

## ON UNIQUELY HOMOGENEOUS SPACES, II

ALEXANDER ARHANGEL'SKII AND JAN VAN MILL

ABSTRACT. It is shown that there is an example of a uniquely homogeneous separable metrizable space that is Abelian but not Boolean. It is also shown that such an example cannot be a Baire space. This answers several problems on (unique) homogeneity.

### 1. INTRODUCTION

All spaces under discussion are Tychonoff. By a homeomorphism of  $X$  we will always mean a homeomorphism of  $X$  onto itself. For a function  $f: X \rightarrow Y$  such that  $f: X \rightarrow f(X)$  is a homeomorphism and  $f(X) \neq Y$ , we use the term *embedding*.

A space  $X$  is called *uniquely homogeneous* provided that for all  $x, y \in X$  there is a unique homeomorphism of  $X$  that takes  $x$  onto  $y$ . This concept is due to Burgess [9] who asked in 1955 whether there exists a non-trivial uniquely homogeneous metrizable continuum. Ungar [19] showed in 1975 that there are no such finite-dimensional metrizable continua and a few years later, Barit and Renaud [5] showed that the assumption on finite-dimensionality is superfluous. A somewhat different argument was given by Keesling and Wilson [13]. A nontrivial uniquely homogeneous Baire space of countable weight was constructed by van Mill [16]. This example is a topological group. There are also uniquely homogeneous spaces that do not admit the structure of a topological group, [17]. It is unknown whether there is a non-trivial Polish uniquely homogeneous space.

In Arhangel'skii and van Mill [2], the authors identified two properties of topological spaces called *skew-2-flexibility* and *2-flexibility* respectively that are useful in studying unique homogeneity. It was shown among other things that every locally compact homogeneous metrizable space is both skew-2-flexible and 2-flexible

---

2000 *Mathematics Subject Classification*. 54C05, 54H11, 54H15, 54G20.

*Key words and phrases*. Homogeneity, unique homogeneity, semitopological group, quasitopological group, topological group.

and that there is an example of a homogeneous Polish space that is skew-2-flexible but not 2-flexible. In addition, in the presence of unique homogeneity, 2-flexibility for  $X$  is equivalent to  $X$  being *Abelian*, i.e., all homeomorphisms of  $X$  commute. Moreover, in the presence of unique homogeneity, skew 2-flexibility for  $X$  implies 2-flexibility and is equivalent to  $X$  being *Boolean*, i.e., all homeomorphisms on  $X$  are involutions. This left open the question whether in the class of uniquely homogeneous spaces, 2-flexibility and skew 2-flexibility are equivalent notions. The aim of this paper is to answer this question in the negative by constructing a uniquely homogeneous (separable metrizable) space  $X$  that is Abelian but not Boolean. In fact, no homeomorphism on  $X$  except for the identity is an involution.

Our example is not a Baire space. We will also prove that such an example cannot be a Baire space so that what we have seems to be optimal.

## 2. PRELIMINARIES

**(A) Groups.** A *semitopological group* (respectively, *paratopological group*) is a group endowed with a topology for which the product is separately (respectively, jointly) continuous. See Bouziad [8], Arhangel'skii and Choban [1], and [4] for conditions guaranteeing that a semitopological group (respectively, paratopological group) is a topological group.

For an Abelian group  $G$  and  $A \subseteq G$  we let  $\langle\langle A \rangle\rangle$  denote the subgroup of  $G$  generated by  $A$ . Moreover, for a subgroup  $A$  of  $G$  we let  $\llbracket A \rrbracket$  denote the subgroup

$$\{x \in G : (\exists n \in \mathbb{Z})(nx \in A)\}$$

of  $G$ . Observe that if  $G$  is a torsion-free Abelian group and  $A \subseteq G$  is a countable subgroup, then  $\llbracket A \rrbracket$  is countable as well.

Let  $G$  be a torsion-free Abelian group. A subset  $A$  of  $G \setminus \{0\}$  is *algebraically independent* if for all pairwise distinct  $a_1, \dots, a_n \in A$  and  $m_1, \dots, m_n \in \mathbb{Z}$  such that  $\sum_{i=1}^n m_i a_i = 0$  we have  $m_1 = \dots = m_n = 0$ . Observe that every uncountable set  $A$  in a torsion-free Abelian group  $G$  contains an uncountable algebraically independent subset. For if  $B \subseteq A$  is countable, then so are  $C = \langle\langle B \rangle\rangle$  and  $D = \llbracket C \rrbracket$ . Hence no maximal algebraically independent subset of  $A$  is countable.

If  $X$  is uniquely homogeneous, for all  $x, y \in X$  we let  $f_y^x$  denote the unique homeomorphism of  $X$  that sends  $x$  to  $y$ . For a fixed  $e \in X$ , define a binary operation  $X \times X \rightarrow X$  by  $x \cdot y = f_x^e(y)$ . This is a group operation on  $X$  having the property that all left translations of  $X$  are homeomorphisms of  $X$ . That is,  $X$  is a left topological group. For details, see [2, Proposition 4.1]. This group operation is called the *standard group operation on  $X$* .

**(B) Topology.** We will need van Douwen's [10, 4.2] generalization of a classical result due to Souslin ([15, p. 437]).

**Theorem 2.1.** *Let  $X$  and  $Y$  be Polish spaces, and let  $\mathcal{F}$  be a countable family of continuous functions from  $X$  to  $Y$  such that:*

*for every countable  $A \subseteq Y : \{f^{-1}(A) : f \in \mathcal{F}\}$  does not cover  $X$ .*

*Then there exists a Cantor set  $K \subseteq X$  such that  $f \upharpoonright K$  is injective for every  $f \in \mathcal{F}$ .*

Let  $X$  be a space. We say that a subset  $A$  of  $X$  is a *bi-Bernstein set* (abbreviated: BB-set) in  $X$  if  $A$  as well as  $X \setminus A$  intersects every Cantor set in  $X$ . Observe that a BB-set in  $X$  intersects every Cantor set in a set of size  $\mathfrak{c}$ , since we can split every Cantor set in a family consisting of  $\mathfrak{c}$  pairwise disjoint Cantor sets.

We let  $K$  denote the standard Cantor set in  $\mathbb{I}$ .

**(C) Measurable functions.** We let  $\mathbb{I}$  denote the closed unit interval  $[0, 1]$ . Let  $X$  be a space. A function  $f: \mathbb{I} \rightarrow X$  is said to be *measurable* if  $f^{-1}(U)$  is a Borel subset of  $\mathbb{I}$  for every open subset  $U$  of  $X$ . We are particularly interested in countable spaces. Observe that if  $X$  is countable, then  $f: \mathbb{I} \rightarrow X$  is measurable if and only if  $f^{-1}(x)$  is Borel for every  $x \in X$ . Measurable functions  $f, g: \mathbb{I} \rightarrow X$  are said to be *equivalent* provided that

$$\lambda(\{t \in \mathbb{I} : f(t) \neq g(t)\}) = 0,$$

where  $\lambda$  denotes Lebesgue measure on  $\mathbb{R}$ .

Let  $\mathcal{F}$  denote the collection of all measurable functions from  $\mathbb{I}$  to  $\mathbb{R}$ .

The sequence  $(f_n)_n$  in  $\mathcal{F}$  *converges to zero in measure* if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \lambda(\{t \in \mathbb{I} : |f_n(t)| \geq \varepsilon\}) = 0.$$

Let  $(f_n)_n$  be a sequence in  $\mathcal{F}$ . Then  $(f_n)_n$  *converges to zero almost everywhere* if there exists a set  $E$  of measure zero such that for every  $x$  not in  $E$  and  $\varepsilon > 0$  there exists  $n_0$  such that  $|f_n(x)| < \varepsilon$  for every  $n \geq n_0$ .

These concepts are known to be related as follows. For completeness sake, we provide a sketch of proof of it below.

**Lemma 2.2.** *A sequence of functions  $(f_n)_n$  in  $\mathcal{F}$  converges to zero in measure if and only if every subsequence of  $(f_n)_n$  contains a subsequence which converges to zero almost everywhere.*

PROOF. Indeed, first assume that  $(f_n)_n$  converges to zero in measure. Every subsequence of  $(f_n)_n$  converges to zero in measure, hence is fundamental in measure by [12, Theorem C on page 92]. Hence some subsequence of it is almost

uniformly fundamental in measure by [12, Theorem D on page 93]. But then this subsequence converges to zero almost everywhere by [12, Theorem B on page 89].

Conversely, assume that every subsequence of  $(f_n)_n$  contains a subsequence which converges to zero almost everywhere. If  $(f_n)_n$  does not converge to zero in measure, then there exist  $\varepsilon > 0$ , a subsequence  $(f_{n_k})_k$  of  $(f_n)_n$  and  $\delta > 0$  such that for every  $k$ ,

$$\lambda(\{t \in \mathbb{I} : |f_{n_k}(t)| \geq \varepsilon\}) \geq \delta.$$

But this subsequence clearly does not have a subsequence that converges to zero almost everywhere.  $\square$

**(D) The spaces  $FM_X$ .**

By  $M_X$  we denote the space consisting of all equivalence classes of measurable functions from  $\mathbb{I}$  into  $X$  endowed with the topology of convergence in measure. For a measurable function  $f$  we let  $[f]$  denote its equivalence class.

The topology on  $M_X$  is induced by the metric

$$(1) \quad \hat{d}([f], [g]) = \sqrt{\int_0^1 d(f(t), g(t))^2 dt},$$

where  $d$  is any admissible bounded metric on  $X$ . The topology on the space  $M_X$  is independent of the (bounded) metric that is chosen to induce its topology. For completeness sake and for later use, we repeat the argument in [6, p. 192].

**Lemma 2.3.** *For a sequence  $([f]_n)_n$  in  $M_X$  and an element  $[f]$  in  $M_X$ , the following statements are equivalent:*

- (1)  $([f]_n)_n$  converges to  $[f]$ ,
- (2) every subsequence of the sequence  $(f_n)_n$  contains a subsequence that converges pointwise to  $f$  almost everywhere.

PROOF. Simply observe that  $\lim_{n \rightarrow \infty} \hat{d}([f]_n, [f]) = 0$  if and only if the sequence of functions  $(\xi_n : t \mapsto d(f_n(t), f(t)))_n$  converges to zero in measure if and only if every subsequence of  $(\xi_n)_n$  contains a subsequence that converges to zero almost everywhere (Lemma 2.2). But this is equivalent to the statement that every subsequence of  $(f_n)_n$  contains a subsequence that converges pointwise to  $f$  almost everywhere.  $\square$

Since the Lemma 2.3(2) does not mention metrics, we see that indeed the topology on  $M_X$  is independent of the chosen (bounded) metric on  $X$ .

**Corollary 2.4.** *Let  $\varphi: X \rightarrow Y$  be a homeomorphism. Then the function  $\bar{\varphi}: M_X \rightarrow M_Y$  defined by  $\bar{\varphi}([f]) = [\varphi \circ f]$  is a homeomorphism.*

Our main interest is in the subspace

$$FM_X = \{[f] \in M_X : (\exists g \in [f])(\text{range}(g) \text{ is finite})\}$$

of  $M_X$ .

The function  $x \mapsto [f_x]$ , where  $f_x: \mathbb{I} \rightarrow X$  is the constant function with value  $x$ , maps  $X$  isometrically onto a closed subset of  $M_X$ . For this we only need to prove that the set  $\{[f_x] : x \in X\}$  is closed in  $M_X$ . But this is easy. For suppose that for  $f: \mathbb{I} \rightarrow X$  we have that  $[f]$  is not the equivalence class of a constant (function). Then there are two distinct elements  $x, y \in X$  such that  $\delta_x = \lambda(f^{-1}(x)) > 0$  and  $\delta_y = \lambda(f^{-1}(y)) > 0$ . Any measurable  $g: \mathbb{I} \rightarrow X$  such that  $\hat{d}([f], [g]) < \min\{\frac{1}{2}d(x, y) \cdot \delta_x, \frac{1}{2}d(x, y) \cdot \delta_y\}$  is not equivalent to a constant function, which does the job.

Bessaga and Pełczyński proved the following fundamental fact about these spaces.

**Theorem 2.5** (Bessaga and Pełczyński [6, Theorem 7.1]). *The space  $M_X$  is homeomorphic to the separable Hilbert space  $\ell^2$  if and only if  $X$  is Polish and contains more than one point.*

### 3. THE GROUP

Let  $G$  be the subgroup of  $\mathbb{R}$  consisting of all rational numbers, i.e.,  $G = \mathbb{Q}$ .

We endow  $G$  with the Sorgenfrey topology. That is, we take the collection of all intervals of the form  $[x, y)$ , where  $x, y \in G$  and  $x < y$ , as an open base. Observe that  $G$  with this topology is an Abelian paratopological group, but that inversion is (badly) discontinuous. Moreover, the Sorgenfrey base is countable since  $G$  is, hence  $G$  is metrizable.

Since  $G$  is obviously dense in itself,  $G$  is homeomorphic to  $\mathbb{Q}$ , but the homeomorphism cannot be chosen to have really nice algebraic properties.

Let  $d$  be a metric bounded by 1 generating the topology on  $G$ .

We now consider the space  $M_G$ . For  $[f], [g] \in M_G$ , define  $[f + g] \in M_X$  by the rule

$$(f + g)(t) = f(t) + g(t) \quad (t \in \mathbb{I}).$$

Clearly,  $f + g$  is measurable, and  $+$  is an Abelian group operation on  $M_X$ .

**Lemma 3.1.**  *$M_G$  is a paratopological group.*

PROOF. Fix  $[f], [g] \in M_G$ , and let  $([f_n])_n$  and  $([g_n])_n$  be sequences converging to  $[f]$  respectively  $[g]$  in  $M_G$ . We have to show that  $[f_n + g_n] \rightarrow [f + g]$  in  $M_G$ . By Lemma 2.3, every subsequence of  $(f_n)_n$  contains a subsequence that

converges pointwise to  $f$  almost everywhere. Similarly for  $g$ . But then since  $G$  is a paratopological group, every subsequence of  $(f_n + g_n)_n$  contains a subsequence that converges pointwise to  $f + g$  almost everywhere. Hence we are done by Lemma 2.3.  $\square$

The constant functions form a closed subgroup of  $M_G$  which is isometric to  $G$ . We write  $G^*$  for this closed subgroup of  $M_G$ . Hence inversion on  $M_G$  is as badly discontinuous on  $M_G$  as it is on  $G$ . Observe that  $FM_G$  is a subgroup of  $M_G$  that obviously contains  $G^*$ .

Since  $M_G$  contains a closed copy of the rational numbers, it is not Polish. In fact, the closed copy of  $\mathbb{Q}$  gives us that  $M_G$  is not hereditarily Baire. It can be shown that  $M_G$  is Borel, hence Čech-analytic. From this it follows from Bouzhiad [8] that  $M_G$  is not a Baire space since  $M_G$  is not a topological group. We do not present the details of this since the group we are after is a subgroup of  $FM_G$ , and for that space it is obvious that it is not a Baire space, as we will now show.

Write  $G$  as  $\bigcup_{n < \omega} G_n$ , where each  $G_n$  is finite and  $G_n \subseteq G_{n+1}$  for every  $n$ . For every  $n$ , put

$$FM_n = \{[f] \in FM_G : (\exists g \in [f])(g(\mathbb{I}) \subseteq G_n)\}.$$

Clearly,  $FM_n$  is naturally homeomorphic to  $M_{G_n}$ ,  $FM_n \subseteq FM_{n+1}$  for every  $n$ , and  $FM_G = \bigcup_{n < \omega} FM_n$ .

**Lemma 3.2.** *For every  $n$ ,  $FM_n$  is a nowhere dense closed subspace of  $FM_G$  which is homeomorphic to  $\ell^2$ .*

PROOF. That  $FM_n$  is homeomorphic to  $\ell^2$  follows from Theorem 2.5. To prove it is closed, take any  $[f] \in FM_G \setminus FM_n$ . Then there is an element  $x \in G \setminus G_n$  such that  $\lambda(f^{-1}(x)) > 0$ . Assume that  $([f_i])_i$  is a sequence in  $FM_n$  converging to  $[f]$ . We assume without loss of generality that  $f_i(\mathbb{I}) \subseteq G_n$  for every  $i$ . By Lemma 2.3 we may assume that  $(f_i)_i$  converges to  $f$  almost everywhere. So there exists  $p \in f^{-1}(x)$  such that  $f_i(p) \rightarrow f(p)$ . Hence there exists  $i$  such that  $f_i(p) \notin G_n$ , which is a contradiction. To prove it is nowhere dense, take any  $[f] \in FM_G$ . Pick  $x \in G$  such that  $\lambda(f^{-1}(x)) > 0$ . Split  $f^{-1}(x)$  into two Borel sets, each of positive measure, say  $A$  and  $B$ . Define a function  $g: \mathbb{I} \rightarrow G$  by  $g|(\mathbb{I} \setminus B) = f|(\mathbb{I} \setminus B)$ , and  $g|B$  is the constant function with value  $a$ , where  $a$  is an element of  $G \setminus G_n$  with very small distance towards  $x$ . Then  $[f]$  and  $[g]$  are very close, and  $[g]$  does not belong to  $FM_n$ .  $\square$

We conclude from the previous lemma, that  $FM_G$  is *strongly  $\sigma$ -complete*, i.e., a countable union of closed Polish subspaces. Hence  $FM_G$  is Borel, but not Baire.

Inversion on  $FM_G$  is badly discontinuous, but not if we consider inversion on one of its building blocks  $FM_n$ .

**Lemma 3.3.** *For a fixed  $n$  define  $i: FM_n \rightarrow FM_G$  by  $i([f]) = [-f]$ . Then  $i$  is an embedding.*

PROOF. Put  $H = \{-s : s \in G_n\}$ . Since  $G_n$  and  $H$  are finite, the function  $\varphi: G_n \rightarrow H$  defined by  $\varphi(s) = -s$  is a homeomorphism. Hence we are done by Corollary 2.4 since the homeomorphism  $\bar{\varphi}: M_{G_n} \rightarrow M_H$  defined there is identical to  $i$ .  $\square$

**Lemma 3.4.**  *$M_G$  is torsion-free.*

PROOF. Take  $[f] \in FM_G$  and assume that  $nf$  is the constant function with value 0 for some  $n \in \mathbb{Z} \setminus \{0\}$ . Let  $x \in \text{range}(f)$ . For  $p \in f^{-1}(x)$  we have that  $nf(p) = 0$ , hence  $x = f(p) = 0$  since  $G$  is torsion-free. We conclude that  $f$  is the constant function with value 0.  $\square$

Hence we almost completed the proof of the following result.

**Theorem 3.5.** *There is a separable metrizable torsion-free Abelian paratopological group  $H$  that can be written as  $\bigcup_{n < \omega} H_n$  such that  $H_n \subseteq H_{n+1}$  for every  $n$ , while moreover:*

- (1)  *$H$  contains a countable subgroup  $G$  on which inversion is discontinuous,*
- (2) *every  $H_n$  is closed in  $H$  and homeomorphic to Hilbert space  $\ell^2$ ,*
- (3) *for every  $n$  there exists  $m \geq n$  such that  $H_n + H_n$  and  $H_n - H_n$  are both contained in  $H_m$ ,*
- (4) *for every  $n$  the function  $i: H_n \rightarrow H$  defined by  $i(x) = -x$  is an embedding.*

PROOF. Of course, we set  $H = FM_G$  and  $H_n = FM_n$  for all  $n$ . By Theorem 2.5, and Lemmas 3.1, 3.2, 3.3 and 3.4, the only thing left to prove is (3). But this is trivial, since for given  $n$ , we may take  $m$  so large that  $G_m$  contains both  $G_n + G_n$  and  $G_n - G_n$ , and then  $m$  is as required.  $\square$

**Corollary 3.6.** *For given  $n$  and  $m \geq n$ , let  $f: S \rightarrow H_m$  be continuous, where  $S \subseteq H_n$ . Then the functions  $\xi, \eta: S \rightarrow H$  defined by  $\xi(x) = x + f(x)$  and  $\eta(x) = x - f(x)$  are continuous. Moreover, there exists  $k$  such that the ranges of both  $\xi$  and  $\eta$  are contained in  $H_k$ .*

PROOF. Simply apply Theorem 3.5(4) and (3) and the fact that  $H$  is a paratopological group.  $\square$

We finish this section by the following technical result, which we will need in the forthcoming §4.

**Lemma 3.7.** *Let  $L \subseteq H$  be any Cantor set. Then  $L$  contains an algebraically independent Cantor set.*

PROOF. In this proof we will say that a subset  $A$  of  $H \setminus \{0\}$  is  $k$ -algebraically independent for some  $k \geq 1$  if for all pairwise distinct  $a_1, \dots, a_n \in A$  and  $m_1, \dots, m_n \in [-k, k]$  such that  $\sum_{i=1}^n m_i a_i = 0$  we have  $m_1 = \dots = m_n = 0$ .

**Claim 1.** Let the pairwise distinct  $a_1, \dots, a_m$  in  $H_n$  be  $k$ -algebraically independent. Then there are pairwise disjoint neighborhoods  $U_1, \dots, U_m$  of  $a_1, \dots, a_m$  in  $H_n$  such that for any choice  $b_1 \in U_1, \dots, b_m \in U_m$  we have that  $b_1, \dots, b_m$  is  $k$ -algebraically independent.

If this is not true, then there are sequences  $(a_i^j)_j$  such that  $a_i^j \rightarrow a_i$  for every  $i \leq m$  and  $a_1^j, \dots, a_m^j$  is not  $k$ -algebraically independent for every  $j$ . This means that there is a linear combination  $\sum_{i=1}^m k_i a_i^j$ , where  $k_1, \dots, k_m \in [-k, k]$ , which is 0 while yet some  $k_{i_0} \neq 0$ . But then infinitely often we have that the  $k_1, \dots, k_m$  and the  $k_{i_0}$  are the same. So we may assume without loss of generality that they are always equal to say  $k_1, \dots, k_m$  and  $k_{i_0}$ . Observe that by Theorem 3.5(3) and (4) the function  $H_n^m \rightarrow H$  defined by  $(t_1, \dots, t_m) \mapsto \sum_{i=1}^m k_i t_i$  is continuous. Hence  $\sum_{i=1}^m k_i a_i = 0$ , and hence  $k_{i_0} \neq 0$  contradicts the  $k$ -algebraic independence of  $a_1, \dots, a_m$ .

Since  $L = \bigcup_{n < \omega} L \cap H_n$ , the Baire Category Theorem implies that we may assume without loss of generality that  $L \subseteq H_n$  for some  $n$ . Since  $L$  is uncountable and  $H$  is torsion-free, it contains an uncountable algebraically independent subset, say  $E$ . By the Cantor-Bendixson Theorem, [11, 1.7.11], we may assume that  $E$  is dense-in-itself. By using disjoint balls about points of  $E$ , we may now construct a Cantor set in the standard manner. A little extra care made possible by Claim 1 will ensure that it will be  $k$ -algebraically independent for every  $k$ , i.e., it will be algebraically independent.  $\square$

#### 4. UNIQUE HOMOGENEITY

Let  $H$  be the group from Theorem 3.5. We now closely follow the construction in van Mill [18], but considerable extra care is needed. As in §3, let  $K$  be the Cantor set in  $\mathbb{I}$ . Consider the collection

$$\mathcal{K} = \{ \langle f, g \rangle : f, g : K \rightarrow H \text{ are embeddings and the functions } f+g \text{ and } f-g \text{ are one-to-one} \}.$$

Observe that if  $\langle f, g \rangle \in \mathcal{K}$ , then also  $\langle g, f \rangle \in \mathcal{K}$ . For every  $\langle f, g \rangle \in \mathcal{K}$  we would like to ‘kill’ the homeomorphism  $g \circ f^{-1}: f(K) \rightarrow g(K)$  or, if this is not possible, the homeomorphism  $f \circ g^{-1}: g(K) \rightarrow f(K)$ . It is clear that  $|\mathcal{K}| \leq \mathfrak{c}$ , hence we may enumerate  $\mathcal{K}$  as  $\{\langle f_\alpha, g_\alpha \rangle : \alpha < \mathfrak{c}\}$  (repetitions permitted).

By transfinite induction on  $\alpha < \mathfrak{c}$ , we will pick a point  $x_\alpha \in K$  and points  $p_\alpha, q_\alpha \in H \setminus \{0\}$  such that

- (1)  $\{p_\alpha, q_\alpha\} = \{f_\alpha(x_\alpha), g_\alpha(x_\alpha)\}$ ,
- (2)  $\langle\langle \{p_\beta : \beta \leq \alpha\} \cup G \rangle\rangle \cap \{q_\beta : \beta \leq \alpha\} = \emptyset$ .

Assume that we picked  $x_\beta, p_\beta$  and  $q_\beta$  for every  $\beta < \alpha$ , where  $\alpha < \mathfrak{c}$  (possibly  $\alpha = 0$ ). Put  $A = \langle\langle \{p_\beta : \beta < \alpha\} \cup G \rangle\rangle$  and  $V = \{q_\beta : \beta \leq \alpha\}$ , respectively. Then  $\max\{|A|, |V|\} \leq |\alpha| \cdot \omega < \mathfrak{c}$ , and  $A \cap V = \emptyset$ . For convenience, put  $f = f_\alpha$  respectively  $g = g_\alpha$ .

**Lemma 4.1.**  $E_f = \{x \in K : \langle\langle \{f(x)\} \cup A \rangle\rangle \cap V \neq \emptyset\}$  has cardinality less than  $\mathfrak{c}$ .

PROOF. For every  $x \in E_f$  there exists  $n_x \in \mathbb{Z}$  such that  $n_x \cdot f(x) \in V - A$ . Since  $A \cap V = \emptyset$ ,  $V - A \subseteq H \setminus \{0\}$ , so always  $n_x \neq 0$ . Suppose that  $|E_f| = \mathfrak{c}$ . Then since  $|V - A| < \mathfrak{c}$ , there are distinct  $x, y \in E_f$  and  $n \in \mathbb{Z} \setminus \{0\}$  such that  $n = n_x = n_y$  and  $n \cdot f(x) = n \cdot f(y)$ . But then  $f(x) = f(y)$  since  $H$  is torsion free, which violates  $f$  being injective.  $\square$

By precisely the same argument, we obtain:

**Lemma 4.2.**  $E_g = \{x \in K : \langle\langle \{g(x)\} \cup A \rangle\rangle \cap V \neq \emptyset\}$  has cardinality less than  $\mathfrak{c}$ .

We now come to the crucial step in our argumentation.

**Lemma 4.3.** If  $F \subseteq K$  has cardinality  $\mathfrak{c}$ , then there exists  $x \in F$  such that  $f(x) \notin \langle\langle \{g(x)\} \cup A \rangle\rangle$  or  $g(x) \notin \langle\langle \{f(x)\} \cup A \rangle\rangle$ .

PROOF. Let  $F \subseteq K$  have size  $\mathfrak{c}$ , and assume that for every  $x \in F$  we have that  $f(x) \in \langle\langle \{g(x)\} \cup A \rangle\rangle$  and  $g(x) \in \langle\langle \{f(x)\} \cup A \rangle\rangle$ . Let  $\kappa = |A| \cdot \omega$ . Then  $|\mathbb{Z} \times A| = \kappa < \mathfrak{c}$ , so there are  $n \in \mathbb{Z}$ ,  $a \in A$  and  $\hat{F} \subseteq F$  of cardinality greater than  $\kappa$  such that for every  $x \in \hat{F}$ ,  $f(x) = n \cdot g(x) + a$ . Since the functions  $f + g$  and  $f - g$  are both one-to-one and  $|A| \leq \kappa$ , we get  $n \notin \{1, -1\}$ . By the same argumentation, there exist a subset  $\tilde{F}$  of  $\hat{F}$  of size bigger than  $\kappa$ ,  $m \in \mathbb{Z} \setminus \{1, -1\}$  and  $\bar{a} \in A$  such that for every  $x \in \tilde{F}$ ,  $g(x) = m \cdot f(x) + \bar{a}$ . For  $x \in \tilde{F}$  we consequently have

$$f(x) = n \cdot g(x) + a = nm \cdot f(x) + (n \cdot \bar{a}) + a,$$

hence

$$(nm - 1) \cdot f(x) = \tilde{a},$$

where  $\tilde{a} = (n \cdot \bar{a}) + a$ . Since  $nm - 1 \neq 0$ ,  $H$  is torsion free and  $f$  is one-to-one, we reached a contradiction.  $\square$

Now let  $E_f$  and  $E_g$  be as in Lemmas 4.1 and 4.2, and put  $F = K \setminus (E_f \cup E_g)$ . By Lemma 4.3 we may assume without loss of generality that there exists  $x \in F$  such that  $f(x) \notin \langle \{g(x)\} \cup A \rangle$ . Now put  $x_\alpha = x$ ,  $p_\alpha = g(x_\alpha)$  and  $q_\alpha = f(x_\alpha)$ . It is clear that our choices are as required. This completes the transfinite induction.

Put  $X = \langle \{p_\alpha : \alpha < \mathfrak{c}\} \cup G \rangle$ . We claim that  $X$  has no homeomorphisms other than translations. This will show that  $X$  is uniquely homogeneous and Abelian, but not Boolean. In fact, no nontrivial translation is an involution since  $X$  is torsion-free, hence  $X$  does not have any involution other than the identity function.

**Lemma 4.4.** *For every  $n < \omega$ ,  $X \cap H_n$  is a BB-set in  $H_n$ .*

PROOF. Let  $L \subseteq H_n$  be a Cantor set. By Lemma 3.7 we may assume without loss of generality that  $L$  is algebraically independent. Now let  $L_0$  and  $L_1$  be disjoint Cantor sets in  $L$ , and let  $\alpha: K \rightarrow L_0$  and  $\beta: K \rightarrow L_1$  be arbitrary homeomorphisms. Then  $\langle \alpha, \beta \rangle \in \mathcal{K}$ , hence  $X$  intersects  $L_0 \cup L_1 \subseteq L$  by construction. Similarly,  $L$  intersects  $H_n \setminus X$ .  $\square$

We now formulate and prove a curious property of  $X$ .

**Lemma 4.5.** *Let  $f: X \rightarrow X$  be a homeomorphism. Then there is a countable subgroup  $A$  of  $X$  such that for every  $x \in X$  there exists  $a \in A$  such that  $f(x) = x + a$  or  $f(x) = -x + a$ .*

PROOF. Let  $E \subseteq X$  be maximal with respect to the properties that the functions  $\xi, \eta: E \rightarrow X$  defined by

$$\xi(x) = x + f(x), \quad \eta(x) = x - f(x)$$

are one-to-one.

We will prove that  $E$  is countable. Striving for a contradiction, assume that  $E$  is uncountable. We have to do some thinning out first. Since  $H = \bigcup_{n < \omega} H_n$ , we may fix an integer  $n$  such that  $E_0 = E \cap H_n$  is uncountable. Since  $f \upharpoonright E_0$  is one-to-one, its range is uncountable. Hence there is  $k \geq n$  such that the set  $E_1 = \{x \in E_0 : f(x) \in H_k\}$  is uncountable. Pick  $m \geq k$  such that  $H_k + H_k$  and  $H_k - H_k$  are both contained in  $H_m$ . Observe that the ranges of the functions  $f \upharpoonright E_1$ ,  $\xi \upharpoonright E_1$  and  $\eta \upharpoonright E_1$  are all contained in  $H_m$ . Since  $H_n$  and  $H_k$  are Polish, being homeomorphic to  $\ell^2$ , there are  $G_\delta$ -subsets  $S$  of  $H_n$  and  $T$  of  $H_k$  such that  $E_1$  is dense in  $S$  and  $f \upharpoonright E_1$  can be extended to a homeomorphism  $\bar{f}: S \rightarrow T$

([11, Theorem 4.3.21]). Define  $\bar{\xi}: S \rightarrow H_m$  by  $\bar{\xi}(x) = x + \bar{f}(x)$ , and, similarly,  $\bar{\eta}: S \rightarrow H_m$  by  $\bar{\eta}(x) = x - \bar{f}(x)$ . Then  $\bar{\xi}$  and  $\bar{\eta}$  are continuous by Corollary 3.6. Since the functions  $\bar{\xi}, \bar{\eta}$  and  $\bar{f}$  are one-to-one on  $E_1$ , there is by Theorem 2.1 a Cantor set  $L$  in  $S$  such that  $\bar{\xi}, \bar{\eta}$  and  $\bar{f}$  are all one-to-one on  $L$ . Let  $\alpha: K \rightarrow L$  be an arbitrary homeomorphism. Consider the pair of functions  $\langle \alpha, \bar{f}|_L \circ \alpha \rangle \in \mathcal{K}$ . By construction there exists  $x \in L$  such that either  $x \in X$  and  $\bar{f}(x) \notin X$ , or  $\bar{f}(x) \in X$  and  $x \notin X$ . Suppose first that  $x \in X$  and  $\bar{f}(x) \notin X$ . Then since  $\bar{f}$  extends  $f$ , we get  $\bar{f}(x) = f(x) \in X$  which is a contradiction. Suppose next that  $\bar{f}(x) \in X$  and  $x \notin X$ . Since  $E_1$  is dense in  $S \supseteq L$ , there is a sequence  $(p_i)_i$  in  $E_1$  which converges to  $x$ . Hence  $\bar{f}(p_i) \rightarrow \bar{f}(x)$ . But  $\bar{f}$  extends  $f$ , hence  $f(p_i) \rightarrow \bar{f}(x)$ . There exists  $p \in X$  such that  $f(p) = \bar{f}(x)$  since  $f$  is a homeomorphism. Hence  $f(p_i) \rightarrow f(p)$  which means that  $p_i \rightarrow p$ . We conclude that  $x = p \in X$ , which is a contradiction. This completes the proof of our claim.

So we conclude that  $E$  is indeed countable. Let  $A = \langle\langle E \cup f(E) \rangle\rangle$ . If  $x \in E$ , then  $f(x) - x \in A$  so there is nothing to prove. Assume that  $x \notin E$ . By maximality of  $E$ ,  $\xi|_{E \cup \{x\}}$  or  $\eta|_{E \cup \{x\}}$  is not one-to-one. Suppose first that there exists  $e \in E$  such that  $\xi(e) = \xi(x)$ . Then  $e + f(e) = x + f(x) \in A$  and hence we are done. If  $\eta|_{E \cup \{x\}}$  is not one-to-one, then we can argue similarly.  $\square$

We now come to our main result.

**Theorem 4.6.** *Every homeomorphism of  $X$  is a translation.*

PROOF. Let  $f: X \rightarrow X$  be a homeomorphism, and let  $A$  be the subgroup of  $X$  we get from Lemma 4.5. For every  $a \in A$  and  $\varepsilon \in \{-1, 1\}$  we put

$$S_a^\varepsilon = \{x \in X : f(x) = \varepsilon \cdot x + a\}.$$

We claim that at most one element of the cover  $\mathcal{S} = \{S_a^\varepsilon : \varepsilon \in \{-1, 1\}, a \in A\}$  of  $X$  is uncountable. Striving for a contradiction, assume that there are at least two elements of  $\mathcal{S}$  that are uncountable. Pick  $n$  such that at least two elements of  $\mathcal{S}$  intersect  $X \cap H_n$  in an uncountable set, say  $S_{a_0}^{\varepsilon_0}$  and  $S_{a_1}^{\varepsilon_1}$ .

For every  $a \in A$  and  $\varepsilon \in \{-1, 1\}$  put  $S_{n,a}^\varepsilon = S_a^\varepsilon \cap H_n$ .

**Claim 1.** For every  $a \in A$  and  $\varepsilon \in \{-1, 1\}$ ,  $S_{n,a}^\varepsilon$  is closed in  $X \cap H_n$ .

This is easy. Let  $(x_i)_i$  be a sequence in  $S_{n,a}^\varepsilon$  converging to some element  $x \in X \cap H_n$ . Then  $f(x_i) = \varepsilon \cdot x_i + a$  for every  $i$ . But  $\varepsilon \cdot x_i + a \rightarrow \varepsilon \cdot x + a$  by Theorem 3.5(4) and (3) and the fact that  $H$  is a paratopological group. Hence  $f(x) = \varepsilon \cdot x + a$ .

For every  $a \in A$  and  $\varepsilon \in \{-1, 1\}$ , let  $T_{n,a}^\varepsilon$  be the closure of  $S_{n,a}^\varepsilon$  in  $H_n$ .

**Claim 2.**  $T = H_n \setminus \bigcup \{T_{n,a}^\varepsilon : \varepsilon \in \{-1, 1\}, a \in A\}$  is countable

This is clear since  $T$  is a  $G_\delta$ -subset of  $H_n$  that misses  $X \cap H_n$ . Hence it has to be countable since if it were uncountable, then it would contain a Cantor set which would intersect  $X \cap H_n$  (Lemma 4.4).

**Claim 3.** If  $a, a' \in A$  are distinct, then  $T_{n,a}^\delta \cap T_{n,a'}^\varepsilon$  is countable for all  $\delta, \varepsilon \in \{-1, 1\}$ .

Striving for a contradiction, assume that for certain distinct  $a, a' \in A$  and  $\delta, \varepsilon \in \{-1, 1\}$  we have that  $T_{n,a}^\delta \cap T_{n,a'}^\varepsilon$  is uncountable. Then  $T_{n,a}^\delta \cap T_{n,a'}^\varepsilon$  contains a Cantor set which consequently intersects  $X \cap H_n$  in a set of size  $\mathfrak{c}$  (Lemma 4.4). So we may pick  $x \in X \setminus \llbracket A \rrbracket$  such that  $f(x) = \varepsilon \cdot x + a = \delta \cdot x + a'$ . If  $\varepsilon = \delta$ , then  $a = a'$ . Hence  $\varepsilon \neq \delta$  which implies that  $2x \in A$ , which is a contradiction.

Now consider the countable subset

$$S = T \cup \bigcup \{T_{n,a}^\delta \cap T_{n,a'}^\varepsilon : a, a' \in A, a \neq a', \delta, \varepsilon \in \{-1, 1\}\}$$

of  $H_n$ . Since  $H_n \approx \ell^2$ ,  $H_n \setminus S$  is path-connected ([6, Theorem 6.4 on Page 166]). Pick arbitrary  $x \in S_{a_0}^{\varepsilon_0} \setminus S$  and  $y \in S_{a_1}^{\varepsilon_1} \setminus S$ . There is an arc  $J$  in  $H_n \setminus S$  which contains both  $x$  and  $y$ . Then the collection

$$\{T_{n,a}^\varepsilon \cap J : a \in A, \varepsilon \in \{-1, 1\}\}$$

is a partition of  $J$  in at least two nonempty and at most countably many nonempty closed sets. But this violates the Sierpiński Theorem, [11, Theorem 6.1.27].

This means that at most one element of the closed cover  $\mathcal{S}$  of  $X$  is uncountable. But  $X$  is locally of cardinality  $\mathfrak{c}$  by Lemma 4.4. As a consequence, since  $\mathcal{S}$  is countable, there is exactly one element of  $\mathcal{S}$  that is nonempty, and hence is equal to  $X$ . There are two cases to consider. First assume that  $S_a^{-1} = X$  for some  $a \in A$ . Then  $f(x) = -x + a$  for every  $x \in X$ . But  $H$  contains the countable group  $G$ , the paratopological group we started with. There is a sequence  $(x_i)_i$  in  $G$  such that  $x_i \rightarrow 0$  but  $(-x_i)_i$  is closed and discrete in  $G$  and hence in  $H$  and hence in  $X$ . Since the translation  $x \mapsto x+a$  is a homeomorphism of  $X$ , this implies that the sequence  $(-x_i + a)_i$  is closed and discrete in  $X$ . But this contradicts the continuity of  $f$ . Hence there is a unique  $a \in A$  such that  $f(x) = x + a$  for every  $x \in X$ , i.e.,  $f$  is a translation.  $\square$

## 5. BAIRE UNIQUELY HOMOGENEOUS SPACES

The example constructed in the previous section is not a Baire space. Here we prove that it cannot be a Baire space.

**Theorem 5.1.** *Let  $X$  be a metrizable Baire space that is uniquely homogeneous. If  $X$  is Abelian, then  $X$  is an Abelian topological group.*

PROOF. There are several ways we can arrive at the desired conclusion.

For example, by [2, Theorem 5.4], the standard group operation on  $X$  is semitopological and Abelian. However, every metrizable semitopological group with the Baire property is a paratopological group, by Bouziad [7, Corollary 5]. But every symmetrizable paratopological group with the Baire property is a topological group, as was shown by Arhangel'skii and Reznichenko [3, Theorem 1.4].

Another route is to use Theorem 2 in Kenderov, Kortezov, and Moors [14].  $\square$

#### REFERENCES

- [1] A. V. Arhangel'skii and M. M. Choban, *Completeness type properties of semitopological groups, and the theorems of Montgomery and Ellis*, Top. Proc. **37** (2011), 1–28.
- [2] A. V. Arhangel'skii and J. van Mill, *On uniquely homogeneous spaces, I*, 2011, to appear in J. Math. Soc. Japan.
- [3] A. V. Arhangel'skii and E. A. Reznichenko, *Paratopological and semitopological groups versus topological groups*, Topology Appl. **151** (2005), 107–119.
- [4] A. V. Arhangel'skii and M. G. Tkachenko, *Topological groups and related structures*, Atlantis Studies in Mathematics, vol. 1, Atlantis Press, Paris, World Scientific, 2008.
- [5] W. Barit and P. Renaud, *There are no uniquely homogeneous spaces*, Proc. Amer. Math. Soc. **68** (1978), 385–386.
- [6] C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, PWN—Polish Scientific Publishers, Warsaw, 1975, Monografie Matematyczne, Tom 58.
- [7] A. Bouziad, *Continuity of separately continuous group actions in  $p$ -spaces*, Topology Appl. **71** (1996), 119–124.
- [8] A. Bouziad, *Every Čech-analytic Baire semitopological group is a topological group*, Proc. Amer. Math. Soc. **124** (1996), 953–959.
- [9] C. E. Burgess, *Homogeneous continua*, Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology, University of Wisconsin, 1955, pp. 75–78.
- [10] E. K. van Douwen, *A compact space with a measure that knows which sets are homeomorphic*, Adv. Math. **52** (1984), 1–33.
- [11] R. Engelking, *General topology*, Heldermann Verlag, Berlin, second ed., 1989.
- [12] P. Halmos, *Measure theory*, Springer-Verlag, New York-Heidelberg-Berlin, 1974.
- [13] J. E. Keesling and D. C. Wilson, *An almost uniquely homogeneous subgroup of  $\mathbb{R}^n$* , Top. Appl. **22** (1986), 183–190.
- [14] P. S. Kenderov, I. S. Kortezov, and W. B. Moors, *Topological games and topological groups*, Topology Appl. **109** (2001), 157–165.
- [15] K. Kuratowski and A. Mostowski, *Set theory*, North-Holland Publishing Co., Amsterdam, revised ed., 1976, With an introduction to descriptive set theory, Translated from the 1966 Polish original, Studies in Logic and the Foundations of Mathematics, Vol. 86.
- [16] J. van Mill, *A topological group having no homeomorphisms other than translations*, Trans. Amer. Math. Soc. **280** (1983), 491–498.

- [17] J. van Mill, *A uniquely homogeneous space need not be a topological group*, *Fund. Math.* **122** (1984), 255–264.
- [18] J. van Mill, *Sierpiński's Technique and subsets of  $\mathbb{R}$* , *Top. Appl.* **44** (1992), 241–261.
- [19] G. S. Ungar, *On all kinds of homogeneous spaces*, *Trans. Amer. Math. Soc.* **212** (1975), 393–400.

Received February 18, 2011

Revised version received April 19, 2012

(Arhangel'skii) MOSCOW, 121165, RUSSIA  
*E-mail address:* `arhangel.alex@gmail.com`

(Van Mill) FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, VU UNIVERSITY AMSTERDAM, 1081 HV AMSTERDAM, THE NETHERLANDS  
*E-mail address:* `j.van.mill@vu.nl`