Additional exercises

A. Prove that there exist positive irrational numbers a and b such that a^b is rational by considering the pair $\sqrt{2}^{\sqrt{2}}$ and $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$.

B. Give an example of two convergent series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ such that $\sum_{k=1}^{\infty} x_k y_k$ diverges. Can this happen if one of the series is absolutely convergent?

C. Prove that if $\sum_{n=1}^{\infty} |a_{mn}|$ is finite for every m and $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} |a_{mn}|)$ is finite, then $\sum_{m=1}^{\infty}(\sum_{n=1}^{\infty} a_{mn}) = \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty} a_{mn})$.

D. *Summation by parts.* Let $A_n = \sum_{k=1}^{n-1} a_k$ and $B_n = \sum_{k=1}^{n} b_k$ (notice that these sums do not have the same number of terms). Suppose the sequence $A_{m+1}B_m$ converges as $m \to \infty$. Show that $\sum_{n=1}^{\infty} a_n B_n$ converges if and only if $\sum_{n=1}^{\infty} A_n b_n$ converges, and relate the two sums (use that $a_n = A_{n+1} - A_n$).

E. Suppose $x_1, x_2, \dots$ and $y_1, y_2, \dots$ are two sequences of rational numbers. Define the shuffled sequence to be $x_1, y_1, x_2, y_2, \dots$. Prove that the shuffled sequence is a Cauchy sequence if and only if $x_1, x_2, \dots$ and $y_1, y_2, \dots$ are equivalent Cauchy sequences.

F. If f is a continuous function, is it necessarily true that $\limsup_{n\to\infty} f(x_n) = f(\limsup_{n\to\infty} x_n)$?

G. A function f is Lipschitz on its domain D if there is an $M > 0$ (independent of x and y) such that $|f(x)-f(y)| \leq M|x-y|$ for all x and y in D. Show that if f and g are bounded and Lipschitz then $f \cdot g$ is also Lipschitz. Give a counterexample to show that it is necessary to assume boundedness.

H. Show that every function that is Lipschitz is also uniformly continuous. Give an example of a function that is uniformly continuous but not Lipschitz.

I. Suppose $f_n \to f$ pointwise and the functions f_n are all Lipschitz with a Lipschitz constant M that does not depend on n (i.e. $|f_n(x) - f_n(y)| \leq M|x-y|$ for all x, y and n). Prove that f is Lipschitz with the same Lipschitz constant.

J. If f is a C^2 function, prove that f cannot have a local maximum or minimum at an inflection point (note that an inflection point is defined as a point where f'' changes sign; it is not enough that f'' vanishes at the point).

K. Let f be a C^1 function on $\mathbb{R}$, and let $g(x) = \int_0^1 f(xy)y^2 dy$. Prove that $g(x)$ is a C^1 function and establish a formula for $g'(x)$ in terms of f.

L. Compute $\frac{d}{dx}(f(x)^{g(x)})$ if f and g are C^1, $f(x) > 0$.

M. Let $A(x)$ be a continuous function. Show that the differential equation $f'(x) = A(x)f(x)$ has solutions $f(x) = C\exp(\int_0^x A(s)ds)$ with $C \in \mathbb{R}$. Show that all solutions are of this form.

N. Let $A(x)$ and $b(x)$ be continuous functions. Define $g(x) = \int_0^x A(s)ds$. Show that the differential equation $f'(x) = A(x)f(x) + b(x)$ has solutions $f(x) = Ce^{g(x)} + e^{g(x)}\int_0^x e^{-g(s)}b(s)ds$ with $C \in \mathbb{R}$. Show that all solutions are of this form.

O. If $f_n \to f$ uniformly and the functions f_n are all Lipschitz (but the Lipschitz constant may depend on n), does this imply that f is also Lipschitz?

P. If f_n is a uniformly equicontinuous sequence of functions on a compact interval and $f_n \to f$ pointwise, prove that f is continuous.

Q. If $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$ for some fixed M and $\alpha > 0$ and all x, y in a compact interval, show that $\{f_n\}$ is uniformly equicontinuous.

R. Let a be a fixed real number with $1 < a < 3$. Prove that the mapping $f(x) = (x/2) + (a/2x)$ satisfies the hypotheses of the contractive mapping principle on the domain $[1, \infty)$. What is the fixed point?

S. Let $T : C([0,1]) \to C([0,1])$ be defined by $Tf(x) = 1 - xf(x)/3 + \int_0^x tf(t)dt$. Prove that T satisfies the hypotheses of the contractive mapping principle. Derive the differential equation satisfied by the fixed point. Conclude that the fixed point is $f(x) = e^{3x}/(1+\frac{x}{3})^{10}$.

T. Show that all solutions of $x''(t) = -x(t)$ are of the form $x(t) = A\cos t + B\sin t$. Using this, decide for which values of t_1 and t_2 the o.d.e. $x''(t) = -x(t)$ with boundary conditions $x(t_1) = a_1$, $x(t_2) = a_2$ has a unique solution on $[t_1, t_2]$ for any choice of a_1, a_2.

U. Consider the o.d.e. $x'(t) = A(t)x(t) + b(t)$, with $x(t_0) = x_0 \in \mathbb{R}^n$ and $A : I \to \mathbb{R}^{n\times n}$ a square matrix of continuous functions and $b : I \to \mathbb{R}^n$ a vector of continuous functions. Prove that this Cauchy problem has a unique solution $x(t)$ on $\mathbb{R}$.

Most of these exercises are taken from "The way of analysis" by Robert S. Strichartz.