Cooperative Game Theory

Non-cooperative game theory specifies the strategic structure of an interaction:

- The participants (players) in a strategic interaction.
- Who can do what and when, and what they know when they do it.
- The payoffs of players as a function of the choices of all players.

Solution concepts (Dominance, Rationalisability, Nash, Perfectness etc.) were strategically based (sometimes incorporating notions of beliefs). A focus on *single-player deviations*.

In contrast, cooperative game theory specifies no such strategic structure, just:

- The participants (players) in a given interaction.
- What each subset of players (or "coalition") can *jointly* achieve.
- Cooperative solution concepts focus on the deviations of coalitions of players…
- …and are based on what payoffs players can achieve rather than what they do.

Cooperative Games with Transferable Utility

So: define TU and NTU cooperative games, the core, the Shapley value, and some examples.

Cooperative Games. A cooperative game with transferable utility is $\mathcal{G} = \langle N, v \rangle$ where:

1. N is the set of players with typical player i.
2. A payoff function $v(S)$, defined for every collection of players $S \subseteq N$.

Very simple! S is a "coalition" and hence these are sometimes referred to as coalitional games. $v(S)$ is sometimes called the worth of coalition S. Some examples of such games are:

1. **Treasure Hunting.** The players are a group of treasure seekers. The worth of each coalition is the amount of treasure that it can carry.
2. **Intra-Firm Bargaining.** The players are a firm and a group of workers. The worth is either their outside option (for workers) or the amount that can be produced in the firm.
3. **Production.** The players are a group of people owning endowments of inputs and production technologies. The worth is the amount of output that can be produced given these constraints.

Superadditivity and Cohesiveness

Throughout this lecture restrict to games which are cohesive:

Cohesiveness. A game with transferable utility $\mathcal{G} = \langle N, v \rangle$ is cohesive if

$$v(N) \geq \sum_{S \in \mathcal{S}} v(S) \qquad \text{for every partition } \mathcal{S} \text{ of } N.$$

The whole is greater than the sum of its parts. More generally, *each* whole might be greater than the sum of its parts. This is a condition called superadditivity. Formally:

Superadditivity. A game with transferable utility $\mathcal{G} = \langle N, v \rangle$ is superadditive if

$$v(S \cup S') \geq v(S) + v(S') \qquad \text{for all coalitions } S \text{ and } S' \text{ where } S \cap S' = \emptyset.$$

Clearly, a superadditive game is cohesive. Cooperative (or coalitional) games simply define what each group of individuals can jointly achieve. How to predict what each player will get?

Note. Recall that a partition of a set is any exhaustive collection of its subsets that contain no elements in common.

The Core

A payoff profile is a vector of payoffs $\mathbf{x} = (x_i)_{i \in N}$. It is said to be S-feasible if

$$\sum_{i \in S} x_i = v(S),$$

and $\mathbf{x}$ is feasible if the same is true when setting $S = N$. Two equivalent definitions:

The Core. The core of a TU game $\mathcal{G} = \langle N, v \rangle$ is the set of feasible payoff profiles $\mathbf{x}$ such that there is no $S \subseteq N$ where $\mathbf{y}$ is S-feasible and $y_i > x_i$ for all $i \in S$.

The core is robust to deviations by coalitions of players. Indeed, it is robust to *all* such deviations. Therefore the following provides an equivalent definition for the core.

The core of a TU game $\mathcal{G} = \langle N, v \rangle$ is the set of feasible payoff profiles $\mathbf{x}$ where

$$\sum_{i \in S} x_i \geq v(S) \qquad \text{for every coalition } S \subseteq N.$$

A Treasure Hunt

"An expedition of n people discover some treasure in the mountains. Each *pair* can carry out a *single* piece of treasure."

Players. These are $N = \{1, 2, \ldots, n\}$, the treasure seekers.

Payoffs. The *worth* of a coalition $S \subseteq N$ is

$$v(S) = \begin{cases} |S|/2 & \text{if } |S| \text{ is even,} \\ (|S|-1)/2 & \text{if } |S| \text{ is odd.} \end{cases}$$

- Suppose that $n = |N|$ is even, and $n \geq 4$. If there are unequal payoffs, then the two players with the lowest payoffs may deviate and obtain more. The core is payoffs of $\frac{1}{2}$ for everyone.
- Suppose that $n = |N| \geq 3$ is odd. If all players receive something, then $n - 1$ could choose to abandon the nth and increase their payoffs. Any zero-payoff player can form a coalition with the lowest positive payoff player, and obtain a better payoff for both. The core is empty!

Intra-Firm Bargaining

"There is a firm and a pool of n workers. Workers have a reservation wage of $\bar{w}$. Production with i workers yields revenue, net of non-labour costs, of $F(i)$."

The marginal product of the ith employee is

$$\Delta F(i) = F(i) - F(i-1).$$

Assume that this is decreasing in i, so that this represents a concave production function. Writing this as a cooperative game with transferable utility, $\mathcal{G} = \langle N, v \rangle$:

Players. These are $N = \text{Firm} \cup \{n \text{ workers}\}$.

Payoffs. The worth of a coalition $S \subseteq N$ is

$$v(S) = \begin{cases} 0 & S = \{\text{Firm}\}, \\ i\bar{w} & S = \{i \text{ workers}\}, \\ F(i) & S = \{\text{Firm}\} \cup \{i \text{ workers}\}. \end{cases}$$

The Core in Intra-Firm Bargaining

So, $x_i \geq \bar{w}$. Otherwise a single worker could deviate and receive their reservation wage.

$x_i \leq \Delta F(n)$, otherwise the remaining $n-1$ workers (and the firm) could abandon i, form a coalition together, and thus receive $F(n-1) > F(n) - x_i$.

This is sufficient. Suppose that k workers are removed from the coalition N, then it follows that

$$\sum_{i=1}^{k} x_i \leq k\Delta F(n) \leq F(n) - F(n-k).$$

The core contains payoffs where firm receives $F(n) - \sum_{i=1}^{n} x_i$ and worker i receives x_i such that

$$\sum_{i=1}^{n} x_i \leq F(n) \quad \text{and} \quad \bar{w} \leq x_i \leq \Delta F(n).$$

Workers are paid above their reservation wage, $\bar{w}$, but below their marginal productivity $\Delta F(n)$. The core may not be a unique payoff profile, and may be empty (e.g. the treasure hunt).

Values in Cooperative Games

The core has some problems in cooperative games with transferable utility:

- Sometimes the core is empty, and sometimes it is large.
- It does not consider the *credibility* of deviating coalitions.

Look for a cooperative solution which exists, is unique, and is robust to “credible” deviations.

A *value* is function that assigns a unique feasible payoff profile to every game $\mathcal{G} = \langle N, v \rangle$. Write $\psi(N, v)$ for the value of game $\mathcal{G}$, where $\psi_i(N, v)$ is the payoff assigned to player i.

Other (non-value based) possibilities for “credible” solutions are vNM stable sets, the bargaining set, the kernel, and the nucleolus. Here, focus on the *Shapley value*, where:

- A player is paid their marginal contribution to a coalition, but…
- …this contribution depends on their *position* in the coalition, so…
- …pay them their *expected* marginal contribution.

The Shapley value incorporates the property that gains from participation are *balanced*.

Motivating the Shapley Value

Returning to the Intra-Firm Bargaining game, consider a firm and a single worker. Denote by $\pi(i)$ and $w(i)$ the payoff the firm and worker respectively get from a coalition of i workers and the firm.

- Alone the worker earns a payoff of $\bar{w}$ and the firm earns $\pi(0) = 0$.
- Together, the firm can produce $F(1) = \Delta F(1)$. Thus, gains from trade are:

$$\text{Gain from Trade} = \Delta F(1) - \bar{w}.$$

- Splitting the gains from trade 50:50 (balanced gains) yields:

$$\pi(1) = \frac{F(1)}{2} - \frac{\bar{w}}{2} \quad \text{and} \quad w(1) = \frac{F(1)}{2} + \frac{\bar{w}}{2}.$$

- Notice that $\pi(1) - \pi(0) = w(1) - \bar{w} = (\Delta F(1) - \bar{w})/2$: "Split the gains".

More generally, this is a "balanced contributions" requirement for coalition building.

A Firm with Two Workers

Suppose that a second worker becomes available, and, as a result, if both workers are hired they receive a wage $w(2)$. The payoff to the firm of hiring i people is $\pi(i) = F(i) - iw(i)$. Now

$$\begin{aligned}\pi(2) - \pi(1) &= [F(2) - 2w(2)] - [F(1) - w(1)]\,, \\ &= \Delta F(2) - 2w(2) + w(1).\end{aligned}$$

For "balanced gains" to negotiating parties, this must equal $w(2) - \bar{w}$. So,

$$\begin{aligned}\Delta F(2) - 2w(2) + w(1) &= w(2) - \bar{w}, \\ \Rightarrow \quad 3w(2) &= \Delta F(2) + w(1) + \bar{w}, \\ &= \Delta F(2) + \left[\frac{\Delta F(1) + \bar{w}}{2}\right] + \bar{w}, \\ \Rightarrow \quad w(2) &= \left[\frac{1}{3} \times \Delta F(2)\right] + \left[\frac{1}{3 \times 2} \times \Delta F(1)\right] + \frac{\bar{w}}{2}.\end{aligned}$$

Three Workers and a Firm

The gains for a firm retaining its ith employee, and to the i workers, are respectively

$$\pi(i) - \pi(i-1) = \Delta F(i) - iw(i) + (i-1)w(i-1) \quad \text{and} \quad w(i) - \bar{w}.$$

Balanced contributions yields $(i+1)w(i) = \Delta F(i) + \bar{w} + (i-1)w(i-1)$. Suppose $i = 3$:

$$\begin{aligned} w(3) &= \left[\frac{1}{4} \times \Delta F(3)\right] + \frac{\bar{w}}{4} + \frac{2}{4} \times \left\{ \left[\frac{1}{3} \times \Delta F(2)\right] + \left[\frac{1}{3 \times 2} \times \Delta F(1)\right] + \frac{\bar{w}}{2} \right\}, \\ &= \frac{3\Delta F(3)}{4 \times 3} + \frac{2\Delta F(2)}{4 \times 3} + \frac{1\Delta F(1)}{4 \times 3} + \frac{\bar{w}}{2}. \end{aligned}$$

Iterating this argument, the general formula for a firm with n workers yields:

$$w(n) = \frac{1}{n(n+1)} \sum_{j=1}^{n} j\Delta F(j) + \frac{\bar{w}}{2}.$$

The Shapley Value

Suppose i is the last member of a coalition S, what is their marginal contribution?

$$m_i(S) = \underbrace{v(S)}_{\text{value with } i} - \underbrace{v(S\backslash\{i\})}_{\text{value without } i} \quad \text{where} \quad i \in S \subseteq N.$$

Now arrange all of the agents at random. Consider the following arrangement:

- Suppose that the group S occurs first, with player i at the end.
- $(|S|-1)!$ arrangements of others in S before i, $(|N|-|S|)!$ arrangements of remainder.
- There are $|N|!$ ways of arranging people in total, so probability that i is marginal to S is

$$p_i(S) = (|S|-1)!(|N|-|S|)!/|N|!$$

The Shapley Value. The Shapley value of a game $\mathcal{G} = \langle N, v \rangle$ is given by $\psi(N, v)$ where

$$\psi_i(N, v) = \sum_{S: i \in S} p_i(S) m_i(S) = \sum_{S: i \in S} \frac{(|S|-1)!(|N|-|S|)!}{|N|!} \left[v(S) - v(S\backslash\{i\}) \right].$$

Balanced Contributions

Marginal contribution story is one way to interpret the Shapley value. There are (many) others.

Define a *subgame* $\mathcal{G}^S = \langle S, v^S \rangle$ of $\mathcal{G} = \langle N, v \rangle$, to be the cooperative game with transferable utility where $S \subseteq N$ and $v^S(T) = v(T)$ for all $T \subseteq S$. Then:

Balanced Contributions. A value ψ has balanced-contributions if for all $\mathcal{G} = \langle N, v \rangle$, and $i, j \in N$

$$\psi_i(N, v) - \psi_i\left(N \backslash \{j\}, v^{N \backslash \{j\}}\right) = \psi_j(N, v) - \psi_j\left(N \backslash \{i\}, v^{N \backslash \{i\}}\right).$$

"The payoff I lose if you leave the game is equal to the payoff you lose if I leave the game."

- The Shapley value is the unique value with balanced contributions.
- Can also provide an axiomatic foundation for Shapley value…
- …and foundation based on "objections" and "counter-objections".

Rather than examine these here, consider further applications of the Shapley value…

The Shapley Value for a Worker

Calculate Shapley value for players in intra-firm bargaining game. Arrange players randomly.

- Suppose the firm comes after the worker in the line up of participants.
- This happens half of the time: the expected marginal contribution is $\bar{w}/2$.

There is a possibility that the worker is marginal to a coalition involving the firm.

- The probability that the worker takes position $j+1$ is $1/(n+1)$.
- There are n remaining places, and j places before the worker.
- The probability that the firm takes one of the places before the worker is j/n.
- In this case, the worker adds to the working firm a contribution of $\Delta F(j)$.
- This yields an expected contribution of $1/(n+1) \times j/n \times \Delta F(j)$.

Thus the Shapley value for a worker in the intra-firm bargaining game is

$$\psi_{\text{worker}}(N, v) = \frac{\bar{w}}{2} + \frac{1}{n(n+1)} \sum_{j=1}^{n} j \Delta F(j).$$

The Shapley Value for the Firm

A firm takes one out of the $n + 1$ positions in the line up of players.

It takes any particular position $j + 1$ with probability $1/(n + 1)$. Suppose it takes position $j + 1$:

- There are j workers before it.
- The workers are no longer earning their outside wage, a loss of $j\bar{w}$.
- Its membership of the coalition enables production to take place, yielding $F(j)$.
- Hence the marginal contribution of the firm is $F(j) - j\bar{w}$.

The Shapley value for the firm in the intra-firm bargaining game is thus

$$\psi_{\text{firm}}(N, v) = \frac{1}{n+1} \sum_{j=0}^{n} \left[F(j) - j\bar{w} \right] .$$

Note. It is possible to apply the definition given earlier directly to find Shapley values. In this case that would involve a good deal of algebra (there are a lot of coalitions S to consider). It is often easier to use shortcuts such as the above.

“Bargaining” Firms versus “Neoclassical” Firms

A “neoclassical” firm with j employees makes profits $\pi_{\text{neo}}(j) = F(j) - j\bar{w}$.

Hence the “bargaining” firm enjoys the *average* neoclassical profits, or

$$\psi_{\text{firm}} = \frac{1}{n+1}\sum_{j=0}^{n} \pi_{\text{neo}}(j) \approx \frac{1}{n}\int_0^n \pi_{\text{neo}}(j)dj.$$

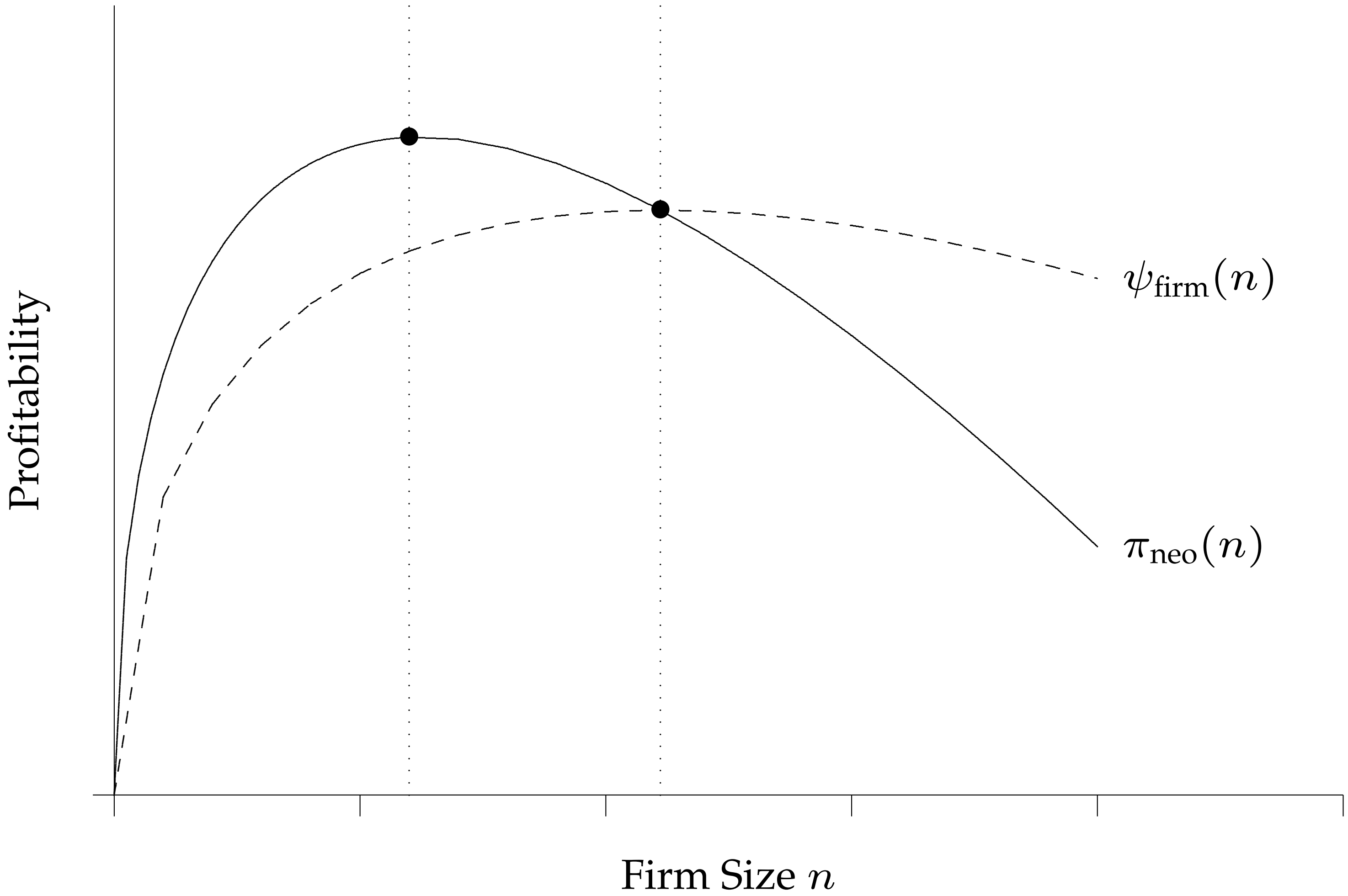
Suppose the “bargaining” firm chose scale (n) optimally. First-order conditions imply

$$\psi'_{\text{firm}}(n) = \frac{1}{n}\pi_{\text{neo}}(n) - \frac{1}{n^2}\int_0^n \pi_{\text{neo}}(j)\,dj = 0 \quad \Leftrightarrow \quad \psi_{\text{firm}}(n) = \pi_{\text{neo}}(n).$$

But of course, the “neoclassical” firm would choose scale (n) to set $\pi'_{\text{neo}}(n) = 0$. The falling marginal cuts the average at its maximum (standard economics intuition) so:

The “bargaining” firm over-hires labour (to bid down the bargained wage).

The Shapley Value and Labour Over-Hire



A Voting Game

"A corporation has four stockholders, holding respectively 10, 20, 30, and 40 shares of stock. A decision is settled by any coalition of shareholders holding a simple majority of the shares."

Players. The players are the stockholders: $N = \{1, 2, 3, 4\}$.

Payoffs. The worth of each coalition $S \subseteq N$ is $v(S) = 0$ except for:

$$v(2,4) = v(3,4) = v(1,2,3) = v(1,2,4) = v(1,3,4) = v(2,3,4) = v(1,2,3,4) = 1.$$

These coalitions might be described as "winning" coalitions. Such games are called "simple".

What is the core of this game? What is the Shapley value?

The core is empty. Consider any payoff profile where each player receives a positive amount. A coalition S where $|S| = 3$ such that $v(S) = 1$ could do better for its members. If only three members receive a positive amount, a coalition S where $|S| = 2$ such that $v(S) = 1$ could do better. But now player 4 can do better by switching to the other two-member coalition such that $v(S) = 1$. Can player 4 receive 1? No. 1,2, and 3 could group together to receive 1.

The Shapley Value of the Voting Game

Start with ψ_1. Notice that player 1 is part of a winning coalition S such that $S\backslash\{1\}$ is losing only for $S = \{1, 2, 3\}$. So $v(S) - v(S\backslash\{1\}) \neq 0$ only for $S = \{1, 2, 3\}$.

Now $|S| = 3$ and $|N| = 4$, so the Shapley value (for player 1) is:

$$\psi_1(N, v) = \sum_{S \in 2^N : 1 \in S} \frac{(|S| - 1)!(|N| - |S|)!}{|N|!} [v(S) - v(S\backslash\{1\})] = \frac{2!1!}{4!} = \frac{1}{12}.$$

Winning coalitions that would lose if player 2 were removed are $\{2, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$. Thus:

$$\psi_2(N, v) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{4}.$$

Calculate ψ_3 and ψ_4 similarly. As a result, the Shapley value is the payoff vector:

$$\psi(N, v) = \left(\frac{1}{12}, \frac{1}{4}, \frac{1}{4}, \frac{5}{12} \right) .$$

General Cooperative Games

Notice that simply thinking about the shares of stock each of the players have gives a vector $(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$. The Shapley value is a measure of the "voting power" the players actually have.

Finally, cooperative games can be generalised for the "non-transferable utility" case:

NTU Cooperative Games. A cooperative game $\mathcal{G} = \langle N, X, V, \{u_i\}_{i \in N} \rangle$ consists of:

1. *Players*. A set of players N with a typical player i.
2. *Consequences*. A set X of consequences (or outcomes).
3. *Consequence Function*. $V(S)$ maps each coalition S to a subset of consequences.
4. *Payoffs*. A payoff function u_i for each player i, defined over the set of consequences.

TU cooperative games may be written as NTU cooperative games. Exchange and production economies may be written in this form, as can bargaining games (e.g. Nash bargaining), and more.

NEXT WEEK: PÉTER ESŐ AND VINCE CRAWFORD. ENJOY THE REST OF THE COURSE!