Oscillatory solutions of fourth order conservative systems[1]

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Abstract. In this paper we investigate periodic solutions of second order Lagrangian systems which oscillate around equilibrium points of center type. The main ingredients are the discretization of second order Lagrangian systems that satisfy the twist property and the theory of discrete braid invariants developed in [5]. The problem with applying this topological theory directly is that the braid types in our analysis are so-called *improper*. This implies that the dynamical implications of the braid invariants do not entirely depend on the topology. In first part of this paper we develop the theory of braid invariant for improper braid classes and in the second part this theory is applied to second order Lagrangian system and in particular to the Swift-Hohenberg equation.

1. **Introduction**

Fourth order conservative systems were studied by several authors (e.g. [4],[13],[2], etc.). These equations may exhibit complicated dynamical behavior and the dependence of the dynamics on parameters can be complex. In this paper the equation

$$u'''' + \alpha u'' - u + u^3 = 0, \tag{1.1}$$

is our main example, although the result also apply to a more general class of fourth order conservative systems which occur as the Euler-Lagrange equations of second order Lagrangians $\int_I L(u, u', u'')dt$. For Equation (1.1) the Lagrangian density is given by $L(u, v, w) = \frac{1}{2}w^2 - \frac{\alpha}{2}v^2 + \frac{1}{4}(u^2 - 1)^2$. Related to this variational structure (through Noether's theorem) is a conserved quantity. Solutions of Equation (1.1) satisfy the energy identity $\mathbb{E}[u] = -u'u''' + \frac{1}{2}(u'')^2 - \frac{\alpha}{2}(u')^2 - \frac{1}{4}(u^2 - 1)^2 = E$. In the case $\alpha < 0$ Equation (1.1) is referred to as the eFK equation (see e.g. [6, 7, 8]) and for $\alpha \geq 0$ it is

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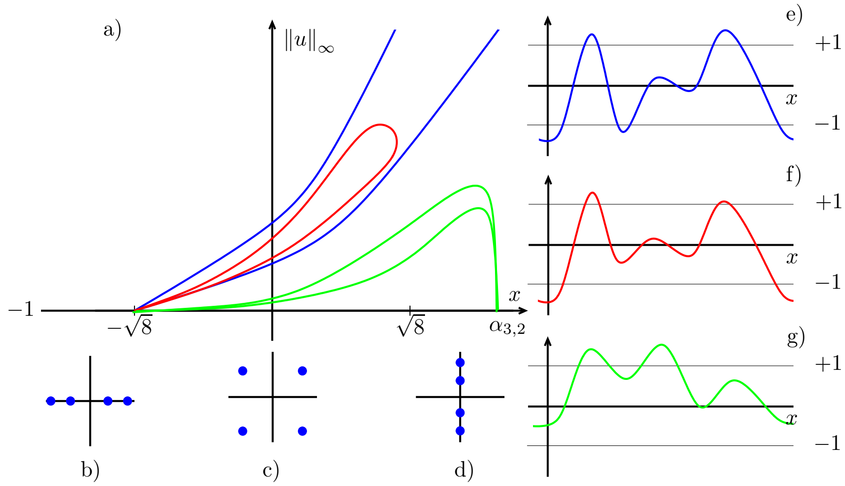

FIGURE 1. Bifurcation diagram a) shows three different types of branches, in the plane $(\alpha, \|u\|_\infty)$, which bifurcate at $\alpha = -\sqrt{8}$. Solutions on the branches that extend beyond the boundary of the diagram are of the first type, see e) for an example; branches that form closed loops consist of solutions of the second type, see f) for an example; branches collapsing on $\|u\|_\infty = 1$ consist of solutions of the third type, see g) for an example. Also depicted is the spectrum of the linearization around P_+ and P_- for b) $\alpha \leq -\sqrt{8}$; c) $\alpha \in (-\sqrt{8}, \sqrt{8})$; d) $\sqrt{8} \leq \alpha$.

referred to as the Swift-Hohenberg equation [13]. Equation (1.1) appears in physical models for phase transitions, Rayleigh-Bénard convection, non-linear optics, etc. For a more extensive survey see [14].

For Equation (1.1) the energy level $E = 0$ contains the two homogenous states $u_\pm = \pm 1$ and this energy level acts as an organizing center for the dynamics. Intuitively, homoclinic solutions to $u_\pm = \pm 1$ and/or a heteroclinic cycle between -1 and $+1$ will, if they exist, lie in this energy level and it is well known that such connecting orbits may be the source of complicated dynamics. This makes for a natural choice to the study of solutions in this *singular* energy level, which leads to analytical difficulties. The focus on the singular energy level is not new and we will summarize some of the known results below.

To introduce the central question of this paper, let us summarize the known results on (1.1) most relevant to our problem. The structure of the set of periodic solutions of Equation (1.1) depends to a large extend on the linearization around the constant solutions $u_\pm = \pm 1$ and hence on the value of the parameter α. In particular, one can identify two critical values of α : $\alpha = +\sqrt{8}$ and $\alpha = -\sqrt{8}$. At these values the linearization around the constant solutions $u_\pm$, i.e. the points $P_\pm = (\pm 1, 0, 0, 0)$ in (u, u', u'', u''') phase space, changes type, as indicated in Figure 1. In fact, for $\alpha \le -\sqrt{8}$, the equilibria $u_\pm$ are real saddles and there are no periodic solutions on the zero energy level. The set of *all* bounded solutions is very limited, and consists of the three equilibrium points, two monotone antisymmetric heteroclinic loops and (modulo transitions) a one parameter family of single bump periodic solutions, which are even with respect to their extrema and odd with respect to their zeros. These periodic solutions can be parameterized by the energy $E \in \left(-\frac{1}{4}, 0\right)$, see [14]. As α increases beyond $-\sqrt{8}$ the equilibria $u_\pm$ become saddle-foci and the set of periodic solutions becomes much richer. There is a plethora of periodic solutions on the energy level zero bifurcating from the heteroclinic loop at $\alpha = -\sqrt{8}$. It has been proved that for $-\sqrt{8} < \alpha \le 0$ the zero energy level contains a great variety of multi-bump periodic solutions. For detailed results we refer to [6, 7, 8]. For $0 < \alpha < \sqrt{8}$ the results are more tentative and less complete. For $\alpha > \sqrt{8}$ the equilibria change to centers and small periodic oscillations around equilibria $u_\pm$ appear. This is the parameter regime of primary interest in the present paper.

It was shown in [4] that at the regular energy levels every solution is a concatenation of monotone laps between extrema and the number of the monotone laps is finite and even per period. Figure 1 shows the bifurcation diagram, where we graph the L^∞-norm of the solutions u^α of (1.1) against the value α. Three branches with very different geometry appear in the bifurcation diagram. Two essential properties are preserved for the solutions laying on the same branch of the bifurcation diagram:

(1) the number of monotone laps;
(2) the number of crossings of the solution with the u_+ and u_-.

The counting of laps and crossings is done with the following conventions. For a regular monotone lap u' does not change sign i.e. $u' < 0$ or $u' > 0$, and a degenerate monotone lap is an inflexion point. We have to count both non-degenerate and degenerate monotone laps in order to obtain the invariant along the bifurcation branch, see [12]. The number of crossings of solution u with $u_\pm$ is the number of zero points of the function $u - u_\pm$ counted over one period *without* multiplicity, i.e. every zero point is counted just once even if it is a multiple zero. The zero points of the function are isolated thus this number is well defined, finite and preserved along the continuous branches, see [12]. We can make a three way classification of solution branches, making use of the two invariants described above.

1. Solutions of the first type cross the constant solutions $u_\pm$ in such a way that two crossings with u_+ are followed by two crossings with u_- and vice versa.

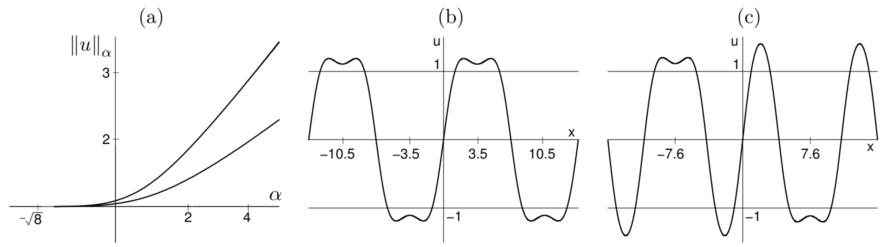

FIGURE 2. Bifurcation diagram for the solutions of the first type and corresponding solutions for $\alpha = 1.5$. The number of monotone laps for the solution pictured at (b) is six and it intersects $u_{\pm}$ two times. For solution (c) number of monotone laps is ten and crossing number with $u_{\pm}$ is six.

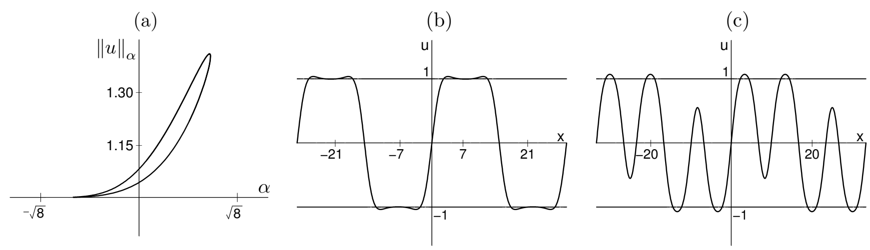

FIGURE 3. Bifurcation diagram for the solutions of the second type and corresponding solutions. Both solutions in (b) and (c) lie on the same bifurcation branch and have six monotone laps and crosses $u_{\pm}$ four times. These solutions corresponds to the parameter value $\alpha = -\frac{1}{10}$.

The existence of an infinite family of periodic solutions of the first type which extend for all $\alpha > -\sqrt{8}$ was proved in [5]. Two examples are shown in Figure 2. In the bifurcation diagram we graph the supremum norm $||u||_{\infty}$ against α.

2. The second type consists of solutions which cross the constant solutions $u_{\pm}$ but crossings do not alternate as for the first type.

Existence of different solutions of the second type was proven for $\alpha \in (-\sqrt{8}, 0)$. Actually, there is a countable infinity of second type solutions with different number of monotone laps. Numerical evidence suggests that these solution continue to exist until some positive α^* and two branches of solutions in the bifurcation diagram form a loop (see Figure 3). Therefore there are two solutions of the second type with the same crossing number and number of monotone laps for $\alpha \in (-\sqrt{8}, \alpha^*)$ and they coalesce at α^*. For more detailed results we refer to [7, 9, 10, 11].

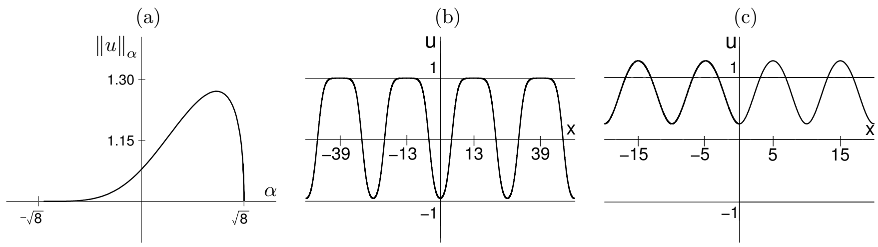

FIGURE 4. Bifurcation diagram for the solutions of the third type and corresponding solutions which are in the class $\mathbf{u}_{1,1}$. Solution (b) corresponds to $\alpha = -1$ and (c) to $\alpha = 1$.

3. The third kind of periodic solutions crosses only one of the constant solution u_+ or u_-.

In this paper we are interested in solutions of the third type, see Figure 4.

◀ **1.1 Definition.** A a periodic solution is of class $\mathbf{u}_{p,q}$ if it is a solution of the third type with $2p$ monotone laps per period and intersects $2q$ times the constant solution $u_+ = 1$. ▶

Solutions of the third type come as a family of countably many distinct periodic solutions which bifurcate from the heteroclinic loop at $\alpha = -\sqrt{8}$. However, this family does not extend to infinity (as the first type) in parameter space nor do they lie on loops (as the second type). Instead, numerical results indicate that these periodic solutions bifurcate from the constant solution u_+ as α tends to a critical value $\alpha_{p,q}$ of the form

$$\alpha_{p,q} = \sqrt{2}\left(\frac{p}{q} + \frac{q}{p}\right), \quad p,q \in \mathbb{N}, \quad (p \geq q), \tag{1.2}$$

see Figure 4a. For $q = 1$ and $p \in \mathbb{N}$ it was analytically shown in [13] that there exists a family of solutions in the class $\mathbf{u}_{p,1}$ for $\alpha \in (-\sqrt{8}, \alpha_{p,1})$. Moreover, for $p \geq 2$ these solutions come in pairs. Numerically computed graphs of two solutions of class $\mathbf{u}_{1,1}$ are shown in Figure 4.

The shooting technique used in [13] to prove existence of a solution of class $\mathbf{u}_{1,1}$ depends very strongly on the particular equation. The method which we develop here generalizes this result in two ways. The application of our method to Equation (1.1) proves the existence of solutions of the class $\mathbf{u}_{p,q}$ for every pair $p,q \in \mathbb{N}$ that is relatively prime. The other aspect is that this technique is not limited to this specific equation. It can be applied to conservative equations with the variational formulation as discussed above. The idea is to use already known solutions of the equation in order to force existence of additional solutions. This idea goes back to [5] where it was shown that a solution of Euler-Lagrange equation of Lagrangian system with a *twist property* corresponds to a fixed point of a flow Ψ^t generated by a parabolic recurrence relation which is defined on an appropriate space of braids. The space of braids is not connected and its connected components are called braid classes. The braid classes used in [5] are isolating neighborhoods for the flow Ψ^t. Therefore the Conley index can be used to show the existence of a fixed point within the class. We will give a more detailed account in the next section. In trying to use these ideas for solutions of the third type the associated braid classes fail to be isolating. This is due to the fact that $u_\pm$ are always fixed points on the boundary. This type of braid classes is called improper. Using the ideas from [1] we show that local information near these fixed points allows us to define modified braid classes which are isolating neighborhoods and for which the invariant set inside the braid class is the same. Based on local information about $u_\pm$ we define topological invariants for the modified proper braid classes. We use the non-triviality of this invariant to prove the existence of a solution which corresponds to a fixed point in an improper braid class.

By applying this result to Equation (1.1) we show the existence of different solutions of the third type. Namely for any $p \geq q$ we prove that there is a solution $u \in \mathbf{u}_{p,q}$, for $\alpha \in [\sqrt{8}, \alpha_{p,q})$. On the other hand we cannot use this approach to extend the result for $\alpha \geq \alpha_{p,q}$ because the local behavior in the fixed point on the boundary of the braid class changes character for this parameter value. Indeed, numerics suggest that the branch of the solutions in the class $\mathbf{u}_{p,q}$ bifurcates from the constant solution $u_+ = 1$ at $\alpha = \alpha_{p,q}$.

◀ **1.2 Theorem.** Let $p, q \in \mathbb{N}$ be relatively prime and $q < p$. Then there exists a solution $u^\alpha \in \mathbf{u}_{p,q}$ of Equation (1.1) with $\mathbb{E}[u^\alpha] = 0$ for every $\alpha \in [\sqrt{8}, \alpha_{p,q})$. ▶

◀ **1.3** *Remark.* One should be able to extend the previous theorem to the parameter range $[0, \sqrt{8}]$, where the eigenvalues of equilibria $u_\pm = \pm 1$ are saddle-foci, by perturbing the potential $F = \frac{1}{4}(u^2 - 1)^2$ as explained in [5]. ▶

2. **Reduction to a finite dimensional problem**

In this section we give a brief survey of the reduction of the problem of finding periodic solutions for Equation (1.1) to the problem of finding fixed points of a vector field generated by a parabolic recurrence relation. We present this approach in the context of general second order Lagrangians.

If we seek closed characteristics i.e., a periodic solution of Equation (1.1) at a given energy level E we can invoke the following variational principle:

$$\text{Extremise } \{J_E[u] : u \in \Omega_{\text{per}}, \tau > 0\}, \tag{2.1}$$

where $\Omega_{\text{per}} = \cup_{\tau>0} C^2(S^1,\tau)$, the periodic functions with period τ, and

$$J_E[u] = \int_0^\tau (L(u,u',u'') + E)dt. \tag{2.2}$$

The function $L \in C^2(\mathbb{R}^3,\mathbb{R})$ is assumed to satisfy $\frac{\partial^2 L}{\partial w^2}(u,v,w) \geq \delta > 0$ for all $(u,v,w) \in \mathbb{R}^3$. For the general second order Lagrangian system the (conserved) energy is given by

$$\mathbb{E}[u] = \left(\frac{\partial L}{\partial u'} - \frac{d}{dt}\frac{\partial L}{\partial u''}\right)u' + \frac{\partial L}{\partial u''}u'' - L(u,u',u''). \tag{2.3}$$

It follows from [4] that the variations in τ guarantee that any critical point u of (2.1) has energy $\mathbb{E}[u] = E$. An energy value E is called regular if $\frac{\partial L}{\partial u}(u,0,0) \neq 0$ for all u that satisfy $L(u,0,0) + E = 0$. The energy manifold $M_E \subset \mathbb{R}^4$ for a regular energy value E is a smooth non-compact manifold without boundary. For a fixed regular energy value E, the extrema of a closed characteristic are contained in the closed set $\{u : L(u,0,0) + E \geq 0\}$. The connected components I_E of this set are called interval components. Moreover, it follows from [4] that solutions on a regular energy level do not have inflexion points. For a singular energy level the interval component I_E contains critical points and the situation is more complicated.

First, we restrict to the regular energy levels. It was shown in [4] that for Lagrangian systems $J[u] = \int_I L(u,u',u'')dt$, where $L(u,u',u'') = \frac{1}{2}u''^2 + K(u,u')$ at energy levels E which satisfy

$$\frac{\partial K}{\partial v}v - K(u,v) - E \leq 0 \text{ for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{2.4a}$$

$$\frac{\partial^2 K}{\partial v^2}v^2 - \frac{5}{2}\left\{\frac{\partial K}{\partial v} - K(u,v) - E\right\} \geq 0 \text{ for all } u \in I_E \text{ and } v \in \mathbb{R}, \tag{2.4b}$$

there is a unique pair (τ, u_τ) minimizing

$$\inf_{u\in X_\tau,\tau\in\mathbb{R}^+} \int_0^\tau (L(u,u',u'') + E)dt,$$

where $X_\tau(u_1,u_2) = \{u \in C^2([0,\tau]) : u(0) = u_1, u(\tau) = u_2, u'(0) = u'(\tau) = 0, u|_{(0,\tau)} > 0 \text{ if } u_1 < u_2 \text{ and } u|_{(0,\tau)} < 0 \text{ if } u_1 > u_2\}$ for $(u_1,u_2) \in I_E \times I_E \setminus \Delta$ and $\Delta = \{(u_1,u_2) \in I_E \times I_E : u_1 = u_2\}$. Moreover, the function defined by

$$S_E(u_1,u_2) = \inf_{u\in X_\tau,\tau\in\mathbb{R}^+} \int_0^\tau (L(u,u',u'') + E)dt, \tag{2.5}$$

for $(u_1,u_2) \in I_E \times I_E \setminus \Delta$ and $S_E|_\Delta = 0$ has the following properties:

(a) $S_E \in C^2(I_E \times I_E \setminus \Delta)$.
(b) $\partial_1\partial_2 S_E(u_1,u_2) > 0$ for all $u_1 \neq u_2 \in I_E$.
(c) $\lim_{u_1 \nearrow u_2} -\partial_1 S_E(u_1,u_2) = \lim_{u_2 \searrow u_1} \partial_2 S_E(u_1,u_2) =$
$= \lim_{u_1 \searrow u_2} \partial_1 S_E(u_1,u_2) = \lim_{u_2 \nearrow u_1} -\partial_2 S_E(u_1,u_2) = +\infty$.

The function S_E is a *generating function* and a Lagrangian system possessing such a generating function is called a *twist system*. The second order Lagrangian system associated to Equation (1.1) is a twist system for $\alpha \geq 0$. For more examples see [4].

The question of finding closed characteristics for a twist system can now be formulated in terms of S_E. Any periodic solution u is a concatenation of monotone laps. Let us take an arbitrary $2p$ periodic sequence $\{u_i\}$ and define u as a concatenation of monotone laps (minimizers $u_\tau(u_i, u_{i+1})$) between the consecutive extremal points u_i solving the Euler-Lagrange equation in between any two extrema. The concatenation u does not have to be a solution on $\mathbb{R}$ because the third derivatives of two monotone laps do not have to match at the extremal point u_i. It was proved in [4] that the third derivatives match if and only if the extrema sequence $\{u_i\}$ is a critical point of discrete action

$$W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1}). \tag{2.6}$$

Critical points of W_{2p} satisfy equations

$$\mathcal{R}_i(u_{i-1}, u_i, u_{i+1}) = \partial_2 S_E(u_{i-1}, u_i) + \partial_1 S_E(u_i, u_{i+1}) = 0, \tag{2.7}$$

where $\mathcal{R}_i(s,t,r)$ is, according to property (a), well-defined and C^1 on the following domains

$$\Omega_i = \{(r,s,t) \in I_E^3 : (-1)^{i+1}(s-r) > 0, (-1)^{i+1}(s-t) > 0\}. \tag{2.8}$$

The functions $\mathcal{R}_i$ and domains Ω_i satisfy $\mathcal{R}_i = \mathcal{R}_{i+2}$ and $\Omega_i = \Omega_{i+2}$ for $i \in \mathbb{Z}$. Property (b) implies that $\partial_1 \mathcal{R}_i = \partial_1 \partial_2 S(u_{i-1}, u_i) > 0$, and $\partial_3 \mathcal{R}_i = \partial_1 \partial_2 S(u_i, u_{i+1}) > 0$. Property (c) provides information about the behavior of $\mathcal{R}_i$ at the diagonal boundaries of Ω_i, namely,

$$\lim_{s \searrow r} \mathcal{R}_i(r,s,t) = \lim_{s \searrow t} \mathcal{R}_i(r,s,t) = +\infty, \tag{2.9}$$

$$\lim_{s \nearrow r} \mathcal{R}_i(r,s,t) = \lim_{s \nearrow t} \mathcal{R}_i(r,s,t) = -\infty. \tag{2.10}$$

Above-mentioned properties of $\mathcal{R}_i$ give us that $\mathcal{R}_i$ is parabolic recurrence relation of up-down type as defined below. First, we define parabolic recurrence relations.

◀ **2.1 Definition.** A parabolic recurrence relation $\mathcal{R}$ on $\mathbb{R}^{\mathbb{Z}}$ is a sequence of real-valued functions $\mathcal{R} = (\mathcal{R}_i)_{i \in \mathbb{Z}}$ satisfying

(A1): [monotonicity] $\partial_1 \mathcal{R}_i > 0$ and $\partial_3 \mathcal{R}_i > 0$ for all $i \in \mathbb{Z}$

(A2): [periodicity] for some $d \in \mathbb{N}, \mathcal{R}_{i+d} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$.

▶

We see that our $\mathcal{R}$ is not a parabolic recurrence relation in the strict sense because it is not defined on whole space $\mathbb{R}^{\mathbb{Z}}$. It is not defined for any sequence satisfying $u_i = u_{i+1}$ for some $i \in \mathbb{Z}$. This corresponds to the nature of solutions of Equation (1.1), namely that minima and maxima alternate.

◀ **2.2 Definition.** A parabolic recurrence relation $\mathcal{R}$ defined on domain given by (2.8) is said to be of up-down type if (2.9) and (2.10) are satisfied. ▶

These results can be summarized in terms of parabolic recurrence relation as follows.

◀ **2.3 Proposition.** Let $J[u] = \int L(u,u',u'')dt$ be a second order Lagrangian twist system. Suppose that W_{2p} is the discrete action defined through (2.5) and (2.6) at the regular energy level E. Then

(a) the functions $\mathcal{R}_i = \partial_i W_{2p}$ defined on Ω_i are components of a parabolic recurrence relation $\mathcal{R}$ of up-down type,

(b) solutions of $\mathcal{R} = 0$ correspond to periodic solutions on the energy level E.

The parabolic recurrence relation is both exact and up-down type. ▶

In order to find solutions of $\mathcal{R} = 0$ we will employ the Conley index. Conley index theory gives information about the invariant set of a flow inside an isolating neighborhood for this flow. In the case of a gradient vector field invariant sets have special structure and thus information about critical points can be obtained. There is a natural way to define a flow generated by an up-down parabolic recurrence relation on the set

$$\Omega^{2p} = \{\mathbf{u} \in \mathbb{R}^{\mathbb{Z}} : \mathbf{u} \text{ is } 2p \text{ periodic and } (u_{i-1}, u_i, u_{i+1}) \in \Omega^i, \text{ for } i \in \mathbb{Z}\}. \tag{2.11}$$

Consider the differential equations

$$\frac{d}{dt} u_i(t) = \mathcal{R}_i(\mathbf{u}(t)), \quad \mathbf{u}(t) \in \Omega^{2p}, \quad t \in \mathbb{R}. \tag{2.12}$$

Equation (2.12) defines a (local) C^1 flow ψ^t on Ω^{2p}. This flow is not defined at the boundary of Ω^{2p}, but conditions (2.9) and (2.10) give us information about the flow close to this boundary. Finding a periodic solution within the class $\mathbf{u}_{p,q}$ can be reduced to constructing an appropriate isolating neighborhood for the flow ψ^t and calculating its (nontrivial) Conley index. We will use the concept of up-down discretized braid diagrams to construct this isolating neighborhood. For any $2p$-periodic extrema sequence we can construct a piecewise linear graph by connecting the consecutive points $(i, u_i) \in \mathbb{R}^2$ by straight line segments. The piecewise linear graph, called a strand, is cyclic: one restricts to $0 \le i \le 2p$ and identifies the end points abstractly. A collection of n closed characteristics of period $2p$ then gives rise to a collection of n strands. For multiple strands we can replace the periodicity of a single sequence to a *braid structure* by assigning a crossing type (positive) to every transverse intersection of the graphs: larger slope crosses over smaller slope, see Figure 5. We represent sequences of extrema in the space of closed, positive, piecewise linear braid diagrams. We briefly recall some basic facts from (discrete) braid theory (for more details see [5]).

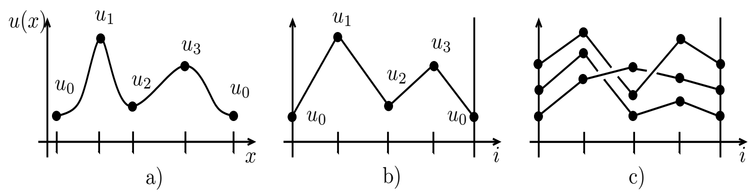

FIGURE 5. (a) A periodic function and (b) its piecewise linear graph (c) a braid consisting of 3 strands.

3. **Braid invariants and the Conley Index**

We recall now the basic theory of proper braid classes and the Conley type braid invariants, and the implications for parabolic recurrence relations [5]. The parabolic recurrence relations coming from fourth order conservative systems as explained in the previous section can be put into this framework.

3a. Braid invariants.

Definition 3.1. Denote by $\mathcal{D}_d^n$ the space of all closed piecewise linear braid diagrams (PL-braid diagrams) on n strands with period d. That is, the space of all (unordered) collections $\beta = \{\beta^k\}_{k=1}^n$ of continuous maps $\beta^k : [0,1] \to \mathbb{R}$ such that

(a) β^k is affine linear on $[\frac{i}{d}, \frac{i+1}{d}]$ for all k and for all $i = 0, \ldots, d-1$;
(b) $\beta^k(0) = \beta^{\tau(k)}(1)$ for some permutation τ;
(c) for any s such that $\beta^k(s) = \beta^l(s)$ with $k \neq l$, the crossing is transversal: for ε sufficiently small

$$(\beta^k(s-\varepsilon) - \beta^l(s-\varepsilon))(\beta^k(s+\varepsilon) - \beta^l(s+\varepsilon)) < 0.$$

Any PL-braid diagram corresponds to some n-collection $\mathbf{u} = \{\mathbf{u}^k\}_{k=0}^{n-1}$ of anchor points $\mathbf{u}^k = \{u_i^k\}$, where

$$u_i^k = \beta^k\left(i/d \bmod 1\right), \tag{3.1}$$

The converse to this statement is not true because condition (c) of Definition 3.1 is not satisfied for arbitrary collection of sequences. A collection $\mathbf{u}$ for which this condition is violated corresponds to a singular PL-braid diagram. We switch between the notation u_i^k of the anchor points and β^k of the piecewise linear braid diagrams throughout this section, using β only if necessary. Discretized braid diagrams will primarily be denote by $\mathbf{u}$. Given anchor points $\mathbf{u}$, the associated piece wise linear braid diagram is given by $\beta(\mathbf{u})$.

Two representatives $\mathbf{u}, \mathbf{u}' \in \mathcal{D}_d^n$ are of the same discretized braid class $[\mathbf{u}] = [\mathbf{u}']$, if and only if they are in the same connected component of $\mathcal{D}_d^n$. Note that if $[\mathbf{u}] = [\mathbf{u}']$, then $\beta(\mathbf{u})$ and $\beta(\mathbf{u}')$ are isotopic as closed positive topological braid diagrams (and braids), see [5]. However, two discretizations of a topological braid are not necessarily

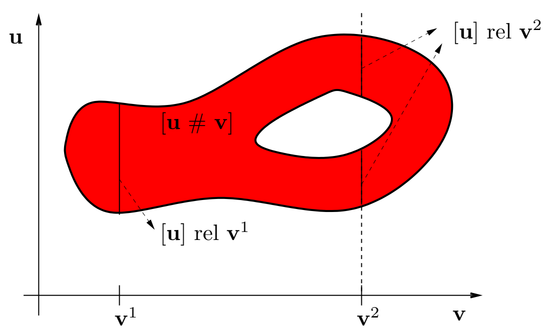

FIGURE 6. Relative braid classes and there fibers.

equivalent in $\mathcal{D}_d^n$, i.e. connected in $\mathcal{D}_d^n$. The connected component $[\mathbf{u}]$ of $\mathcal{D}_d^n$ are called *braid classes* of period d. Define the singular braids $\mathbf{u}$ are defined by $\Sigma_d^n := \overline{\mathcal{D}}_d^n \backslash \mathcal{D}_d^n$ and consists of braids $\mathbf{u}$ failing (c) in Definition 3.1. We suppress the indices and denote the semi-algebraic sub-variety of singular braids by Σ. The set $\Sigma^- \subset \Sigma$ denotes the collapsed singularities, see [5].

For pairs of braids we can define the space of braid pairs using the fact that the union of two braid diagrams is a again a braid diagram satisfying (a) and (b) of Definition 3.1. Consider

$$\mathcal{D}_d^{n,m} := \left\{ (\mathbf{u}, \mathbf{v}) \in \mathcal{D}_d^n \times \mathcal{D}_d^m \mid \mathbf{u} \cup \mathbf{v} \in \mathcal{D}_d^{n+m} \right\}. \tag{3.2}$$

If a pair $(\mathbf{u}, \mathbf{v}) \in \mathcal{D}_d^{n,m}$ we write $\mathbf{u} \# \mathbf{v}$. Note that for $\mathbf{u} \# \mathbf{v} \in \mathcal{D}_d^{n,m}$ it holds $\mathbf{u} \in \mathcal{D}_d^n$ and $\mathbf{v} \in \mathcal{D}_d^m$. As before the connected components of $\mathcal{D}_d^{n,m}$ are denoted by $[\mathbf{u} \# \mathbf{v}]$ and are called *relative braid classes* (of period d). Associated with $[\mathbf{u} \# \mathbf{v}]$ we have the projection

$$\pi : \mathcal{D}_d^{n,m} \to \mathcal{D}_d^m, \quad \mathbf{u} \# \mathbf{v} \mapsto \mathbf{v}.$$

For each $\mathbf{v}' \in \pi([\mathbf{u} \# \mathbf{v}])$ we can define the fiber $[\mathbf{u}']$ rel $\mathbf{v}' := \{\mathbf{u}' \in \mathcal{D}_d^n \mid \mathbf{u}' \# \mathbf{v}' \in [\mathbf{u} \# \mathbf{v}]\}$. The fiber $[\mathbf{u}']$ rel $\mathbf{v}'$ is called a *relative braid class with fixed skeleton* $\mathbf{v}'$. Depending on the period d a fiber $[\mathbf{u}']$ rel $\mathbf{v}'$ may consists of more than one connected component. The set of connected components relative to a fixed braid $\mathbf{v} \in \mathcal{D}_d^m$ is denoted by $\mathcal{D}_d^n$ rel $\mathbf{v}$.

◀ **3.1 Definition.** A relative braid class $[\mathbf{u} \# \mathbf{v}] \subset \mathcal{D}_d^{n,m}$ is called *bounded* if every fiber $[\mathbf{u}']$ rel $\mathbf{v}'$, with $\mathbf{v}' \in \pi([\mathbf{u} \# \mathbf{v}])$, is a bounded set. ▶

As before we can define the singular relative braids as Σ_d^n rel $\mathbf{v} := \overline{\mathcal{D}}_d^n$ rel $\mathbf{v} \backslash \mathcal{D}_d^n$ rel $\mathbf{v}$ and Σ_- rel $\mathbf{v} := \Sigma_-^{n+m} \cap (\mathcal{D}^n$ rel $\mathbf{v})$.

◀ **3.2 Definition.** A relative braid class $[\mathbf{u} \# \mathbf{v}]$ is called *proper* if for every fiber $[\mathbf{u}']$ rel $\mathbf{v}'$, with $\mathbf{v}' \in \pi([\mathbf{u} \# \mathbf{v}])$, it holds that $\mathrm{cl}(([\mathbf{u}'] \text{ rel } \mathbf{v}') \cap (\Sigma_- \text{ rel } \mathbf{v}') = \varnothing$. If $[\mathbf{u} \# \mathbf{v}]$ is not proper it is called improper. ▶

For each fiber of a bounded proper relative braid class $[\mathbf{u} \# \mathbf{v}]$ we define a topological invariant. Fix a fiber $[\mathbf{u}']$ rel $\mathbf{v}'$, with $\mathbf{v}' \in \pi([\mathbf{u} \# \mathbf{v}])$ and let $N = \mathrm{cl}(([\mathbf{u}'] \text{ rel } \mathbf{v}')$. By assumption N is compact and $\partial N \cap (\Sigma_- \text{ rel } \mathbf{v}') = \varnothing$. The the $N^- \subset \partial N$ as follows: For each $\mathbf{u}' \in \partial N$ there exists a small enough neighborhood W in $\overline{\mathcal{D}}^n_d$ such that $W - \Sigma$ rel $\mathbf{v}'$ consists consists of finitely many components W_j. Set $W_0 = W \cap N$, then

$$N^- := \mathrm{cl}\big\{\mathbf{u}' \in \partial N \mid |W_0|_{\text{word}} \geq |W_j|_{\text{word}},\ \forall j > 0\big\}.$$

For a fiber $[\mathbf{u}']$ rel $\mathbf{v}'$ there are finitely many components (N_i, N_i^-). Now define the index $\mathbf{h}(\mathbf{u}' \text{ rel } \mathbf{v}') = \bigvee_i [N_i/N_i^-]$, where $[N_i, N_i^-]$ denotes the homotopy type of the pointed space $(N_i/N_i^-, [N_i^-])$. It was proved in [5] this is indeed an invariant.

◀ **3.3 Proposition.** The homotopy type $\mathbf{h}(\mathbf{u}' \text{ rel } \mathbf{v}') = \bigvee_i [N_i/N_i^-]$ is independent of the fiber $[\mathbf{u}']$ rel $\mathbf{v}'$ in $[\mathbf{u} \# \mathbf{v}]$. ▶

Due to Proposition 3.3 we can define

$$\mathbf{H}(\mathbf{u} \# \mathbf{v}; d) = \bigvee_i [N_i/N_i^-], \tag{3.3}$$

The homological analogue is defined as $\mathbf{CH}([\mathbf{u} \# \mathbf{v}], d) = \bigoplus_k H_k(N, N^-; \mathbb{Z})$. It was proved in [5] this is indeed an invariant. Proposition 3.3 was proved in [5] by associating discrete relative braids to parabolic recurrence relations.

3b. Parabolic recurrence relations. Let $\mathcal{R}$ be a parabolic recurrence relation (see Definition 2.1). For any braid $\mathbf{v} \in \mathcal{D}^m_d$ one can choose a parabolic recurrence relation such that all strands $\mathbf{v}^k$ in $\mathbf{v}$ satisfy $\mathcal{R}_i(v^k_{i-1}, v^k_i, v^k_{i+1}) = 0$, or $\mathcal{R}(\mathbf{v}) = 0$ for short. Denote by Ψ^t the local flow generated by the vector field $\mathcal{R}$ and as such Ψ^t becomes a flow in $\overline{\mathcal{D}}^n_d$. Given a proper bounded relative braid class $[\mathbf{u} \# \mathbf{v}]$ and fix a fiber $[\mathbf{u}']$ rel $\mathbf{v}'$. Choose a parabolic recurrence relation such that $\mathcal{R}(\mathbf{v}') = 0$. Then, by the structure of parabolic flows the set $N = \mathrm{cl}([\mathbf{u}'] \text{ rel } \mathbf{v}')$ is an isolating neighborhood of Ψ^t in the sense of Conley [3]. It holds that the Conley index is given by

$$h(N; \Psi^t) = \mathbf{h}(\mathbf{u}' \text{ rel } \mathbf{v}') = \bigvee_i [N_i/N_i^-].$$

Continuation properties of the Conley index yield Proposition 3.3.

A more fundamental result is that the invariant $\mathbf{H}$ is independent of the period d in the following sense. Define the operator $\mathbf{E} : \overline{\mathcal{D}}^n_d \to \overline{\mathcal{D}}^n_{d+1}$ as follows:

$$(\mathbf{E}(\mathbf{u}))^k_i := \begin{cases} u^k_i & \text{for } i = 0, \cdots, d, \\ u^k_d & \text{for } i = d+1. \end{cases}$$

Given a bounded proper relative braid class $[\mathbf{u} \# \mathbf{v}]$ in $\mathcal{D}_d^{n+m}$, then $[\mathbf{E}(\mathbf{u}) \# \mathbf{E}(\mathbf{v})]$ is a bounded proper relative braid class in $\mathcal{D}_{d+1}^{n+m}$. The main result in [5] is:

◀ **3.4 Proposition.** It holds that $\mathbf{H}(\mathbf{u} \# \mathbf{v}; d) = \mathbf{H}(\mathbf{E}(\mathbf{u}) \# \mathbf{E}(\mathbf{v}); d+1)$. ▶

One conclusion from Proposition 3.4 is that given an equivalence class of of continuous positive relative braid diagams of $[\beta(\mathbf{u}) \# \beta(\mathbf{v})]_{C^0}$, determined by the representative $\beta(\mathbf{u})$ rel $\beta(\mathbf{v})$, then the index $\mathbf{H}$ is independent of the chosen discretization d, see [5]. Therefore define the topological invariant

$$\mathbf{H}\big(\beta(\mathbf{u}) \# \beta(\mathbf{v})\big) := \mathbf{H}(\mathbf{u} \# \mathbf{v}; d), \tag{3.4}$$

for any discretization d as described above. The index $\mathbf{H}\big(\beta(\mathbf{u}) \# \beta(\mathbf{v})\big)$ is an invariant for topological bounded proper relative braid classes $[\beta(\mathbf{u}) \# \beta(\mathbf{v})]_{C^0}$.

◀ **3.5** *Remark.* For more details we refer to [5] where definitions of properness, boundedness, etc. for topological classes are given. ▶

The braid invariant $\mathbf{H}$ has Morse theoretical implications for parabolic recurrence relations. Let Ψ^t is a parabolic flow on $\mathcal{D}_d^n$ which fixes a skeleton $\mathbf{v} \in \mathcal{D}_d^m$ and let $[\mathbf{u} \# \mathbf{v}]$ be a bounded and proper relative braid class. If $\mathbf{H}(\beta(\mathbf{u}) \# \beta(\mathbf{v})) \neq 0$ (homotopically non-trivial), then the relative braid class $[\mathbf{u}]$ rel $\mathbf{v}$ has at least one fixed point for the parabolic flow, and thus a zero for the associated parabolic recurrence relation.

4. Parabolic recurrence relations for convervative systems

4a. Braid classes of up-down type. By Proposition 2.3 closed characteristics correspond to sequences of local minima and maxima satisfy a parabolic recurrence relation of up-down type. The extrema alternate in the sense that $(-1)^i(u_{i\pm 1} - u_i) > 0$ — the (natural) up-down restriction — and therefore an n-collection of extrema sequences $\{\mathbf{u}^k\}_{k=0}^{n-1}$ can be seen as a point in the space of up-down piecewise linear braid diagrams.

◀ **4.1 Definition.** The space $\mathcal{E}_{2p}^n$ of up-down PL-braid diagrams on n strands with period $2p$ is the subset of $\mathcal{D}_{2p}^n$ determined by the relation $(-1)^i(u_{i+1}^k - u_i^k) > 0$ for $k = 1, \ldots, n$ and $i = 0, \ldots, 2p-1$. Let $\overline{\mathcal{E}}_{2p}^n$ be the subset of all braid diagrams in $\overline{\mathcal{D}}_{2p}^n$ satisfying $(-1)^i(u_{i+1}^k - u_i^k) > 0$ and as before the singular braid diagrams are defined as $\Sigma_{\mathcal{E}} = \overline{\mathcal{E}}_{2p}^n \setminus \mathcal{E}_{2p}^n$. ▶

The set $\overline{\mathcal{E}}_{2p}^n$ has a boundary in $\overline{\mathcal{D}}_{2p}^n$ which can be characterized as follows:

$$\partial\overline{\mathcal{E}}_{2p}^n = \{\mathbf{u} \in \overline{\mathcal{E}}_{2p}^n : u_i^k = u_{i+1}^k \text{ for at least one } i \text{ and } k\}. \tag{4.1}$$

Such braids, called horizontal singularities, are not included in Definition of $\overline{\mathcal{E}}_{2p}^n$ since the recurrence relation (2.7) does not induce a well-defined flow on the boundary $\partial\overline{\mathcal{E}}_{2p}^n$. Up-down parabolic recurrence relations therefore define a well-defined parabolic (semi-)flow Ψ^t on $\overline{\mathcal{E}}_{2p}^n$. This has the important property that $\overline{\mathcal{E}}_{2p}^n$ is forward invariant with respect to Ψ^t, i.e. $\Psi^t(\overline{\mathcal{E}}^n_{2p}) \subset \overline{\mathcal{E}}^n_{2p}$ for all $t \geq 0$. The properties can be summarized as follows.

◀ **4.2 Lemma.** Let Ψ^t be a parabolic flow of up-down type on $\overline{\mathcal{E}}^n_{2p}$.

(a) For each point $\mathbf{u} \in \Sigma_{\mathcal{E}} - \Sigma^-_{\mathcal{E}}$, the local orbit $\{\Psi^t(\mathbf{u}) : t \in [-\varepsilon, \varepsilon]\}$ intersects $\Sigma_{\mathcal{E}}$ uniquely at $\mathbf{u}$ for all ε sufficiently small.
(b) For any such $\mathbf{u}$, the word metric of the braid diagram $\Psi^t(\mathbf{u})$ for $t > 0$ is strictly less than that of the diagram $\Psi^t(\mathbf{u})$, $t < 0$.
(c) The flow blows up in a neighborhood of $\partial\overline{\mathcal{E}}^n_{2p}$ in such a manner that the vector field points into $\overline{\mathcal{E}}^n_{2p}$.
(d) The flow is forward invariant: $\Psi^t(\overline{\mathcal{E}}^n_{2p}) \subset \overline{\mathcal{E}}^n_{2p}$ for all $t \geq 0$.

The boundary $\partial\overline{\mathcal{E}}^n_{2p}$ can be regarded as a repelling set. ▶

If v is a closed characteristic of a second order Lagrangian system, then its sequence of extrema $\mathbf{v} = \{v_i\}$ is a zero of the associated parabolic recurrence relation (up-down) $\mathcal{R}$ and thus a fixed point for parabolic flow Ψ^t generated by $\mathcal{R}$. In the case of braids with the up-down restriction we can again define braid classes and relative braid classes, see [5]. Define the space of relative braids of up-down type

$$\mathcal{E}^{n,m}_{2p} = \{(\mathbf{u}, \mathbf{v}) \in \mathcal{E}^n_{2p} \times \mathcal{E}^m_{2p} \mid \mathbf{u} \cup \mathbf{v} \in \mathcal{E}^{n+m}_{2p}\}.$$

Elements in this space are again denoted by $\mathbf{u} \,\#\, \mathbf{v}$ and the connected components, or relative braid classes by $[\mathbf{u} \,\#\, \mathbf{v}]_{\mathcal{E}}$. The space of relative braids with a fixed skeleton $\mathbf{v} \in \mathcal{E}^m_{2p}$ is denoted by $\mathcal{E}^n_{2p}$ rel $\mathbf{v}$. The fibers in $[\mathbf{u} \,\#\, \mathbf{v}]_{\mathcal{E}}$ for a fixed skeleton $\mathbf{v}' \in \pi([\mathbf{u} \,\#\, \mathbf{v}]_{\mathcal{E}})$ are denoted by $[\mathbf{u}']_{\mathcal{E}}$ rel $\mathbf{v}' \in \mathcal{E}^n_{2p}$ rel $\mathbf{v}'$. The notions of boundedness and properness are in defined in the same way, see [5].

Parabolic recurrence relations of up-down type and the associated braid classes satisfy an important universality principle. Let Ψ^t fix a skeleton $\mathbf{v} \in \mathcal{D}^m_d$ and let $[\mathbf{u} \,\#\, \mathbf{v}]_{\mathcal{E}}$ be a bounded and proper relative braid class. Then $N_{\mathcal{E}} := \mathrm{cl}\big([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\big)$ is an isolating neighborhood in the sense of Conley and therefore its Conley index $h(N_{\mathcal{E}}; \Psi^t)$ is well-defined. Define the extended skeleton $\mathbf{v}^* = \mathbf{v} \cup \mathbf{v}^+ \cup \mathbf{v}^-$, where

$$v_i^+ = \max_{k,i} v_i^k + 1 + (-1)^{i+1}, \quad v_i^- = \min_{k,i} v_i^k - 1 + (-1)^{i+1}.$$

The following crucial property was proved in [5].

◀ **4.3 Proposition.** It holds that $h(N_{\mathcal{E}}; \Psi^t) = \mathbf{H}(\beta(\mathbf{u}) \,\#\, \beta(\mathbf{v}^*))$. ▶

If $\mathbf{H}(\beta(\mathbf{u}) \,\#\, \beta(\mathbf{v}^*)) \neq 0$ (homotopically non-trivial), then the relative braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ has at least one fixed point for the parabolic flow, and thus a zero for the associated parabolic recurrence relation of up-down type. In [5] it was also proved that this Proposition 4.3 can also be used in the setting of braid invariants for up-down type relative braid classes. In the up-down case we can also define $\mathbf{H}(\mathbf{u} \,\#\, \mathbf{v}, \mathcal{E}; 2p)$, and

$\mathbf{H}(\mathbf{u}\ \#\ \mathbf{v}, \mathcal{E}; 2p) = \mathbf{H}(\mathbf{u}\ \#\ \mathbf{v}^*; 2p)$. This principle gives us a powerful tool to compute the Conley index of isolating neighborhood given by bounded proper relative braid classes of up-down type via universal braid class invariants.

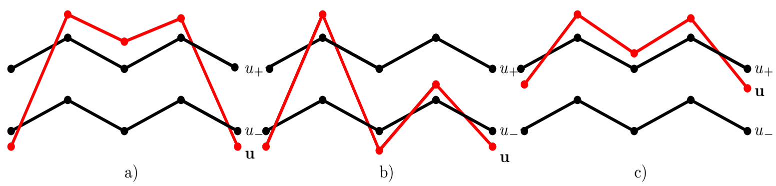

FIGURE 7. Representants of the three different relative braid classes. A fixed point in the relative braid class defined by its representant a) corresponds to the solution of the type I, b) type II and c) type III. Braid classes (a) and (b) are proper but (c) is not.

4b. Fourth order equations. Let us go back now to the classification of solutions of Equation (1.1) and relate the three types of solutions in Figure 1 to braid classes and put them in the context of the definitions presented in this section. The three types of solutions are distinguished according to their intersections with the constant solutions $u_\pm = \pm 1$. The most straightforward way of relating a solution to a relative braid class is to take the two constant strands $\pm\mathbf{1}$ as a skeleton and define the relative braid class by the free strand $\mathbf{u}$ which intersects the constant strands $\pm\mathbf{1}$ in the same manner as the solution u intersects $u_\pm$. However, the flow Ψ^t is well defined only for the braids with up-down restriction. Hence instead of taking the constant strands we have to use the skeleton $\mathbf{v} = \mathbf{u}_+ \cup \mathbf{u}_-$, where the strands $\mathbf{u}_\pm$ correspond to the solutions of Equation (1.1) which oscillate around $u_\pm$ with a small amplitude (on a slightly positive energy level) and the free strand $\mathbf{u}$ intersects the skeleton strands in the same manner as u intersects $u_\pm$. Figure 7 shows the three different braid classes which correspond to the three different types of solutions. The first two braid classes are proper and the third one is not. All these braid classes are obviously unbounded. It was shown in [5] how to use properties of Equation (1.1) to find extra skeletal strands which make the class bounded. We will give more details in Section 7.

According to [5] the braid invariant $\mathbf{H}$ for any braid class corresponding to a solution of the first type is non trivial. Conley index theory then guarantees the existence of a fixed point for Ψ^t in this class. A fixed point in this braid class corresponds to the solution of Equation (1.1) of the first type. Thus there are many different solutions of the first type and their bifurcation branches exist for all $\alpha \geq 0$ as we can see in Figure 2.

For the second braid class the braid invariant $\mathbf{H}$ is trivial and thus does not provide information about fixed points. However, if we know that there exists a non-degenerate (hyperbolic) solution of the second type then it corresponds to a fixed point in the braid class with a trivial Conley index. Hence there must be another fixed point in this class which corresponds to a different solution of the same type. This explains that the bifurcation curves form loops in Figures 1 and 3. We should point out that the existence of a local minimum of second type as was shown in [7, 6] is enough to find a second fixed point.

In the third case, the braid class is not proper (not an isolating neighborhood), since the free strand can collapse on a skeletal strand $\mathbf{u}_+$. Using the information about the flow Ψ^t near the strand $\mathbf{u}_+$, we will perturb the parabolic recurrence relation on a neighborhood of the boundary of the improper braid class $[\mathbf{u}]_\mathcal{E}$ rel $\mathbf{v}$ and construct some new fixed strands which will make the class proper without changing the invariant set inside the class. In Figure 8 we schematically demonstrate the behavior of the vector field $\mathcal{R}$ and its perturbation on the boundary of the improper braid class and boundary of a new proper braid class $[\mathbf{u}]_\mathcal{E}$ rel $\mathbf{v}$ created by adding extra strands which are fixed points of this perturbed vector field. We will show that the Conley index $h(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v}); \Psi^t)$ is non trivial via the invariant $\mathbf{H}(\mathbf{u} \# \overline{\mathbf{v}}^*)$.

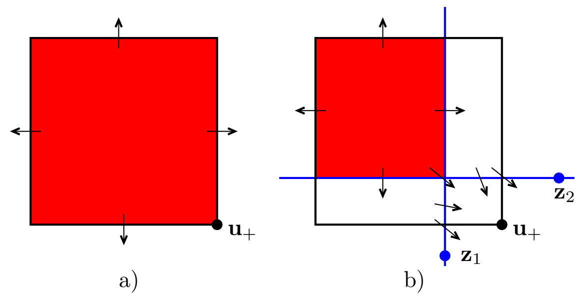

FIGURE 8. Figure (a) schematically shows behavior of the vector field $\mathcal{R}$ on the boundary of the improper braid class $[\mathbf{u}]_\mathcal{E}$ rel $\mathbf{v}$ corresponding to the third type of solution. The strand $\mathbf{u}_+$ is a fixed point for the flow Ψ^t and some trajectories approach this point as $t \longrightarrow -\infty$. Figure (b) shows the behavior of the perturbed vector field on the boundary of the proper braid class $[\mathbf{u}]_\mathcal{E}$ rel $\overline{\mathbf{v}}$, where $\overline{\mathbf{v}} = \mathbf{v} \cup \mathbf{z}_1 \cup \mathbf{z}_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are fixed strands for this perturbed vector field.

In the next section we study linearisation of W_{2p} at the fixed points and introduce notion of a rotation number. Proofs of the results listed in this section can be found in [1].

◀ **4.4** *Remark.* In this paper we restrict ourselves to improper braid classes with one free strand i.e., $[\mathbf{u} \# \mathbf{v}] \subset \mathcal{D}^{1,m}_{2p}$. In this case the free strand $\mathbf{u}$ of an improper braid class can collapse only onto one skeletal strand. Hence the set Σ_- consists of isolated points which are fixed points for the flow Ψ^t. ▶

5. Linearisation of the discrete action

Let $\mathbf{u} \in \mathcal{E}^1_{2p}$ be a critical point of $W_{2p} = \sum_{i=0}^{2p-1} S_E(u_i, u_{i+1})$ and define $P_i = (u_i, w_i)$, where $w_i = \partial_1 S_E(u_i, u_{i+1})$. It was shown in [1] that we can define the differentiable functions F_i on some neighborhood of P_i by the relation

$$(u', w') = F_i(u, w) \Leftrightarrow w = \partial_1 S_E(u, u') \text{ and } w' = -\partial_2 S_E(u, u').$$

It holds that $P_{i+1} = F_i(P_i)$ because $\mathbf{u} \in \mathcal{E}^1_{2p}$ is a critical point of W_{2p}.

We define the rotation number as follows. Take a vector $Q_0 \in T_{P_0}\mathbb{R}^2$ such that $Q_0 \neq 0$, and define $Q_i \in T_{P_i}\mathbb{R}^2$ by

$$Q_i = dF_i(P_{i-1})Q_{i-1}, \quad \text{for all } i.$$

Identify the tangent spaces $T_{P_i}\mathbb{R}^2$ with $\mathbb{R}^2$ in the obvious way, and let the vector Q_i have components (ξ_i, η_i). For each integer i we define θ_i to be the angle between Q_{i-1} and Q_i, oriented in the clockwise sense. This angle is only defined up to a multiple of 2π, so we have to specify which multiple we mean. For this we use the following rule:

$$\text{if} \quad \xi_{i-1}\xi_i \geqslant 0, \text{ then } -\pi < \theta_i \leq \pi, \tag{5.1a}$$

$$\text{if} \quad \xi_{i-1}\xi_i < 0, \text{ then } \ 0 < \theta_i < 2\pi. \tag{5.1b}$$

Then we define the rotation number of the orbit $\mathbf{u}, \tau(\mathbf{u})$, to be

$$\tau(\mathbf{u}) = \lim_{n\to\infty} (2n)^{-1} \sum_{i=-2pn}^{+2pn} \theta_i / 2\pi. \tag{5.2}$$

Roughly speaking, $2\pi\tau(\mathbf{u})$ is the average angle about which $dF(P_0)$ rotates the vector u_0, where $F = F_{2p-1} \circ \ldots \circ F_0$. Or, alternatively, $2\tau(\mathbf{u})$ is the average number of times the sequence ξ_n changes sign, in interval of the length $2p$. This holds due to the choice done in (5.1a) and (5.1b)

If we differentiate ∇W_{2p} at the point $\mathbf{u}$ we get the following expression for the i-th component of the linearization L

$$(L\boldsymbol{\xi})_i = \alpha_i \xi_{i-1} + \beta_i \xi_i + \alpha_{i+1}\xi_{i+1}, \tag{5.3}$$

where $\boldsymbol{\xi} = (\xi_0, \cdots, \xi_{2p-1})$ and

$$\alpha_i = \partial_1\partial_2 S_E(u_{i-1}, u_i) > 0, \tag{5.4a}$$

$$\beta_i = \partial_2^2 S_E(u_{i-1}, u_i) + \partial_1^2 S_E(u_i, u_{i+1}), \tag{5.4b}$$

The fact that $\alpha_i > 0$ follows from the monotonicity property $\partial_1\partial_2 S_E > 0$ of the generating function. Thus L is a Jacobi matrix, and the following is known (see [15]).

◀ **5.1 Proposition.** The spectrum of L is given by

$$\mathrm{spec}(L) = \{\lambda_0 > \lambda_1 \geqslant \lambda_2 > \lambda_3 \geqslant \ldots \geqslant \lambda_{2p-1}\}.$$

In particular, for all i we have $\lambda_{2i} > \lambda_{2i+1}$. ▶

Let us summarize the results obtain for the linearization L in [1]. We use a symbol $[a]$ for the lower integer part of the number a.

◀ **5.2 Lemma.** Let $\boldsymbol{\xi}^j$ be an eigenvector of L corresponding to the eigenvalue λ_j and $1 \le k \le l \le 2p$ be given. Then any nonzero linear combination of $\boldsymbol{\xi}^k, \boldsymbol{\xi}^{k+1}, \ldots, \boldsymbol{\xi}^l$ has at least $2[(k+1)/2]$ and at most $2[(l+1)/2]$ sign changes. ▶

◀ **5.3 Lemma.** The linearization L has at least $2[\tau(\mathbf{u})]+1$ positive eigenvalues. ▶

It follows from Equation (5.3) that if $L\xi = 0$ then all ξ_i can be computed from (ξ_0, ξ_1) and

$$\begin{pmatrix} \xi_{2p} \\ \xi_{2p+1} \end{pmatrix} = M(\mathbf{u}) \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \tag{5.5}$$

with

$$M(\mathbf{u}) = \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_{2p-1}}{\alpha_{2p}} & \frac{-\beta_{2p-1}}{\alpha_{2p}} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_0}{\alpha_1} & \frac{-\beta_0}{\alpha_1} \end{pmatrix}, \tag{5.6}$$

where α_i, β_i are given by (5.4a), (5.4b). One can see that

$$\det(M(\mathbf{u})) = \frac{\alpha_{2p-1}}{\alpha_{2p}} \frac{\alpha_{2p-2}}{\alpha_{2p-1}} \cdots \frac{\alpha_0}{\alpha_1} = \frac{\alpha_0}{\alpha_{2p}} = 1.$$

◀ **5.4** *Remark.* The matrix $M(\mathbf{u})$ is conjugate to the matrix

$$dF(P_0) = dF_{2p-1}(P_{2p-1}) \circ \ldots \circ dF_0(P_0),$$

see Lemma 3.1. in [1]. ▶

6. The invariant set of an improper braid class

The closure of a proper braid class is an isolating neighborhood and Conley index theory provides information about qualitative properties of the maximal invariant set within the braid class. Therefore, if we show that a certain invariant set inside a improper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ is identical to the maximal invariant set in the closure of some proper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^*$, then we can use the Conley index, and thus the global braid invariant $\mathbf{H}$, to study qualitative properties of this invariant set in

$[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$. We will apply these ideas to parabolic recurrence relations of up-down type and thus fourth order equations. These ideas easily extend to general braid classes.

6a. Main results. As outlined in Subsection *4b* the basic idea behind creating a corresponding proper braid class from an improper braid class is to add skeletal strands which will prevent the free strand from collapsing onto the skeleton, see Figure 9. In order to compare the invariant sets of parabolic flows on both braid classes we perturb the parabolic recurrence $\mathcal{R}$ in such a way that we can find the new skeletal strands and the invariant sets are the same. We start with bounded improper braid class $[\mathbf{u} \# \mathbf{v}]_{\mathcal{E}} \subset \mathcal{E}_{2p}^{1,m}$ with skeleton $\mathbf{v} \in \mathcal{E}_{2p}^{m}$. We assume that $\mathbf{u}$ consists of one strand and $\mathbf{u}$ can collapse only on the skeletal strand $\mathbf{v}^1$. This implies the boundary conditions $u_0 = u_{2p}$ and $v_0^1 = v_{2p}^1$.

In general define the distance between braids $\mathbf{u}', \mathbf{u}''$ (with n and m strands respectively) by

$$\sigma(\mathbf{u}', \mathbf{u}'') := \min(|u_i'^k - u_i''^l| > 0 : 0 \le i \le 2p-1,\ 1 \le k \le n,\ 1 \le l \le m), \tag{6.1}$$

An up-down braid $\mathbf{z} = \{\mathbf{z}^k\}$, $\mathbf{z}^k \neq \mathbf{v}^1$ satisfying the following properties:

(i) $\sigma(\mathbf{z}, \mathbf{v}^1) < \min\{\sigma(\mathbf{u}, \mathbf{v}^1), \sigma(\mathbf{v}, \mathbf{v}^1)\}$;
(ii) for all k it holds that $z_0^k \neq z_{2p}^k$, or when $z_0^k = z_{2p}^k$, then $I(\mathbf{z}^k, \mathbf{u}) \neq I(\mathbf{u}, \mathbf{v}^1)$,

is called an admissible augmentation of the skeleton $\mathbf{v}$. The associated augmentation of $\mathbf{v}$ by is defined by

$$\overline{\mathbf{v}} := \mathbf{v} \cup \mathbf{z}. \tag{6.2}$$

By definition the braid class $[\mathbf{u} \# \overline{\mathbf{v}}^*]$ is a bounded proper relative braid class. The augmentation procedure can be carried out for any improper braid class, but the meaning of the braid invariant $\mathbf{H}(\mathbf{u} \# \overline{\mathbf{v}}^*)$ with respect to the dynamics of Ψ^t on fibers of $[\mathbf{u} \# \mathbf{v}]_{\mathcal{E}}$ depends strongly on the dynamical behavior of the skeletal strand $\mathbf{v}^1$, i.e. the rotation number $\tau = \tau(\mathbf{v}^1)$. For improper braid classes we use the following notion of maximal invariant set:

$$\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}) := \Big\{\mathbf{u}' \text{ rel } \mathbf{v} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v} \mid \mathrm{cl}(\Psi^t(\mathbf{u}')) \subset [\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}\Big\}. \tag{6.3}$$

The next theorems give a relation between the invariant set defined in (6.3) and the dynamics of a perturbed parabolic flow Φ_ε^t.

◀ **6.1 Theorem.** Let $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v} \subset \mathcal{E}_{2p}^1$ rel $\mathbf{v}$ be a bounded improper braid class and let $\mathcal{R}$ be a parabolic recurrence relation of up-down type fixing the skeleton $\mathbf{v}$. Suppose that $\mathbf{u}$ can collapse only on the skeletal strand $\mathbf{v}^1$ and the intersection number satisfies $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1)$. Then,

(a) the invariant set $S = \overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v})$ is an isolated invariant set;

(b) there exists an $\varepsilon_0 > 0$ such that for all positive integers p', q', with $(p', q', 2p)$ relative prime and $\left|\frac{q'}{p'} - \tau(\mathbf{v}^1)\right| < \varepsilon < \varepsilon_0$, there exists an admissible augmented skeleton $\overline{\mathbf{v}}^\varepsilon = \mathbf{v} \cup \mathbf{z}^\varepsilon$ and a perturbed parabolic flow Φ^t_ε, with $\Phi^t_\varepsilon(\overline{\mathbf{v}}_\varepsilon) = \overline{\mathbf{v}}_\varepsilon$ and

$$S = \overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}) = \mathrm{INV}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon)).$$

If $N' \subset [\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ is any isolating neighborhood for S then $h(N'; \Psi^t) \cong h(N'; \Phi^t_\varepsilon)$. ▶

Depending on the intersection number $I(\mathbf{u}, \mathbf{v}^1)$ we can associate a universal braid type to $[\mathbf{u} \# \mathbf{v}]_{\mathcal{E}}$. Consider two augmentations of $\mathbf{v}$.

(i) $0 \leq I(\mathbf{u}, \mathbf{v}^1) < 2p$.
Set $k = 1$ and define $\mathbf{z}_I = \{\mathbf{z}^1\}$ as follows. Choose any $\mathbf{z}^1$ such that $I(\mathbf{z}^1, \mathbf{v}^1) = 2p$ and $\sigma(\mathbf{z}, \mathbf{v}^1) < \min\{\sigma(\mathbf{u}, \mathbf{v}^1), \sigma(\mathbf{v}, \mathbf{v}^1)\}$, which makes $\mathbf{z}_I$ and admissible augmentation.

(ii) $I(\mathbf{u}, \mathbf{v}^1) = 2p$.
Set $k = 2$ and define $\mathbf{z}_I = \{\mathbf{z}^1, \mathbf{z}^2\}$ as follows. Choose the strands $\mathbf{z}^1$ and $\mathbf{z}^2$ such that $I(\mathbf{z}^1, \mathbf{v}^1) + I(\mathbf{z}^2, \mathbf{v}^1) = 2$, $I(\mathbf{z}^1, \mathbf{z}^2) = 1$ and $\sigma(\mathbf{z}, \mathbf{v}^1) < \min\{\sigma(\mathbf{u}, \mathbf{v}^1), \sigma(\mathbf{v}, \mathbf{v}^1)\}$, which also makes $\mathbf{z}_I$ and admissible augmentation.

Note that in both cases it holds that $I(\mathbf{u}, \mathbf{z}^k) = I(\mathbf{u}, \mathbf{v}^1)$. Define the associated augmented skeleton by

$$\overline{\mathbf{v}}_I := \mathbf{v} \cup \mathbf{z}_I.$$

◀ **6.2 Theorem.** Let $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1)$, then the Conley index is given by

$$h(N'; \Psi^t) \cong \mathbf{H}(\mathbf{u} \# \overline{\mathbf{v}}_I^*; 2p) \cong \mathbf{H}\big(\beta(\mathbf{u}) \# \beta(\overline{\mathbf{v}}_I^*)\big). \tag{6.4}$$

In particular, when $\mathbf{H} \neq 0$, then $S \neq \varnothing$. ▶

Theorem 6.2 will be proved in Subsection *6e*. The arguments can be extended to the case when the free strand can collapse on several skeletal strands.

◀ **6.3** *Remark.* The same result as Theorem 6.1 can be proved for arbitrary improper braid classes in $\mathcal{D}^{n,m}_d$ with some minor modifications. ▶

6b. Perturbation of the parabolic recurrence relation. The parabolic recurrence relation $\mathcal{R}$, generated by Equation (1.1) is of up-down type and 2-periodic. However, we will deal with a more general setting, namely that $\mathcal{R}$ is $2p$-periodic i.e. $\mathcal{R}_{i+2p} = \mathcal{R}_i$ for all $i \in \mathbb{Z}$. Every component $\mathcal{R}_i$ depends only on (u_{i-1}, u_i, u_{i+1}) and we use the notation $\mathbf{u}_i = (u_{i-1}, u_i, u_{i+1})$. Throughout this section we use a smooth bump function $\omega^\varepsilon : \mathbb{R}^3 \to \mathbb{R}$ which satisfies

$$\omega^\varepsilon(x_1, x_2, x_3) = \begin{cases} 1 & \text{for } \|x\| \leq \frac{\varepsilon}{2}, \\ 0 & \text{for } \|x\| \geqslant \varepsilon, \end{cases}$$

where $\|\mathbf{x}\| = \|(x_1, x_2, x_3)\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Moreover we suppose that $\left|\frac{\partial \omega^\varepsilon}{\partial x_i}\right| < \frac{A}{\varepsilon}$ and $\left|\frac{\partial^2 \omega^\varepsilon}{\partial u_i \partial u_j}\right| < \frac{B}{\varepsilon^2}$, for $1 \leq i, j \leq 3$, for some $A, B > 0$ (independent of ε).

We construct a perturbation of the vector field $\mathcal{R}$ which is linear near $\mathbf{v}^1$. Due to a technical reason which will become clear later, we do not replace the vector field $\mathcal{R}$ by its linearization at $\mathbf{v}^1$ but with a linear function which is sufficiently close to this linearization.

◀ **6.4 Definition.** Let $\varepsilon > 0$ and $\alpha, \beta \in \mathbb{R}^{2p}$ such that

$$\left\|\alpha - \partial_1\mathcal{R}(\mathbf{v}^1)\right\| < \varepsilon \text{ and } \left\|\beta - \partial_2\mathcal{R}(\mathbf{v}^1)\right\| < \varepsilon,$$

where $\partial_i\mathcal{R}(\mathbf{v}^1) = (\partial_i\mathcal{R}_0(\mathbf{v}^1), \ldots, \partial_i\mathcal{R}_{2p-1}(\mathbf{v}^1))$. Then

$$\mathcal{N}_i^{\varepsilon\alpha\beta}(\mathbf{u}_i) = \omega^\varepsilon(\mathbf{u}_i - \mathbf{v}_i^1)\mathcal{L}_i^{\varepsilon\alpha\beta}(\mathbf{u}_i) + (1 - \omega^\varepsilon(\mathbf{u}_i - \mathbf{v}_i^1))\mathcal{R}_i(\mathbf{u}_i), \tag{6.5}$$

where $\mathcal{L}_i^{\varepsilon\alpha\beta}(\mathbf{u}_i) = \alpha_{i-1}(u_{i-1} - v_{i-1}^1) + \beta_i(u_i - v_i^1) + \alpha_{i+1}(u_{i+1} - v_{i+1}^1)$. ▶

◀ **6.5** *Remark.* If there is no ambiguity in choosing (α, β), or if results do not depend on their values but just on the distance form $\partial_i\mathcal{R}(\mathbf{v}^1)$, then we use the notation $\mathcal{N}^\varepsilon$. ▶

The following two lemmas summarize the properties of $\mathcal{N}^\varepsilon$.

◀ **6.6 Lemma.** There exists an $\varepsilon_0 > 0$ such that $\mathcal{N}^\varepsilon$ is a parabolic recurrence relation of up-down type for any $0 < \varepsilon < \varepsilon_0$. ▶

Proof. Every $\mathcal{N}_i^\varepsilon$ is well defined on the set Ω_i and $\mathcal{N}_i^\varepsilon(\mathbf{u}_i) = \mathcal{R}(\mathbf{u}_i)$ if $\mathbf{u}_i \notin B_i(\varepsilon)$, where

$$B_i(\varepsilon) := \{\mathbf{v} \in \mathbb{R}^3 : \left\|\mathbf{v} - \mathbf{v}_i^1\right\| \leq \varepsilon\}. \tag{6.6}$$

Thus $\mathcal{N}_i$ has all required properties on the complement of the set $B_i(\varepsilon)$. The up-down restriction for the braid $\mathbf{v}^1$ implies that

$$\sigma(\mathbf{v}, \mathbf{v}^1) = \sigma(\mathbf{v}) := \min(|v_i^1 - v_{i-1}^1|,\ i \in \{0, \ldots, 2p-1\}) > 0. \tag{6.7}$$

If we choose $\varepsilon < \frac{\sigma}{3}$ then a sufficiently small neighborhood of $\partial\Omega_i$ is in the complement of $B_i(\varepsilon)$ and $\mathcal{N}^\varepsilon$ is of up-down type, since the limits (2.9) and (2.10) for $\mathcal{N}_i^\varepsilon$ are the same as for $\mathcal{R}_i$.

In order to prove the monotonicity conditions on $\mathcal{N}$ we need to show that both $\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ and $\partial_3\mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ on Ω_i. We carry the proof for the first inequality as the second follows in exactly the same way. For the the first inequality we show that there exists a universal constant $C_i > 0$ such that for

$$\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) \geqslant \partial_1\mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon C_i, \quad \mathbf{u}_i \in B_i(\varepsilon)' \quad \forall\varepsilon < \frac{\sigma}{3}. \tag{6.8}$$

Monotonicity of $\mathcal{R}$ ($\partial_1\mathcal{R}_i > 0$) combined with inequality (6.8) implies that $\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) > 0$ for $\mathbf{u}_i \in B^i(\varepsilon)$, where $0 < \varepsilon < \delta_i$, and $\delta_i = \min\left\{\frac{\sigma}{3}, \frac{\partial_1\mathcal{R}_i(\mathbf{v}_i^1)}{C_i}\right\}$. In order to prove inequality (6.8) we use that $\mathcal{R}_i(\mathbf{v}_i^1) = 0$ to estimate

$$\begin{aligned}|\mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)| \leq{}& |(\alpha_i - \partial_1\mathcal{R}_i(\mathbf{v}_i^1))(u_{i-1} - v_{i-1}^1) + (\beta_i - \partial_2\mathcal{R}_i(\mathbf{v}_i^1))(u_i - v_i^1) \\ &+ (\alpha_{i+1} - \partial_3\mathcal{R}_i(\mathbf{v}_i^1))(u_{i+1} - v_{i+1}^1)| + \tfrac{1}{2}\left\|d^2\mathcal{R}_i(\mathbf{v}_i)(\mathbf{u}_i - \mathbf{v}_i^1, \mathbf{u}_i - \mathbf{v}_i^1)\right\|,\end{aligned}$$

where $\mathbf{v}_i = (1-t)\mathbf{u}_i + t\mathbf{v}_i^1$, for some $t \in (0,1)$ and therefore

$$|\mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)| \leq 3\varepsilon \left\|\mathbf{u}_i - \mathbf{v}_i^1\right\| + \frac{D_i}{2}\left\|\mathbf{u}_i - \mathbf{v}_i^1\right\|^2, \tag{6.10}$$

for $\mathbf{u}_i \in B_i(\varepsilon)$, where $D_i = \max_{\mathbf{v}\in B_i(\varepsilon)} \left\|d^2\mathcal{R}_i(\mathbf{v})\right\|$. By the same token we show that

$$\left|\frac{\partial_1 \mathcal{L}_i^\varepsilon(\mathbf{u}_i)}{\partial u_i} - \frac{\partial_1 \mathcal{R}_i(\mathbf{u}_i)}{\partial u_i}\right| < \varepsilon + D_i \left\|\mathbf{u}_i - \mathbf{v}_i^1\right\|, \tag{6.11}$$

for every $\mathbf{u}_i \in B_i(\varepsilon)$. We write $\mathcal{N}_i^\varepsilon = \mathcal{L}_i^\varepsilon + (1-\omega^\varepsilon)(\mathcal{R}_i - \mathcal{L}_i^\varepsilon)$. Using the estimate $|\partial_1\omega^\varepsilon| < \frac{A}{\varepsilon}$, for some $A > 0$, we obtain

$$\begin{aligned}
\partial_1 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) &= \partial_1 \mathcal{N}_i^\varepsilon(\mathbf{u}_i) = \partial_1 \mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \partial_1\omega^\varepsilon(\mathbf{u}_i)(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)) \\
&\quad + (1-\omega^\varepsilon(\mathbf{u}_i))\partial_1(\mathcal{R}(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)) \\
&\geq \alpha_i - \frac{A}{\varepsilon}|\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)| - |\partial_1(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i))| \\
&\geq \partial_1\mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon - \frac{A}{\varepsilon}|\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i)| - |\partial_1(\mathcal{R}_i(\mathbf{u}_i) - \mathcal{L}_i^\varepsilon(\mathbf{u}_i))|,
\end{aligned}$$

and therefore $\partial_1\mathcal{N}_i^\varepsilon(\mathbf{u}_i) \geqslant \partial_1\mathcal{R}_i(\mathbf{v}_i^1) - \varepsilon - \frac{A}{\varepsilon}\varepsilon^2\left(3 + \frac{D_i}{2}\right) - \varepsilon\left(1 + \frac{D_i}{2}\right)$ for any $\mathbf{u}_i \in B_i(\varepsilon)$ and $\varepsilon < \frac{\sigma}{3}$. The last inequality guarantees the existence of the universal constant C_i in (6.8). Positivity of $\partial_3\mathcal{N}_i$ can be shown in the same way. Therefore, the monotonicity condition for parabolic recurrence relation is satisfied. □

◀ **6.7** *Remark.* Inequality (6.10) implies that for ε_0 small enough there exists a constant C^{ε_0} such that $\|\mathcal{N}^\varepsilon(\mathbf{u}) - \mathcal{R}(\mathbf{u})\| < \varepsilon C^{\varepsilon_0}$, for all $\mathbf{u} \in \mathcal{E}_{2p}^1$ and $0 < \varepsilon < \varepsilon_0$. ▶

◀ **6.8 Lemma.** There exists an $\varepsilon_0 > 0$ and a positive constants K^{ε_0} such that $\mathcal{N}^\varepsilon$ can be written in the form

$$\mathcal{N}^\varepsilon(\mathbf{u}) = \mathcal{L}^\varepsilon(\mathbf{u}) + P^\varepsilon(\mathbf{u}), \tag{6.12}$$

where

$$\|P^\varepsilon(\mathbf{u})\| \leq K^{\varepsilon_0}\|\mathbf{u} - \mathbf{v}\|^2, \tag{6.13}$$

for all $\mathbf{u} \in \mathcal{E}_{2p}^1$ such that $\left\|\mathbf{u} - \mathbf{v}^1\right\| < \varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$. ▶

Proof. We show that there exists P_i^ε and $K_i^{\varepsilon_0}$ such that (6.12) holds for every component $\mathcal{N}_i^\varepsilon$ and (6.13) holds for every P_i^ε. Then the lemma holds for $P^\varepsilon = (P_1^\varepsilon, \ldots, P_{2p}^\varepsilon)^T$ and $K^{\varepsilon_0} = \sqrt{2p}\max_{i\in\{0,\ldots,2p-1\}} K_i^{\varepsilon_0}$.

Due to the usual estimate on the remainder of the Taylor series it is enough to show that for every $k, l \in \{0, \ldots, 2p-1\}$ there exists a constant $K_{i,k,l}^{\varepsilon_0}$ with the property

$$\left|\frac{\partial^2\mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\right| \leq K_{i,k,l}^{\varepsilon_0}, \tag{6.14}$$

for $\mathbf{u}_i \in B_i(\varepsilon_0)$. Then,

$$\frac{\partial^2 \mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i) = \omega^\varepsilon(\mathbf{u}_i)\frac{\partial^2 \mathcal{L}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i) + (1-\omega^\varepsilon(\mathbf{u}_i))\frac{\partial^2 \mathcal{R}_i}{\partial u_k \partial u_l}(\mathbf{u}_i)$$
$$+ \frac{\partial \omega^\varepsilon}{\partial u_l}(\mathbf{u}_i)\left(\frac{\partial \mathcal{L}_i^\varepsilon}{\partial u_k}(\mathbf{u}_i) - \frac{\partial \mathcal{R}_i}{\partial u_k}(\mathbf{u}_i)\right) +$$
$$+ \frac{\partial \omega^\varepsilon}{\partial u_k}(\mathbf{u}_i)\left(\frac{\partial \mathcal{L}_i^\varepsilon}{\partial u_l}(\mathbf{u}_i) - \frac{\partial \mathcal{R}_i}{\partial u_l}(\mathbf{u}_i)\right)$$
$$+ \frac{\partial^2 \omega^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\left(\mathcal{L}_i^\varepsilon(\mathbf{u}_i) - \mathcal{R}_i(\mathbf{u}_i)\right).$$

Using the estimates $\left|\frac{\partial \omega^\varepsilon}{\partial u_l}\right| < \frac{A}{\varepsilon}$ and $\left|\frac{\partial^2 \omega^\varepsilon}{\partial u_l \partial u_l}\right| < \frac{B}{\varepsilon^2}$, the bounds in (6.10), (6.11) imply that

$$\left|\frac{\partial^2 \mathcal{N}_i^\varepsilon}{\partial u_k \partial u_l}(\mathbf{u}_i)\right| \le D_i + 2\frac{A}{\varepsilon}\varepsilon D_i + \frac{B}{\varepsilon^2}\varepsilon^2\frac{D_i}{2},$$

for $\mathbf{u}_i \in B_i(\varepsilon_0)$, where $D_i = \max_{\mathbf{v}\in B_i(\varepsilon_0)} \left\|d^2\mathcal{R}_i(\mathbf{v})\right\|$. □

6c. The construction of proper braid classes. In the previous subsection we defined the perturbation $\mathcal{N}^\varepsilon$ of the parabolic recurrence relation $\mathcal{R}$. Now we show that for every $\varepsilon > 0$ the braid class $[\mathbf{u}]_\mathcal{E}$ rel $\mathbf{v}$ can be associated with a proper braid class $[\mathbf{u}]_\mathcal{E}$ rel $\mathbf{v}^\varepsilon$ where $\mathcal{N}^\varepsilon(\mathbf{v}^\varepsilon) = 0$ for any $0 < \varepsilon < \varepsilon_0$. Before we define the skeleton $\mathbf{v}^\varepsilon$ we will show how to employ local information about the parabolic recurrence relation $\mathcal{R}$ near $\mathbf{v}^1 \in \mathcal{E}_{2p}^1$ in order to construct a fixed point for $\mathcal{N}^\varepsilon$.

◀ **6.9 Lemma.** Let $\mathbf{v}^1 \in \mathcal{E}_{2p}^1$ be a fixed point of Ψ^t with positive rotation number $\tau(\mathbf{v}^1)$ and assume that the matrix $M(\mathbf{v}^1)$ given by (5.6) is conjugate to the matrix

$$\begin{pmatrix} \cos 2\pi\tau(\mathbf{v}^1) & -\sin 2\pi\tau(\mathbf{v}^1) \\ \sin 2\pi\tau(\mathbf{v}^1) & \cos 2\pi\tau(\mathbf{v}^1) \end{pmatrix}.$$

Then for every $\varepsilon_0 > 0$ there exist $\varepsilon > 0$, $p', q' \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}^{2p}$ such that $p', q', 2p$ are relatively prime, $0 \le \left|\frac{q'}{p'} - \tau(\mathbf{v}^1)\right| < \varepsilon < \varepsilon_0$ and $\mathcal{N}^{\varepsilon\alpha\beta}$ possesses a $2pp'$-periodic zero $\mathbf{c}^\varepsilon$ with up-down restriction satisfying:

(a) $|c_i^\varepsilon - v_i^1| < \varepsilon/4$, for all i,
(b) the sequence $(c_0^\varepsilon - v_0^1, \ldots, c_{2pp'-1}^\varepsilon - v_{2pp'-1}^1)$ changes the sign $2q'$ times and intersects the zero sequence transversally,
(c) $c_{2ip}^\varepsilon = c_{2jp}^\varepsilon$ only if $|i-j| = kp'$ for some $k \in \mathbb{N}$.

▶

Proof. To prove the lemma we need to construct a parabolic recurrence relation $\mathcal{N}^{\varepsilon\alpha\beta}$ and a point $\mathbf{c}^\varepsilon$ satisfying conditions (a), (b) and (c) such that $\mathcal{N}^{\varepsilon\alpha\beta}(\mathbf{c}^\varepsilon) = 0$. However, $\mathcal{N}^{\varepsilon\alpha\beta} = \mathcal{L}^{\varepsilon\alpha\beta}$ near $\mathbf{v}^1$. Therefore we construct a zero point of $\mathcal{L}^{\varepsilon\alpha\beta}$.

As we mentioned in Section 5, finding a $2pp'$-periodic zero point of $\mathcal{L}^{\varepsilon\alpha\beta}$ is equivalent to solving the equation

$$\begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = M^{p'}_{\alpha,\beta} \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \tag{6.16}$$

where

$$M_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_{2p-1}}{\alpha_{2p}} & \frac{-\beta_{2p-1}}{\alpha_{2p}} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \frac{-\alpha_0}{\alpha_1} & \frac{-\beta_0}{\alpha_1} \end{pmatrix}.$$

It holds that $\alpha \in \mathbb{R}^{2p}$ and the entry α_{2p} is not defined. In order to keep the notation from Section 5 we set $\alpha_{2p} = \alpha_0$. Then $\det(M_{\alpha\beta}) = 1$ for arbitrary (α, β).

First suppose that $\tau(\mathbf{v}^1) = \frac{q'}{p'}$, for some $p', q' \in \mathbb{N}$, such that $p', q', 2p$ are relatively prime. Then for $\alpha = \partial_1\mathcal{R}(\mathbf{v}^1)$ and $\beta = \partial_2\mathcal{R}(\mathbf{v}^1)$ the matrix $M_{\alpha\beta} = M(\mathbf{v}^1)$ is conjugate to a rotation about the angle $2\pi\frac{q'}{p'}$. Therefore Equation (6.16) is satisfied for arbitrary values of (ξ_0, ξ_1) and there is a $2pp'$-periodic zero point of the linearization $\mathcal{L}^{\varepsilon\alpha\beta}$ corresponding to (ξ_0, ξ_1), for arbitrary $\varepsilon < \varepsilon_0$. The fact that the matrix $M(\mathbf{v}^1)$ is conjugate to a rotation about the angle $2\pi\frac{q'}{p'}$ and $p', q', 2p$ are relatively prime, allows us to choose (ξ_0, ξ_1) in such a way that the sequence $\boldsymbol{\xi} = \{\xi_0, \cdots, \xi_{2pp'-1}\}$ is not constant and $\xi_{2ip} = \xi_{2jp}$ only if $|i-j| = kp'$ for some $k \in \mathbb{N}$. We may assume that $|\xi_i| < \frac{\varepsilon}{4}$ for all $i \in \mathbb{Z}$ otherwise we take the sequence $\{c\xi_i\}$ where $c = \frac{\varepsilon}{4}\max_i(\xi_i)$, $i = 0, \ldots, 2pp'-1$. According to Section 5 the sequence $\boldsymbol{\xi} = (\xi_0, \xi_1, \ldots, \xi_{2pp'-1})$ changes sign $2q'$ times. To prove that the nonconstant sequence $\boldsymbol{\xi}$ intersects zero transversally suppose that $\xi_i = 0$. The equation $(\mathcal{L}^{\varepsilon\alpha\beta}\boldsymbol{\xi})_i = 0$ then implies that $\xi_{i+1} = -\frac{\alpha_i}{\alpha_{i+1}}\xi_{i-1}$, where $\alpha_i = \partial_1\mathcal{R}_i(\mathbf{v}^1)$ is positive for all i, and thus $\xi_{i-1}\xi_{i+1} \le 0$. Hence, for a non transversal intersection it holds that $\xi_{i-1} = \xi_i = \xi_{i+1} = 0$, and the sequence $\boldsymbol{\xi}$ is identically equal zero sequence which is a contradiction. We conclude that in the first case $\tau(\mathbf{v}^1) = \frac{q'}{p'}$ we can define

$$c_i^{\varepsilon} = v_i^1 + \xi_i, \text{ for } i \in \mathbb{Z}. \tag{6.17}$$

The second case is when $\tau(\mathbf{v}^1)$ is not rational, or p', q' and $2p$ are not relative prime. We show that for arbitrary $\varepsilon_0 > 0$ there exists an $\varepsilon < \varepsilon_0$ and $\alpha, \beta \in \mathbb{R}^{2p}$ such that $\left\|\alpha - \partial_1\mathcal{R}(\mathbf{v}^1)\right\| < \varepsilon$, $\left\|\beta - \partial_3\mathcal{R}(\mathbf{v}^1)\right\| < \varepsilon$ and the matrix $M_{\alpha\beta}$ is conjugate to

$$\begin{pmatrix} \cos 2\pi\frac{q'}{p'} & -\sin 2\pi\frac{q'}{p'} \\ \sin 2\pi\frac{q'}{p'} & \cos 2\pi\frac{q'}{p'} \end{pmatrix}$$

for some $p', q' \in \mathbb{N}$, where $p', q', 2p$ are relatively prime and

$$\left|\frac{q'}{p'} - \tau(\mathbf{v}^1)\right| < \varepsilon.$$

Then following the previous construction we get a zero point of $\mathcal{N}^{\varepsilon\alpha\beta}$ with properties (a), (b) and (c). For $\alpha = \partial_1\mathcal{R}(\mathbf{v}^1)$, $\beta = \partial_3\mathcal{R}(\mathbf{v}^1)$ the matrix $M_{\alpha\beta}$ is conjugate to a rotation matrix about the angle $2\pi\tau(\mathbf{v}^1)$. It is enough to show that by an arbitrary small perturbation in (α,β) we can slightly change the eigenvalues of the matrix $M_{\alpha\beta}$ to make the rotation angle $\theta = \frac{q'}{p'}$ rational and such that $p', q', 2p$ are relatively prime. For every $(\alpha,\beta) \in \mathbb{R}^{2p}$ it holds that $\det(M_{\alpha\beta}) = 1$. The equation for the eigenvalues of $M_{\alpha\beta}$ is given by

$$\lambda^2 - \mathrm{trace}(M_{\alpha\beta})\lambda + 1 = 0.$$

Since $\mathrm{trace}(M_{\alpha\beta})$ is a rational function of (α,β), it suffices to show that $\mathrm{trace}(M_{\alpha\beta})$ is not a constant function of (α,β). It holds that $\mathrm{trace}(M_{\alpha\beta}) = 2(-1)^p$ for $\alpha = (1,\dots,1), \beta = (0,\dots,0)$ and $\mathrm{trace}(M_{\alpha\beta}) = (-1)^p$ for $\alpha = (1,\dots,1), \beta = (1,1,0,\dots,0)$. This proves that the rational function $\mathrm{trace}(M_{\alpha\beta})$ is not constant. Therefore by an arbitrary small perturbation of (α,β) we can continuously change the eigenvalues of the matrix $M_{\alpha\beta}$. □

Using the zero point $\mathbf{c}^\varepsilon$ of $\mathcal{N}^\varepsilon$ we define $\mathbf{v}^\varepsilon$ as

$$\overline{\mathbf{v}}^\varepsilon := \mathbf{v} \cup \mathbf{z}^\varepsilon, \tag{6.18}$$

where $(z^\varepsilon)_i^k := c^\varepsilon_{2pk+i}$, for $k \in \{0,\dots,p'-1\}$ and $i \in \{0,\dots,2p\}$.

◀ **6.10 Lemma.** For $\varepsilon < \sigma(\mathbf{v})$, where $\sigma(\mathbf{v})$ is given by (6.1), it holds that $\overline{\mathbf{v}}^\varepsilon \in \mathcal{E}_{2p}^{n+p'}$ and $\mathcal{N}^\varepsilon(\overline{\mathbf{v}}^\varepsilon) = 0$. ▶

Proof. The only condition which needs to be checked is a transversality condition. We start by proving that strands in $\mathbf{z}^\varepsilon$ intersect transversally. For $p' = 1$ it is trivial. Lets suppose that $(z^\varepsilon)_i^k = (z^\varepsilon)_i^l$, for some i and $0 \le l < k < p'$, where $p' > 1$. Then the equation

$$\mathcal{L}_i^{\varepsilon\alpha\beta}((z^\varepsilon)_{i-1}^l - (z^\varepsilon)_{i-1}^k, (z^\varepsilon)_i^l - (z^\varepsilon)_i^k, (z^\varepsilon)_{i+1}^l - (z^\varepsilon)_{i+1}^k) = 0 \text{ for all } i,k, \tag{6.19}$$

implies

$$-\tfrac{\alpha_i}{\alpha_{i+1}}((z^\varepsilon)_{i-1}^l - (z^\varepsilon)_{i-1}^k) = ((z^\varepsilon)_{i+1}^l - (z^\varepsilon)_{i+1}^k).$$

Thus if $(z^\varepsilon)_{i-1}^l \neq (z^\varepsilon)_{i-1}^k$ then the transversality condition

$$((z^\varepsilon)_{i-1}^l - (z^\varepsilon)_{i-1}^k)((z^\varepsilon)_{i+1}^l - (z^\varepsilon)_{i+1}^k) < 0$$

is satisfied. If $(z^\varepsilon)_{i-1}^l = (z^\varepsilon)_{i-1}^k$, applying (6.19) we get $(z^\varepsilon)_i^l = (z^\varepsilon)_i^k$ for all i. Therefore $c^\varepsilon_{2kp} = c^\varepsilon_{2lp}$, where $0 < |k-l| < p'$ and we have a contradiction with the condition (c) of the fixed point $\mathbf{c}^\varepsilon$ in Lemma 6.9.

We are left with showing that $\mathbf{z}^\varepsilon$ transversally intersects strands in $\mathbf{v}$. Condition (b) of the fixed point $\mathbf{c}^\varepsilon$ implies that $\mathbf{z}^\varepsilon$ transversally intersects the strand $\mathbf{v}^1$. Condition (a) of the fixed point $\mathbf{c}^\varepsilon$ implies that $|(z^\varepsilon)_i^k - v_i^1| < \frac{\varepsilon}{4}$ for all k,i. If the anchor point v_i^m, $m \neq 1$ satisfies $(z^\varepsilon)_i^l = v_i^m$, for some l, then $|v_i^m - v_i^1| < \frac{\varepsilon}{4} < \sigma(\mathbf{v})$ and it follows, from the definition of $\sigma(\mathbf{v})$, that $v_i^m = v_i^1$. All intersections of the strands $\mathbf{v}^1$ and $\mathbf{v}^m$ are transversal. Hence $(v_{i-1}^m - v_{i-1}^1)(v_{i+1}^m - v_{i+1}^1) < 0$. The previous inequality combined with $|v_{i\pm1}^m - v_{i\pm1}^1| \geq \sigma(\mathbf{v}) > |(z^\varepsilon)_{i\pm1}^l - v_{i\pm1}^1|$ implies that $(v_{i-1}^m - (z^\varepsilon)_{i-1}^l)(v_{i+1}^m - (z^\varepsilon)_{i+1}^l) < 0$.

According to the definition $\mathcal{N}^\varepsilon(\mathbf{v}^k) = \mathcal{R}(\mathbf{v}^k) = 0$ for all k. The previous lemma implies that $\mathcal{N}^\varepsilon(\mathbf{z}^\varepsilon) = 0$, and therefore $\mathcal{N}^\varepsilon(\overline{\mathbf{v}}^\varepsilon) = 0$. □

With the improper braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ we now associate a proper braid class as follows. Choose a representative $\mathbf{u}$ rel $\mathbf{v} \in [\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ which satisfies

$$|u_i - v_i^1| \geq \frac{\varepsilon}{2}, \quad i \in \{0, \ldots 2p-1\}.$$

◀ **6.11 Lemma.** For $\varepsilon < \sigma(\mathbf{v})$ the fiber $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^\varepsilon$ is a well-defined relative braid class in $\mathcal{E}_{2p}^1$ rel $\overline{\mathbf{v}}^\varepsilon$. Moreover, if the braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ is bounded and $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1)$, then $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^\varepsilon$ is bounded and proper for $\varepsilon < |I(\mathbf{u}, \mathbf{v}^1) - 2\tau(\mathbf{v}^1)|$. ▶

Proof. Every up-down braid $\mathbf{u}$ rel $\mathbf{v} \in [\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ which satisfies $|u_i - v_i^1| \geq \frac{\varepsilon}{2}$, for all i, does not have a common anchor point with the strands $\mathbf{z}^\varepsilon$. Thus the the fiber $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^\varepsilon$ is indeed a well-defined relative braid class in $\mathcal{E}_{2p}^1$ rel $\overline{\mathbf{v}}^\varepsilon$.

First we show that the braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^\varepsilon$ is well-defined. Let $\mathbf{u}^1$ rel $\mathbf{v}$ and $\mathbf{u}^2$ rel $\mathbf{v}$ be arbitrary braids in $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ which satisfy $|u_i^{1,2} - v_i^1| \geq \frac{\varepsilon}{2}$. Let $\mathbf{u}(t)$ rel $\mathbf{v} \in \mathcal{E}_{2p}^1$ rel $\mathbf{v}$ be the path between them, which, without loss of generality, evolves just one anchor point at the time. This means that there is a partition of the interval $[0,1]$, given by $0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = 1$, such that only anchor point u_{i_j} evolves for $t \in (t_j, t_{j+1}]$ where $i_j \in \{0, \ldots, 2p\}$. The path $\mathbf{u}(t)$ rel $\mathbf{v}$ does not have to be in $\mathcal{E}_{2p}^1$ rel $\overline{\mathbf{v}}^\varepsilon$, because non-transverse crossing with some strand in $\mathbf{z}^\varepsilon$ may occur. We will modify the path $\mathbf{u}(t)$ rel $\mathbf{v}$ in order to avoid this. If $|u_{i_j}(t_{j+1}) - v_{i_j}^1| < \frac{\varepsilon}{2}$, then we perturb the function $\widetilde{u}_{i_j}(t) : (t_j, t_{j+1}] \to \mathbb{R}$ as follows

$$\widetilde{u}_{i_j}(t) = \begin{cases} u_{i_j}(t_j)(1-\bar{t}) + (v_{i_j} + \varepsilon/2)\bar{t}, & \text{if } u_{i_j}(t_{j+1}) \geq v_{i_j}, \\ u_{i_j}(t_j)(1-\bar{t}) + (v_{i_j} - \varepsilon/2)\bar{t}, & \text{otherwise,} \end{cases}$$

where $\bar{t} = \frac{t-t_j}{t_{j+1}-t_j}$. We set $\widetilde{u}_{i_j}(t) = \widetilde{u}_{i_j}(t_{j+1})$, for all $t > t_{j+1}$, until the original path moves u_{i_j} again. The fact that $u_{i_j}(1) \notin (v_i^1 - \varepsilon/2, v_i^1 + \varepsilon/2)$ implies that there is a j' such that $\mathbf{u}(t)$ rel $\mathbf{v}$ evolves the point u_{i_j} for $t \in (t_{j'}, t_{j'+1}]$. We then define $\widetilde{u}_{i_j}(t) : (t_{j'}, t_{j'+1}] \to \mathbb{R}$ as a linear function connecting $\widetilde{u}_{i_j}(t_{j+1})$ with $u_{i_j}(t_{j'+1})$. We repeat this procedure for any anchor point ending within $\frac{\varepsilon}{2}$ from $\mathbf{v}^1$. This perturbation does not create non-transverse intersections with $\mathbf{v}^1, \mathbf{v}^2, \ldots \mathbf{v}^n$. Along the perturbed path only one anchor point u_i can be in the interval $(v_i^1 - \varepsilon/2, v_i^1 + \varepsilon/2)$ at a time. If u_i passes through this interval then $u_{i-1} < v_i^1 - \varepsilon/2 < v_i^1 + \varepsilon/2 < u_{i+1}$ or $u_{i+1} < v_i^1 - \varepsilon/2 < v_i^1 + \varepsilon/2 < u_{i-1}$. Hence a non-transverse crossing with the strands in $\mathbf{z}^\varepsilon$ is not possible because all their anchor points are within distance $\frac{\varepsilon}{4}$ of $\mathbf{v}^1$. This shows that $\mathbf{u}^1$ rel $\mathbf{v}$ and $\mathbf{u}^2$ rel $\mathbf{v}$ define the same braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^{\varepsilon}$.

The braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\overline{\mathbf{v}}^{\varepsilon}$ is clearly bounded. To prove the properness we have to show that the free strand $\mathbf{u}$ cannot collapse on the skeletal strands in $\mathbf{z}^{\varepsilon}$ and $\mathbf{v}^1$. If $\mathbf{z}^{\varepsilon}$ consists only of one strand then Lemma 6.9 implies that for the crossing number satisfies $I(\mathbf{v}^1,\mathbf{z}^{\varepsilon}) = 2q'$ and $|2q' - 2\tau(\mathbf{v}^1)| < \varepsilon$. If $\mathbf{u}$ can collapse onto $\mathbf{z}^{\varepsilon}$ then $I(\mathbf{u},\mathbf{v}^1) = I(\mathbf{z}^{\varepsilon},\mathbf{v}^1) = 2q'$ and $|I(\mathbf{u},\mathbf{v}^1) - 2\tau(\mathbf{v}^1)| < \varepsilon$. This contradicts the assertion of the lemma. It holds that $I(\mathbf{u},\mathbf{v}^1) = I(\mathbf{u},\mathbf{z}^{\varepsilon})$, and hence a similar contradiction proves that $\mathbf{u}$ cannot collapse onto $\mathbf{v}^1$.

If $\mathbf{z}^{\varepsilon}$ contains of more than one strand ($p' > 1$), then it follows from Lemma 6.9 that $(z^{\varepsilon})^i_0 \neq (z^{\varepsilon})^i_{2p}$ for any i. This ensures that $\mathbf{u}$ cannot collapse onto any of the strands of $\mathbf{z}^{\varepsilon}$. We will show that $\mathbf{u}$ cannot collapse on $\mathbf{v}^1$ by contradiction. First $\sum_{\mathbf{z}^k \in \mathbf{z}^{\varepsilon}} I(\mathbf{u},\mathbf{z}^k) = \sum_{\mathbf{z}^k \in \mathbf{z}^{\varepsilon}} I(\mathbf{u},\mathbf{v}^1) = p' I(\mathbf{u},\mathbf{v}^1)$. However, if $\mathbf{u}$ can collapse onto $\mathbf{v}^1$, then $\sum_{\mathbf{z}^k \in \mathbf{z}^{\varepsilon}} I(\mathbf{u},\mathbf{z}^k) = \sum_{\mathbf{z}^k \in \mathbf{z}^{\varepsilon}} I(\mathbf{v}^1,\mathbf{z}^k) = 2q'$. The previous two equalities imply that $I(\mathbf{u},\mathbf{v}^1) = 2\frac{q'}{p'}$. By applying Lemma 6.9 one infers that $|I(\mathbf{u},\mathbf{v}^1) - 2\tau(\mathbf{v}^1)| < \varepsilon$, a contradiction. □

6d. The invariant set in $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$. The following lemma establishes a connection between the maximal invariant sets $\mathrm{INV}_{\Phi^t_{\varepsilon}}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^{\varepsilon}))$ and $\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ defined in (6.3), where the flow Φ^t_{ε} is generated by the parabolic recurrence relation $\mathcal{N}^{\varepsilon}$.

◀ **6.12 Theorem.** If $I(\mathbf{u},\mathbf{v}^1) < 2\tau(\mathbf{v}^1)$, then there exists an $\varepsilon_0 > 0$ such that

$$\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}) = \mathrm{INV}_{\Phi^t_{\varepsilon}}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^{\varepsilon}))$$

for $0 < \varepsilon < \varepsilon_0$. ▶

◀ **6.13** *Remark.* Similar arguments prove the assertion of the previous theorem when $I(\mathbf{u},\mathbf{v}^1) > 2\tau(\mathbf{v}^1)$. ▶

Proof. We start by proving the inclusion

$$\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}) \subset \mathrm{INV}_{\Phi^t_{\varepsilon}}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^{\varepsilon})).$$

The sets $\mathrm{INV}_{\Psi^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$ and $\partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ are compact and disjoint. Thus there exists an $\varepsilon_1 < \sigma(\mathbf{v})$ such that their distance satisfies

$$\rho(\overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}), \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})) > \varepsilon_1. \tag{6.20}$$

For any point $\mathbf{z} \in \mathrm{INV}_{\Psi^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$ with $|z_i - v^1_i| < \varepsilon_1$ it holds that

$$|z_{i\pm1} - v^1_{i\pm1}| > \varepsilon_1, \tag{6.21}$$

$$(z_{i-1} - v^1_{i-1})(z_{i+1} - v^1_{i+1}) < 0. \tag{6.22}$$

Indeed, if

$$\mathbf{s} = (z_0,\dots,z_{i-2},v^1_{i-1},v^1_i,z_{i+1},\dots,z_{2p-1}) \in \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}),$$

and if $|z_{i-1}-v^1_{i-1}| \le \varepsilon_1$, or $|z_{i+1}-v^1_{i+1}| \le \varepsilon_1$, then by fact that $\mathbf{z}$ is in the invariant set we conclude that

$$\rho(\overline{\mathrm{INV}_{\Psi^t}}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}), \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})) \le \rho(\mathbf{z},\mathbf{s}) \le \varepsilon_1.$$

This contradicts (6.20) and thus 6.21 holds. If we assume that

$$(z_{i-1}-v^1_{i-1})(z_{i+1}-v^1_{i+1}) \ge 0$$

then we obtain a similar contradiction for

$$\mathbf{s} = (z_0,\dots,z_{i-1},v^1_i,z_{i+1},\dots,z_{2p-1}) \in \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}),$$

and therefore (6.22) holds as well.

We show first that $\mathbf{z} \in \mathrm{INV}_{\Psi^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})) \subset [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon$ for $\varepsilon < \varepsilon_1$. If $|z_i - v_i| \ge \frac{\varepsilon}{2}$ for all i, then $\mathbf{z} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon$. If $|z_i - v_i| < \frac{\varepsilon}{2}$ for some i, then it follows from Equations (6.21) and (6.22) that z_i can be moved out of interval $(v_i - \frac{\varepsilon}{2}, v_i + \frac{\varepsilon}{2})$ without changing intersection number with the skeletal strands $\bar{\mathbf{v}}^\varepsilon$. Therefore $\mathbf{z} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon$. If $\mathbf{z} \in \mathrm{INV}_{\Psi^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$, then it follows form (6.20) that $\Psi^t(\mathbf{z})$ stays away from the boundary $\partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$. This implies that $\Psi^t(\mathbf{z}) = \Phi^t_\varepsilon(\mathbf{z})$ for $t \in \mathbb{R}$ and $\Phi^t_\varepsilon(\mathbf{z}) \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon$. Therefore $\mathbf{z} \in \mathrm{INV}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon))$ which proves the inclusion for for $\varepsilon < \varepsilon_1$.

We are left with proving the opposite inclusion. Suppose that

$$\mathbf{z} \in \mathrm{INV}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon))$$

and $\rho(\Phi^t_\varepsilon(\mathbf{z}), \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})) > \varepsilon$ for all t, then $\Phi^t_\varepsilon(\mathbf{z}) = \Psi_t(\mathbf{z})$ for $t \in \mathbb{R}$ and $\mathbf{z} \in \mathrm{INV}_{\Psi^t}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}))$. It therefore suffices to prove that there exists $\varepsilon_2 > 0$ such that

$$\rho(\mathrm{INV}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon)), \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})) > \varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon_2. \tag{6.23}$$

We show that for every singular braid $\mathbf{y} \in \partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ there exists an $\varepsilon_{\mathbf{y}}$ such that for $\mathbf{x} \in B_{\varepsilon_{\mathbf{y}}}(\mathbf{y}) = \{\mathbf{x} \in \mathcal{E}^1_{2p} : \|\mathbf{x}-\mathbf{y}\| < \varepsilon_{\mathbf{y}}\}$ it holds that $\mathbf{x} \notin \mathrm{INV}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \bar{\mathbf{v}}^\varepsilon))$. The compact set $\partial\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ can by covered by a finite covering $U = \{B_{\varepsilon_{\mathbf{y}_i}}(\mathbf{y}_i)\}$. Hence (6.23) holds for $\varepsilon_2 := \min \varepsilon_{\mathbf{y}_i}$.

We start with the boundary point $\mathbf{v}^1$. Identify $\mathcal{E}^1_{2p}$ and $\mathbb{R}^{2p}$ via

$$\mathbf{u} \leftrightarrow \mathbf{x} = (u_0 - v^1_0, \dots, u_{2p-1} - v^1_{2p-1}) \in \mathbb{R}^{2p},$$

so that $\mathbf{v}^1$ becomes the origin. By following the ideas in the proof of Lemma 7.2 in [1] we write the linear part $\mathcal{L}^\varepsilon$ of $\mathcal{N}^\varepsilon$ at $\mathbf{v}^1$ as $\mathcal{L}^\varepsilon = \mathcal{L}^\varepsilon_+ + \mathcal{L}^\varepsilon_-$. It holds that $\mathcal{L}^\varepsilon_+\mathcal{L}^\varepsilon_- = \mathcal{L}^\varepsilon_-\mathcal{L}^\varepsilon_+ = 0$ and

$$(\mathbf{x}, \mathcal{L}^\varepsilon_+\mathbf{x}) \geqslant 0, \quad (\mathbf{x}, \mathcal{L}^\varepsilon_-\mathbf{x}) \le 0,$$

for all nonzero $\mathbf{x} \in \mathbb{R}^{2p}$. Let $\{\boldsymbol{\xi}_0, \ldots, \boldsymbol{\xi}_{2p-1}\}$ and $\{\lambda_0 > \lambda_1 \geq \lambda_2, \ldots, \lambda_{2p-1}\}$ be the eigenvectors and eigenvalues of $\mathcal{L}^0$, where $\mathcal{L}^0$ is the linearization of $\mathcal{R}$ at $\mathbf{v}^1$. The null space of $\mathcal{L}^0_+$ is spanned by $\{\boldsymbol{\xi}_m, \boldsymbol{\xi}_{m+1}, \ldots, \boldsymbol{\xi}_{2p-1}\}$, where $m > I(\mathbf{u}, \mathbf{v}^1)$. Indeed, $I(\mathbf{u}, \mathbf{v}^1) < 2\tau(\mathbf{v}^1)$ implies that $\mathcal{L}^0$ must have at least $I(\mathbf{u}, \mathbf{v}^1) + 1$ positive eigenvalues, see Lemma 5.3 and [1]. Hence Lemma 5.2 implies that if $\mathbf{x} \neq 0$ and $\mathcal{L}^0_+ \mathbf{x} = 0$, then $\mathbf{x}$ has at least $I(\mathbf{u}, \mathbf{v}^1) + 2$ sign changes and therefore $\mathbf{x}$ does not lie in $\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$. Therefore, there is a constant $K^0 > 0$ such that

$$(\mathbf{x}, \mathcal{L}^0_+ \mathbf{x}) \geq K^0 \|\mathbf{x}\|^2$$

holds for all $\mathbf{x} \in \mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$, and thus $\left\|\mathcal{L}^0_+ \mathbf{x}\right\| \geq K^0 \|\mathbf{x}\|$. It follows from the continuous dependence of the eigenvalues of $\mathcal{L}^\varepsilon$ on ε that there exists a constant $K > 0$ such that

$$(\mathbf{x}, \mathcal{L}^\varepsilon_+ \mathbf{x}) \geq K \|\mathbf{x}\|^2,$$

for ε small enough.

Consider the function $G^\varepsilon(\mathbf{x}) = \frac{1}{2}(\mathbf{x}, \mathcal{L}^\varepsilon_+ \mathbf{x})$. Lemma 6.8 implies that close to $\mathbf{x} = 0$, then the flow Φ^ε_t is given by

$$\mathbf{x}'(t) = \mathcal{L}^\varepsilon(\mathbf{x}(t)) + P^\varepsilon(\mathbf{x}(t)), \tag{6.24}$$

where

$$|P^\varepsilon(\mathbf{x}(t))| < K_1 \|\mathbf{x}(t)\|^2, \tag{6.25}$$

for some $K_1 > 0$ and ε sufficiently small. Thus

$$\frac{d}{dt} G^\varepsilon(\mathbf{x}) = (\mathcal{L}^\varepsilon_+ \mathbf{x}, \mathcal{L}^\varepsilon_+ \mathbf{x}) + o(\|\mathbf{x}\|^2) \geq (K^2 + o(1)) \|\mathbf{x}\|^2 > 0,$$

for all $\mathbf{x} \in \mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ close to the origin and $G^\varepsilon(\mathbf{0}) = 0$. Now, let $\varepsilon_3 > 0$ be small such that $\frac{d}{dt} G^\varepsilon > 0$, whenever $G^\varepsilon(\mathbf{x}) < \varepsilon_3$. Define

$$U = \{\mathbf{x} : G^\varepsilon(\mathbf{x}) \ < \varepsilon_3\}.$$

We choose ε_4 sufficiently small such that the ball with radius ε_4 is a subset of U. We now track points in $[\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon \cap B_{\varepsilon_4}(\mathbf{v}^1)$ back in time for the flow Φ^ε_t. If orbit $\Phi^\varepsilon_t(\mathbf{x})$ stays in $[\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon$, for $t < 0$, then $\Phi^\varepsilon_t(\mathbf{x})$ stays in U and $\frac{d}{dt} G^\varepsilon(\Phi^t_\varepsilon(\mathbf{x})) > 0$ for all $t < 0$. It follows that $\Phi^t_\varepsilon(\mathbf{x}) \to \mathbf{0}$ as $t \to -\infty$. Hence, $\sum_{\mathbf{z}^k \in \mathbf{z}^\varepsilon} I(\Phi^t_\varepsilon(\mathbf{x}), \mathbf{z}^k) \to 2q'$ as $t \to -\infty$. On the other hand if $\Phi^t_\varepsilon(\mathbf{x}) \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon$ then $\sum_{\mathbf{z}^k \in \mathbf{z}^\varepsilon} I(\Phi^t_\varepsilon(\mathbf{x}), \mathbf{z}^k) = p' I(\mathbf{u}, \mathbf{v}^1)$ and $p' I(\mathbf{u}, \mathbf{v}^1) < 2q'$. Therefore $\Phi^t_\varepsilon(\mathbf{x})$ leaves the class $[\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon$, for some $t_0 < 0$ and $\mathbf{x} \notin \mathrm{INV}_{\Phi_\varepsilon}(\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon))$ for any $0 < \varepsilon < \varepsilon_4$ and all $\mathbf{x} \in [\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}^\varepsilon \cap B_{\varepsilon_4}(\mathbf{v}^1)$.

Now, suppose that $\mathbf{y} \in \partial \mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ and $\mathbf{y} \neq \mathbf{v}^1$. The flow Ψ^t is transverse to the set $\partial \mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v}) \setminus \{\mathbf{v}^1\}$. We may assume that it points out of the set $\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ at $\mathbf{y}$. Otherwise we get the same result for the reversed time direction. According to Lemma 4.2 the flow Ψ^t cannot enter the class $\mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \mathbf{v})$ after leaving it. This combined with the transversality of the flow implies that there exists $\varepsilon_5 > 0$ such that for every $\mathbf{x} \in \mathrm{cl}(B_{\varepsilon_5}(\mathbf{y})) \cap \mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v})$ there is a constant $T_\mathbf{x}$ with properties $\Psi^{T_\mathbf{x}}(\mathbf{x}) \in \partial\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v})$ and $\Psi^{T_\mathbf{x}+1}(\mathbf{x}) \notin \mathrm{cl}(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v}))$. Moreover $T = \sup_{\mathbf{x}\in C} T_\mathbf{x}$ is finite and

$$\delta = \min_{\mathbf{x}\in C} \rho(\Psi^{T_\mathbf{x}+1}(\mathbf{x}), \partial\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v})) > 0.$$

One can estimate

$$\left\|\Phi^t_\varepsilon(\mathbf{x}) - \Psi^t(\mathbf{x})\right\| \le \int_0^t \|\mathcal{N}^\varepsilon(\Phi^s_\varepsilon(\mathbf{x})) - \mathcal{R}(\Psi^s(\mathbf{x}))\|\, ds \le$$

$$\le \int_0^t \|\mathcal{N}^\varepsilon(\Phi^s_\varepsilon(\mathbf{x})) - \mathcal{R}(\Phi^s_\varepsilon(\mathbf{x}))\|\, ds + \int_0^t \|\mathcal{R}(\Phi_s(\mathbf{x})) - \mathcal{R}(\Psi^s(\mathbf{x}))\|\, ds.$$

For some constant $C > 0$ and $t \in [0, T+1]$ we get

$$\left\|\Phi^t_\varepsilon(\mathbf{x}) - \Psi^t(\mathbf{x})\right\| \le K\varepsilon(T+1) + C\int_0^t \|\Phi^s_\varepsilon(\mathbf{x}) - \Psi^s(\mathbf{x})\|\, ds,$$

where K is a positive constant (see Remark 6.7) and the estimate on the second term follows from the C^1-regularity on $\mathcal{R}$. Gronwall's theorem implies that

$$\left\|\Phi^t_\varepsilon(\mathbf{x}) - \Psi^t(\mathbf{x})\right\| \le K\varepsilon(T+1)e^{Rt},$$

for $t \in [0, T+1]$ and $0 < \varepsilon < \varepsilon_5$. So for

$$\varepsilon < \varepsilon_\mathbf{y} := \min\left\{\varepsilon_5, \frac{\delta}{2K(T+1)} e^{-R(T+1)}\right\}$$

we have that

$$\left\|\Phi^{T_x+1}_\varepsilon(\mathbf{x}) - \Psi_{T_x+1}(\mathbf{x})\right\| \le \frac{1}{2}\delta.$$

Therefore $\Phi^{T_\mathbf{x}+1}_\varepsilon \notin [\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}^\varepsilon$, for $\mathbf{x} \in B_{\varepsilon_\mathbf{y}}(\mathbf{y})$ and

$$B_{\varepsilon_\mathbf{y}}(\mathbf{y}) \cap \mathrm{I}\scriptstyle{\mathrm{NV}}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}^\varepsilon)) = \varnothing$$

for $\varepsilon < \varepsilon_\mathbf{y}$. This concludes (6.23) which implies that

$$\mathrm{INV}_{\Phi^t_\varepsilon}(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}^\varepsilon)) \subset \overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v})$$

and thus finishes the proof of the theorem. □

6e. Proof of Theorem 6.2. We start with the important observations that $[\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}_I \subsetneq [\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v}$. Let Ξ^t be any parabolic flow of up-down type such that $\Xi^t(\overline{\mathbf{v}}_I) = \overline{\mathbf{v}}_I$ and $I(\mathbf{u}, \mathbf{v}^1) \neq 2\tau(\mathbf{v}^1; \Xi^t) = 2\tau(\mathbf{v}^1; \Psi^t)$. The existence of such a flow follows from [5].[2] From Theorem 6.1 we have that $S' = \overline{\mathrm{INV}}_{\Xi^t}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v})$ is isolated and if $\sigma(\mathbf{z}_I, \mathbf{v}^1)$ is sufficiently small, then $S' \subset \mathrm{INV}_{\Xi^t}(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}_I))$. It also holds that

$$\mathrm{INV}_{\Xi^t}(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}_I)) = \overline{\mathrm{INV}}_{\Xi^t}([\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}_I) \subset \overline{\mathrm{INV}}_{\Xi^t}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v}) = S',$$

which shows that $S' = \mathrm{INV}_{\Xi^t}(\mathrm{cl}([\mathbf{u}]_\mathcal{E} \text{ rel } \overline{\mathbf{v}}_I))$.

[2] A another way to find Ξ^t is to perturb the up-down recurrence relation $\mathcal{R}$ for Ψ^t by local perturbations $\mathcal{R} + \omega_\varepsilon \mathcal{N}$ near $\mathbf{z}_I$.

Let Ξ^t_λ be a homotopy such that $\Xi^t_\lambda(\overline{\mathbf{v}}) = \overline{\mathbf{v}}$ for all $\lambda \in [0,1]$ by considering $\mathcal{R}_\lambda = (1-\lambda)\mathcal{R} + \lambda\mathcal{R}'$, where $\mathcal{R}'$ generates Ξ^t. Since $\tau(\mathbf{v}^1;\Xi^t_\lambda) = \tau$ is constant in λ it follows from the above considerations that the Conley index continuous and there exists a uniform neighborhood $S_\lambda \subset N' \subset [\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$, which is an isolating neighborhood for S_λ for all $\lambda \in [0,1]$. Therefore,

$$h(N';\Psi^t) \cong h(N';\Xi^t_\lambda) \cong h(N';\Xi^t) = h(N_{\mathcal{E}};\Xi^t),$$

where $N_{\mathcal{E}} = \mathrm{cl}([\mathbf{u}]_{\mathcal{E}} \text{ rel } \overline{\mathbf{v}}_I)$. If we now apply Proposition 4.3 we obtain

$$h(N_{\mathcal{E}};\Xi^t) \cong \mathbf{H}(\mathbf{u} \# \overline{\mathbf{v}}_I, \mathcal{E}; 2p) \cong \mathbf{H}(\mathbf{u} \# \overline{\mathbf{v}}^*_I; 2p) \cong \mathbf{H}\big(\beta(\mathbf{u}) \# \beta(\overline{\mathbf{v}}^*_I)\big),$$

which proves the theorem.

7. **The application to fourth order differential equations**

In this section we study solutions of Equation (1.1) on the zero energy level. We concentrate on solutions of the third type i.e., functions which intersect constant the solution $u_+ = +1$, but not the constant solution $u_- = -1$. As we mentioned in Section 1 these fucntions can be classified by the number of monotone loops — $2p$ — and number of intersections — $2q$ — with u_+. To prove Theorem 1.2 we will show existence of solution $u^\alpha \in \mathbf{u}_{p,q}$, for $\alpha \in (\sqrt{8}, \alpha_{p,q})$ and $\alpha_{p,q}$ is given by (1.2).

One obstacle to applying the machinery developed in the previous sections is that strands $\mathbf{u}_\pm$ corresponding to the discretization of the constant solutions $u_\pm = \pm 1$ do not obey the up-down restriction. Hence we cannot not include them in the skeleton $\mathbf{v}$ in order to define the braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$, by taking a free strand $\mathbf{u}$ which intersects $2q$ times $\mathbf{u}_+$ and which does not intersect the strand $\mathbf{u}_-$. To overcome this problem we have to use a more elaborate approach. First we will show that for small positive energy values E there exist two solutions of (1.1) which oscillates around u_+ and u_-. Then we define the braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$. The strands associated to the small oscillations around $u_\pm$ are included in $\mathbf{v}$ and the free strand is braided with them in the way mentioned above. We use the results from the previous section to prove existence of a fixed point within the braid class. This provides a solution u^E of (1.1), for small positive E, such that $\mathbb{E}[u^E] = E$. Finally we will use a limit process $E \to 0$ for solutions u^E to find a solution $u \in \mathbf{u}_{p,q}$ at the zero energy level.

7a. Small oscillations. We show in this section that for small positive energy levels there exist solutions that oscillate around $u_\pm$. The rotation number of these solutions will also be computed.

◀ **7.1 Lemma.** For every $\alpha > \sqrt{8}$ and sufficiently small $E > 0$ there exists a periodic solution u_+^E of Equation (1.1) with two extrema per period such that $\min u_+^E < 1 < \max u_+^E$ and $\mathbb{E}[u_+^E] = E$. Moreover $u_+^E \to +1$ as $E \to 0$. ▶

Proof. The transformation $u(t) = 1 + \varepsilon w(t)$ transforms Equation (1.1) into

$$w'''' + \alpha w'' + 2w + 3\varepsilon w^2 + \varepsilon^2 w^3 = 0. \tag{7.1}$$

The energy functional is given by

$$\mathbb{E}_\varepsilon[w] = -w'w''' + \frac{1}{2}(w'')^2 - \frac{\alpha}{2}(w')^2 - F_\varepsilon(w),$$

where $F_\varepsilon(w) = w^2 + \varepsilon w^3 + \frac{1}{4}\varepsilon^2 w^4$. If $\varepsilon = 0$, then (7.1) reduces to the linear equation

$$w'''' + \alpha w'' + 2w = 0. \tag{7.2}$$

The eigenvalues of the latter are given by

$$\lambda_i^2 = \frac{1}{2}\Big[\alpha - (-1)^i\sqrt{\alpha^2 - 8}\Big].$$

Thus, for $\alpha > \sqrt{8}$, $w_0(t) = -\cos(\lambda_1 t)$ is its solution with two extrema per period and energy $\mathbb{E}_0[w_0] = \frac{\lambda_1^4}{2} - 1 > 0$. Equation (7.1) contains only even derivatives. This implies that for every solution satisfying

$$w'(0) = w'(T) = w'''(0) = w'''(T) = 0, \quad T \in \mathbb{R}^+,$$

is $2T$-periodic. Define $G : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$G(A, T, \varepsilon) = \begin{pmatrix} w'_{\varepsilon,A}(T) \\ w'''_{\varepsilon,A}(T) \end{pmatrix},$$

where $w_{\varepsilon,A}$ is the solution of (7.1) with initial data

$$w_{\varepsilon,A}(0) = A, \quad w'_{\varepsilon,A}(0) = 0,$$
$$w''_{\varepsilon,A}(0) = \sqrt{2(F_\varepsilon(A) + \mathbb{E}_0[w_0])}, \quad w'''_{\varepsilon,A}(0) = 0.$$

If $G(w_{\varepsilon,A}, T, \varepsilon) = (0,0)^T$, then $w_{\varepsilon,A}$ is a $2T$-periodic solution of (7.1). The condition $w''_{\varepsilon,A}(0) = \sqrt{2(F_\varepsilon(A) + \mathbb{E}_0[w_0])}$ implies that $\mathbb{E}_\varepsilon[w_{\varepsilon,A}] = \mathbb{E}_0[w_0]$.

To prove the existence of periodic solutions of (7.1) for $\varepsilon > 0$ we will employ the implicit Function Theorem for the function G. For $\varepsilon = 0$ we have $w_{0,-1} = w_0$ and

$$G\big(-1, \tfrac{\pi}{\lambda_1}, 0\big) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have

$$w_{\varepsilon,A}(t) = C(A)\cos(\lambda_1 t) + D(A)\cos(\lambda_2 t) + g(\varepsilon, A, t), \tag{7.3}$$

where $g = o(\varepsilon)$ and $C(A)$, $D(A)$ satisfy

$$C(A) + D(A) = A, \quad \text{and}$$

$$-\lambda_1^2 C(A) - \lambda_2^2 D(A) = \sqrt{2(F_\varepsilon(A) + \mathbb{E}_0[w_0])}.$$

Using (7.3) we obtain that

$$\det \begin{pmatrix} \frac{\partial G}{\partial A} \frac{\partial G}{\partial T} \end{pmatrix}_{(-1,\frac{\pi}{\lambda_1},0)} = \det \begin{pmatrix} \partial_A w'_{\varepsilon,A}(T) & w''_{\varepsilon,A}(T) \\ \partial_A w'''_{\varepsilon,A}(T) & w''''_{\varepsilon,A}(T) \end{pmatrix}_{(-1,\frac{\pi}{\lambda_1},0)} =$$

$$= \frac{\lambda_1^4}{\lambda_1^2 - \lambda_2^2} \sin \frac{\lambda_2}{\lambda_1}\pi \neq 0.$$

From the Implicit Function Theorem we conclude the existence of two continuous functions $A : (-\delta,\delta) \to \mathbb{R}$ and $T : (-\delta,\delta) \to \mathbb{R}$, $\delta > 0$, such that $A(0) = -1$, $T(0) = \frac{\pi}{\lambda_1}$ and

$$G(A(\varepsilon), T(\varepsilon), \varepsilon) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

for $\varepsilon \in [0,\delta)$. The periodic solutions $w_\varepsilon(t) := w_{A(\varepsilon),\varepsilon}(t)$ converge to $w_0 = -\cos\lambda_1 t$ as $\varepsilon \to 0$ in C^3 norm. Therefore, w_ε has two extrema per period (one negative, one positive) for ε small enough. Let $\varepsilon(E) = \sqrt{\frac{E}{E_0[w_0]}}$. Then the solution $u_+^E(t) = 1 + \varepsilon(E) w_{\varepsilon(E)}(t)$ of Equation (1.1) satisfies energy identity

$$\mathbb{E}[u_+^E(t)] = \varepsilon(E)^2 \mathbb{E}_{\varepsilon(E)}[w_{\varepsilon(E)}(t)] = E,$$

which proves the lemma. □

◀ **7.2** *Remark.* An analogous construction can by carried out to construct u_-^E. The solutions u_-^E have similar properties as u_+^E and $u_-^E \to -1$ as $E \to 0$. ▶

We have to keep in mind that every solution u_+^E of Equation (1.1) is a solution for some value of the parameter α, although we do not indicate this in the notation. We can associate the solution u_+^E with a braid $\mathbf{u}_+^E \in \mathcal{E}_2^1$ via its sequence of extrema. The following lemma estimates the rotation number $\tau(\mathbf{u}_+^E)$.

◀ **7.3 Lemma.** Let $\alpha > \sqrt{8}$, and let $\mathbf{u}_+^E \in \mathcal{E}_2^1$ be the braid corresponding to the solution u_+^E. Then for every $\varepsilon > 0$ there exists $E_0 > 0$ such that

$$\left| \tau(\mathbf{u}_+^E) - \frac{\lambda_2}{\lambda_1} \right| < \varepsilon \quad \text{for all } 0 < E < E_0, \tag{7.4}$$

where $\lambda_i^2 = \frac{1}{2}\left[\alpha - (-1)^i \sqrt{\alpha^2 - 8}\right]$. Moreover, the matrix $M(\mathbf{u}_+^E)$ is conjugate to the matrix

$$\begin{pmatrix} \cos(2\pi\tau(\mathbf{u}_+^E)) & -\sin(2\pi\tau(\mathbf{u}_+^E)) \\ \sin(2\pi\tau(\mathbf{u}_+^E)) & \cos(2\pi\tau(\mathbf{u}_+^E)) \end{pmatrix}. \tag{7.5}$$

▶

Proof. As we mentioned in Section 5, the twist maps $F_i^E(x,y)$ corresponding to the generating function S_E for Lagrangian system with Euler-Lagrange equation given by (1.1), can be defined as follows. Let u_i be a solution of Equation (1.1) with the initial values

$$u_i(0) = x, \quad u_i'(0) = 0,$$

$$u_i''(0) = (-1)^i\sqrt{2E + (x^2-1)^2}, \quad u_i'''(0) = y.$$

Let $t_0 > 0$ be the first nonzero time for which $u'(t_0) = 0$. Then

$$F_i^E(x,y) = (u_i(t_0), u_i'''(t_0)).$$

Remark (5.4) implies that $M(\mathbf{u}_+^E)$ is conjugate to

$$d(F_1^E \circ F_0^E)(u_+^E(0), (u_+^E)'''(0)).$$

Let us compute $dF_0^E(u_+^E(0), (u_+^E)'''(0))$. To do so we will use the transformation $u(t) = 1 + \varepsilon(E)w$ where $\varepsilon(E) = \sqrt{\frac{2E}{\lambda_1^4 - 2}}$. One observes that

$$dF_0^E(u_+^E(0), (u_+^E)'''(0)) = d\widetilde{F}^E(w_E(0), w_E'''(0)),$$

where $\widetilde{F}^E$ is defined in the same manner as F_0^E, but with u a solution of Equation (7.1) with initial data

$$u(0) = x, \quad u'(0) = 0,$$

$$u''(0) = \sqrt{2\left(\frac{\lambda_1^4}{2} - 1 + x^2 + \varepsilon(E)x^3 + \frac{1}{4}\varepsilon(E)^2x^4\right)}, \quad u'''(0) = y.$$

Continuous dependence upon E implies that for every $\varepsilon_1 > 0$ there exists an E_1 such that

$$\left\| D\widetilde{F}^E(w_E(0), w_E'''(0)) - D\widetilde{F}^0(w_0(0), w_0'''(0)) \right\| < \varepsilon_1 \text{ for all } 0 < E < E_1, \tag{7.6}$$

where $w_0 = -\cos(\lambda_1 t)$. The value of $D\widetilde{F}^E(w_0(0), w_0'''(0))$ in the direction $(\cos\theta, \sin\theta)^T$, for $0 \le \theta < 2\pi$, is computed as follows

$$d\widetilde{F}^0(w_0(0), w_0'''(0)) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} =$$

$$= \frac{d}{d\mu}\widetilde{F}^0(w_0(0) + \mu\cos\theta, w_0'''(0) + \mu\sin\theta)_{\mu=0} =$$

$$= \begin{pmatrix} \partial_\mu y_{\mu,\theta}(P_\theta(0)) \\ -\partial_\mu y_{\mu,\theta}'''(P_\theta(0)) - y_{\mu,\theta}''''(P_\theta(0))\frac{d}{d\mu}P_\theta(0)|_{\mu=0} \end{pmatrix},$$

where $P_\theta(\mu)$ is the first positive time for which $y'_{\mu,\theta}(P_\theta(\mu)) = 0$ has a maximum. The function $y_{\mu,\theta}$ is a solution of Equation (7.2) with initial conditions

$$\begin{aligned}
y_{\mu,\theta}(0) &= w_0(0) + \mu\cos\theta, \\
y'_{\mu,\theta}(0) &= 0, \\
y''_{\mu,\theta}(0) &= \sqrt{2\left(\frac{\lambda_1^4}{2} - 1 + (w_0(0) + \mu\cos\theta)^2\right)}, \\
y'''_{\mu,\theta}(0) &= w_0'''(0) + \mu\sin\theta.
\end{aligned}$$

We evaluate $\frac{d}{d\mu}P_\theta(0)|_{\mu=0}$ by differentiating the equation $y'_{\mu,\theta}(P_\theta(\mu)) = 0$, with respect to the parameter μ:

$$\frac{d}{d\mu}P_\theta(\mu)|_{\mu=0} = -\frac{\partial_\mu y'_{\mu,\theta}(P_\theta(0))|_{\mu=0}}{y''_{\mu,\theta}(P_\theta(0))|_{\mu=0}}.$$

Linearity of Equation (7.2) enables us to compute all components of the expression $d\widetilde{F}^E(w_0(0), w_0'''(0))(\cos\theta, \sin\theta)^T$, for any θ. By doing so for $\theta = 0$ and $\theta = \frac{\pi}{2}$ we obtain

$$d\widetilde{F}^E(w_0(0), w_0'''(0)) = \begin{pmatrix} \cos(\frac{\lambda_2}{\lambda_1}\pi) & -\frac{\lambda_1}{\lambda_2}\sin(\frac{\lambda_2}{\lambda_1}\pi) \\ \frac{\lambda_2}{\lambda_1}\sin(\frac{\lambda_2}{\lambda_1}\pi) & \cos(\frac{\lambda_2}{\lambda_1}\pi) \end{pmatrix},$$

which is conjugate to

$$\begin{pmatrix} \cos(\frac{\lambda_2}{\lambda_1}\pi) & -\sin(\frac{\lambda_2}{\lambda_1}\pi) \\ \sin(\frac{\lambda_2}{\lambda_1}\pi) & \cos(\frac{\lambda_2}{\lambda_1}\pi) \end{pmatrix}.$$

From Equation (7.6) it follows that we can choose E_0 in such a way, that for all $0 < E < E_0$, the matrix dF_0^E is conjugate to the rotation matrix

$$\begin{pmatrix} \cos(2\tau_E\pi) & -\sin(2\tau_E\pi) \\ \sin(2\tau_E\pi) & \cos(2\tau_E\pi) \end{pmatrix}, \tag{7.8}$$

where $|\tau_E - \frac{\lambda_2}{2\lambda_1}| < \frac{\varepsilon}{2}$.

By the same token we get the same result for dF_1^E. By composing dF_0^E and dF_1^E one gets that $d(F_1^E \circ F_0^E)$ is also conjugate to the matrix of the form (7.8) for some (different) τ_E which satisfies $\left|\tau_E - \frac{\lambda_2}{\lambda_1}\right| < \varepsilon$. It follows from Equation (5.2) that the rotation number is given by $\tau(\mathbf{u}_+^E) = \tau_E + k$, for some $k \in \mathbb{N}$. Using the fact that $\frac{\lambda_2}{2\lambda_1} < \frac{1}{2}$, for $\alpha > \sqrt{8}$, implies $k = 0$, which concludes the proof. □

◀ **7.4** *Remark.* From now on, if there is no ambiguity, we will indicate a p-fold of $\mathbf{u}_+^E$ by the same symbol. The rotation number $\tau(\mathbf{u}_+^E)$ of the p-fold $\mathbf{u}_+^E \in \mathcal{E}_{2p}^1$ is p times the rotation number of $\mathbf{u}_+^E$. ▶

7b. The solution u^E with positive energy. We will prove the existence of a solution u of Equation (1.1) on the energy level zero as a limit of solutions u^E on positive energy levels, given by the following lemma, for $E \to 0$.

◀ **7.5 Theorem.** Let $p,q \in \mathbb{N}$ be relatively prime such that $0 < q < p$ and $\alpha \in (\sqrt{8}, \alpha_{p,q})$. Then for sufficiently small E there exists a solution u^E of (1.1) with $\mathbb{E}[u^E] = E$ and its sequence of extrema $\mathbf{u}^E$ is $2p$-periodic. Moreover, $I(\mathbf{u}^E, \mathbf{u}^E_+) = 2q$ and $I(\mathbf{u}^E, \mathbf{u}^E_-) = 0$, where $\mathbf{u}^E$, $\mathbf{u}^E_+$ and $\mathbf{u}^E_-$ are the sequences of extrema in $\mathcal{E}^1_{2p}$, corresponding to the solutions u^E, u^E_+ and u^E_- respectively. ▶

Proof. To prove this theorem we will employ the relative braid class $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v} \subset \mathcal{E}^1_{2p}$ rel $\mathbf{v}$. This braid class will turn out to contain a fixed point $\mathbf{u}^E$ which is a sequence of extrema for a solution u^E. Let us start by identifying the skeleton

$$\mathbf{v} = \mathbf{v}^1 \cup \mathbf{v}^2 \cup \mathbf{v}^3 \in \mathcal{E}^3_{2p}.$$

We define $\mathbf{v}^1 = \mathbf{u}^E_+$ and $\mathbf{v}^2 = \mathbf{u}^E_-$. To construct the strand $\mathbf{v}^3$ we use the dissipativity of the Lagrangian system generated by Equation (1.1). Dissipativity implies the existence of $u^*_1, u^*_2 \in \mathbb{R}$ such that $u^*_1 < v^1_i, v^2_i < u^*_2$ for all i and $\mathcal{R}_{2i}(u_{2i-1}, u^*_1, u_{2i+1}) < 0$ for $u^*_1 < u_{2i\pm1} < u^*_2$ while $\mathcal{R}_{2i+1}(u_{2i}, u^*_2, u_{2i+1}) > 0$, for $u^*_1 < u_{2i}, u_{2i+2} < u^*_2$. For more details see [4]. Let

$$\Omega_i = \begin{cases} \{(u_{i-1}, u_i, u_{i+1}) \in \mathbb{R}^3 : u^*_1 < u_{i\pm1} < u_i < u^*_2\}, i \text{ odd}, \\ \{(u_{i-1}, u_i, u_{i+1}) \in \mathbb{R}^3 : u^*_1 < u_i < u_{i\pm1} < u^*_2\}, i \text{ even}. \end{cases}$$

Denote by Ω^{2p} the set of $2p$-periodic sequences $\{u_i\}$ for which $(u_{i-1}, u_i, u_{i+1}) \in \Omega_i$. Furthermore define the set

$$C = \{\mathbf{u} \in \Omega^{2p} : I(\mathbf{u}, \mathbf{v}^1) = I(\mathbf{u}, \mathbf{v}^2) = 2p\}.$$

Since $I(\mathbf{v}^1, \mathbf{v}^2) = 0$ the vector field $\mathcal{R}$ is transverse to ∂C. Moreover, the set C is contractible, compact and $\mathcal{R}$ is pointing outward at the boundary ∂C due to the dissipativity. The set C is therefore negatively invariant for the induced flow Ψ^t. Consequently, there exists a fixed point $\mathbf{v}^3$ of Ψ^t in the interior of C. We define $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v} \in \mathcal{E}^1_{2p}$ rel $\mathbf{v}$ by its representative $\mathbf{u}$ satisfying

(i) $(-1)^i u_i > (-1)^i v^3_i$,
(ii) $u_i > v^2_i$,
(iii) $I(\mathbf{u}, \mathbf{v}^1) = 2q$,

where $0 < 2q < 2p$, see Figure 9. For $p \geq 2$, $[\mathbf{u}]_{\mathcal{E}}$ rel $\mathbf{v}$ is a bounded improper, free[3] up-down braid class where $\mathbf{u}$ can collapse only onto $\mathbf{v}^1$. It follows from Lemma 7.3

[3] A braid class is free if it consists of one connected component, see [5].

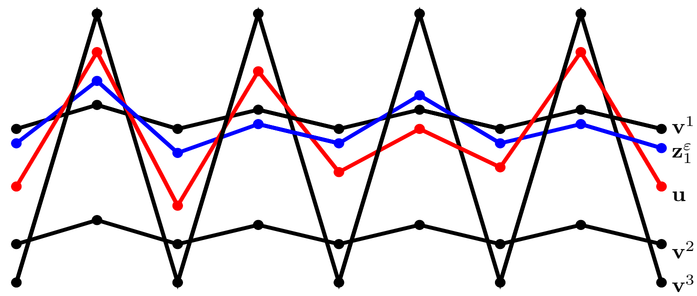

FIGURE 9. A representat of the braid class $[\mathbf{u} \text{ rel } \mathbf{v}^\varepsilon]$ for $p = 4, q = 3, p' = 1$ and $q' = 2$. If we skip the strand $\mathbf{z}_1^\varepsilon$ from the skeleton we get a representant of the braid class $[\mathbf{u} \text{ rel } \mathbf{v}]$.

and Remark 7.4 that for every $\varepsilon_1 > 0$ we can choose $E > 0$ so small that the rotation number of $\mathbf{v}^1 = \mathbf{u}_+^E$ satisfies the inequality

$$\left|\tau(\mathbf{v}^1) - p\frac{\lambda_2}{\lambda_1}\right| < \varepsilon_1,$$

where $\lambda_i^2 = \frac{1}{2}\left[\alpha - (-1)^i\sqrt{\alpha^2 - 8}\right]$. If $\alpha \in (\sqrt{8}, \alpha_{p,q})$, then $0 < \frac{q}{p} < \frac{\lambda_2}{\lambda_1}$ and therefore, for any $\alpha \in (\sqrt{8}, \alpha_{p,q})$, we can choose ε_1 in such a way that $\tau(\mathbf{v}^1) > q$. Hence according to Theorem 6.1, $S = \overline{\mathrm{INV}}_{\Psi^t}([\mathbf{u}]_\mathcal{E} \text{ rel } \mathbf{v})$ is an isolated invariant set and by Theorem 6.2 it follows

$$h(S;\Psi^t) \cong \mathbf{H}(\mathbf{u} \,\#\, \overline{\mathbf{v}}_I^*; 2p) \cong \mathbf{H}\big(\beta(\mathbf{u}) \,\#\, \beta(\overline{\mathbf{v}}_I^*)\big),$$

where $\mathbf{z}_I = \{\mathbf{z}^1\}$ is a 1-strand augmentation as described in Theorem 6.2 with $I(\mathbf{z}^1, \mathbf{v}^1) = 2p$. It follows from Proposition 7.6 that $\mathbf{H}\big(\beta(\mathbf{u}) \,\#\, \beta(\overline{\mathbf{v}}_I^*)\big) \cong S^{2q-1} \vee S^{2q} \neq 0$, which implies that the braid class $[\mathbf{u}]_\mathcal{E}$ rel $\mathbf{v}$ contains a fixed point of the flow Ψ^t. □

◀ **7.6 Proposition.** [4] The proper bounded braid class $[\mathbf{u}]$ rel $\overline{\mathbf{v}}_I^*$ defined in Theorem 7.5 it holds that $\mathbf{H}\big(\beta(\mathbf{u}) \,\#\, \beta(\overline{\mathbf{v}}_I^*)\big) \cong S^{2q-1} \vee S^{2q}$. ▶

Proof. For a detailed proof among other integrable braid classes see [5]. For the homotopy type we derive that $\mathbf{CH}\big(\beta(\mathbf{u}) \,\#\, \beta(\overline{\mathbf{v}}_I^*); \mathbb{R}\big) = \mathbb{R} \oplus \mathbb{R}$. □

7c. The limiting process $E \to 0$. We proved existence of a solution u^E of (1.1) in the parameter range $\alpha \in (\sqrt{8}, \alpha_{p,q})$ on small positive energy levels E. Its sequence of extrema $\mathbf{u}^E$ is $2p$ periodic and $I(\mathbf{u}^E, \mathbf{u}_+^E) = 2q$, while $I(\mathbf{u}^E, \mathbf{u}_-^E) = 0$. We will construct a sequence $\{u_n\}_{n=0}^\infty$ given by $u_n = u^{E_n}$ such that $u = \lim_{n\to\infty} u_n$ is a solution of (1.1)

[4] See [5].

in the periodic class $\mathbf{u}_{p,q}$ and $\mathbb{E}[u] = 0$. First we will show that there is a convergent sequence $\{u_n\}_{n=0}^{\infty}$.

◀ **7.7 Lemma.** There exists a convergent sequence $\{u_n\}_{n=0}^{\infty}$ of solutions of (1.1) such that $u_n \to u$ for $n \to \infty$ in the C^4 norm. Moreover u is a solution of (1.1) on the zero energy level. ▶

Proof. Define the sequence $\{u_m\}_{m=0}^{\infty}$ by $u_m = u^{E_m}$, where $E_m \to 0$ as $m \to \infty$. We will show that sequence $\left\{\frac{d^i}{dt^i}u_m(0)\right\}_{m=0}^{\infty}$ is bounded for $i \in \{0,1,2,3\}$. It follows from the construction of the solutions u^E that $u_1^* < u_m(t) < u_2^*$ for all t and $u_m'(0) = 0$. The energy equation implies that $u_m''(0) = \sqrt{2E_m + \frac{(u_m^2(0)-1)^2}{2}}$ and therefore $\{u_m''(0)\}_{m=0}^{\infty}$ is bounded. By standard estimates on the third derivative one can get that the sequence $\{u_m'''(0)\}_{m=0}^{\infty}$ is bounded as well. Now choose a subsequence $\{u_n\}_{n=0}^{\infty}$ such that $\frac{d^i}{dt^i}u_n(0) \to u^i$ for $n \to \infty$. The sequence $\{u_n\}_{n=0}^{\infty}$ converges in the C^4-norm to a solution u of Equation (1.1) which satisfies the initial conditions $\frac{d^i}{dt^i}u(0) = u^i$. For the energy it holds that $\mathbb{E}[u] = \lim_{n\to\infty}\mathbb{E}[u_n] = 0$. □

The following Lemma shows that if the limit solution u is not constant then it is in the periodic class $\mathbf{u}_{p,q}$.

◀ **7.8 Lemma.** Let u be the limit of the sequence $\{u_n\}_{n=0}^{\infty}$ given by the previous lemma. If $u \not\equiv 1$ then $u \in \mathbf{u}_{p,q}$. ▶

Proof. Let T_n be the period of u_n. Every solution u_n has $2p$ extrema per period, denoted by $\mathbf{u}_n = (u_0^n, \ldots, u_{2p-1}^n)$ and $I(\mathbf{u}_n, \mathbf{u}_+^{E_n}) = 2q$ for $n \in \mathbb{N}$. Let $t_n^i \in [0, T_n)$ be the time in which the solution u_n attains the minimum (maximum) u_i^n. The energy $\mathbb{E}[u_n] = E$ is positive and small and hence any two extremal points u_i^n and u_{i+1}^n are connected by a non-degenerate monotone lap.

According to Lemma 7.7 the limit $u = \lim_{n\to\infty} u_n$ lies in the zero energy level. We show now that $u \in \mathbf{u}_{p,q}$. First we show that u is not a constant solution of (1.1), i.e., $u \not\equiv \pm 1$. We excluded the case $u \equiv 1$ in the assumption of the lemma. On the other hand it follows from $I(\mathbf{u}_n, \mathbf{u}_+^{E_n}) = 2q$ that for every $n \in \mathbb{N}$ there is t_n such that $u_n(\bar{t}_n) > 1$. Thus u_n can not converge to $u_- \equiv -1$.

We will prove periodicity of u by contradiction. If u is not periodic then $T_n \to \infty$ as $n \to \infty$. Then u is non-constant and consists of finitely many monotone laps, and u monotonically converges to $u_\pm = \pm 1$ for $t \to \infty$, which is a contradiction since the equilibrium points ± 1 are centers. Consequently, there exists a constant K such that $T_n < K$ for all $n \in \mathbb{N}$ and T_n converges to some $T \in \mathbb{R}$, which shows that u is T periodic.

Next we show that $u \in \mathbf{u}_{p,q}$. We start with showing that u has $2p$ monotone laps per period. Degenerate monotone laps (inflexion points) can occur on the singular energy level $E = 0$. According to the definition of the solution class we have to count also these degenerate laps. To show that u has $2p$ monotone laps per period it is enough to prove that no sets of more than two extremal points can collapse onto each other and if two extremal points collapse then a degenerated monotone lap is created. Suppose that there are three different extremal points collapsing onto each other. Then sequences $\{t_n^{i-1}\},\{t_n^i\},\{t_n^{i+1}\}$ converge to the same t_0. The equalities $u_n'(t_n^{i-1}) = u_n'(t_n^i) = u_n'(t_n^{i+1}) = 0$ imply that there exist $\widetilde{t}_n \in (t_n^{i-1}, t_n^i)$ and $\widehat{t}_n \in (t_n^i, t_n^{i+1})$ such that $u_n''(\widetilde{t}_n) = u_n''(\widehat{t}_n) = 0$. Finally, there are $\bar{t}_n \in (\widetilde{t}_n, \widehat{t}_n)$ such that $u_n'''(\bar{t}_n) = 0$. By continuity $u'(t_0) = u''(t_0) = u'''(t_0) = 0$. Since $\mathbb{E}[u] = 0$, it holds that $u(t_0) = \pm 1$ and u is a constant solution. However, we already showed that u can not be constant. If there is a collapsing monotone lap (two extremal points collapse on one) then the same argumentation as above implies that it collapses on an inflexed point and the number of monotone laps is preserved.

Now we will show that the solution u intersects the constant solution $u_+ \equiv 1$ exactly $2q$ times per period. Let $s_n^i \geq 0$ be the time at which the i-th intersection of u_n with u_+ occurs. Between any two crossings of $\mathbf{u}_n$ and $\mathbf{u}_+^{E_n}$ is at least one anchor point. Thus for every s_n^i there exists t_n^j such that $s_n^i \leq t_n^j \leq s_n^{i+1}$. If two crossing points s_n^i and s_n^j with $i < j$ collapse i.e., $s_n^i - s_n^j \to 0$, then $u_n(t_n^j) \to 1$ as $n \to \infty$ for all j such that $s_n^i \leq t_n^j \leq s_n^j$. We showed that more than two extremal points can not collapse, and thus more than three crossings cannot collapse.

Now suppose that three crossings collapse (the case of just two collapsing crossings is dealt with later). We may assume that crossings are between extremal points $u_i^n, u_{i+1}^n, u_{i+2}^n, u_{i+3}^n$. It is $s_n^i \leq t_n^{j+1} \leq s_n^{i+1} \leq t_n^{j+2} \leq s_n^{i+2}$ and $s_n^i, s_n^{i+2} \to \bar{t}$ as $n \to \infty$. As before one can show that $u(\bar{t}) = 1$ and $u'(\bar{t}) = u''(\bar{t}) = 0$. We assert that $u(t_n^j) \to A \neq 1$ and $u(t_n^{j+4}) \to B \neq 1$ for $n \to \infty$. Otherwise at least three extremal points would collapse. Let $A < B$ (the other case is analogous), then $u'''(\bar{t}) > 0$. By construction $\left| u_+^{E_n} - \left(1 + \sqrt{\frac{E_n}{E_0}} \cos(\lambda_1 t)\right)\right| \to 0$ for $n \to \infty$, where $E_0 = \frac{\lambda_1^4}{2} - 1$. Hence $|(u_+^{E_n})_i - 1| > \frac{1}{2}\sqrt{\frac{E_n}{E_0}}$ for all i and n sufficiently large. It holds for the maximum of the solution u_n at t_n^{i+1} that $u_n(t_n^{i+1}) > (u_+^{E_n})_{i+1} > 1$, while for the minimum at t_n^{i+2} we get the inequality $u_n(t_n^{i+2}) < (u_+^{E_n})_{i+2} < 1$. We then estimate

$$u_n(t_n^{i+1}) - u_n(t_n^{i+2}) \geq (u_+^{E_n})_{i+1} - (u_+^{E_n})_{i+2} > \sqrt{\frac{E_n}{E_0}}.$$

Let $\delta_n = \sqrt{\frac{E_n}{E_0}}$, then it follows form the mean value theorem that for every n there exists $c_n \in (t_{i+1}^n, t_{i+2}^n)$ such that

$$-u_n'(c_n) = \frac{u_n(t_n^{i+1}) - u_n(t_n^{i+2})}{t_n^{i+2} - t_n^{i+1}}.$$

Due to $t^n_{i+2} - t^n_{i+1} \to 0$, we can estimate

$$-u_n'(c_n) > \delta_n > \sqrt{\frac{E_n}{E_0}}, \tag{7.9}$$

for n large enough. If we divide the energy equation

$$E_n = -u_n'(c_n)u_n'''(c_n) + \frac{1}{2}(u_n''(c_n))^2 - \frac{\alpha}{2}(u_n'(c_n))^2 - \frac{1}{4}(u_n^2(c_n)-1)^2.$$

by the positive number $-u_n'(c_n)$ and use Inequality (7.9), we get

$$\sqrt{E_0E_n} \geq u_n'''(c_n) - \frac{\alpha}{2}|u_n'(c_n)| + \frac{(u_n^2(c_n)-1)^2}{4u_n'(c_n)}. \tag{7.10}$$

We estimate

$$|u_n(c_n) - 1| < u_n(t_n^{i+1}) - u_n(t_n^{i+2}) = -u_n'(c_n)(t_n^{i+2} - t_n^{i+1}) \leq -u_n'(c_n),$$

and

$$\left|\frac{(u_n^2(c_n)-1)^2}{u_n'(c_n)}\right| \leq |u_n'(c_n)|(u^2(c_n)+1).$$

Taking limit for $n \to \infty$ in Inequality (7.10) implies that $u'''(\bar{t}) \leq 0$, which is a contradiction with $u'''(\bar{t}) > 0$. Thus three crossings can not collapse.

Now we show by contradiction that two crossings cannot collapse. If two crossings collapse then there exist $s_n^i \leq t_n^j \leq s_n^{i+1}$ such that $s_n^i, s_n^{i+1} \to \bar{t}$. We can assume that $u_n(t_n^{j\pm1}) \not\to 1$ otherwise the proof is analogous to the case of three collapsing intersections. Then, for $t = \bar{t}$ the solution u has an extremum and $u(\bar{t}) = 1$. As before, this contradicts that $\mathbb{E}[u] = 0$ and $u \not\equiv 1$.

Finally, $u(t) > -1$ for all t because otherwise there would be an extremum point $\bar{t}$ of u with $u(\bar{t}) = -1$ and again $\mathbb{E}[u] > 0$. □

The final step is to show that $\{u_n\}_{n=0}^\infty$ does not converge to the constant solution $u_+ = 1$. Let $\mathbb{E}[u_n] = E_n$ and define the sequences $\{w^n\}_{n=0}^\infty$ and $\{w_+^n\}_{n=0}^\infty$ as follows

$$u_n = 1 + \varepsilon(n)w^n,$$
$$u_+^{E_n} = 1 + \varepsilon(n)w_+^n,$$

where $\varepsilon(n) = \|u_n - 1\|_{L^\infty}$. Then w^n, w_+^n are solutions of equation

$$w'''' + \alpha w'' + 2w + 3\varepsilon(n)w^2 + \varepsilon^2(n)w^3 = 0.$$

Let $\mathbb{E}_\varepsilon$ be the energy functional related to the previous equation. Then $\mathbb{E}_{\varepsilon(n)}[w^n] = \mathbb{E}_{\varepsilon(n)}[w_+^n] > 0$. If $u_n \to 1$ then $\varepsilon(n) \to 0$, $w^n \to w$ and $w_+^n \to w_+$ where w and w_+ are solutions of the linear equation

$$w'''' + \alpha w'' + 2w = 0, \tag{7.11}$$

with $\mathbb{E}_0[w] = \mathbb{E}_0[w_+] = E \geq 0$. By construction $w_+ = \sqrt{\frac{E}{E_0}}\cos(\lambda_1 t)$, where $E_0 = \frac{\lambda_1^4}{2} - 1$. The following two lemmas summarize the properties of linear equation (7.11).

◀ **7.9 Lemma.** Let $\alpha > \sqrt{8}$ be such that $\frac{\lambda_2}{\lambda_1}$ is irrational. Then there is no periodic solution of (7.11) on the energy level zero. The only periodic solution on a positive energy level is w_+. ▶

Proof. Every solution of (7.11) can be written as

$$x(t) = A\cos(\lambda_1 t + \varphi_1) + B\cos(\lambda_2 t + \varphi_2), \tag{7.12}$$

where $A, B, \varphi_1, \varphi_2 \in \mathbb{R}$. The ratio of the frequencies $\frac{\lambda_2}{\lambda_1}$ is irrational. Thus if x is periodic then either $A = 0$ or $B = 0$. Plugging (7.12) into the energy equation proves the lemma. □

◀ **7.10 Lemma.** Let $\alpha > \sqrt{8}$ be such that $\frac{\lambda_2}{\lambda_1}$ is rational i.e. there are $p', q' \in \mathbb{N}$ relatively prime and $\frac{\lambda_2}{\lambda_1} = \frac{q'}{p'}$. Assume that $E > 0$ and $w_+ = \sqrt{\frac{E}{E_0}\cos(\lambda_1 t)}$ where $E_0 = \frac{\lambda_1^4}{2} - 1$. Then every solution w of (7.11), with $\mathbb{E}[x] = E$, which is not equal to w_+, has the property that its sequence of extrema $\mathbf{v}$ is $2p'$-periodic and intersects $\mathbf{v}_+$ exactly $2q'$ times per period. ▶

Proof. It is enough to prove the statement for the solutions w which attain a minimum for $t = 0$. Since $\frac{\lambda_2}{\lambda_1}$ is rational, it follows from (7.12) that all solutions on the positive energy level E are periodic with the period $\frac{2\pi}{\lambda_1}p'$.

First we show that the number of extremal points per period $\frac{2\pi}{\lambda_1}p'$ is $2p'$ for all solutions of (7.11). Let w_1 and w_2 be two different solutions. Weinterpolate between them and let $y(s,t)$ be a solution of (7.11) for every fixed $s \in [0,1]$ with initial conditions

$$\begin{aligned}
y(s,0) &= sw_1(0) + (1-s)w_2(0),\\
y'(s,0) &= 0,\\
y''(s,0) &= \sqrt{2(E + (sw_1(0) + (1-s)w_2(0))^2)},\\
y'''(s,0) &= sw_1'''(0) + (1-s)w_2'''(0).
\end{aligned}$$

For every fixed $s \in [0,1]$ it holds that $\mathbb{E}_0[y(s,t)] = E$. The fact that the energy level $E > 0$ is regular implies that $y(s,t)$ is a concatenation of regular monotone laps (degenerate monotone lap cannot occur) for any fixed s. If two extremal points would collapse or a new one would be created along the path $y(s,t)$ then the degenerate monotone lap occurs which is impossible. Therefore the number of extremal points per period $\frac{2\pi}{\lambda_1}p'$ is constant along the path $y(s,t)$. This implies that w_1 and w_2 have the same number of extremal points per period $\frac{2\pi}{\lambda_1}p'$. By counting the number of extremal points of the solution w_+ on the interval $[0, \frac{2\pi}{\lambda_1}p')$ one gets that this number is $2p'$. Hence the extremal sequence of any solution is $2p'$-periodic.

It follows from the proof of Lemma 7.3 that the rotation number $\tau(\mathbf{v}_+) = \frac{\lambda_2}{\lambda_1} = \frac{q'}{p'}$. This combined with the fact that the extremal sequence $\mathbf{v}$ of an arbitrary solution is $2p'$

periodic implies that $I(\mathbf{v},\mathbf{v}_+) = 2q'$ for all solutions whose initial data are sufficiently close to the initial data of the solution w but $w \not\equiv w_+$.

Again by interpolating between the solutions we will prove that $I(\mathbf{v},\mathbf{v}_+) = 2q'$ for an arbitrary solution not equal to w_+. Let w_1 and w_2 be two solutions such that $w_1, w_2 \not\equiv w_+$ and $y(s,t)$ be the connecting path between them defined as above. It may happen that $y(s,t) = w_+$ for some s_0, but by small perturbation of the path of initial conditions, say varying $y'''(s,0)$ slightly, we can avoid it. Therefore, we suppose that $y(s,t)$ is not equal to w_+ for any s. Let $\mathbf{y}(s)$ be an extremal sequence of $y(s,t)$. Now we show that $I(\mathbf{v},\mathbf{y}(s))$ is constant by contradiction. If it is non-constant, then there exists $s_0 \in [0,1]$, for which $\mathbf{v}_+$ and $\mathbf{y}(s)$ have a non-transversal intersection. However according to Lemma 4.2 two stationary points $\mathbf{v}_+$ and $\mathbf{y}(s)$ of the flow Ψ^t generated by Equation (7.11) can not have a non-transversal intersection. Hence we proved that $I(\mathbf{v},\mathbf{v}_+) = 2q'$ for an arbitrary solution w not equal to w_+. □

The final lemma completes the proof of Theorem 1.2. Recall the parameter range $\alpha \in [\sqrt{8}, \alpha_{p,q})$, where $\frac{q}{p} < \frac{\lambda_2}{\lambda_1}$.

◀ **7.11 Lemma.** The sequence $\{u_n\}_{n=0}^{\infty}$ does not converge to the constant solution. ▶

Proof. We will prove this by contradiction. Suppose that $u_n \to 1$. Then $w^n \to w$, where w is a solution of the linear equation (7.11) and $\mathbb{E}[w] \geq 0$. Moreover, $\|w\|_{L^\infty} = \lim_{n\to\infty} \|w^n\|_{L^\infty} = 1$. Let T_n be the period of w^n.

First we assume that $\frac{\lambda_2}{\lambda_1}$ is irrational. Let us start with the case $\mathbb{E}[w] = 0$. It follows from Lemma 7.9 that w is not periodic and as we showed in the proof of Lemma 7.8 it holds that $T_n \to \infty$. Therefore solution w has at most $2p$ extrema on $\mathbb{R}$ and according to (7.12) it holds $w \equiv 0$, which is a contradiction with $\|w\|_{L^\infty} = 1$.

If $\mathbb{E}[w] > 0$, then w has to be periodic, otherwise we obtain a contradiction as above. It follows from Lemma 7.9 that the only periodic solution on this energy level is w_+. Thus $w^n \to w_+$ and $\left\|w^n - w_+^n\right\|_{L^\infty} \to 0$. The fact that $\tau(\mathbf{v}_+^n) \to \tau(\mathbf{v}_+) = \frac{\lambda_1}{\lambda_2} > \frac{q}{p}$ contradicts the assumption that $\mathbf{v}^n$ is $2p$ periodic and $I(\mathbf{v}^n,\mathbf{v}_+^n) = 2q$.

In the rational case $\frac{\lambda_2}{\lambda_1}$ we argue as follows. If $\mathbb{E}[w] = 0$, then $w^+ \equiv 0$ and w^n can not converge to w^+ because $\|w\|_{L^\infty} = 1$. Hence, by repeating the ideas in the proof of Lemma 7.8 one gets that $\mathbf{v}$ is $2p$ periodic and intersects zero $2q$ times per period. We will obtain a contradiction by showing that $\mathbf{v}$ is $2p'$ periodic and it intersects zero $2q'$ times per period, where $p', q' \in \mathbb{N}$ such that $\frac{q'}{p'} = \frac{\lambda_2}{\lambda_1} > \frac{q}{p}$. To prove the previous statement about the extremal sequence $\mathbf{v}$ we employ solutions w_L^n of the linear equation (7.11) with the same initial conditions as solutions w^n. These solutions converge to w and the energy $\mathbb{E}[w_L^n] > 0$ for all $n \in \mathbb{N}$. It follows from Lemma 7.10 that $\mathbf{v}_L^n$ is $2p'$ periodic and $I(\mathbf{v}_L^n,\mathbf{v}_+^n) = 2q'$. Hence as before the limit process for w_L^n implies that $\mathbf{v}$

is $2p'$ periodic and intersects zero $2q'$ times per period. This contradicts the inequality $\frac{q'}{p'} > \frac{q}{p}$.

Finally, if $\mathbb{E}[w] > 0$, then solutions w^n can not converge to w_+, otherwise we arrive at the same contradiction as in the irrational case. So Lemma 7.10 implies that $\mathbf{v}$ is $2p'$ periodic and $I(\mathbf{v}, \mathbf{v}^+) = 2q'$. On the other hand $w^n \to w$ and by repeating the ideas in the proof of Lemma 7.8 we get that $\mathbf{v}$ is $2p$ periodic and $I(\mathbf{v}, \mathbf{v}^+) = 2q$, which is a contradiction. □

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