1\lambda_2<\mu_1\mu_2-\lambda_1\mu_2-\lambda_2\mu_1+\lambda_1\lambda_2=(\mu_1-\lambda_1)(\mu_2-\lambda_2)$. Hence, the relation $\lambda_1/(\mu_2-\lambda_2)<(\mu_1-\lambda_1)/\lambda_2$ holds. Thus, indeed there is an area of pairs (k_1,k_2) such that the Markov chain is w-GR(M), and thus also w-GE. For these points it holds that $(1+k_i)<1/\rho_i$ for $i=1,2$.

It is clear that the cost function $c\in\mathbb{B}_w(\mathbb{N}_0^2)$ with $c(x,y)=h_1x+h_2y$. To meet the final requirement that $V\in\mathbb{B}_w(\mathbb{N}_0^2)$, i.e., $\|V\|_w<\infty$, we need that

$$\frac{1}{\mu_1(1-\rho_1)^2}\varphi_1-\frac{\rho_1}{\mu_1(1-\rho_1)^3}h_1=0,\qquad \frac{1}{\mu_2(1-\rho_2)^2}\varphi_2-\frac{\rho_2}{\mu_2(1-\rho_2)^3}h_2=0.$$

Finally, we obtain the unique pair (φ,V) by solving this equation. The result is as expected

$$\varphi=\frac{\rho_1}{1-\rho_1}h_1+\frac{\rho_2}{1-\rho_2}h_2.$$

Observe that in this example we required the stronger condition $\rho_1+\rho_2<1$ instead of $\rho_1<1$ and $\rho_2<1$. In case of $\rho_1+\rho_2>1$ the curves $f_1(k_1,k_2)=1$ and $f_2(k_1,k_2)=1$ do not determine a common area of points such that f_1 and f_2 are simultaneously less than 1. Note that the stronger condition is superfluous. Hordijk and Spieksma [13] obtain geometric ergodicity for which stability of the Markov chain suffices. However, the expressions are cumbersome. For our purpose of proving geometric ergodicity for the priority queue, it suffices to consider the stronger condition, which yields a simpler expression.

A priority queue with switching costs. Consider a queueing system with two classes of customers arriving according to a Poisson process. There is only one server available serving either a class-1 or a class-2 customer with exponentially distributed service times. A class-i customer has arrival rate λ_i, and is served with rate μ_i for $i=1,2$. The system is subject to holding costs and switching costs. The cost of holding a class-i customer in the system for one unit of time is h_i for $i=1,2$. The cost of switching from serving a class-1 to a class-2 customer (from a class-2 to a class-1 customer) is s_1 (s_2, respectively).

The system follows a priority discipline indicating that class-1 customers have priority over class-2 customers. The priority is also preemptive, i.e., when serving a class-2 customer, the server switches immediately to serve a class-1 customer upon arrival of a class-1 customer. Upon emptying the queue of class-1 customers, the service of class-2 customers, if any present, is resumed from the point where it was interrupted. Due to the exponential service times, this is equivalent to restarting the service for this customer.

Let state (x, y, z) for $x, y \in \mathbb{N}_0, z \in \{1, 2\}$ denote that there are x class-1 and y class-2 customers present in the system, with the server serving a class-z customer, if present. Let $\rho_i = \lambda_i/\mu_i$ for $i = 1, 2$ and assume that the stability condition $\rho_1 + \rho_2 < 1$ holds. Then the Markov chain is stable and $\varphi < \infty$ holds. Furthermore, the Markov chain satisfies the unichain condition, hence the Poisson equations are given by

$$\begin{aligned}
\varphi + (\lambda_1 + \lambda_2 + \mu_1)V(x, y, 1) &= h_1 x + h_2 y + \lambda_1 V(x + 1, y, 1) + \lambda_2 V(x, y + 1, 1) \\
&\quad + \mu_1 V(x - 1, y, 1), \quad x > 0, y \geq 0, \\
V(0, y, 1) &= s_1 + V(0, y, 2), \quad y > 0, \\
V(x, y, 2) &= s_2 + V(x, y, 1), \quad x > 0, y \geq 0, \\
\varphi + (\lambda_1 + \lambda_2 + \mu_2)V(0, y, 2) &= h_2 y + \lambda_1 V(1, y, 2) + \lambda_2 V(0, y + 1, 2) \\
&\quad + \mu_2 V(0, y - 1, 2), \quad y > 0, \\
\varphi + (\lambda_1 + \lambda_2)V(0, 0, z) &= \lambda_1 V(1, 0, z) + \lambda_2 V(0, 1, z), \quad z = 1, 2.
\end{aligned}$$

Koole and Nain [14] study this queueing model in the context of total discounted costs. Groenevelt, Koole and Nain [10] study the average cost case of this model, and determine the value function by numerical experimentation. Furthermore, they provide some intuition into the form of the value function. However, they do not derive all solutions to the optimality equations, and they do not show that their solution is the unique solution to the Poisson equations. In this example we will provide all solutions, and show that the norm of the previous example yields the unique solution.

First observe that, given the values of $V(x, y, 1)$, the values of $V(x, y, 2)$ are easily obtained by considering the difference equations: $V(x, y, 2) = V(x, y, 1) + (\lambda_1 s_2 - \lambda_2 s_1)/(\lambda_1 + \lambda_2)\,\mathbb{1}(x = 0, y = 0) - s_1\,\mathbb{1}(x = 0, y > 0) + s_2\mathbb{1}(x > 0, y \geq 0)$. Therefore, we will only show the expression for $V(x, y, 1)$, as $V(x, y, 2)$ is easily derived from it. Let $V = V_\varphi + V_{h_1} + V_{h_2} + V_{s_1} + V_{s_2}$ be the decomposition of the value function. Then the previous observation directly prescribes that V_φ, V_{h_1}, and V_{h_2} are independent of z. Furthermore, the value function V_{h_1} equals the value function of the single server queue due to the preemptive priority behaviour of the system.

The other value functions can be derived by setting $V(x, y) = c_1 x^2 + c_2 x + c_3 y^2 + c_4 y + c_5 xy + c_6$ with constants c_i for $i = 1, \ldots, 6$ to be determined. This quadratic form is to be expected, when one considers the solutions given in Groenevelt, Koole and Nain [10], and Koole and Nain [14]. Substitution of this equality into the optimality equations yields a new set of equations that is easier to solve. However, in order to correct for the boundary $x = 0$, one needs to add the term $c_7[(1/\rho_1)^x - 1]$, which is the homogeneous solution to the equations for $x > 0$ and $y > 0$. Let θ be the unique root $\theta \in (0, 1)$ of the equation

$$\lambda_1 x^2 - (\lambda_1 + \lambda_2 + \mu_1)x + \mu_1 = 0.$$

Then the correction term for the boundary $y = 0$ is given by $c_8(1 - \theta^x)$. Observe that for the boundary $x = 0$, we do not have correction terms with powers of y. This has to do with the fact that there are no jumps towards the boundary $y = 0$ from the interior of the state space. Solving for the constants yields the solution to the optimality equations, which is given by

$$\begin{aligned}
V_\varphi(x,y,z) &= -\frac{x}{\mu_1(1-\rho_1)}\varphi + \left[\left(\frac{1}{\rho_1}\right)^x - 1\right]\frac{1}{\mu_1(1-\rho_1)^2}\varphi,\\
V_{h_1}(x,y,z) &= \frac{x(x+1)}{2\mu_1(1-\rho_1)}h_1 + \frac{\rho_1 x}{\mu_1(1-\rho_1)^2}h_1 - \left[\left(\frac{1}{\rho_1}\right)^x - 1\right]\frac{\rho_1}{\mu_1(1-\rho_1)^3}h_1,\\
V_{h_2}(x,y,z) &= \frac{\lambda_2 x(x+1)}{2\mu_1^2(1-\rho_1)(1-\rho_1-\rho_2)}h_2 + \frac{\rho_2(\mu_1-\lambda_1+\mu_2\rho_1)x}{\mu_1^2(1-\rho_1)^2(1-\rho_1-\rho_2)}h_2\\
&\quad + \frac{y(y+1)}{2\mu_2(1-\rho_1-\rho_2)}h_2 + \frac{xy}{\mu_1(1-\rho_1-\rho_2)}h_2\\
&\quad - \left[\left(\frac{1}{\rho_1}\right)^x - 1\right]\frac{\rho_2(\mu_1-\lambda_1+\mu_2\rho_1)}{\mu_1^2(1-\rho_1)^3(1-\rho_1-\rho_2)}h_2,
\end{aligned}$$

$$\begin{aligned}
V_{s_i}(x,y,1) &= \frac{\lambda_1\theta}{\lambda_1+\lambda_2}\frac{\rho_1\rho_2 x}{1-\rho_1}s_i + \frac{\lambda_1\theta}{\lambda_1+\lambda_2}\frac{\lambda_1 y}{\mu_2}s_i - \left[\left(\frac{1}{\rho_1}\right)^x - 1\right]\frac{s_i}{\mu_1(1-\rho_1)^2}\\
&\quad\times\left[\lambda_1\left\{\rho_1\left(\frac{\lambda_1\theta}{\lambda+\lambda_2} - 1\right) + \frac{\lambda_1}{\lambda_1+\lambda_2}(1-\theta)\right\}\right.\\
&\quad\left. + \lambda_2\left\{\frac{\lambda_1\theta}{\lambda_1+\lambda_2}\frac{\lambda_1}{\mu_2} + \frac{\lambda_1}{\lambda_1+\lambda_2}\right\}\right] + \frac{\lambda_1}{\lambda_1+\lambda_2}s_i\mathbb{1}(y>0)\\
&\quad + \frac{\lambda_1}{\lambda_1+\lambda_2}(1-\theta^x)s_i\mathbb{1}(x>0,\ y=0), \quad i = 1,2.
\end{aligned}$$

Observe that, regardless of the position of the server, the system dynamics satisfy equations with the same transition rates. Hence, when the case $z = 1$ is studied separately from $z = 2$, the transition rates for both cases will be the same. Thus, we can expect that $w(x,y) = (1+k_1)^x(1+k_2)^y$ for some k_1 and k_2 to be a suitable weight function (independent of z).

Due to the stability condition $\rho_1 + \rho_2 < 1$, the average cost $\varphi < \infty$. Furthermore, it makes the Markov chain stable. Assume without loss of generality that $\lambda_1 + \lambda_2 + \max\{\mu_1, \mu_2\} < 1$ (this can always be obtained through scaling). Then the system has the following transition rate matrix.

$$\begin{aligned}
&P_{(x,y)(x+1,y)} = \lambda_1,\ P_{(x,y)(x,y+1)} = \lambda_2,\\
&P_{(x,y)(x-1,y)} = \mu_1\mathbb{1}(x>0), P_{(x,y)(x,y-1)} = \mu_2\mathbb{1}(x=0,y>0),\\
&P_{(x,y)(x,y)} = 1 - P_{(x,y)(x+1,y)} - P_{(x,y)(x-1,y)} - P_{(x,y)(x,y+1)} - P_{(x,y)(x,y-1)}.
\end{aligned}$$

Let $M = \{(0,0)\}$, and assume that $w(x,y) = (1+k_1)^x(1+k_2)^y$ for some k_1 and k_2. Now consider

$$\sum_{(x',y')\notin M}\frac{P_{(x,y)(x',y')}w(x',y')}{w(x,y)},$$

which is given by

$$\begin{cases} \lambda_1(1+k_1)+\lambda_2(1+k_2), & (x,y)=(0,0), \\ \lambda_1(1+k_1)+\lambda_2(1+k_2)+(1-\lambda_1-\lambda_2-\mu_1), & (x,y)=(1,0), \\ \lambda_1(1+k_1)+\lambda_2(1+k_2)+(1-\lambda_1-\lambda_2-\mu_2), & (x,y)=(0,1), \\ \lambda_1(1+k_1)+\lambda_2(1+k_2)+\frac{\mu_1}{1+k_1}+(1-\lambda_1-\lambda_2-\mu_1), & x>1,\ y=0, \\ \lambda_1(1+k_1)+\lambda_2(1+k_2)+\frac{\mu_2}{1+k_2}+(1-\lambda_1-\lambda_2-\mu_2), & x=0,\ y>1, \\ \lambda_1(1+k_1)+\lambda_2(1+k_2)+\frac{\mu_1}{1+k_1}+(1-\lambda_1-\lambda_2-\mu_1), & x>0,\ y>0. \end{cases}$$

We need to choose k_1 and k_2 such that all expressions are strictly less than 1. However, this problem is equivalent to the problem studied in the previous example. All the assumptions used in that example are also satisfied. Hence, we can use the same weight function w. Then, it is clear that the cost function $c \in \mathbb{B}_w(\mathbb{N}_0^2 \cup \{1,2\})$ with $c(x,y) = h_1x + h_2y + s_1\mathbb{1}(x=0,\ y>0,\ z=1) + s_2\mathbb{1}(x>0, y\geq 0, z=2)$. To meet the final requirement that $V \in \mathbb{B}_w(\mathbb{N}_0^2 \cup \{1,2\})$, i.e., $\|V\|_w < \infty$, we need that

$$\frac{\varphi}{\mu_1(1-\rho_1)^2} - \frac{\rho_1 h_1}{\mu_1(1-\rho_1)^3} - \frac{\rho_2(\mu_1-\lambda_1+\mu_2\rho_1)h_2}{\mu_1^2(1-\rho_1)^3(1-\rho_1-\rho_2)} - \frac{s_1+s_2}{\mu_1(1-\rho_1)^2}$$
$$\times\left[\lambda_1\left\{\rho_1\left(\frac{\lambda_1\theta}{\lambda+\lambda_2}-1\right)+\frac{\lambda_1}{\lambda_1+\lambda_2}(1-\theta)\right\}+\lambda_2\left\{\frac{\lambda_1\theta}{\lambda_1+\lambda_2}\frac{\lambda_1}{\mu_2}+\frac{\lambda_1}{\lambda_1+\lambda_2}\right\}\right]=0.$$

Finally, we obtain the unique pair (φ, V) by solving this equation. The result is

$$\varphi = \frac{\rho_1}{1-\rho_1}h_1 + \frac{\rho_2(\mu_1-\mu_1\rho_1+\mu_2\rho_1)}{\mu_1(1-\rho_1)(1-\rho_1-\rho_2)}h_2 + (s_1+s_2)$$
$$\times\left[\lambda_1\left\{\rho_1\left(\frac{\lambda_1\theta}{\lambda+\lambda_2}-1\right)+\frac{\lambda_1}{\lambda_1+\lambda_2}(1-\theta)\right\}+\lambda_2\left\{\frac{\lambda_1\theta}{\lambda_1+\lambda_2}\frac{\lambda_1}{\mu_2}+\frac{\lambda_1}{\lambda_1+\lambda_2}\right\}\right],$$

as stated in Theorem 3.1 of [10].

References

1. Bertsekas DP (1995) Dynamic programming and optimal control, volume one. Athena Scientific, Belmont, Massachusetts
2. Bertsekas DP (1995) Dynamic programming and optimal control, volume two. Athena Scientific, Belmont, Massachusetts
3. Bhulai S, Koole GM (2003) On the structure of value functions for threshold policies in queueing models. To appear in Journal of Applied Probability 40
4. Chen R-R, Meyn S (1999) Value iteration and optimization of multiclass queueing networks. Queueing Systems 32:65–97
5. Chung KL (1967) Markov chains with stationary transition probabilities. Springer-Verlag, Berlin
6. Dekker R (1985) Denumerable Markov decision chains: optimal policies for small interest rates. PhD thesis, Leiden University
7. Dekker R, Hordijk A (1989) Average, sensitive and Blackwell optimality in denumerable state Markov decision chains with unbounded rewards. Mathematics of Operations Research 13:395–421

8. Dekker R, Hordijk A (1992) Recurrence conditions for average and Blackwell optimality in denumerable state Markov decision chains. Mathematics of Operations Research 17:271–290
9. Dekker R, Hordijk A, Spieksma FM (1994) On the relation between recurrence and ergodicity properties in denumerable Markov decision chains. Mathematics of Operations Research 19:1–21
10. Groenevelt R, Koole GM, Nain P (2002) On the bias vector of a two-class preemptive priority queue. Mathematical Methods of Operations Research 55:107-120
11. Hernández-Lerma O, Lasserre JB (1996) Discrete-time Markov control processes: basic optimality criteria. Springer-Verlag, New York, NY
12. Hernández-Lerma O, Lasserre JB (1999) Further topics on discrete-time Markov control processes. Springer-Verlag, New York, NY
13. Hordijk A, Spieksma FM (1992) On ergodicity and recurrence properties of a Markov chain with an application to an open Jackson network. Advances in Applied Probability 24:343–376
14. Koole GM, Nain P (2000) On the value function of a priority queue with an application to a controlled polling model. Queueing Systems 34:199–214
15. Kumar PR, Varaiya P (1986) Stochastic systems: estimation, identification and adaptive control. Prentice-Hall, Englewood Cliffs, NJ
16. Lippman SA (1975) On dynamic programming with unbounded rewards. Management Science 21:1225–1233
17. Meyn S (1997) The policy improvement algorithm for Markov decision processes with general state space. IEEE Transactions on Automatic Control 42:191–196
18. Ott TJ, Krishnan KR (1992) Separable routing: A scheme for state-dependent routing of circuit switched telephone traffic. Annals of Operations Research 35:43–68
19. Puterman ML (1994) Markov decision processes: discrete stochastic dynamic programming. John Wiley & Sons, New York, NY
20. Ross SM (1993) Introduction to stochastic dynamic programming. Academic Press, New York, NY
21. Sennott L (1999) Stochastic dynamic programming and the control of queueing systems. John Wiley & Sons, New York, NY
22. Spieksma FM (1990) Geometrically ergodic Markov chains and the optimal control of queues. PhD thesis, Leiden University
23. Tijms HC (1994) Stochastic models: an algorithmic approach. John Wiley & Sons, Chichester, UK