Coupled Oscillator Networks: Structure, Interactions, and Dynamics

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... but thanks also go to various other students and collaborators.
Coupled Oscillator Networks

Clocks

Brain

Power grids

Chris Bick
Coupled Metronomes are a Network Dynamical Systems

https://www.youtube.com/watch?v=T58lGKREuego
Network of oscillatory units: \( \bullet \in \{ \bullet, \bullet, \ldots \} \)

**Network structure (‘topology’):**
Who interacts with whom?

**Network interaction:**
How does one oscillator influence the other?

**Network dynamics:**
Collective dynamics of all nodes.

Q: How do structure and interactions shape the network dynamics?
From Oscillators to Phase Oscillators

Weakly coupled **nonlinear oscillators** with state $x_k \in \mathbb{R}^Q$

$$\dot{x}_k = F_k(x_k) + \varepsilon \sum_{j=1}^{N} G_{kj}(x_j, x_k).$$

**Phase reduction**, phase response curve (PRC) $Z(\phi)$, interactions $h_{kj}(t)$

**Average** over fast oscillations

(Averaged) **Phase oscillator network** with state $\theta_k \in T = \mathbb{R}/2\pi\mathbb{Z}$

$$\dot{\theta}_k = \omega_k + \sum_{j=1}^{N} g_{kj}(\theta_j - \theta_k)$$

Interaction: **Coupling functions** $g_{kj}$. *Kuramoto model* with $g_{kj} = \sin$.

1. Population
2. Populations
3. Populations
Consider $N$ symmetric oscillators with $z_k \in \mathbb{C}$ close to a Hopf bifurcation

$$\dot{z}_k = F_\lambda(z_k) + \epsilon G_\lambda(z_k; z_1, \ldots, z_N).$$
Consider $N$ symmetric oscillators with $z_k \in \mathbb{C}$ close to a Hopf bifurcation

$$\dot{z}_k = F_\lambda(z_k) + \varepsilon G_\lambda(z_k; z_1, \ldots, z_N).$$

Phase approximation, $\theta_k \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, valid for $t = O(\varepsilon^{-1} \lambda^{-1})$ is

$$\dot{\theta}_k = \tilde{\omega}(\theta, \varepsilon) + \frac{\varepsilon}{N} \sum_{j=1}^{N} g_2(\theta_j - \theta_k),$$

where $\tilde{\omega}(\theta, \varepsilon)$ is a $S_N$-symmetric function in the phases and

$$g_2(\phi) = \xi_1^0 \cos(\phi + \chi_1^0)$$
Consider \( N \) symmetric oscillators with \( z_k \in \mathbb{C} \) close to a Hopf bifurcation
\[
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Phase approximation, \( \theta_k \in T = \mathbb{R}/2\pi \mathbb{Z} \), valid for \( t = O(\varepsilon^{-1} \lambda^{-2}) \) is
\[
\dot{\theta}_k = \tilde{\omega}(\theta, \varepsilon) + \frac{\varepsilon}{N} \sum_{j=1}^{N} g_2(\theta_j - \theta_k) + \frac{\varepsilon}{N^2} \sum_{j,l=1}^{N} g_3(\theta_j + \theta_l - 2\theta_k)
+ \frac{\varepsilon}{N^2} \sum_{j,l=1}^{N} g_4(2\theta_j - \theta_l - \theta_k) + \frac{\varepsilon}{N^3} \sum_{j,l,m=1}^{N} g_5(\theta_j + \theta_l - \theta_m - \theta_k)
\]

where \( \tilde{\omega}(\theta, \varepsilon) \) is a \( S_N \)-symmetric function in the phases and
\[
\begin{align*}
g_2(\phi) &= \xi_1^0 \cos(\phi + \chi_1^0) + \lambda \xi_1^1 \cos(\phi + \chi_1^1) + \lambda \xi_2^1 \cos(2\phi + \chi_2^1), \\
g_3(\phi) &= \lambda \xi_3^1 \cos(\phi + \chi_3^1), \quad g_4(\phi) = \lambda \xi_4^1 \cos(\phi + \chi_4^1), \\
g_5(\phi) &= \lambda \xi_5^1 \cos(\phi + \chi_5^1).
\end{align*}
\]
What is Important?

First-Order Approximation

\[ \dot{\theta}_k = \omega + \sum_{j=1}^{N} \sin(\theta_j - \theta_k + \alpha) \]

*Kuramoto–Sakaguchi equations*: Integrable with 2 degrees of freedom.

Second-Order Approximation

\[ \dot{\theta}_k = \omega + \sum_{j=1}^{N} \sin(\theta_j - \theta_k + \alpha) + \cdots + \sum_{j,l=1}^{N} \sin(\theta_j + \theta_l - 2\theta_k + \hat{\alpha}) + \cdots \]

Network dynamical system with *higher-order interactions*.

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Network dynamical system with *higher-order interactions*.

**Advertisement!**  
*What are higher-order networks?* To go to SIAM Review (any day).

Symmetric Consequences

Invariant phase configurations

\[ S = \{\theta_1 = \cdots = \theta_N\} = \bullet \quad \text{D} = \{\theta_{k+1} = \theta_k + 2\pi / N\} = \circ \]

Phase ordering is preserved

\[ N = 3 \quad \text{N} = 4 \]

Phase Oscillators with Higher-Order Interactions

**Q:** Are there chaotic attractors with nonpairwise coupling?

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Fix $N = 4$, $\lambda = 1$, $\xi = (-0.3, 0.3, 0.02, 0.8, 0.02)$ and parametrize

$$g_2(\phi) = \xi_1 \cos(\phi + \chi_1) + \xi_2 \cos(2\phi + \chi_2), \quad g_3(\phi) = \xi_3 \cos(\phi + \chi_3),$$

$$g_4(\phi) = \xi_4 \cos(\phi + \chi_4), \quad g_5(\phi) = \xi_5 \cos(\phi + \chi_5).$$
A: There chaotic attractors with nonpairwise coupling!

Fix $N = 4$, $\lambda = 1$, $\xi = (\xi_1, 0.3, 0.3, 0.02, 0.8, 0.02)$ and parametrize

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\end{align*}
\]

Parameters

$\chi = (0.108, 0.27, 0, 1.5, 0)$.

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Parameters

$\chi = (0.154, 0.318, 0, 1.74, 0)$. 

**A:** There chaotic attractors with nonpairwise coupling!

Fix $N = 4$, $\lambda = 1$, $\xi = (-0.3, 0.3, 0.02, 0.8, 0.02)$ and parametrize

\[
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    g_2(\phi) &= \xi_1 \cos(\phi + \chi_1) + \xi_2 \cos(2\phi + \chi_2), \\
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    g_3(\phi) &= \xi_3 \cos(\phi + \chi_3), \\
    g_5(\phi) &= \xi_5 \cos(\phi + \chi_5).
\end{align*}
\]

Parameters

$\chi = (0.2, 0.316, 0, 1.73, 0)$. 

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Frequencies

**Instantaneous frequency**

\[ \dot{\theta}_k(t) \]

**Asymptotic average frequency**

\[ \Omega_k = \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{\theta}_k(t) \, dt \]

**Identical all-to-all coupling, \( S_N \) symmetry**: for all \( k, j \)

\[ \Omega_k = \Omega_j. \]

**No frequency separation for identical (1st order) phase oscillators.**
1. Population
2. Populations
3. Populations
Oscillator $k$ in population $\sigma$ has phase

$$\theta_{\sigma,k} \in T.$$ 

**Identical oscillators:** can exchange any two oscillators while preserving the equations of motion. Have $\omega_{\sigma,k} = \omega$.

**Phase configurations**

$$S = \{\theta_{\sigma,1} = \cdots = \theta_{\sigma,N}\} \quad D = \{\theta_{\sigma,k+1} = \theta_{\sigma,k} + 2\pi / N\}$$

Write

$$S, D \quad SS, SD, DS, DD \quad SSS, SSD, SDD, \ldots$$
Frequencies

\[ \dot{\theta}_{\sigma,k}(t) \quad \Omega_{\sigma,k} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{\theta}_{\sigma,k}(t) \, dt \]

**Frequency synchrony** in population \( \sigma \): for \( k \neq j \)

\[ \Omega_{\sigma,k} = \Omega_{\sigma,j} \]

**Weak chimera** characterized by **localized frequency synchrony**

\[ \Omega_{\sigma,k} = \Omega_{\sigma,j} \quad \text{for any } \sigma \text{ and } j \neq k \]
\[ \Omega_{\sigma,k} \neq \Omega_{\tau,k} \quad \text{for } \sigma \neq \tau. \]

**Dynamics of identical oscillators show distinct frequencies.**
Ashwin and Burylko: There is localized frequency synchrony for weakly (globally and identically) coupled populations.

1. **Two uncoupled populations**: SD is invariant.
2. SD has **localized frequency synchrony**.
3. **Persistence** for small $\varepsilon > 0$. 

---

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1. **Two uncoupled populations**: SD is invariant.
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3. **Persistence** for small $\varepsilon > 0$.

B and Ashwin: Generalization and larger populations.

1. Population
2. Populations
3. Populations
The final results consist of a match/trial ratio and Z probability matches (the exception is max_isi and max_gap, since max_isi and max_gap would have allowed a single letter to represent more than one word). Significant matching in POST occurs for longer, lower-wave/ripple occurrence rate of approximately 0.5–1 Hz compared to 270/24 expected matches, Z<sub>0</sub> = 7.2, Z<sub>0</sub> = 0.13, i.e., 35 matches out of 270 words that could have had a low-probability match (note that Z<sub>0</sub> corresponds to fewer matches than expected based on chance). PRE SWS p<sub>4</sub> (note that Z<sub>0</sub>) corresponds to fewer matches than expected based on chance).

This activity that extended into the following word (Figure 2) shows significant similarity for a wide range of sequences for all parameter values, while POST SWS activity shows no significant similarity to RUN sequences for all parameter values. Furthermore, the most significant matching in POST occurs for longer, lower-wave/ripple occurrence rate of approximately 0.5–1 Hz compared to 270/24 expected matches, Z<sub>0</sub> = 7.2, Z<sub>0</sub> = 0.13, i.e., 35 matches out of 270 words that could have had a low-probability match (note that Z<sub>0</sub> corresponds to fewer matches than expected based on chance). PRE SWS p<sub>4</sub> (note that Z<sub>0</sub>) corresponds to fewer matches than expected based on chance).
Chris Bick  
Coupled Oscillator Networks
Q: Can one observe transitions of frequencies over time?
A: Yes, one may observe transitions of frequencies.
How to Get Transitions

Transitions induced by Heteroclinics Orbits

1. Have a finite collection of saddles $A_q$.
2. Suppose that the unstable manifold of $A_q$ has a nontrivial intersection with the stable manifold of $A_{q+1}$—there are heteroclinic connections.
3. Impose additional stability conditions.

Dynamics: Transitions from one saddle to the next along the cycle/network.

Robust heteroclinic cycles/networks may arise in

- Lotka–Volterra type systems,
- Systems with symmetries.

Phase reduction, $\hat{g}_2$, higher harmonics, $\hat{g}_3$, $\hat{g}_4$, $\hat{g}_5$ one harmonic

$$\dot{\theta}_k = \omega + \sum_{j=1}^{Q} \hat{g}_2(\theta_j - \theta_k) + \sum_{j,l=1}^{Q} \hat{g}_3(\theta_j + \theta_l - 2\theta_k)$$

$$+ \sum_{j,l=1}^{Q} \hat{g}_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{Q} \hat{g}_5(\theta_j + \theta_l - \theta_m - \theta_k)$$
Coupled Populations with Higher-Order Interactions

Phase reduction, $\hat{g}_2$ higher harmonics, $\hat{g}_3, \hat{g}_4, \hat{g}_5$ one harmonic

$$\dot{\theta}_k = \omega + \sum_{j=1}^{Q} a_2^{(jk)} \hat{g}_2(\theta_j - \theta_k) + \sum_{j,l=1}^{Q} a_3^{(ljk)} \hat{g}_3(\theta_j + \theta_l - 2\theta_k)$$

$$+ \sum_{j,l=1}^{Q} a_4^{(ljk)} \hat{g}_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{Q} a_5^{(mljk)} \hat{g}_5(\theta_j + \theta_l - \theta_m - \theta_k)$$
Phase reduction, \( \hat{g}_2 \) higher harmonics, \( \hat{g}_3, \hat{g}_4, \hat{g}_5 \) one harmonic

\[
\dot{\theta}_k = \omega + \sum_{j=1}^{Q} a_{2}^{(jk)} \hat{g}_2(\theta_j - \theta_k) + \sum_{j,l=1}^{Q} a_{3}^{(ljk)} \hat{g}_3(\theta_j + \theta_l - 2\theta_k) \\
+ \sum_{j,l=1}^{Q} a_{4}^{(ljk)} \hat{g}_4(2\theta_j - \theta_l - \theta_k) + \sum_{j,l,m=1}^{Q} a_{5}^{(mjk)} \hat{g}_5(\theta_j + \theta_l - \theta_m - \theta_k)
\]

**Special case:** \( M = 3 \) populations of \( N = 2 \) oscillators, \( j = 3 - k \)

\[
\dot{\theta}_{\sigma,k} = \sin(\theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) + r \sin(2(\theta_{\sigma,j} - \theta_{\sigma,k} + \alpha)) \\
- K \cos(\theta_{\sigma-1,1} - \theta_{\sigma-1,2} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) \\
- K \cos(\theta_{\sigma-1,2} - \theta_{\sigma-1,1} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) \\
+ K \cos(\theta_{\sigma+1,1} - \theta_{\sigma+1,2} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha) \\
+ K \cos(\theta_{\sigma+1,2} - \theta_{\sigma+1,1} + \theta_{\sigma,j} - \theta_{\sigma,k} + \alpha)
\]

The system is \( T^M \) equivariant: one phase-shift symmetry per population.
Transitions of Localized Frequency Synchrony

Theorem

For $M = 3$ populations with $N = 2, 3$ oscillators, there are coupling parameters such that there is a robust and dissipative heteroclinic cycle between distinct patterns of localized frequency synchrony.

Idea of proof

1. No coupling: Separate frequencies of $S, D$.
2. Ensure stability of $DSS, DDS, \ldots$.
3. Heteroclinic connections, e.g., in $D\psi S$.

$D\psi S$ for $N = 3$:

$D\psi S$ for $N = 2$:

Heteroclinic Cycles in Action!

Dynamics of $M = 3$ populations of $N = 2, 3$ oscillators.

**Project** (J. Mujica, VU): Bifurcations of heteroclinic cycles under forced symmetry breaking.

**Project** (T. Böhle, TUM): Dynamics of the mean-field limit.

From Heteroclinic Cycles to Networks

Dynamics of $M = 4$ populations of $N = 2$ oscillators.

**Theorem**

*Coupled populations of phase oscillators support the heteroclinic network between distinct patterns of frequency synchrony below which contains two cycles.*

![Diagram of heteroclinic network](image-url)

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More Action!

Dynamics of $M = 4$ populations of $N = 2$ oscillators.


Chris Bick  Coupled Oscillator Networks
1. Population
2. Populations
3. Populations
4. Back to the Real World
Weakly coupled nonlinear oscillators

Phase oscillator network
Synchronization Engineering

**Nonlinear oscillator network** with state $x_k$

$$\dot{x}_k = F(x_k) + \varepsilon \sum_{j=1}^{N} G_{kj}(x_j, x_k)$$

Calculate feedback parameters for $h(x)$ to match phase reduction.

Apply (delayed) feedback $p_k(x)$ to oscillators with known PRC $Z(\phi)$

$$\dot{x}_k = F(x_k) + Kp_k(x) \quad p_k(t) = \sum_{j=1}^{N} K_{kj} h(x(t - \tau))$$

**Phase oscillator network** with state $\theta_k$

$$\dot{\theta}_k = \omega + \sum_{j=1}^{N} g_{kj}(\theta_j - \theta_k)$$


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Experiments

Pairwise Interactions

Nonpairwise Interactions

Project (B. Liefting, Exeter). Generalization of Synchronization Engineering to Networks with Higher-Order Interactions.

Conclusions and outlook

Conclusions

!* Higher-order interactions* yield interesting phase dynamics.

!* Identical oscillators* can give rise to *distinct frequencies* through network interactions.

!* Transitions of frequencies* can arise through heteroclinic cycles and networks.

!* Describe such phenomena* *mathematically* but can be seen in *experiments*

Outlook (i.e., more questions)

? Experimental realization of frequency transitions?

? Rigorous analysis of what happens as symmetries are broken.
Thank you for your attention!

References
Get in touch!

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