A non-commutative approach to the topology of circle and sphere bundles
Algebra is but written geometry and geometry is but figured algebra.

Sophie Germain (1776–1831)

The duality between algebra and geometry dates back to the work of Descartes: coordinate system.

Algebraic geometry in the early XX century: Noether, Hilbert (Nullstellensatz, Basissatz):

\[
\text{polynomial equations} \leftrightarrow \text{algebraic variety}
\]

Brought forward by Grothendieck.
For $X$ a compact Hausdorff space, consider

$$C(X) := \{ f : X \to \mathbb{C} : f \text{ is continuous} \}.$$ 

The set $C(X)$ comes with

- vector space structure: for $f, g \in C(X)$ and $\lambda \in \mathbb{C}$
  $$\lambda f + g)(x) := \lambda f(x) + g(x), \quad \forall x \in X;$$

- commutative product: for $f, g \in C(X)$:
  $$(fg)(x) := f(x)g(x), \quad \forall x \in X;$$

- unit: the function identically equal to 1; and

- an involution $*: C(X) \to C(X)$ given by
  $$f^*(x) := \overline{f(x)}.$$
There is a natural norm on the space $C(X)$, given by

$$
\|f\| = \sup_{x \in X} |f(x)|. \tag{1}
$$

with respect to which $C(X)$ is a Banach $\ast$-algebra.

The norm satisfies

$$
\|f^* f\| = \|f\|^2.
$$

$C(X)$ is a commutative $C^*$-algebra.

**Example**

Let $X$ consist of $n$-points. $C(X) \simeq \mathbb{C}^n$ with the usual vector space structure, coordinate-wise multiplication and complex conjugation, and norm

$$
\|(z_1, \ldots, z_n)\|^2 = \max\{|z_i^* z_i| \mid i = 1, \ldots, n\}
$$
Any point \( P \in X \) can be thought of as a functional

\[
\sigma_P : C(X) \to \mathbb{C}, \quad \sigma_P(f) := f(P),
\]

and it satisfies

\[
\sigma_P(fg) = \sigma_P(f)\sigma_P(g), \quad \sigma_P(1) = 1,
\]

i.e. \( \sigma_P \) is a character (also, a pure state).

All characters on \( C(X) \) are of this form and the set of characters \( \Sigma(C(X)) \) is homeomorphic to \( X \).

**Theorem (Gelfand Duality)**

Let \( A \) be a commutative unital \( C^* \)-algebra. Then there is a \( * \)-isomorphism

\[
A \simeq C(\Sigma(A))
\]

of commutative \( C^* \)-algebras.
Definition

A C*-algebra is a Banach *-algebra $A$ with the property that

$$\|a^*a\| = \|a\|^2,$$

for all $a \in A$.

Some examples

- The algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices with conjugate transpose and the operator norm

  $$\|A\| = \sup_{x \in \mathbb{C}^n, \|x\|=1} \|Ax\|;$$

- The algebra $B(H)$ of bounded operators on a Hilbert space, with operator adjoint, and operator norm

  $$\|A\| = \sup_{x \in H, \|x\|=1} \|Ax\|;$$
$B(H)$ is the prototypical example of $C^*$-algebra.

**Theorem (Gelfand–Naimark–Seagal)**

Let $A$ be a $C^*$-algebra. Then there exist a Hilbert space $H$ and an injective $*$-homomorphism $\pi : A \to B(H)$.

Every $C^*$-algebra can be embedded into the bounded operators on a Hilbert space.

**Idea**

Motivated from Gelfand duality, look at noncommutative $C^*$-algebras of operators as algebras of functions on some noncommutative space.
The circle:

\[ S^1 := \{ z \in \mathbb{C} \mid \bar{z}z = 1 \} . \]

The C*-algebra \( C(S^1) \) is the closure of the \textit{Laurent polynomials}

\[
\mathbb{C}[\zeta, \bar{\zeta}] / \langle \bar{\zeta}\zeta = 1 \rangle.
\]

We represent \( C(S^1) \) via multiplication operators on the Hilbert space

\[ H = L^2(S^1) \cong \ell^2(\mathbb{Z}). \]

Under this isomorphism, multiplication by \( e^{2\pi i \theta} \) is mapped to the bilateral shift

\[ U(e_n) = (e_{n+1}), \quad U^*(e_n) = e_{n-1}. \]

\( C(S^1) \) is the smallest C*-subalgebra of \( B(\ell^2(\mathbb{Z})) \) that contains the \textit{unitary} \( U \).
Now instead consider the Hilbert space $\ell^2(\mathbb{N})$ and the shift operator

$$T(e_n) = (e_{n+1})$$

Its adjoint is not invertible

$$T^*(e_n) = \begin{cases} 
  e_{n-1} & n \geq 1 \\
  0 & n = 0 
\end{cases}.$$

The *Toeplitz algebra* $\mathcal{T}$ is the smallest $C^*$-subalgebra of $B(\ell^2(\mathbb{N}))$ that contains $T$. It is not commutative since $T^*T = \text{Id}$ and $TT^* = 1 - P_{\ker(T^*)}$. 


Elements of $\mathcal{T}$ commute up to compact operators:

$$
\begin{array}{c}
0 & \longrightarrow & \mathcal{K}(\ell^2(\mathbb{N})) & \longrightarrow & \mathcal{T} & \overset{\pi}{\longrightarrow} & C(S^1) & \longrightarrow & 0.
\end{array}
$$

The spectrum $\Sigma(\mathcal{T})$ (defined as the set of pure states) is the disk $\mathbb{D} \subseteq \mathbb{C}$.

The algebra $C(S^1)$ is the "boundary" of a noncommutative disk.
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Principal circle bundle

\[ S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2 \]

Look at \( S^3 \) inside \( \mathbb{C}^2 \):

\[ S^3 := \{ (z_1, z_2) \in \mathbb{C}^2 \mid \overline{z_1}z_1 + \overline{z_2}z_2 = 1 \}. \]

Circle action defined component-wise: for every \( \lambda \in S^1 \),

\[ \alpha_\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2). \]

The orbit space is the two sphere \( S^2 \).

In physics: connections on the Hopf bundle describe magnetic monopole potentials.
The Hopf projection $\pi : S^3 \to S^2$ dualises to an inclusion of C*-algebras

$$C(S^2) \looparrowright C(S^3).$$

Circle action on $C(S^3)$, such that $C(S^2)$ is the fixed point algebra. The coordinate algebra

$$C(S^3) \supseteq \mathcal{O}(S^3) := \frac{\mathbb{C}[z_1, z_2, \overline{z}_1, \overline{z}_2]}{\langle z_1 \overline{z}_1 + \overline{z}_2 z_2 = 1 \rangle}$$

admits a vector space decomposition

$$\mathcal{O}(S^3) \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

where each $\mathcal{L}_n$ is the space of elements of $\mathcal{O}(S^3)$ that transform under the circle action as

$$\phi \mapsto \lambda^{-n} \phi, \quad \forall \lambda \in S^1$$
Each $\mathcal{L}_n$ is a bimodule over $\mathcal{L}_0 \simeq \mathcal{O}(S^2)$ and it is finitely generated projective.

The condition that the bundle is principal translates into the algebraic condition that the grading

$$\mathcal{O}(S^3) \simeq \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$$

is strong, i.e.

$$\mathcal{L}_n \otimes_{\mathcal{L}_0} \mathcal{L}_m \simeq \mathcal{L}_{n+m}.$$

This is in turn equivalent to invertibility of the module $\mathcal{L}_1$:

$$\mathcal{L}_1 \otimes_{\mathcal{L}_0} \mathcal{L}_{-1} \simeq \mathcal{L}_0 \simeq \mathcal{L}_{-1} \otimes_{\mathcal{L}_0} \mathcal{L}_1.$$

The Peter–Weyl decomposition allows to decompose the coordinate algebra of a circle bundle into sums of powers of line bundles and to characterise principal circle bundles. Many $C^*$-algebras have this structure.
Hilbert modules generalize the notion of Hilbert space with the field $\mathbb{C}$ replaced by a $C^*$-algebra $B$.

A Hilbert module is a pair $(E, \langle \cdot, \cdot \rangle_B)$, where

- $E$ is a right $B$-module with an Hermitian $B$-valued inner product; and
- $E$ is complete in the norm

$$\|\xi\|^2 := \|\langle \xi, \xi \rangle_B\|^2.$$ 

Operations on Hilbert modules: direct sums, tensor products.

The adjointable operators

$$\text{End}^*(E) := \{ T : E \to E \mid \exists T^* : E \to E : \langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle \},$$

form a $C^*$-algebra.
Define the \( C^\ast \)-algebraic dual

\[
E^* := \{ \lambda_\xi, \xi \in E \mid \lambda_\xi(\eta) = \langle \xi, \eta \rangle \} \subseteq \text{Hom}^*(E, B).
\]

Let \( E \) be a finitely generated projective Hilbert bimodule over a unital \( C^\ast \)-algebra \( B \). We say that \( E \) is a self-Morita equivalence over \( B \) if

\[
E \otimes_B E^* \cong B \cong E^* \otimes_B E.
\]

Example

Let \( B = C(X) \). Then \( E = \Gamma(L) \), the module of sections of a Hermitian line bundle \( L \to X \) is a self-Morita equivalence over \( B \).
Out of internal tensor products, construct

\[ \mathcal{F}(E) := B \oplus \bigoplus_{n \geq 1} E^\otimes n \]

For every \( \xi \in E \) define the shift operators by

\[ T_\eta(\xi_1 \otimes \cdots \xi_n) = \eta \otimes \xi_1 \otimes \cdots \xi_n, \quad T_\eta b = \eta \cdot b. \]

They are adjointable operators on \( \mathcal{F}(E) \).

**Definition**

The *Toeplitz algebra* \( T_E \) is the smallest \( C^* \)-subalgebra of \( \text{End}^*(\mathcal{F}(E)) \) that contains all the shifts.
If $E$ is a self-Morita equivalence bimodule, we can define the two-sided Fock module

$$\mathcal{F}_\mathbb{Z}(E) := \bigoplus_{n \in \mathbb{Z}} E^{(n)}$$

where $E^{(n)} := E \otimes^n$ for $n > 0$, $E^{(0)} = B$ and $E^{(n)} := (E^*) \otimes^n$ for $n < 0$.

On $\mathcal{F}_\mathbb{Z}(E)$ we consider bilateral shift operators $S_\xi$, $\xi \in E$.

**Definition**

The *Cuntz–Pimsner algebra* of $E$, denoted $\mathcal{O}_E$, is the smallest $C^*$-subalgebra of $\text{End}^*(\mathcal{F}_\mathbb{Z}(E))$ which contains all the bilateral shift operators.

We have an exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{K}(\mathcal{F}(E)) \longrightarrow T_E \overset{\pi}{\longrightarrow} \mathcal{O}_E \longrightarrow 0.$$
Both $\mathcal{T}_E$ and $\mathcal{O}_E$ come endowed with a circle action. 
We denote by $\mathcal{O}_E^\gamma$ the fixed point algebra for this action.

**Proposition (A.–Rennie)**

$E$ is a self-Morita equivalence bimodule if and only if $\mathcal{O}_E^\gamma \simeq B$.

**Theorem (A.–Kaad–Landi)**

Pimsner algebras of self-Morita equivalences are quantum principal circle bundles.

Examples: q-deformations
The six-term exact sequence in **operator** K-theory

\[
\begin{array}{c}
K_0(A) \xrightarrow{1-[X]} K_0(A) \xrightarrow{j_*} K_0(O_X) \\
\uparrow_{[\partial]} & & \downarrow_{[\partial]} \\
K_1(O_X) & \xleftarrow{j_*} & K_1(A) & \xleftarrow{1-[X]} & K_1(A)
\end{array}
\]

is a noncommutative analogue of the **topological** K-theory Gysin sequence for a circle bundle \( P \to X \) coming from the Hermitian line bundle \( L \).

\[
\begin{array}{c}
K^0(X) \xrightarrow{1-[L]} K^0(X) \xrightarrow{j^*} K^0(P) \\
\uparrow_{[\partial]} & & \downarrow_{[\partial]} \\
K^1(P) & \xleftarrow{j^*} & K^1(X) & \xleftarrow{1-[L]} & K^1(X)
\end{array}
\]
1 Trading spaces for algebras

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Extending the Toeplitz extension

\[ \begin{array}{ccccccc}
0 & \rightarrow & \mathcal{K}(\ell^2(\mathbb{N})) & \rightarrow & \mathcal{T} & \xrightarrow{\pi} & C(S^1) & \rightarrow & 0.
\end{array} \]

Cuntz Pimsner algebras of (injective) $C^*$-correspondences.

\[ \begin{array}{ccccccc}
0 & \rightarrow & \mathcal{K}(F(E)) & \xrightarrow{j} & T_E & \xrightarrow{\pi} & \mathcal{O}_E & \rightarrow & 0.
\end{array} \]

Arveson’s Toeplitz extensions for odd-dimensional spheres.

\[ \begin{array}{ccccccc}
0 & \rightarrow & \mathcal{K}(H^2_d) & \xrightarrow{j} & T_d & \xrightarrow{\pi} & C(S^{2d-1}) & \rightarrow & 0.
\end{array} \]

All these are examples of the *defining extensions* for Cuntz–Pimsner algebras of subproduct systems (Shalit and Solel 2009, Viselter 2012).
Let $d \in \mathbb{N}_0$, and $z_0, \ldots, z_{d-1}$ commuting variables, and consider the space of polynomials $\mathbb{C}[z_0, \ldots, z_{d-1}]$. For $z = (z_0, \ldots, z_{d-1})$ and every multi-index $\alpha = (\alpha_0, \ldots, \alpha_{d-1}) \in \mathbb{N}_0^d$ we write

$$z^{\alpha} = z_0^{\alpha_0} \cdots z_{d-1}^{\alpha_{d-1}}.$$ 

The Drury–Arveson space $H^2_d$ is a completion of the polynomials $\mathbb{C}[z_0, \ldots, z_{d-1}]$, w.r.t. the inner product

$$\langle z^{\alpha}, z^{\beta} \rangle = \delta_{\alpha, \beta} \frac{\alpha!}{|\alpha|!}.$$ 

It can be identified with the space of holomorphic functions $f : \mathbb{B}^d \subseteq \mathbb{C}^d \to \mathbb{C}$ which have a power series $f(z) = \sum_\alpha c_\alpha z^{\alpha}$ satisfying

$$\|f\|^2_d := \sum_\alpha |c_\alpha|^2 \frac{\alpha!}{|\alpha|!} < \infty.$$ 

Clearly, $H^2_d \simeq F_{\text{sym}}(\mathbb{C}^d) := \bigoplus_{n \geq 0} \text{Sym}^n(\mathbb{C}^d)$, the $d$-symmetric Fock space.
On $H_d^2$, we consider the $d$-shift, a $d$-tuple of multiplication operators given by

$$Mz = (Mz_0, \ldots, Mz_{d-1}).$$

Through $H_d^2 \cong F_{\text{sym}}(\mathbb{C}^d)$, the shift operator is identified with a compression of the shift on the full Fock space, that we denote by $T = (T_0, \ldots, T_{d-1})$.

The $d$-shift satisfies the following properties:

- $T$ is commuting: $T_i T_j = T_j T_i$.
- $\sum_{i=0}^{d-1} T_i T_i^* = 1 - P_\mathbb{C}$
- $T$ is essentially normal:

$$T_i^* T_j - T_j T_i^* = (1 + N)^{-1}(\delta_{ij} 1 - T_j T_i^*),$$

where $N$ is the number operator: $N\xi = n\xi$ for $\xi \in \text{Sym}^n(\mathbb{C}^d)$. 
Theorem (Arveson 1998)

Let $\mathbb{T}_d = C^* (1, T)$ be the $C^*$-algebra generated by the $d$-shift. We have an exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{K} (H^2_d) \longrightarrow \mathbb{T}_d \longrightarrow C (S^{2d-1}) \longrightarrow 0,$$

(2)

where $C (S^{2d-1})$ is the commutative $C^*$-algebra of continuous functions on the $(2d - 1)$-sphere $S^{2d-1} = \partial \mathbb{B}^d \subseteq \mathbb{C}^d$.

Odd-dimensional sphere as "boundaries" of a noncommutative $C^*$-algebras of operators.

Example of a subproduct system extension.
Work in progress with J. Kaad (SDU Odense)

For a given $n \geq 0$, consider the irreducible representation $\rho_n : SU(2) \to U(L_n)$. Where $L_n = (\mathbb{C}^2)^{\otimes s^n}$.

We define the determinant of the representation:

$$\det(\tau, H) = \{\xi \in H \otimes H \mid (\tau(g) \otimes \tau(g))\xi = \xi \quad \forall g \in SU(2)\}.$$

We inductively construct a family of Hilbert spaces where

- $E_0 = \mathbb{C}$;
- $E_1 = L_n$;
- $E_m := K_m \subseteq (L_n)^{\otimes m}$, where

$$K_m = \sum_{i=0}^{m-2} L_n^i \otimes D \otimes L_n^{m-i-2}, \quad D := \det(\rho_n, L_n).$$
Subproduct systems from $SU(2)$-representations

We construct the Fock space $F_E := \bigoplus_{m \geq 0} E_m(\rho_n, L_n)$. We let $\{e_j\}_{j=0}^n$ denote the orthonormal basis for $L_n$ and consider the associated Toeplitz operators:

$$T_i := T_{e_i} : F_E \rightarrow F_E \quad T_i(\zeta) := \iota_{1,m}^*(e_i \otimes \zeta), \quad \zeta \in E_m(\rho_n, L_n).$$

where $\iota_{1,m} : E_{m+1} \rightarrow E_1 \otimes E_m$, for $m \in \mathbb{N}_0$.

**Definition**

The Toeplitz algebra of the subproduct system $\mathbb{T}_E$ the unital C*-algebra generated by the Toeplitz operators.

It comes with a natural $SU(2)$-action so that we have an equivariant $SU(2)$-extension of C*-algebras:

$$0 \rightarrow \mathbb{K}(F_E) \rightarrow \mathbb{T}_E \xrightarrow{q} \mathbb{O}_E \rightarrow 0. \quad (3)$$
Theorem (A–Kaad 2020)

Let $T_E$ be the Toeplitz algebra of the $SU(2)$-product system of an irreducible representation. Then $T_E$ and $\mathbb{C}$ are KK-equivalent (i.e. the same in K-theory and K-homology) in an $SU(2)$-equivariant way.

We have Gysin-type exact sequence

\[ 0 \longrightarrow K_1(\mathcal{O}) \xrightarrow{([F] \hat{\otimes} \mathcal{K}(F) \cdot) \circ \partial} K_0(\mathbb{C}) \xrightarrow{1_C - [L_n] + [\det(\rho_n, L_n)]} K_0(\mathbb{C}) \xrightarrow{i_*} K_0(\mathcal{O}) \longrightarrow 0 \]

for every $n \in \mathbb{N}$.

*Note that the Euler class comprises of three terms, as we would expect classically!*
- $C^*$-algebras provide an elegant setting for problems in geometry and topology.

- Within the NCG dictionary, Cuntz–Pimsner algebras are a model for circle bundles.

- Cuntz–Pimsner algebras of subproduct systems are suitable to encode spherical symmetries.