HOMEOMORPHISM GROUPS OF HOMOGENEOUS COMPACTA NEED NOT BE MINIMAL

JAN VAN MILL

Abstract. It is shown that the homeomorphism group of the n-dimensional Menger universal continuum is not minimal. This answers a question by Stojanov from about 1984.

1. Introduction

All spaces under discussion are Hausdorff.

A topological group G is called *minimal* if its topology cannot be properly weakened to another group topology. It is known that a minimal Abelian topological group is precompact (Prodanov and Stojanov [22]), and that for non-Abelian groups this need not hold (Gaughan [16]). For information on minimal groups, see e.g., Dikranjan, Prodanov and Stojanov [10], Dikranjan and Megrelishvili [9] and Lukacs [18].

It was asked by Stojanov (see Arhangel′skiĭ [3, VI.7] or Comfort, Hofmann and Remus [7, 3.3.3(a)]), whether the homeomorphism group $\mathscr{H}(X)$ of a homogeneous compactum is minimal. As usual, $\mathscr{H}(X)$ is endowed with the compact-open topology. It is known that this is the case for X the Cantor set (Gamarnik [15]; see also Uspenskiĭ [23]), but it is not known for X the Hilbert cube (this is a question of Uspenskiĭ [24]). The aim of this note is to answer Stojanov's question in the negative.

A topological group is *non-archimedean* if it has a local base at the identity consisting of open subgroups. A non-archimedean topological group is clearly zero-dimensional. The group of rational numbers with its usual topology is an example of a zero-dimensional group which is not non-archimedean.

The aim of this note is to prove the following result.

Theorem 1.1. *For $n \geq 1$, let X be an n-dimensional compact space such that for every nonempty open subset U of X there is a compact subset A of U that homotopically dominates the n-sphere. Then $\mathscr{H}(X)$ admits a weaker non-archimedean group topology whose weight does not exceed the weight of X.*

For the proof of Theorem 1.1 we make good use of the proof of Theorem 5 in Oversteegen and Tymchatyn [21]. Similar arguments were also used by Anderson [1] (for details, see [6, Theorem 1.3]).

Date: September 3, 2010.

1991 *Mathematics Subject Classification.* 22A05, 54H11, 54F15.

Key words and phrases. Homogeneous compactum, homeomorphism group, minimal topology.

For $n \geq 1$, let μ^n denote the n-dimensional universal Menger continuum (Menger [19]). These spaces are obtained from finite-dimensional cubes by drilling holes in them in a way similar to the creation of the Cantor ternary set by repeatedly deleting the open middle thirds of a set of line segments. See [13, §1.11] for details. From the definition of μ^n it is clear that every nonempty open subset of it contains a copy of $\mathbb{S}^n$. Hence μ^n satisfies the conditions mentioned in Theorem 1.1.

Bestvina [5] provided elegant characterizations of these spaces and proved their homogeneity (for $n = 1$ this was done earlier by Anderson [2]). We denote the group of homeomorphisms of μ^n by $\mathscr{H}^n$. It was shown in Oversteegen and Tymchatyn [21, Theorem 5] that $\dim \mathscr{H}^n \leq 1$. Dijkstra [8, Theorem 7] established that $\mathscr{H}^n$ contains a copy of the famed Erdős space $\mathscr{E}$ from [14] which is 1-dimensional. The surprising and highly counterintuitive conclusion of these results is that $\dim \mathscr{H}^n = 1$.

By Theorem 1.1, $\mathscr{H}^n$ admits a weaker (separable metrizable) non-archimedean group topology. This topology is strictly weaker than the 1-dimensional compact-open topology on $\mathscr{H}^n$ and so μ^n solves Stojanov's problem in the negative.

I am indebted to Dikran Dikranjan and Michael Megrelishvili for helpful comments.

2. Preliminaries

For $n \in \mathbb{N}$, let $\mathbb{S}^n$ denote the euclidean sphere $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$. As usual, by $f \simeq g$ we mean that f and g are homotopic functions. It is well-known, and easy to prove, that if $f, g\colon X \to \mathbb{S}^n$ are such that for each $x \in X$, $f(x)$ and $g(x)$ are not antipodal, then $f \simeq g$ (Dugundji [11, XV.1.2(1)]. In particular, if $\|f(x) - g(x)\| < 1$ for every $x \in X$, then $f \simeq g$.

Let X and Y be spaces. We say that X *homotopically dominates* Y if there exist continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g$ is homotopic to the identity function on Y.

Let $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$. If $f \simeq f'$ and $g \simeq g'$, then $g \circ f \simeq g' \circ f'$. This elementary fact about the homotopy relation will be used without explicit reference from now on.

If X and Y are topological spaces, then $C(X, Y)$ denotes the set of all continuous functions from X to Y endowed with the compact-open topology. Moreover, let $\mathscr{H}(X, Y)$ denote $\{h \in C(X, Y) : h \text{ is a homeomorphism}\}$. If $X = Y$, then $\mathscr{H}(X)$ abbreviates $\mathscr{H}(X, X)$. Hence $\mathscr{H}(X)$ is the *group of homeomorphisms of* X *with the compact-open topology*. It is not necessarily a topological group with function composition as the group operation. But for a compact space X, $\mathscr{H}(X)$ *is* a topological group with the relative topology from $C(X, X)$ and function composition as the group operation.

Let G be a group, and let $\mathscr{V}$ be a collection of subsets of G with the following properties:

(G1) for every $V \in \mathscr{V}$, there exists $W \in \mathscr{V}$ such that $W^2 \subseteq V$;
(G2) for every $V \in \mathscr{V}$, there exists $W \in \mathscr{V}$ such that $W^{-1} \subseteq V$;
(G3) for every $V \in \mathscr{V}$ and $x \in V$, there is $W \in \mathscr{V}$ such that $Wx \subseteq V$;
(G4) for every $V \in \mathscr{V}$ and $x \in G$, there is $W \in \mathscr{V}$ such that $xWx^{-1} \subseteq V$;
(G5) for $V, W \in \mathscr{V}$ there is $X \in \mathscr{V}$ such that $X \subseteq V \cap W$;

(G6) $\{e\} = \bigcap \mathscr{V}$.

It is well-known that the family $\{Vx : x \in G, V \in \mathscr{V}\}$ is a base for a T_1-topology on G. With this topology, G is a topological group, and the family $\{xV : x \in G, V \in \mathscr{V}\}$ is a base for the same topology on G. For details, see e.g. [17, II.4.5] or [4, 1.3.12]. Observe that a T_1-topological group is Tychonoff, see e.g. [17, II.8.4] or [4, 3.3.11].

The identity function on a set X is denoted by id_M or 1_M.

3. Proof of Theorem 1.1

Let X be a compact space satisfying the hypotheses stated in Theorem 1.1. In addition, let U be a dense subset of $C(X, \mathbb{S}^n)$ with extra conditions to be specified later. For every $u \in U$ we put

$$C_u = \{h \in \mathscr{H}(X) : u \circ h \simeq u\}.$$

Lemma 3.1. *For $u \in U$, C_u is a clopen subgroup of $\mathscr{H}(X)$.*

Proof. Let $f, g \in C_u$. Then $f \circ g \in C_u$ since $u \circ (f \circ g) = (u \circ f) \circ g \simeq u \circ g \simeq u$. Moreover, $f^{-1} \in C_u$ since $u = u \circ f \circ f^{-1} \simeq u \circ f^{-1}$. Hence C_u is a subgroup. To prove it is clopen, take an arbitrary $h \in \mathscr{H}(X)$. There exists a neighborhood N of h in $\mathscr{H}(X)$ such that for every $g \in N$, $\|(u \circ h) - (u \circ g)\| < 1$. Hence if $g \in N$, then $u \circ g \simeq u \circ h$, and so $g \in C_u$ if and only if $h \in C_u$. This clearly implies that C_u is open and that $\mathscr{H}(X) \setminus C_u$ is open. □

Lemma 3.2. *For every $u \in U$ and $g \in \mathscr{H}(X)$ there exists $v \in U$ such that $gC_vg^{-1} \subseteq C_u$.*

Proof. Pick $v \in U$ such that $\|v - (u \circ g)\| < 1$. Now if $h \in C_v$, then $v \circ h \simeq v$, and hence

$$u \circ g \circ h \simeq v \circ h \simeq v \simeq u \circ g.$$

So $u \circ g \circ h \circ g^{-1} \simeq u \circ g \circ g^{-1} = u$, as required. □

Let $\mathscr{V}$ denote the collection of all finite intersections of elements of $\{C_u : u \in U\}$.

Lemma 3.3. *$\mathscr{V}$ satisfies the conditions* (G1) *through* (G5) *in* §2.

Proof. Since $\mathscr{V}$ consists of subgroups of G (Lemma 3.1), (G1), (G2) and (G3) are obvious. Moreover, (G5) is obvious. It remains to verify (G4). To this end, let F be an arbitrary finite nonempty subset of U, and fix $g \in \mathscr{H}(X)$. By Lemma 3.2, there exists for every $u \in F$ an element $v(u) \in U$ such that $gC_{v(u))}g^{-1} \subseteq C_u$. Hence

$$g\Big(\bigcap_{u\in F} C_{v(u)}\Big)g^{-1} \subseteq \bigcap_{u \in F} C_u,$$

as required. □

Remark 3.4. From Lemma 3.3 we conclude that the collection $\mathscr{V}$ determines a group topology on $\mathscr{H}(X)$. Observe that the conditions on X were not used so far. In addition, with respect to homotopies the only thing we used is that 'close' maps into $\mathbb{S}^n$ are homotopic. So we can replace $\mathbb{S}^n$ by any ANR. What we described is actually a (simple) method for constructing potentially interesting non-archimedean group topologies on homeomorphism groups $\mathscr{H}(X)$ for compact spaces X. The problem with these topologies is of course that they may not be Hausdorff. Consider for example the case that $\mathscr{H}(X)$ is connected. Below we use the conditions on the space X in Theorem 1.1 to prove Hausdorffness. Different ANR's and different arguments may yield Hausdorffness in different situations.

Now assume that U is a dense subset of $C(X,\mathbb{S}^n)$ whose cardinality does not exceed the weight of X.

Lemma 3.5. *Let $g \in \mathscr{H}(X)$ not be the identity. Then there exists $u \in U$ such that $g \notin C_u$.*

Proof. Since g is not the identity, there is a nonempty open subset V of X such that $V \cap g(V) = \emptyset$. Let A be a compact subset of V which homotopically dominates $\mathbb{S}^n$. Let $\xi\colon \mathbb{S}^n \to A$ and $\eta\colon A \to \mathbb{S}^n$ be continuous functions such that $\eta \circ \xi$ is homotopic to the identity function on $\mathbb{S}^n$. Define $\alpha\colon A \cup g(A) \to \mathbb{S}^n$ as follows:

$$\alpha(x) = \begin{cases} \eta(x) & (x \in A), \\ (1,0,\dots,0) & \big(x \in g(A)\big). \end{cases}$$

Since $\dim X = n$, α can be extended to a continuous function $\bar{\alpha}\colon X \to \mathbb{S}^n$ ([13, 3.2.10]). Pick $u \in U$ such that $\|\bar{\alpha} - u\| < 1$. We claim that $g \notin C_u$. Striving for a contradiction, assume that $u \circ g \simeq u$. Since $\bar{\alpha} \simeq u$, we have

$$\bar{\alpha} \circ g \simeq u \circ g \simeq u \simeq \bar{\alpha},$$

hence $\bar{\alpha} \circ g \circ \xi \simeq \bar{\alpha} \circ \xi$. But $\bar{\alpha} \circ g \circ \xi$ is the constant function with value $(1,0,0,\dots)$, and $\bar{\alpha} \circ \xi = \eta \circ \xi$ is homotopic to the identity function on $\mathbb{S}^n$. This violates the Brouwer Fixed-Point Theorem. ☐

Hence $\mathscr{V}$ satisfies condition (G6) in §2. Since $\mathscr{V}$ consists of clopen subgroups of $\mathscr{H}(X)$, we consequently conclude that there is a T_1-group topology $\mathscr{T}$ on $\mathscr{H}(X)$ such that $\mathscr{V}$ is a neighborhood base at e in $(\mathscr{H}(X),\mathscr{T})$. Hence $\mathscr{T}$ is contained in the original topology on $\mathscr{H}(X)$, and the elements of $\mathscr{V}$ are clopen in $(\mathscr{H}(X),\mathscr{T})$. As a consequence, $(\mathscr{H}(X),\mathscr{T})$ is non-archimedian.

Lemma 3.6. *The weight of $(\mathscr{H}(X),\mathscr{T})$ does not exceed the weight of X.*

Proof. Let $\kappa \geq \omega$ be the weight of X. As we observed in §2, the weight and hence the Lindelöf number of $\mathscr{H}(X)$ does not exceed κ. This implies that the Lindelöf number of $(\mathscr{H}(X),\mathscr{T})$ does not exceed κ. But $|\mathscr{V}| \leq \kappa$, hence the neutral element of $(\mathscr{H}(X),\mathscr{T})$ has a neighborhood base of size at most κ. This clearly implies that the weight of $(\mathscr{H}(X),\mathscr{T})$ is at most $\kappa{\cdot}\kappa = \kappa$. ☐

It is natural to ask whether $\mathscr{T}$ is a 'nice' topology in the sense that the natural action

$$X \times (\mathscr{H}(X),\mathscr{T}) \to X$$

is continuous. We will show that for the spaces μ^n, this is not the case.

Proposition 3.7. *Let C be a clopen subgroup of $\mathscr{H}(X)$, where X is a homogeneous compact space. Then for every $x \in X$ we have that Cx is clopen in X.*

Proof. By the Effros Theorem from [12] (see also [20]), Cx is open in X for every $x \in X$. Now pick an arbitrary $x \in X$, and take $y \in \overline{Cx}$. Then $Cy \cap Cx \neq \emptyset$ since Cy is open. Pick $\alpha, \beta \in C$ such that $\alpha x = \beta y$. Then $(\beta^{-1}\alpha)x = y$, i.e., $y \in Cx$ since $\beta^{-1}\alpha \in C$. ☐

Hence if the space X in Proposition 3.7 is a nontrivial continuum, then for every clopen subgroup C of $\mathscr{H}(X)$ and every $x \in X$ we have that $Cx = X$. This evidently implies that for a weaker non-archimedean topology $\mathscr{T}$ on $\mathscr{H}(X)$, the natural action $X \times \big(\mathscr{H}(X), \mathscr{T}\big) \to X$ is badly discontinuous. Simply observe that if U is a proper nonempty open subset of X, then the preimage of U under the natural action is not open.

References

[1] R. D. Anderson, *The group of all homeomorphisms of the universal curve*, Abstract 323, Bull. Amer. Math. Soc. **63** (1957), p. 43.

[2] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. (2) **67** (1958), 313–324.

[3] A. V. Arhangel′skiĭ, *Topological homogeneity, topological groups and their continuous images*, Russian Math. Surveys **42** (1987), 69–105.

[4] A. V. Arhangel′skiĭ and M. G. Tkachenko, *Topological groups and related structures*, Atlantis Studies in Mathematics, vol. 1, Atlantis Press, Paris, World Scientific, 2008.

[5] M. Bestvina, *Characterizing k-dimensional universal Menger compacta*, Mem. Amer. Math. Soc. **71** (1988), no. 380, vi+110.

[6] B. L. Brechner, *On the dimensions of certain spaces of homeomorphisms*, Trans. Amer. Math. Soc. **121** (1966), 516–548.

[7] W. W. Comfort, K.-H. Hofmann, and D. Remus, *Topological Groups and Semigroups*, Recent Progress in General Topology (M. Hušek and J. van Mill, eds.), North-Holland Publishing Co., Amsterdam, 1992, pp. 58–144.

[8] J. J. Dijkstra, *On homeomorphism groups of Menger continua*, Trans. Amer. Math. Soc. **357** (2005), 2665–2679.

[9] D. N. Dikranjan and M. Megrelishvili, *Relative minimality and co-minimality of subgroups in topological groups*, Top. Appl. **157** (2010), 62–76.

[10] D. N. Dikranjan, I. R. Prodanov, and L. N. Stojanov, *Topological groups*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 130, Marcel Dekker Inc., New York, 1990, Characters, dualities and minimal group topologies.

[11] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[12] E. G. Effros, *Transformation groups and C^*-algebras*, Annals of Math. **81** (1965), 38–55.

[13] R. Engelking, *Theory of dimensions finite and infinite*, Heldermann Verlag, Lemgo, 1995.

[14] P. Erdős, *The dimension of the rational points in Hilbert space*, Annals of Math. **41** (1940), 734–736.

[15] D. Gamarnik, *Minimality of the group* Aut(C), Serdika **17** (1991), 197–201.

[16] E. D. Gaughan, *Topological group structures of infinite symmetric groups*, Proc. Nat. Acad. Sci. U.S.A. **38** (1967), 907–910.

[17] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[18] G. Lukacs, *Compact-like topological groups*, Heldermann Verlag, Lemgo, 2009, Research and Exposition in Math. 31.

[19] K. Menger, *Kurventheorie*, Teubner, Leipzig, 1932.

[20] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly. **111** (2004), 801–806.

[21] L. G. Oversteegen and E. D. Tymchatyn, *On the dimension of certain totally disconnected spaces*, Proc. Amer. Math. Soc. **122** (1994), 885–891.

[22] I. Prodanov and L. N. Stojanov, *Every minimal Abelian group is precompact*, Comptes Rendus de l'Académie Bulgare des Sciences **37** (1984), 23–26.

[23] V. V. Uspenskiĭ, *The Roelcke compactification of groups of homeomorphisms*, Top. Appl. **111** (2001), 195–205.

[24] V. V. Uspenskiĭ, *On subgroups of minimal topological groups*, Top. Appl. **155** (2008), 1580–1606.

Faculty of Sciences, Department of Mathematics, VU University Amsterdam, De Boelelaan 1081^a, 1081 HV Amsterdam, The Netherlands
E-mail address: `vanmill@few.vu.nl`