Optimal weighing schemes

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Abstract

We study the problem of determining the masses of a set of weights, given one standard weight, based on comparing two disjoint subsets of those weights with approximately equal mass. The question is how to choose a weighing scheme, i.e., different pairs of subsets, such that the masses can be determined as accurately as possible within a given number of measurements. In this paper we discuss a new way of using the so-called STS method of comparing two approximately equal masses, and we will give optimal weighing schemes which turn out to outperform schemes that are currently used by national metrology institutes.

1 Introduction

In this paper we study the following problem presented to us by the Dutch metrology institute NMi (the 'Nederlands Meetinstituut'). Consider the following set of weights $M_0, \ldots, M_5$: one weight of 1000 g, one of 500 g, two of 200 g, and two of 100 g, respectively. The specified masses of these six weights are approximate, since their true masses, let us call them $\mu_0, \mu_1, \ldots, \mu_5$, are unknown. We want to estimate the true masses by comparing the weights in a certain way to each other and to the standard 1 kg, which is a platinum-iridium cylinder stored at the NMi. The comparison of weights is done using an electronic scale. This scale is capable of measuring small differences in mass quite accurately. Hence, we will only compare two sets of weights that have approximately the same total mass, and measure the difference in mass (a direct comparison). Consequently, we have to devise a weighing scheme that tells us which combination of weights we have to compare to which other combination of weights. For example, we could compare weight M_0 ($\mu_0 \approx 1000$) to the combination of weights M_1, M_2, M_3, and M_5 ($\mu_1 + \mu_2 + \mu_3 + \mu_5 \approx 1000$). For practical reasons, the two combinations in a direct comparison may not both contain the same weight. The problem now is to find a weighing scheme for a given number of measurements such that the relative error in the masses is as small as possible. To solve this problem, we will first take a closer look at the weighing procedure.

2 The STS weighing procedure

Suppose that we have selected two sets of weights with approximately the same total mass. How do we measure the difference in mass? The NMi uses what is called an STS-procedure.

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One set of weights is called the Standard set (S), and the other set is called the Test set (T). The set S is placed on the scale, at which time the scale is set to 0. Then the set S is removed and placed on the scale again, resulting in the first measurement, which we call x_1. After this, the set S is removed, and the set T is placed on the scale, resulting in measurement x_2. We then continue by alternating sets S and T, so the third measurement is of set S, the fourth of T, and so on; hence the name STS-procedure.

The data gathered by the STS-procedure will consist of k measurements $x_1, \ldots, x_k$. If we denote the mass of set S by μ_S, and the mass of set T by μ_T, we model the measurements x_i as realizations of the following random variables:

$$x_i = 1_{\{i=\text{even}\}}(\mu_T - \mu_S) + D(i) + V_i. \tag{2.1}$$

Here, V_i is a random effect (a measurement error), and we model the V_i's as independent and identically distributed random variables. The function $D(i)$ models the *drift* of the electronic scale, which is observed in practice: after each measurement the scale is set off by a small amount. We will now make some additional important assumptions:

(A1) We assume that the V_i's are independent, and that

$$V_i \sim N(0, \alpha^2 \mu_S^2).$$

Here $\alpha > 0$ is an unknown constant.

(A2) We assume that for all $1 \leq j \leq k-2$,

$$\tfrac{1}{2}\big(D(j) + D(j+2)\big) - D(j+1) \approx 0,$$

i.e., this quantity is negligibly small.

Assumption (A1) implies that the measurement error V_i is proportional to the mass being weighed. Here we use that $\mu_T \approx \mu_S$, so the difference is small compared to the total mass. Assumption (A2) is exactly fulfilled when the drift is linear. In fact, we only assume that the drift between two consecutive measurements is almost equal.

Having obtained the measurements $x_1, \ldots, x_k$, how should we use them to estimate the difference in mass $\Delta\mu$ given by

$$\Delta\mu := \mu_T - \mu_S?$$

To benefit from Assumption (A2), we define the following auxiliary variables for $1 \leq j \leq k-2$:

$$\Delta m_j = (-1)^{j+1}\big(x_{j+1} - \tfrac{1}{2}(x_j + x_{j+2})\big),$$

so

$$\begin{aligned} \Delta m_1 &= x_2 - \tfrac{1}{2}(x_3 + x_1), \\ \Delta m_2 &= \tfrac{1}{2}(x_4 + x_2) - x_3, \end{aligned}$$

and so forth. Using Equation (2.1) and Assumption (A2), we find that

$$\begin{pmatrix} \Delta m_1 \\ \Delta m_2 \\ \vdots \\ \Delta m_{k-2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \Delta\mu + \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mp\frac{1}{2} & \pm 1 & \mp\frac{1}{2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{pmatrix}.$$

If we define, for $k \geq 3$

$$B_k = \begin{pmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \frac{1}{2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \mp\frac{1}{2} & \pm 1 & \mp\frac{1}{2} \end{pmatrix} \text{ and } \underline{\mathbf{1}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

we see that

$$\Delta m = \underline{\mathbf{1}}\Delta\mu + B_k V. \tag{2.2}$$

Let Z' denote the transpose of an arbitrary matrix or vector Z. To illustrate how we should calculate an estimate for $\Delta\mu$ we choose an invertible $(k-2) \times (k-2)$ matrix D_k such that

$$D_k D_k' = B_k B_k' =: \Sigma_k.$$

This is always possible, because Σ_k is symmetric and positive definite. Now we multiply Equation (2.2) by D_k^{-1}:

$$D_k^{-1}\Delta m = D_k^{-1}\underline{\mathbf{1}}\Delta\mu + D_k^{-1} B_k V. \tag{2.3}$$

We know that

$$D_k^{-1} B_k V \sim N_{k-2}(0, \alpha^2 \mu_S^2 D_k^{-1} B_k B_k' D_k'^{-1}) \stackrel{\text{d}}{=} N_{k-2}(0, \alpha^2 \mu_S^2 I),$$

where I is the identity matrix. This shows that Model (2.3) corresponds to a standard linear model

$$Y = X\beta + U,$$

with $Y = D_k^{-1}\Delta m$, $X = D_k^{-1}\underline{\mathbf{1}}$, $\beta = \Delta\mu$, and $U = D_k^{-1} B_k V$. For this model the least squares estimator is given by

$$\hat{\beta} = (X'X)^{-1} X'Y,$$

which gives us

$$\widehat{\Delta\mu} = \frac{\underline{\mathbf{1}}'\Sigma_k^{-1}\Delta m}{\underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}}}. \tag{2.4}$$

This estimator differs from the estimator that is normally used in the STS-procedure, namely the average of all the Δm_i's. In fact, Estimator (2.4) is a weighted average of the Δm_i's, where the weights may be negative! For example, when $k = 10$ (i.e., there are ten measurements in the STS procedure), we get that

$$\widehat{\Delta\mu} = (0.30, -0.10, 0.25, 0.05, 0.05, 0.25, -0.10, 0.30) \cdot \Delta m.$$

We can show that in this case, the variation in the least squares estimator is almost 10% smaller than the variation in the average of the Δm_i's. It is true, however, that as k grows, the ratio of the two variances tends to 1.

We know that the variance of the least squares estimator is given by

$$\mathrm{Var}(\widehat{\Delta\mu}) = \frac{\alpha^2 \mu_S^2}{\underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}}}. \tag{2.5}$$

The usual unbiased estimator of this variance is given by

$$S^2 = \left(\Delta m' \Sigma_k^{-1} \Delta m - \frac{(\underline{\mathbf{1}}'\Sigma_k^{-1}\Delta m)^2}{\underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}}} \right) / (k-3).$$

This follows directly from the fact that Model (2.3) is a standard linear model.

3 Weighing schemes using the STS-procedure

Now that we know how to deal with the STS-procedure, we would like to have some idea of which weighing scheme we should use to accurately determine the masses $\mu_0, \ldots, \mu_5$. Since all the weights are unknown, we need to use the standard 1 kg weight in our weighing scheme. However, we do not want to use this precious weight very often, so we only use it to determine μ_0, which is approximately 1 kg. We proceed with the STS-procedure until we have determined μ_0 up to a certain accuracy. From then on, we will use different combinations of weights to determine $\mu_1, \ldots, \mu_5$ in terms of μ_0. In those combinations we will not use the standard 1 kg.

A combination of weights that can be used for the STS-procedure, can be described by a vector containing 5 entries, each of which is either -1, 0, or $+1$. Each entry corresponds to one of the weights $\mu_1, \ldots, \mu_5$ in the following way: a 0 indicates that the weight is not included, a -1 indicates that the weight is included in the Standard set of the STS-procedure, and a $+1$ indicates that the weight is included in the Test set of the STS-procedure. The reason that there is no entry for μ_0 is because the total mass of the Standard set has to be approximately equal to the total mass of the Test set. This means that the entry for μ_0 is determined by the other entries. Also, we define new parameters

$$\Delta\mu = \begin{pmatrix} \Delta\mu_1 \\ \Delta\mu_2 \\ \Delta\mu_3 \\ \Delta\mu_4 \\ \Delta\mu_5 \end{pmatrix} = \begin{pmatrix} \mu_1 - 0.5\,\mu_0 \\ \mu_2 - 0.2\,\mu_0 \\ \mu_3 - 0.2\,\mu_0 \\ \mu_4 - 0.1\,\mu_0 \\ \mu_5 - 0.1\,\mu_0 \end{pmatrix}.$$

Note that $\Delta\mu$ is small compared to μ.

Now suppose we have a combination of weights characterized by a (row) vector $w = (w_1, \ldots, w_5)$. Define for convenience

$$(M_0, M_1, M_2, M_3, M_4, M_5) = (1, 0.5, 0.2, 0.2, 0.1, 0.1),$$

so we have that $\mu_i \approx M_i$, for $0 \leq i \leq 5$. Define

$$w_0 = -\sum_{i=1}^{5} M_i w_i. \tag{3.1}$$

Then w_0 has to be either -1, 0 or 1. This means that there are essentially only ten possible choices of w, not taking into account interchanging the Standard set and the Test set (this corresponds to taking $-w$). These ten possible combinations are given in the following matrix W:

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Given a combination w and using Equation (3.1), we have for our STS-procedure that

$$\mu_S = \sum_{0 \leq i \leq 5: w_i = -1} \mu_i \quad \text{and} \quad \mu_T = \sum_{0 \leq i \leq 5: w_i = +1} \mu_i.$$

This means that

$$\mu_T - \mu_S = \sum_{i=0}^{5} w_i \mu_i = \sum_{i=1}^{5} w_i \Delta\mu_i = w\Delta\mu.$$

If we take k measurements in the STS-procedure, we see that Model (2.3) corresponds to

$$D_k^{-1}\Delta m = D_k^{-1}\underline{\mathbf{1}}w\Delta\mu + D_k^{-1}B_k V, \tag{3.2}$$

where

$$D_k^{-1}B_k V \sim N_{k-2}(0, \alpha^2 \mu_S^2 I). \tag{3.3}$$

It is of course impossible to estimate the full vector $\Delta\mu$ from these data, we can only estimate the linear combination $w\Delta\mu$. If we repeat the STS-procedure for a suitable set of different combinations, we can estimate $\Delta\mu$ as well. A choice of different combinations of weights is called a *weighing scheme*. A weighing scheme can be represented by a matrix A, consisting of different rows $A(l)$, which correspond to rows from the matrix W, i.e., the possible combinations of weights. For now we will assume that each row of A corresponds to $k = 20$ STS measurements (the number used by the NMi), using that particular combination of weights.

Equation (3.3) shows that we have to be a bit careful: changing the combination of weights might change μ_S, which would imply that in the full model, that takes all chosen combinations of weights into account, we would not have a measurement error with a constant variance. The way to handle this is quite straightforward: we rescale all STS measurements using a weight-combination w by dividing them by the total mass of the Standard set. With a slight abuse of notation, the total mass $\mu_S(w)$ of the Standard set given w is in good approximation given by

$$\mu_S(w) = \sum_{i=0}^{5} M_i 1_{\{w_i=+1\}}.$$

Model (3.2) then becomes

$$\frac{D_k^{-1}\Delta m}{\mu_S(w)} = \frac{D_k^{-1}\underline{\mathbf{1}}w\Delta\mu}{\mu_S(w)} + \frac{D_k^{-1}B_kV}{\mu_S(w)},$$

where

$$\frac{D_k^{-1}B_kV}{\mu_S(w)} \sim N_{k-2}(0, \alpha^2 I).$$

Now if our weighing scheme A consists of s rows $A(1), \dots, A(s)$, then our full linear model becomes

$$\begin{pmatrix} \frac{(D_k^{-1}\Delta m^{(1)})_1}{\mu_S(A(1))} \\ \vdots \\ \frac{(D_k^{-1}\Delta m^{(1)})_{k-2}}{\mu_S(A(1))} \\ \frac{(D_k^{-1}\Delta m^{(2)})_1}{\mu_S(A(2))} \\ \vdots \\ \frac{(D_k^{-1}\Delta m^{(2)})_{k-2}}{\mu_S(A(2))} \\ \vdots \\ \frac{(D_k^{-1}\Delta m^{(s)})_1}{\mu_S(A(s))} \\ \vdots \\ \frac{(D_k^{-1}\Delta m^{(s)})_{k-2}}{\mu_S(A(s))} \end{pmatrix} = \begin{pmatrix} \frac{(D_k^{-1}\underline{\mathbf{1}})_1\, A(1)\Delta\mu}{\mu_S(A(1))} \\ \vdots \\ \frac{(D_k^{-1}\underline{\mathbf{1}})_{k-2}\, A(1)\Delta\mu}{\mu_S(A(1))} \\ \frac{(D_k^{-1}\underline{\mathbf{1}})_1\, A(2)\Delta\mu}{\mu_S(A(2))} \\ \vdots \\ \frac{(D_k^{-1}\underline{\mathbf{1}})_{k-2}\, A(2)\Delta\mu}{\mu_S(A(2))} \\ \vdots \\ \frac{(D_k^{-1}\underline{\mathbf{1}})_1\, A(s)\Delta\mu}{\mu_S(A(s))} \\ \vdots \\ \frac{(D_k^{-1}\underline{\mathbf{1}})_{k-2}\, A(s)\Delta\mu}{\mu_S(A(s))} \end{pmatrix} + U, \tag{3.4}$$

where $\Delta m^{(l)}$ is the vector of $k-2$ measurements from the STS-procedure using the weight combination $A(l)$ and $U \sim N_{s(k-2)}(0, \alpha^2 I)$. Now define

$$\Delta\tilde{m}^{(l)} = \frac{\Delta m^{(l)}}{\mu_S(A(l))} \quad \text{and} \quad \tilde{A}(l) = \frac{A(l)}{\mu_S(A(l))}.$$

Use a matrix-block notation to see that Model (3.4) becomes

$$\begin{pmatrix} D_k^{-1} & & \\ & \ddots & \\ & & D_k^{-1} \end{pmatrix} \begin{pmatrix} \Delta\tilde{m}^{(1)} \\ \vdots \\ \Delta\tilde{m}^{(s)} \end{pmatrix} = \begin{pmatrix} D_k^{-1}\underline{\mathbf{1}} & & \\ & \ddots & \\ & & D_k^{-1}\underline{\mathbf{1}} \end{pmatrix} \tilde{A}\Delta\mu + U.$$

This is a standard linear model and the least squares estimator is given by

$$\widehat{\Delta\mu} = \left(\tilde{A}' \begin{pmatrix} \underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}} & & \\ & \ddots & \\ & & \underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}} \end{pmatrix} \tilde{A}\right)^{-1} \tilde{A}' \begin{pmatrix} \underline{\mathbf{1}}'\Sigma_k^{-1} & & \\ & \ddots & \\ & & \underline{\mathbf{1}}'\Sigma_k^{-1} \end{pmatrix} \Delta\tilde{m}.$$

The covariance matrix of this estimator is given by

$$\mathrm{Cov}(\widehat{\Delta\mu}) = \alpha^2 \left(\tilde{A}' \begin{pmatrix} \underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}} & & \\ & \ddots & \\ & & \underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}} \end{pmatrix} \tilde{A}\right)^{-1}. \tag{3.5}$$

The diagonal of the covariance matrix gives us the variances of the separate estimators for $\Delta\mu_1, \ldots, \Delta\mu_5$. Clearly, we would like to choose our weighing scheme A such that these variances are minimized in some way. We believe that it is most sensible to minimize the relative error in each weight, which is why we choose the sum of squares of the relative errors as a measure of inaccuracy. Thus, we wish to find a weighing scheme A that minimizes the "loss function"

$$L(A) = \sum_{i=1}^{5} \frac{\mathrm{Var}(\widehat{\Delta\mu_i})}{M_i^2}.$$

In the next sections we will discuss our findings and make a comparison with schemes that are actually used by national metrology institutes, such as the NMi, the Slovak metrology institute SMU, and the German metrology institute PTB. We already wish to note that Equation (3.5) can be easily generalized to the case where each row of A has a different number of STS measurements: simply use the appropriate k for each Σ_k in the block matrix.

For the sake of completeness, we will write down the estimate for α^2. Define

$$S = \begin{pmatrix} \underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}} & & \\ & \ddots & \\ & & \underline{\mathbf{1}}'\Sigma_k^{-1}\underline{\mathbf{1}} \end{pmatrix}.$$

Then

$$\hat{\alpha}^2 = \left(\Delta\tilde{m}'S\Delta\tilde{m} - \widehat{\Delta\mu}'\tilde{A}'S\tilde{A}\widehat{\Delta\mu}\right)/(n-1),$$

where n is the length of $\Delta\tilde{m}$.

4 Optimal weighing schemes for the NMi

The Dutch metrology institute NMi currently uses a weighing scheme with 8 combinations of weights (i.e., eight rows of W) and 20 STS measurements for each combination (i.e., $k = 20$). Let $W(l)$ denote the l-th row of the matrix W. Then the NMi weighing scheme A_{NMi} is given by

$$A_{\mathrm{NMi}} = \big(W(1), W(2), W(3), W(4), W(5), W(6), W(8), W(9)\big),$$

with the following uncertainty associated with it

$$L(A_{\mathrm{NMi}}) = 1.1812 \cdot \alpha^2 = 1.1812,$$

if we assume that $\alpha = 1$. We can do this without loss of generality, since we are comparing different weighing schemes with each other that all have the same factor α^2. Therefore, we shall assume that $\alpha = 1$ in the following.

In order to improve upon this weighing scheme we have certain degrees of freedom. Firstly, we can choose different weighing schemes A by choosing different combinations of weights from the matrix W. We shall describe matrix A by listing the indices of the chosen rows $W(l)$, e.g., the indices of A_{NMi} are 1, 2, 3, 4, 5, 6, 8, and 9. Secondly, we can change the number of combinations we choose from W, this number will, as before, be denoted by s. Finally, we can change the number of STS measurements per combination, this number was already denoted by k.

Let us now study what the optimal weighing scheme is for the NMi, and how this can be improved by increasing the number of combinations s in the weighing scheme. Table 1 summarizes these experiments (calculated using MatLab) and shows that the optimal weighing scheme with the same parameters used at the NMi (i.e., $s = 8$ and $k = 20$) already gives a reduction in uncertainty of around 28 percent. Note that there can be more weighing schemes with the same uncertainty which for simplicity we have not mentioned in Table 1. The uncertainty can be reduced even more by adding more combinations of weights to the scheme.

As a side remark, the Slovak metrology institute SMU (the 'Slovenský Metrologický Ústav') has the same set of weights. However, they use a weighing scheme with $s = 14$ of which the indices are given by 1, 2, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9, 10, and 10. The associated uncertainty with this weighing scheme is given by $L(A_{\mathrm{SMU}}) = 0.7285$. Table 1 shows that the optimal weighing scheme reduces the uncertainty with 36 percent compared to current practice.

Note that the optimal weighing schemes do not include measurements $W(5)$ and $W(6)$, which the NMi does include. Instead, the optimal schemes include measurements $W(7)$ and $W(10)$. One could ascribe this to rounding errors in the calculation, however, all solutions within 1 percent of the optimal solution have this property as well.

s	indices of $A = \big(W(1), \ldots, W(s)\big)$	$L(A)$
8	1, 1, 2, 4, 7, 8, 9, 10	0.8468
9	1, 1, 2, 3, 4, 7, 8, 9, 10	0.7242
10	1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.6441
11	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.5963
12	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 10, 10	0.5507
13	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 9, 10, 10	0.5054
14	1, 1, 1, 2, 2, 3, 4, 4, 7, 8, 8, 9, 10, 10	0.4655

Table 1: Optimal weighing schemes for different s

$s \times k$	s	k	indices of $A = \big(W(1), \dots, W(s)\big)$	$L(A)$
280	14	20	1, 1, 1, 2, 2, 3, 4, 4, 7, 8, 9, 9, 10, 10	0.4655
280	10	28	1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.4659
280	8	35	1, 1, 2, 4, 7, 8, 9, 10	0.4940
286	13	22	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 9, 10, 10	0.4613
286	11	26	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.4633
288	9	32	1, 1, 2, 3, 4, 7, 8, 9, 10	0.4602
288	12	24	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 10, 10	0.4623
288	8	36	1, 1, 2, 4, 7, 8, 9, 10	0.4799
290	10	29	1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.4513
294	14	21	1, 1, 1, 2, 2, 3, 4, 4, 7, 8, 9, 9, 10, 10	0.4461
296	8	37	1, 1, 2, 4, 7, 8, 9, 10	0.4679
297	9	33	1, 1, 2, 3, 4, 7, 8, 9, 10	0.4474
297	11	27	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.4479
299	13	23	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 9, 10, 10	0.4435
300	10	30	1, 1, 2, 2, 3, 4, 7, 8, 9, 10	0.4358
300	12	25	1, 1, 1, 2, 2, 3, 4, 7, 8, 9, 10, 10	0.4458

Table 2: Optimal weighing schemes for different s and k

This observation can be explained if one realizes that $W(5)$ and $W(6)$ provide the same information as $W(7)$ and $W(10)$, only with a greater uncertainty associated with them. This can be seen by adding and subtracting the rows: $W(5) + W(6) = 2 \times W(7)$, and $W(5) - W(6) = 2 \times W(10)$. Hence, both sets provide the same information, however, the set with $W(7)$ and $W(10)$ only uses one weight one the scale and thus adds less uncertainty to the measurements.

Although Table 1 lists optimal weighing schemes, it disregards the total number of measurements. For fixed s and k we need in total $s \times k$ measurements to determine the masses. The current weighing scheme needs $8 \times 20 = 160$ measurements. The measurements used to be done by hand, constraining the maximum number of measurements in a weighing scheme. However, due to the introduction of automatic weighing devices at the NMi the maximum number of measurements has been increased to around 280 to 300. In Table 2 we have computed the optimal weighing schemes as a function of the total number of measurements from 280 to 300. From the table we can see that with these parameters a reduction of around 62 percent in uncertainty can be achieved. We again mention that there might be more weighing schemes that achieve the same uncertainty, but which for simplicity we have not included into Table 2.

5 Improved weighing schemes for the PTB

Let us consider the set of weights used by the German metrology institute PTB (the 'Physikalisch-Technische Bundesanstalt'), as reported in Kochsiek and Gläser [1]. Apart from the standard 1000 g, this set has another weight of 1000 g, two of 500 g, two of 200 g, and two of 100 g. For this set there are 104 possible combinations of weights in the matrix W instead of 10. Identifying by full enumeration the optimal weighing scheme with say $s = 10$ out of all 104 combinations requires enormous computational resources. Therefore we only consider a reduced class of possible combinations, namely those combinations involving on one side only one weight. Note that this is a reasonable choice by the same argument we used to exclude $W(7)$ and $W(10)$ in the previous section. The matrix W now becomes

$$W = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 & -1 & 0 & -1 \\ 1 & 0 & -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & -1 & -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Having specified the matrix W, we can repeat the calculations for this set of weights using the parameters $s = 10$ and $k = 20$. Table 3 shows the weighing schemes A_{opt1} and A_{opt2} that are optimal in the reduced class, and compares them to the scheme A_{PTB} of the PTB. We can see that the optimal weighing schemes even in the reduced class leads to a 30 percent reduction in uncertainty compared to the PTB weighing scheme presently used.

	indices of $A = \big(W(1), \ldots, W(10)\big)$	$L(A)$
A_{opt1}	1, 3, 6, 6, 8, 13, 16, 18, 19, 20	1.1356
A_{opt2}	1, 4, 5, 5, 9, 14, 15, 18, 19, 20	1.1356
A_{PTB}	1, 2, 7, 12, 13, 16, 17, 18, 19, 20	1.6244

Table 3: Improved weighing schemes for the PTB set of weights

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References

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