PDE Excercise series

NOTE: These exercise are extra. Although solutions can be handed in for correction, this will not in any way contribute to a higher grade for the course.

1 Poisson equation with Neumann boundary condition

(PDE 2006, series 3, exercise 3) Let U be a bounded domain in $\mathbb{R}^m$ with smooth boundary ∂U. Denote the outward normal on ∂U by ν. The divergence theorem says that for $v \in C^1(\overline{U}, \mathbb{R}^m)$

$$\int_U \nabla \cdot v = \int_{\partial U} v \cdot \nu$$

Consider the problem

$$-\Delta u = f \qquad \text{in } U, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial U. \tag{2}$$

(i) Suppose that $u \in C^2(\overline{U})$ is a classical solution of Problem (1, 2). Derive an integral condition (IC) that f must satisfy. Show that Problem (1, 2) also has a solution which satisfies (IC).

(ii) Suppose that also $\phi \in C^2(\overline{U})$. Evaluate

$$\int_U \nabla u \cdot \nabla \phi \tag{3}$$

Remember that $H^1(U)$ is the space of all $L^2(U)$ functions which have first order weak partial derivatives in $L^2(U)$, equipped with the norm

$$\|u\|_{H^1(U)} = \left(\int_U \left(|u|^2 + |\nabla u|^2 \right) \right)^{\frac{1}{2}}$$

Remember that this space is compactly embedded in $L^2(U)$, meaning that a sequence that is bounded in $H^1(U)$ has a subsequence that converges in $L^2(U)$.

(iii) Which Sobolev space and which integral equality for u and arbitrary ϕ would you suggest as the defining property for a function u to be a weak solution of Problem (1, 2)?

(iv) Show that (3) defines an inner product on

$$\tilde{H}^1(U) = \left\{ u \in H^1(U) : u \text{ satisfies (IC)} \right\}$$

The inner product norm corresponding to (3) will be equivalent to the norm $\|.\|_{H^1(U)}$ on $\tilde{H}^1(U)$, provided there exists a constant C such that for all $u \in \tilde{H}^1(U)$ the following inequality holds:

$$\int_U |u|^2 \leq C \int_U |\nabla u|^2$$

(v) Show, arguing by contradiction and using the compactness of the embedding $\tilde{H}^1(U) \to L^2(U)$, that there is no sequence $u_n \in \tilde{H}^1(U)$ which has $\int_U |u_n|^2 = 1$ and $\int_U |\nabla u|^2 \to 0$. Deduce that indeed both norms are equivalent on $H^1(U)$.

(vi) Let $f \in L^2(U)$ satisfy (IC). Show applying the Riesz representation theorem in $\tilde{H}^1(U)$, that Problem (1, 2) has a weak solution $u \in H^1(U)$ which is unique up to an additive constant.

(vii) The construction of u in the proof of (vi) works equally well without the assumption that f satisfies (IC). Which problem does u solve then?

2 Abstract aspects of the Riesz approach (Challenge)

(PDE 2006, series 3, exercise 1) Recall that, if H is a Hilbert space and $f : H \to \mathbb{R}$ is a linear functional, continuity of f is equivalent to

$$\|f\|_{H'} = \sup_{\|x\|_H=1} \{|f(x)|\} < \infty.$$

The Riesz Representation theorem states that the space H', of continuous linear functionals on H, also known as the dual space of H, is isometrically isomorphic to H itself: the continuous linear functionals on H are precisely the mappings $x \mapsto (x, y_f)$, where $\|y_f\|_H = \|f\|_{H'}$.

Now let H and V be Hilbert spaces such that $V \subset H$. The inner products on V and H are denoted by $(.,.)_V$ and $(.,.)_H$, respectively. We shall write

$$(u,u)_H = \|u\|_H \quad \text{for } u \in H \qquad \text{and} \qquad (u,u)_V = \|u\|_V \quad \text{for } u \in V$$

We assume that V is dense in H and that the inclusion map $i : V \to H$ is continuous.

(i) Let $f \in H$. Prove that there exists a unique $u \in V$ such that $(u,v)_V = (f,v)_H$ for all $v \in V$. Define the mapping $A : H \to V$ by $u = Af$.

(ii) Prove that A is injective.

(iii) Prove that A is linear, symmetric (meaning $(Au,v) = (u,Av)$) and continuous with respect to $(.,.)_H$. (Hint: consider $i \circ A : H \to H$)

(iv) Prove that $A|_V$ is linear, symmetric and continuous with respect to V. (Hint: consider $A \circ i : V \to V$)

We also assume that V is compactly embedded in H, meaning that bounded sequences in V have convergent subsequences in H.

(vi) Prove that A is compact as an operator from $H \to H$ (if u_n is a bounded sequence in H, then Au_n has a convergent subsequence in H). Prove also that $A|_V$ is compact.

If $A : H \to H$ is a positive, symmetric, compact linear operator, then H has an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of eigenvectors of A corresponding to positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots$ with $\lambda_n \to 0$ as $n \to \infty$, where

$$\lambda_1 = \max_{f \in H} \frac{(Af,f)_H}{(f,f)_H}$$

and, more generally, for $n > 1$,

$$\lambda_n = \max_{f \in H : (f,\phi_1)_H = \cdots = (f,\phi_{n-1})_H = 0} \frac{(Af,f)_H}{(f,f)_H}$$

The proof of this statement is based on the existence of a maximizing vector ϕ_1 for

$$\max_{f \in H} \frac{(Af,f)_H}{(f,f)}$$

combined with the observation that every such maximizing vector is an eigenvector of A. This produces λ_1 and ϕ_1. The proof is completed by induction. Since A maps the space H_n to itself, where $H_n = \{u \in H : (u,\phi_1) = \cdots = (f,\phi_{n-1})_H = 0\}$ to it self, the same argument produces λ_{n+1} and ϕ_{n+1}.

(vii) The above statements applies to A, seen as a mapping from $H \to H$, but also to $A|_V$. Relate the resulting orthonormal bases to oneanother. Evaluate the eigenvalue formula's for $A|_V$ in terms of norms only, i.e. without A appearing in the formula's.