Closed characteristics on non-compact mechanical contact manifolds

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Abstract. This paper is concerned with the existence of closed characteristics for a class of non-compact contact manifolds: mechanical contact manifolds. In [3] it was proved that, provided certain geometric assumptions are satisfied, regular mechanical hypersurfaces in \( \mathbb{R}^{2n} \), in particular non-compact ones, contain a closed characteristic if one homology group among the top half does not vanish. In the present paper, we extend the above mentioned existence result to the case of non-compact mechanical contact manifolds via embeddings in cotangent bundles of Riemannian manifolds.

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1. Introduction

The question of existence of periodic orbits of a Hamiltonian vector field $X_H$ on a given regular energy level, i.e. a level set $\Sigma = H^{-1}(0)$ of the Hamiltonian function $H$, with $dH \neq 0$ on $\Sigma$, has been a central question in Hamiltonian dynamics and symplectic topology which has generated some of the most interesting recent developments in those areas. The existence of a periodic orbit does not depend on the Hamiltonian itself, but only on the geometry of the energy level that the Hamiltonian defines. For this reason one also speaks of closed characteristics of the energy level.

After the first pioneering existence results of Rabinowitz [18] and [19] and Weinstein [23] for star-shaped and convex hypersurfaces respectively, Viterbo [21] proved the existence of closed characteristics on all compact hypersurfaces of $\mathbb{R}^{2n}$ of so called contact type. The latter notion was introduced by Weinstein as a generalization of both convex and star-shaped [24]. These first results were obtained by variational methods applied to a suitable (indefinite) action functional.

More recently, Floer, Hofer, Wysocki [6] and Viterbo [20], provided an alternative proof of the same results (and much more) using the powerful tools of symplectic homology or Floer homology for manifolds with boundary.

Up to now though, very little is known about closed characteristics on non-compact energy hypersurfaces: even the Floer homology type of technique mentioned above breaks down when one drops the compactness assumptions. It is clear that additional geometric and topological assumptions are needed in order to make up for the lack of compactness. In [3] we were able to formulate a set of such assumptions that led to an existence result for the case of mechanical hypersurfaces in $\mathbb{R}^{2n}$, that is, hypersurfaces arising as level sets of Hamiltonian functions of the form kinetic plus potential energy.

Mechanical hypersurfaces in cotangent bundles are an important class of contact manifolds since they occur naturally in conservative mechanical dynamics. In the case of compact mechanical hypersurfaces Bolotin [4], Benci [2], and Gluck and Ziller [7] show the existence of a closed characteristic on $\Sigma$ via closed geodesics of the Jacobi metric on the configuration manifold. A more general existence result for cotangent bundles is proved by Hofer and Viterbo in [10] and improved in [22]: Any connected compact hypersurface of contact type over a simply connected manifold has a closed characteristic, which confirms the Weinstein Conjecture in cotangent bundles of simply connected manifolds. However, the existence of closed characteristics for non-compact mechanical
hypersurfaces is not covered by the result of Hofer and Viterbo and fails without additional geometric conditions. In this paper we address the question for non-compact mechanical hypersurfaces, in cotangent bundles over arbitrary non-compact (smooth) Riemannian manifolds \((M, g)\). There is a freedom of choice in the definition of the hypersurface, which we would like to emphasize. This leads to the definition of *mechanical contact manifolds*.

1.1. Mechanical contact manifolds. An odd, say \(2n - 1\), dimensional manifold \(\Sigma\) with a one form \(\Theta\), whose differential \(d\Theta\) has maximal rank, defines dynamical systems in a variational manner. The characteristic line bundle of \((\Sigma, \Theta)\) is defined to be \(\mathcal{L} = \ker d\Theta\). Critical points of the action functional \(\mathcal{A}(\gamma) = \int_{\gamma} \Theta\), defined on the loop space of \(\Sigma\), are exactly the periodic orbits of non-zero sections of \(\mathcal{L}\). The requirement that \(d\Theta\) has maximal rank is weaker than the contact condition, but this setting naturally occurs, for example in fourth order systems [1]. We now introduce a special class of these manifolds, which arise in mechanical Hamiltonian dynamics.

**Definition 1.** Let \(\Sigma\) be a smooth manifold of dimension \(2n - 1\) and let \(\Theta\) be a 1-form on \(\Sigma\). The pair \((\Sigma, \Theta)\) is called a *mechanical contact manifold*, if it is a regular energy level set of a mechanical Hamiltonian, i.e.

(i) There exists an \(n\)-dimensional Riemannian manifold \((N, g)\), possibly with boundary \(\partial N\), and a potential function \(V : N \to \mathbb{R}\), which satisfies

\[
V|_{N \setminus \partial N} < 0, \quad V|_{\partial N} = 0, \quad \text{and} \quad \text{grad } V|_{\partial N} \not= 0.
\]

(ii) There exists an embedding \(j : \Sigma \to T^*N\), such that

\[
j(\Sigma) = \left\{(q, \varphi_q) \in T^*N \left| H(q, \varphi_q) = \frac{1}{2} g^*(\varphi_q, \varphi_q) + V(q) = 0\right.\right\}.
\]

(iii) The dynamics on \(\Sigma\) is induced by the embedding into \(T^*N\), thus \(j^*\Lambda = \Theta\), where \(\Lambda\) is the Liouville form.

The Hamiltonian vector field defined by \(H\), thus \(i_{X_H} d\Lambda = -dH\), restricted to \(\Sigma\), corresponds to a non-zero section of the characteristic line bundle defined by \(\Theta\). Note that we do not require any compactness of \(N\). In Section 7 we describe the topology of mechanical contact manifolds. In the special case that \(\partial N = \emptyset\), then \(\Sigma\) is a sphere bundle over \(N\). If \(\partial N \not= \emptyset\) then \(\Sigma\) is a *pinched sphere bundle*. The fibers over the interior points of \(N\) are spheres, and at the boundary \(\partial N\) the fibers collapse to points, cf. Figure 7.2. Condition (iii) implies that the boundary \(\partial N\) is diffeomorphic to the degeneracy locus \(\Theta = 0\). The nomenclature suggests that the following theorem is true.

**Theorem 1.** A mechanical contact manifold \(\Sigma\) is a contact manifold.
This theorem is well known [11] when $\Sigma$ is compact, and we will discuss its proof in the non-compact case in Section 2. The form $\Theta$ needs to be perturbed by an exact form to make $\Sigma$ a contact manifold, but the existence of closed characteristics does not change under this perturbation.

1.2. Conformal Geometry. The dynamics on $\Sigma$ does not change under specific conformal transformations, which we discuss below. We recall some notions of conformal geometry. Two Riemannian metrics $g$ and $h$ on a manifold $M$ are said to be (locally) conformally equivalent if there exists a smooth function $f : M \rightarrow \mathbb{R}$ such that $h = e^{2f}g$. This defines an equivalence relation on the space on Riemannian metrics on $M$ and a class is called a conformal structure on $M$, which we denote by $[g]$. A conformal structure is also referred to as a conformal metric or conformal class on $M$. A manifold $M$ with a conformal structure $[g]$ is called a conformal (Riemannian) manifold and is denoted by $(M, [g])$.

A conformal structure on $M$ is conformally flat if for each point $q \in M$ there exists a representative $h \in [g]$ which is flat on a neighborhood of $q$, i.e. the curvature tensor vanishes identically. Thus a metric $g$ is conformally flat, if for each $q \in M$, there exists a neighborhood $U \ni q$, and a representative $h \in [g]$ such that $h$ is flat on $U$.

The above mentioned conformally flat structures are also called locally conformally flat as opposed to globally conformally flat structures. For the latter a representative $h \in [g]$ exists for which the curvature tensor vanishes globally. We are interested in a special class of conformal structures, which are conformally flat outside a compact set. A conformal structure $[g]$ on $M$ has conformal flat ends if there exists a compact set $K \subset M$, such that there exists a representative $h \in [g]$ whose curvature tensor vanishes on $M \setminus K$. A representative $h \in [g]$ for which the curvature vanishes on $M \setminus K$ is said to have flat ends. Manifolds that admit conformally flat structures are for instance generalized cylinders $S^k \times \mathbb{R}^\ell$, and in general connected sums of manifolds with conformal flat ends, when the gluing operations are performed in a compact set, admit asymptotically conformally flat structures.

Remark 1. Suppose a conformal structure $[g]$ has a representative $h \in [g]$, which is complete and has flat ends. If $M$ has only a finite number of components, then $(M, h)$ is a Riemannian manifold of bounded geometry, i.e. the injectivity radius of $M$ is positive and the curvature, and its covariant derivatives, are uniformly bounded. A proof of this fact is given by S. Ivanov on MathOverflow [12].

1.3. Conformal freedom. After $N$ is fixed, there is a conformal freedom in choosing the metric $g$ and the potential $V$. A different choice of metric $h$ in the same conformal class $h \in [g]$, thus $h = e^{2f}g$ for some function $f$, induces the same energy level if the potential $V$ is replaced by the potential $e^{-2f}V$. The existence of closed characteristics does not depend on this choice.
For the variational approach we need to embed $N$ into a manifold without boundary $M$, and we require that the metric and potential function extend to $M$ in a compatible manner.

**Definition 2.** Let $N$ be a $n$-dimensional manifold with boundary, with metric $g_N$, and potential $V_N$, and $M$ a complete $n$-dimensional orientable manifold with without boundary, with metric $g_M$, and potential $V_M$. An embedding $i : N \to M$, is called *admissible* if there exists a function $f : M \to \mathbb{R}$ such that $g_N = i^*(e^{-2f}g_M)$, and $V_N = i^*(e^{2f}V_M)$. Outside of $i(N)$ we demand the potential $V_M$ to be positive.

We will also call the induced embedding $\iota : \Sigma \to T^*M$ admissible. An admissible embedding always exists, by attaching a collar along $\partial N$, and extending the metric $g_M$ and potential $V_M$ via partitions of unity.

In [3] the geometric assumption *asymptotic regularity* was introduced for mechanical hypersurfaces in $\mathbb{R}^{2n}$. This geometric assumption controls the asymptotic behavior of the potential at infinity. This condition can be formulated for mechanical contact manifolds.

**Definition 3.** Let $\Sigma$ be a mechanical contact manifold. An admissible embedding $\iota : \Sigma \to T^*M$, is *asymptotically regular* if

i) The metric $g$ on $M$ has flat ends.

ii) The potential function $V$ on $M$ satisfies

$$\|\text{grad } V(q)\| \geq V_\infty, \text{ for } q \in M \setminus K \text{ and } \frac{\|\text{Hess } V(q)\|}{\|\text{grad } V(q)\|} \to 0, \text{ as } d_M(q, K) \to \infty,$$

for a compact set $K \subset M$ and a constant $V_\infty > 0$.

We will say that $\Sigma$ is asymptotically regular if an asymptotically regular embedding exists. We will also call the potential $V$ asymptotically regular. In the above notation grad $V$ is the gradient vector field of $V$ and Hess $V$ is the *Hessian* 2-tensor field defined by $\text{Hess } V(x, y) := (\nabla dv)(y) = x \cdot (y \cdot V) - (\nabla_x y) \cdot V$, for vector fields $x, y$ on $M$, where $\nabla$ is the Levi-Civita connection. The norm $\|\text{grad } V(q)\|$ on $T_qM$ is defined via $g$ and $\|\text{Hess } V(q)\| = \sup_{\|x\|=1,\|y\|=1} |\text{Hess } V(q)(x, y)|$.

When we consider mechanical contact manifolds we have a number of parameters that we can choose, which have no influence on the existence of closed characteristics:

- the embedding $i : N \to M$;
- the extension of the metric $g_N$ to a compatible complete structure on $M$ with conformal flat ends;
- the choice of extension of the potential $V_N$ to the potential $V$ to all of $M$.

We are now able to state the main result of this paper:

**Theorem 2.** Let $(\Sigma, \Theta)$ be a mechanical contact manifold. Assume there exists an asymptotically regular embedding $\iota : \Sigma \to T^*M$, and an integer $0 \leq k \leq n - 1$, such that
Assume and Theorem 3. Theorem 2 reduces to: 

\[ H_{k+1}(\Lambda M) = 0 \text{ and } H_{k+2}(\Lambda M) = 0, \text{ and} \]

\[ H_k(\Sigma) \neq 0. \]

Then \( \Sigma \) contains a closed characteristic, which is contractible in \( M \).

In this theorem \( \Lambda M \) denotes the free loop space of \( H^1 \) loops into \( M \). Functional analytical properties of the loop space are discussed in Section 3 and some topological and geometrical properties of the loop space are discussed in Section 6.

The proof of Theorem 2 follows the scheme of the proof of the existence result for non-compact hypersurfaces in \( \mathbb{R}^{2n} \) presented in [3]. We regard closed characteristics as critical points of a suitable action functional \( \mathcal{A} \). The functional does not satisfy the Palais-Smale condition. Therefore we introduce a sequence of approximating functionals \( \mathcal{A}_\varepsilon \), for \( \varepsilon > 0 \), and we prove that these functionals satisfy the Palais-Smale condition, by using the assumptions of asymptotic regularity. Next, based on the assumptions on the topology of \( \Sigma \) and \( M \), we construct linking sets in \( M \) and lift these to linking sets in the free loop space, where we apply a linking argument to produce critical points for the approximating functionals. Finally, we show that these critical points converge to a critical point of \( \mathcal{A} \) as \( \varepsilon \to 0 \). Because we construct the linking sets in the component of the loop space containing the contractible loops, we find contractible loops. In this paper we choose not to consider non-contractible loops.

One of the main issues in cotangent bundles (as opposed to \( \mathbb{R}^{2n} \)) is that the functional is defined on a Hilbert manifold rather than a Hilbert space. Another difficulty is that curvature terms appear in the analysis of the functional, which require some care.

Theorem 2 directly generalizes the results of [3]. Suppose that \( \Sigma \) admits an asymptotically regular embedding into \( T^*M \), with \( M = \mathbb{R}^n \). The standard metric on \( \mathbb{R}^n \) is flat, and the the loop space is a Hilbert space, hence contractible. Thus \( H_{k+1}(\Lambda M) = H_{k+2}(\Lambda M) = 0 \) for all \( k \geq 0 \), and we obtain the theorem as stated in [3]. In [3] examples are given that show that both topological and geometric assumptions on \( \Sigma \) are necessary. Theorem 2 also improves the result in the \( \mathbb{R}^n \) case, because it requires weaker assumptions on the metric.

Often \( \Sigma \) is directly given as the zero level set of a Hamiltonian \( H \). In this we can state Theorem 2 reduces to:

**Theorem 3.** Let \((M, g)\) be an \( n \)-dimensional complete orientable Riemannian manifold and \( H : T^*M \to \mathbb{R} \) the Hamiltonian defined by

\[
H(q, \vartheta_q) = \frac{1}{2} g^*(\vartheta_q, \vartheta_q) + V(q).
\]

Assume \( dH \neq 0 \) on \( \Sigma \), and assume \( V \) is asymptotically regular, i.e. there exist a compact \( K \subset M \), and a constant \( V_\infty > 0 \), such that

\[
|\text{grad } V(q)| \geq V_\infty, \text{ for } q \in M \setminus K \quad \text{and} \quad \frac{\|\text{Hess } V(q)\|}{|\text{grad } V(q)|} \to 0, \text{ as } d(q, K) \to \infty.
\]
Assume moreover that there exists an integer \(0 \leq k \leq n - 1\) such that

(i) \(H_{k+1}(\Lambda M) = 0\) and \(H_{k+2}(\Lambda M) = 0\), and  
(ii) \(H_{k+n}(\Sigma) \neq 0\).

Then there is a periodic orbit, contractible in \(M\), with energy 0.

1.4. Some examples. We give an example of a manifold with non-trivial topology that satisfies the conditions of Theorem 2. Let \(M = \mathbb{R}^6 - \{\text{pt}\} \cong \mathbb{S}^5 \times \mathbb{R}\). Let \((\varphi, r)\) be coordinates on \(\mathbb{S}^5 \times \mathbb{R}\), and define the metric on \(M\) via: \(g_M = f(r)g_{\mathbb{S}^5} + dr^2\), with \(f\) equal to \(r^2\) outside \([-1, 1]\) and positive everywhere, and \(g_{\mathbb{S}^5}\) is the round metric on the 5-sphere. This manifold is asymptotically conformally flat and admits an asymptotically flat and complete representative via \(g_M\) itself. This manifold can also be visualized as two copies of \(\mathbb{R}^6\), with a ball of radius 1 cut out. These are then connected via a tube \(\mathbb{S}^5 \times [0, 1]\), cf. Figure 1.4. In these coordinates we take the potential function

\[
V(q) = \frac{\rho(q)}{2}(q_1^2 + q_2^2 - q_3^2 - q_4^2 - q_5^2 - q_6^2) - C,
\]

with \(\rho\) equal to zero within the ball of radius 1 and equal to 1 outside the ball of radius 2. For \(C > 0\) chosen large enough, we can study the topology of \(\Sigma\). The topology of the projection of \(\Sigma\) to the base manifold, and its boundary

\[
N = \pi(\Sigma) = \{q \in M \mid V(q) \leq 0\}, \quad \text{and} \quad \partial N = \{q \in M \mid V(q) = 0\},
\]

are easily visualized, \(N\) deformation retracts to \(\mathbb{S}^5\), and the boundary \(\partial N\) deformation retracts to two disjoint circles \(\mathbb{S}^1\). From this, using Proposition 20, we can compute the the homology of \(\Sigma\). From the long exact sequence of the pair we read off (for \(k = 1\)) that \(H_2(N, \partial N) = \mathbb{Z}^2\), and hence \(H_1(\Sigma) \neq 0\). The free loop space of \(M\) is homotopic to the
free loop space of the 5-sphere. This is well studied, and from the path and loop space fibrations, and a spectral sequence argument, we see that $H_2(\Lambda M) = H_3(\Lambda M) = 0$. Thus $\Sigma$ contains a contractible closed characteristic.

We now study another example. Consider the question of existence of periodic orbits of the Hamiltonian system on the cylinder $M = S^1 \times \mathbb{R}$, with an asymptotically regular potential $V$, which is negative on $N$, a set homeomorphic to $S^1 \times [0, \frac{1}{2}] \times \mathbb{R}_{>0}$, where the corners are sufficiently smoothed, cf. Figure 1.5. The boundary $\partial N$ consists of two components, one diffeomorphic to $\mathbb{R}$ and one component diffeomorphic to $S^1$. Using Proposition 20 we find that $H_{k+n}(\Sigma) \neq 0$, for $k = 0$. The free loop space of $M$ is homotopic to $S^1 \wedge \mathbb{Z}$, and we see that $H_1(\Lambda M) \neq 0$. It seems Theorem 2 is not applicable. However, it is not hard to find an admissible embedding of $\Sigma$ into $T^*\mathbb{R}^2$, cf. Figure 1.5, which shows the existence of closed characteristics on $\Sigma$.

1.5. Convention. For the remainder of the paper, we assume an admissible embedding $\iota : \Sigma \to T^*M$ is given, which always exists by the remark after Definition 2, and we identify $\Sigma$ with its image $\iota \Sigma$. After the proof of Theorem 1 in Section 2 we will always assume that this embedding is asymptotically regular. The Hamiltonian defined by the embedding is

$$H(q, \vartheta_q) = \frac{1}{2} \vartheta_q^a(q, \vartheta_q) + V(q).$$

Furthermore, without loss of generality, we assume $M$ to be connected, so that it is of bounded geometry. If $M$ is not connected, each component of $M$ is of bounded geometry. The arguments in the proofs go through ad verbatim.

2. Mechanical Contact Manifolds

In this section we prove Theorem 1. We give two proofs. One proof shows that a mechanical contact manifold always is of contact type. If the mechanical contact manifold
is asymptotically regular, it is possible to write down an explicit contact form which has good asymptotic properties at infinity. This is the content of Proposition 4.

Let us fix some notation. We will use vectors and covectors on the base manifold, as well as vectors and covectors on the cotangent bundle. In an attempt to reduce possible confusion, we denote elements of these bundles by different fonts. We use lowercase roman for vectors, thus a vector on \( M \) is denoted by \( x = (q, x_q) \in TM \), where \( x_q \in T_q M \). A covector on \( M \) is denoted in lowercase greek, i.e. \( \vartheta = (q, \vartheta_q) \in T^*M \) with \( \vartheta_q \in T^*_q M \). Vectors on the cotangent bundle, e.g. the Hamiltonian vector field \( X_H \), are denoted in uppercase roman. In uppercase greek we denote covectors on \( T^*M \), e.g. the Liouville form \( \Lambda \). The Riemannian metric \( g \) on \( M \) gives the musical isomorphism \( \flat : TM \to T^*M \), which is defined by

\[
\flat(q, x_q) = (q, g_q(x_q, \cdot)),
\]

where \( g_q(x_q, \cdot) \in T^*_q M \). Its inverse is \( \# : T^*M \to TM \). We also write \( \flat \) and \( \# \) for the fiber-wise isomorphisms. The Riemannian metric \( g \) induces a non-degenerate symmetric bilinear form \( g^* \) on \( T^*M \) via

\[
g^*_q(\vartheta_q, \vartheta'_q) = g_q(x_q, x'_q),
\]

where \( (q, x_q) = \#(q, \vartheta_q) \) and \( (q, x'_q) = \#(q, \vartheta'_q) \). By abuse of notation, both for \( (q, x_q), (q, x'_q) \in TM \) and for \( (q, \vartheta_q), (q, \vartheta'_q) \in T^*M \) we write

\[
|x_q|^2 = g_q(x_q, x_q), \quad \langle x_q, x'_q \rangle = g_q(x_q, x'_q), \quad |\vartheta_q|^2 = g^*_q(\vartheta_q, \vartheta_q), \quad \langle \vartheta_q, \vartheta'_q \rangle = g^*_q(\vartheta_q, \vartheta'_q).
\]

We need some standard constructions in symplectic geometry, which can be found for example in Hofer and Zehnder [11]. The tangent map of the bundle projection \( \pi : T^*M \to M \) is a mapping \( T \pi : TT^*M \to TM \). The Liouville form \( \Lambda \), a one-form on \( T^*M \), is defined by

\[
\Lambda_{(q, \vartheta_q)} = \vartheta_q \circ T_{(q, \vartheta_q)} \pi.
\]

The cotangent bundle \( T^*M \) has a canonical symplectic structure via the Liouville form given by the 2-form \( \Omega = d\Lambda \). Recall the definition of a contact type hypersurface.

**Definition 4.** A hypersurface \( \iota : \Sigma \to T^*M \) is of contact type if there exists a 1-form \( \Theta \) on \( \Sigma \) such that

(i) \( d\Theta = \iota^* \Omega \),

(ii) \( \Theta(X) \not= 0 \), for all \( X \in \mathcal{L}_\Sigma \) with \( X \not= 0 \),

where \( \mathcal{L}_\Sigma \) is the characteristic line bundle for \( \Sigma \) in \( (T^*M, \Omega) \), defined by \( \mathcal{L}_\Sigma = \ker \iota^* \Omega \).

The Hamiltonian vector field \( X_H \) on \( \Sigma \), is defined by the relation

\[
i_{X_H} \Omega = -dH,
\]

and is a smooth non-vanishing section in the bundle \( \mathcal{L}_\Sigma \).
Theorem 1 states that a mechanical contact manifold always is of contact type. The proof is an adaptation of [1, Lemma 3.3], which is in turn inspired by a contact type theorem for mechanical hypersurfaces in $\mathbb{R}^{2n}$ in [11].

**Proof of Theorem 1.** The restriction of the Liouville form $i^*(\Lambda)$ to $\Sigma$ vanishes on

$$S = \{(q, \partial_q) \in \Sigma \mid \partial_q = 0\},$$

which is a submanifold of $\Sigma$ of dimension $n - 1$. The form $i^*\Lambda$ satisfies condition (i) of Definition 4, but does not satisfy condition (ii). Any exact perturbation $\Theta = i^*\Lambda + df$, for functions $f : \Sigma \to \mathbb{R}$, obviously satisfies condition (i) as well. We now construct an $f$ such that the condition (ii) is also satisfied. This is first done locally. We observe that $i_X i^*\Lambda \geq 0$ for all positive multiples $X$ of $X_H$. We therefore only need to prove that $i_X H \Theta > 0$.

Let $x \in S$. Because $\Sigma$ is a regular hypersurface, that is $dH \neq 0$ on $\Sigma$, the Hamiltonian vector field is transverse to $S$, i.e. $X_H \notin TS$. Thus there exists a chart $U \ni x$, with coordinates $y = (y_1, \ldots, y_{2n-1})$, with $S \cap U = \{(y_1, \ldots, y_{2n-2}, 0)\}$, and $X_H = \frac{\partial}{\partial y_{2n-1}}$. Construct a smooth and positive bump function $h$ around $x$, supported in $U$, which is a product of different bump functions of the form

$$h(y) = h_1(y_{2n-1}) h_2(y_1, \ldots, y_{2n-2}).$$

Assume $h$ is equal to 1 on a neighborhood $W \subset U$ of $x$. The function $f_x : \Sigma \to \mathbb{R}$ defined by

$$f_x = \begin{cases} y_{2n-2} h(y) & \text{for } y \in U \\ 0 & \text{otherwise,} \end{cases}$$

then satisfies

$$i_X H d f_x = h + y_{2n-1} \frac{\partial h_1}{\partial y_{2n-1}} h_2 \quad \text{on } U.$$

Because $y_{2n-1} = 0$ on $S$, this shows that $i_X H d f_x \geq 0$ on $S$, and equal to 1 at $x$, hence positive on a small neighborhood around $x$. By the product structure of the bump function $i_X H d f_x < 0$ only on a compact set outside a neighborhood of $S$. The quantity

$$A_x = \sup_{\Sigma \cap S} \frac{\max(-i_X H d f_x, 0)}{i_X H i^* \Lambda},$$

is therefore finite and non-zero.

Now we patch the local constructions together. Take a countable and dense collection of points $\{x_k\}_{k \geq 1}$ in $S$, and construct $f_{x_k}$ as above. Define $f : \Sigma \to \mathbb{R}$ via

$$f = \frac{1}{2} \sum_{k=1}^{\infty} 2^{-k} \frac{f_{x_k}}{A_k + \|f_{x_k}\| C^k}.$$
This series converges in $C^\infty$ to a smooth function. On $S$ we have $i_{X_H} df x_k \geq 0$ for all $k$, and for any $x \in S$, there exists a $k$ such that $i_{X_H} f x_k(x) > 0$ by the density of the sequence $\{x_k\}_{k=1}^\infty$. Thus $i_{X_H} f |S) > 0$. We compute this quantity outside of $S$

$$-i_{X_H} df = \frac{1}{2} \sum_{k=1}^\infty 2^{-k} \frac{-i_{X_H} df x_k}{A_k + \|f x_k\|_{C_k}},$$

$$\leq \frac{1}{2} \sum_{k=1}^\infty 2^{-k} \max(-i_{X_H} df x_k, 0),$$

$$\leq \frac{1}{2} \sum_{k=1}^\infty 2^{-k} i_{X_H} \Lambda < \frac{1}{2} i_{X_H} \Lambda.$$  

On $\Sigma \setminus S$, we therefore have that $i_{X_H} df \geq -\frac{1}{2} i_{X_H} \Lambda$. From this we see that $i_{X_H} \Theta = i_{X_H} (\star \Lambda + df) \geq 0$ on $\Sigma$. Thus $\Theta$ is a contact form.

Theorem 1 shows that the contact type condition is a general property of mechanical contact manifolds, regardless of asymptotic behavior at infinity. In the case that $\Sigma$ is asymptotically regular, and an asymptotically regular embedding has been chosen, an explicit contact form can be constructed and a stronger contact type condition holds. Consider the vector field:

$$v(q) = -\frac{\text{grad } V(q)}{1 + |\text{grad } V(q)|^2},$$

and the function $f : T^* M \to \mathbb{R}$ defined by $f(x) = \vartheta_q(v(q))$, for all $x = (q, \vartheta_q) \in T^* M$. For $\kappa > 0$, define the 1-form $\Theta = \Lambda + \kappa df$ on $T^* M$. Clearly, $d\Theta = \Omega$. Define the energy surfaces $\Sigma_\epsilon = \{x \in T^* M \mid H(x) = \epsilon\}$.

**Proposition 4.** Let $\Sigma \to T^* M$ be an asymptotically regular embedding. There exists $\epsilon_0, \kappa_0 > 0$, such that for every $-\epsilon_0 < \epsilon < \epsilon_0$, $\Theta = \Lambda + \kappa df$ restricts to a contact form on $\Sigma_\epsilon$, for all $0 < \kappa \leq \kappa_0$. Moreover, for every $\kappa$, there exists a constant $a_\kappa > 0$ such that

$$\Theta(X_H) \geq a_\kappa > 0, \quad \text{for all } x \in \Sigma_\epsilon \quad \text{and for all } -\epsilon_0 < \epsilon < \epsilon_0.$$

The energy surfaces $\Sigma_\epsilon$ are said to be of uniform contact type.

**Proof.** A tedious, but straightforward, computation, which we have moved to an appendix, Section 12, reveals that

$$X_H(f)(q, \vartheta_q) = \frac{|\text{grad } V(q)|^2}{1 + |\text{grad } V(q)|^2} - \frac{\text{Hess } V(q)(\# \vartheta_q, \# \vartheta_q)}{1 + |\text{grad } V(q)|^2}$$

$$+ \frac{2 \vartheta_q(\text{grad } V(q)) \text{Hess } V(q)(\text{grad } V, \# \vartheta_q)}{(1 + |\text{grad } V(q)|^2)^2}.$$
The reverse triangle inequality, and Cauchy-Schwarz directly give
\[
X_H(f) \geq \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} - \left( \frac{\|\text{Hess } V(q)\| \|\vartheta_q\|^2}{1 + \|\text{grad } V(q)\|^2} + \frac{2 \|\text{Hess } V(q)\| \|\text{grad } V(q)\|^2 \|\vartheta_q\|^2}{(1 + \|\text{grad } V(q)\|^2)^2} \right)
\]
\[
\geq \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} - \frac{3 \|\text{Hess } V(q)\| \|\vartheta_q\|^2}{1 + \|\text{grad } V(q)\|^2}.
\]
By asymptotic regularity there exists a constant \(C\) such that \(\frac{3 \|\text{Hess } V(q)\|}{1 + \|\text{grad } V(q)\|^2} \leq C\), hence
\[
X_H(f) \geq \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} - C \|\vartheta_q\|^2.
\]
This yields the following global estimate
\[
\Theta_x(X_H)(q, \vartheta_q) \geq \left( 1 - \kappa C \right) \|\vartheta_q\|^2 + \kappa \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2}
\]
\[
\geq \frac{1}{2} \|\vartheta_q\|^2 + \kappa \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} > 0, \quad \text{for all } x \in T^*M,
\]
for all \(0 < \kappa \leq \kappa_0 = 1/2C\). The final step is to establish a uniform positive lower bound on \(a_\kappa\) for \((q, \vartheta_q) \in \Sigma_\epsilon\), independent of \((q, \vartheta_q)\) and \(\epsilon\).

If \(d_M(q, K) \geq R\) is sufficiently large, then asymptotic regularity gives that \(\|\text{grad } V(q)\| > V_\infty\). Thus, in this region,
\[
\frac{1}{2} \|\vartheta_q\|^2 + \kappa \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} \geq \kappa \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} \geq \frac{\kappa V_0^2}{1 + V_\infty^2}.
\]
On \(d_M(q, K) < R\) we can use standard compactness arguments. For \((q, \vartheta_q) \in \Sigma_\epsilon\), we have the energy identity
\[
\frac{1}{2} \|\vartheta_q\|^2 + V(q) = \epsilon.
\]
Suppose that \(\frac{1}{2} \|\vartheta_q\|^2 < \epsilon_0\), then \(|V(q)| < \epsilon + \epsilon_0\). If \(\epsilon_0\) is sufficiently small, this implies that \(\|\text{grad } V(q)\| \geq V_0 > 0\) for some constant \(V_0\), because \(\text{grad } V \not\equiv 0\) at \(V(q) = 0\). Therefore in this case
\[
\frac{1}{2} \|\vartheta_q\|^2 + \kappa \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} > \frac{\kappa V_0^2}{1 + V_\infty^2}.
\]
If \(\|\vartheta_q\|^2 > \epsilon_0\), then obviously
\[
\frac{1}{2} \|\vartheta_q\|^2 + \kappa \frac{\|\text{grad } V(q)\|^2}{1 + \|\text{grad } V(q)\|^2} > \epsilon_0.
\]
We have exhausted all possibilities and
\[
a_\kappa = \min \left( \frac{\kappa V_\infty^2}{1 + V_\infty^2}, \frac{\kappa V_0^2}{1 + V_\infty^2}, \epsilon_0 \right) > 0,
\]
is a uniform lower bound for \(i_{X_H} \Theta\).
3. The Variational Setting

Closed characteristics on $M$ can be regarded as critical points of the action functional

$$\mathcal{B}(q, T) = \int_0^T \left\{ \frac{1}{2} |q'(t)|^2 - V(q(t)) \right\} \, dt,$$

for mappings $q : [0, T] \to M$. Via the coordinate transformation

$$(q(t), T) \mapsto (c(s), \tau) = (q(sT), \log(T)).$$

we obtain the rescaled action functional

$$\mathcal{A}(c, \tau) = \frac{e^{-\tau}}{2} \int_0^1 |c'(s)|^2 \, ds - e^\tau \int_0^1 V(c(s)) \, ds,$$

for mappings $c : [0, 1] \to M$ and $\tau \in \mathbb{R}$. For closed loops we impose the boundary condition $c(0) = c(1)$. By defining the parametrized circle $S^1 = [0, 1]/\{0, 1\}$, the mappings $c : S^1 \to M$ satisfy the appropriate boundary condition.

In order to treat closed characteristics as critical points of the action functional $\mathcal{A}$ we need an appropriate functional analytic setting. We briefly recall this setting. Details can be found in the books of Klingenberg [13] and [14]. A map $c : S^1 \to M$ is called $H^1$ if it is absolutely continuous and the derivative is square integrable with respect to the Riemannian metric $g$ on $M$, i.e. $\int_0^1 |c'(s)|^2 \, ds < \infty$. The space of $H^1$ maps is denoted by $H^1(S^1, M)$, or $\Lambda M$, the free loop space of $H^1$-loops. An equivalent way to define $\Lambda M$ is to consider continuous curves $c : S^1 \to M$ such that for any chart $(U, \varphi)$ of $M$, the function $\varphi \circ c : I = e^{-1}(U) \to \mathbb{R}^n$ is in $H^1(I, \mathbb{R}^n)$. There is a natural sequence of continuous inclusions

$$C^\infty(S^1, M) \subset H^1(S^1, M) \subset C^0(S^1, M),$$

and the free loop space $C^0(S^1, M)$ is complete with respect to the metric $d_{C^0}(c, \bar{c}) = \sup_{s \in S^1} d_M(c(s), \bar{c}(s))$, where $d_M$ is the metric on $M$ induced by $g$. The smooth loops $C^\infty(S^1, M)$ are dense in $C^0(S^1, M)$. The free loop space $\Lambda M = H^1(S^1, M)$ can be given the structure of a Hilbert manifold. Let $c \in C^\infty(S^1, M)$ and denote by $C^\infty(c^*TM)$ the space of smooth sections in the pull-back bundle $c^*TM \to S^1$. For sections $\xi, \eta \in C^\infty(c^*TM)$ consider the norms and inner products:

$$\|\xi\|_{C^0} = \sup_{s \in S^1} |\xi(s)|,$$

$$\langle \xi, \eta \rangle_{L^2} = \int_0^1 \langle \xi(s), \eta(s) \rangle \, ds,$$

$$\langle \xi, \eta \rangle_{H^1} = \int_0^1 \langle \xi(s), \eta(s) \rangle \, ds + \int_0^1 \langle \nabla_s \xi(s), \nabla_s \eta(s) \rangle \, ds,$$
with $\nabla_s$ the induced connection on the pull-back bundle (from the Levi-Civita connection on $M$). The spaces $C^0(e^*TM)$, $L^2(e^*TM)$, and $H^1(e^*TM)$ are the completions of $C^\infty(e^*TM)$ with respect to the norms $\|\cdot\|_0$, $\|\cdot\|_2$, and $\|\cdot\|_{H^1}$, respectively. The loop space $\Lambda M$ is a smooth Hilbert manifold locally modeled over the Hilbert spaces $H^1(e^*TM)$, with $c$ any smooth loop in $M$. For each smooth loop $c$, the space $H^1(e^*TM)$ is a separable Hilbert space, hence these are all isomorphic. The tangent space $T_c\Lambda M$ at a smooth loop $c$ consists of $H^1$ vector fields along the loop and is canonically isomorphic to $H^1(e^*TM)$.

For fixed $\tau \in \mathbb{R}$ the functional is well-defined on the loop space $\Lambda M$. The kinetic energy term $E'(c) := \frac{1}{2} \int_0^1 |c'(s)|^2 ds$ is well-defined and continuous. The embedding of $\Lambda M$ into $C^0(S^1, M)$ and the continuity of the potential $V$ imply that the potential energy is also well-defined and continuous. The latter implies the continuity of the functional $\mathcal{A} : \Lambda M \times \mathbb{R} \to \mathbb{R}$. We now discuss differentiability and the first variation formula. The details are in the book of Klingenberg [14].

We also denote the weak derivative of $c \in \Lambda M$ by $\partial c = c'$. It is possible to extend the Levi-Civita connection on $\Lambda M$ to differentiate $\xi \in H^1(e^*TM)$ with respect to $\eta \in L^2(e^*TM)$. We denote this connection with the same symbol $\nabla_\eta \xi$, and we write, for $\xi \in T_c\Lambda M$,

$$\nabla \xi = \nabla_{\partial c} \xi.$$ 

In general $\nabla \xi \in L^2(e^*TM)$. If $\xi$ and $c$ are smooth, then $\nabla \xi(s) = \nabla_s \xi(s)$. For $c \in \Lambda M$, and $\xi, \eta \in T_c\Lambda M$, the metric on $\Lambda M$ can be written as

$$\langle \xi, \eta \rangle = \langle \xi, \eta \rangle_{L^2} + \langle \nabla \xi, \nabla \eta \rangle_{L^2}.$$ 

Observe that the kinetic energy is given by $E'(c) = \frac{1}{2} \|\partial c\|_{L^2}^2$. Let $\xi \in T_c\Lambda M$ and $\gamma : (-\varepsilon, \varepsilon) \to \Lambda M$ a smooth curve with $\gamma(0) = c$ and $\gamma'(0) = \xi$. Then,

$$\left. \frac{d}{dt} E'(\gamma(t)) \right|_{t=0} = \langle \partial \gamma(t), \nabla_{\partial \gamma(t)} \frac{\partial \gamma(t)}{\partial t} \rangle_{L^2} \bigg|_{t=0} = \langle \partial c, \nabla \xi \rangle_{L^2}.$$ 

For the function $W(c) = \int_0^1 V(c(s)) ds$ a similar calculation gives:

$$\left. \frac{d}{dt} W(\gamma(t)) \right|_{t=0} = \langle \text{grad } V \circ c, \xi \rangle_{L^2}.$$ 

For convenience we write $\mathcal{A}(c, \tau) = e^{-\tau} E'(c) - e^\tau W(c)$. 


Lemma 5. The action $A : \Lambda M \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. For any $(\xi, \sigma) \in T_{(c, \sigma)} \Lambda M \times \mathbb{R}$ the first variation is given by
\[
d A(c, \tau)(\xi, \sigma) = e^{-\tau} \int_0^1 \langle \partial c(s), \nabla \xi(s) \rangle ds - e^{\tau} \int_0^1 \langle \text{grad} V(c(s)), \xi(s) \rangle ds
- \int_0^1 \left[ \frac{e^{-\tau}}{2} |\partial c(s)|^2 + e^{\tau} V(c(s)) \right] \sigma ds
= d_c A(c, \tau) \xi - \left( e^{-\tau} \mathcal{E}(c) + e^{\tau} W(c) \right) \sigma,
\]
where the gradient $\text{grad} V$ is taken with respect to the metric $g$ on $M$.

Proof. The first term in $A$ is $e^{-\tau} \mathcal{E}(c)$ of which the derivative is given by
\[
- e^{-\tau} \mathcal{E}(c) \sigma + e^{-\tau} d\mathcal{E}(c) \xi = - e^{-\tau} \mathcal{E}(c) \sigma + e^{-\tau} \langle \partial c, \nabla \xi \rangle_{L^2}.
\]
Writing out the $L^2$-metric gives the desired result. The second and third term follow directly from the definition of derivative.

A critical point of $A$ is a pair $(c, \tau) \in \Lambda M \times \mathbb{R}$ such that $d A(c, \tau) = 0$. Critical points are contained in $C^\infty(S^1, M) \times \mathbb{R}$ and satisfy the second order equation $e^{-\tau} \nabla_x c'(s) + e^{\tau} \text{grad} V(c(s)) = 0$. One can see this by taking variations of the form $(\xi, 0)$. Consider the expression $H(s) = \frac{e^{-2\tau}}{2} |c'(s)|^2 + V(c(s))$, then
\[
\frac{d}{ds} H(s) = e^{-2\tau} \langle c'(s), \nabla_x c'(s) \rangle + \langle \text{grad} V(c(s)), c'(s) \rangle = 0,
\]
which implies that $H(s)$ is constant. From the first variation formula, with variations $(0, \sigma)$ it follows that $\int_0^1 H(s) ds = 0$, and therefore $H \equiv 0$ for critical points of $A$.

Since $\Lambda M \times \mathbb{R}$ has a Riemannian metric we can define the gradient of $A$. The metric on $\Lambda M \times \mathbb{R}$ is denoted by $\langle \cdot, \cdot \rangle_{H^1 \times \mathbb{R}} = \langle \cdot, \cdot \rangle_{H^1} + \langle \cdot, \cdot \rangle_{\mathbb{R}}$. The gradient $\text{grad} A(c, \tau)$ is the unique vector such that
\[
\langle \text{grad} A(c, \tau) \xi, \sigma \rangle_{H^1 \times \mathbb{R}} = d A(c, \tau) \xi, \sigma,
\]
for all $(\xi, \sigma) \in T_{(c, \sigma)} \Lambda M \times \mathbb{R}$, and $\text{grad} A$ defines a vector field on $\Lambda M \times \mathbb{R}$.

We conclude with some basic inequalities for the various metrics which will be used in this analysis. The proofs are in Klingenberg [14]. Let $c, e \in \Lambda M$, then
\[
d_M(c(s), c(s')) \leq \sqrt{|s - s'| \sqrt{2} \delta(c)},
\]
\[
d_{C^0}(c, e) \leq \sqrt{2} d_{\Lambda M}(c, e),
\]
where $d_{\Lambda M}$ is the metric induced by the Riemannian metric on $\Lambda M$, and $d_{C^0}(c, e) = \sup_{s \in S^1} d_M(c(s), e(s))$. Furthermore, for $\xi \in T_c \Lambda M$, we have
\[
\|\xi\|_{L^2} \leq \|\xi\|_{C^0} \leq \sqrt{2} \|\xi\|_{H^1}.
\]
4. THE PALAIS-SMALE CONDITION

The functional $\mathcal{A}$ does not satisfy the Palais-Smale condition. We therefore approximate this functional by functionals $\mathcal{A}_\varepsilon$, and show that the approximating functionals do satisfy PS. We then find critical points of the approximating functionals using a linking argument. Finally we show that these critical points converge to a critical point of $\mathcal{A}$ as $\varepsilon \to 0$. The approximating, or penalized, functionals are defined by

$$\mathcal{A}_\varepsilon(c, \tau) = \mathcal{A}(c, \tau) + \varepsilon(e^{-\tau} + e^{\tau/2}).$$

The term $\varepsilon e^{-\tau}$ penalizes orbits with short periods, and $\varepsilon e^{\tau/2}$ penalizes orbits with long periods. Recall, that for $\varepsilon > 0$ fixed, a sequence $\{(c_n, \tau_n)\} \in \Lambda M \times \mathbb{R}$ is called a Palais-Smale sequence for $\mathcal{A}_\varepsilon$, if:

(i) there exist constants $a_1, a_2 > 0$ such that $a_1 \leq \mathcal{A}_\varepsilon(c_n, \tau_n) \leq a_2$;

(ii) $d\mathcal{A}_\varepsilon(c_n, \tau_n) \to 0$ as $n$ tends to $\infty$.

Condition (ii) can be equivalently rewritten as

$$d\mathcal{A}_\varepsilon(c_n, \tau_n)(\xi, \sigma) = \langle \text{grad} \mathcal{A}_\varepsilon(c_n, \tau_n), (\xi, \sigma) \rangle_{H^1 \times \mathbb{R}} = o(1)\left(\|\xi\|_{H^1} + |\sigma|\right), \quad (4.1)$$

as $n \to \infty$ and $(\xi, \sigma) \in T_{(c_n, \tau_n)} \Lambda M \times \mathbb{R}$. Condition (i) implies that, by passing to a subsequence if necessary, $\mathcal{A}_\varepsilon(c_n, \tau_n) \to a_\varepsilon$, with $a_1 \leq a_\varepsilon \leq a_2$. In what follows we tacitly assume we have passed to such a subsequence.

**Remark 2.** We will only consider Palais-Smale sequences that are positive, thus $a_1 > 0$. The functionals $\mathcal{A}_\varepsilon$ satisfy the Palais-Smale condition for critical levels $a_\varepsilon > a_1 > 0$.

The relation between $\mathcal{A}$ and $\mathcal{A}_\varepsilon$ gives:

$$d\mathcal{A}_\varepsilon(c_n, \tau_n)(\xi, \sigma) = d\mathcal{A}(c_n, \tau_n)(\xi, \sigma) - \varepsilon\left(e^{-\tau} - \frac{1}{2}e^{\tau/2}\right)\sigma.$$

**Lemma 6.** A Palais-Smale sequence $(c_n, \tau_n)$ satisfies

$$2e^{-\tau_n}\mathcal{E}(c_n) + \varepsilon\left(2e^{-\tau_n} + \frac{1}{2}e^{\tau_n/2}\right) = a_\varepsilon + o(1), \quad (4.2)$$

$$e^{\tau_n}\mathcal{W}(c_n) - \varepsilon\frac{3}{4}e^{\tau_n/2} = -\frac{a_\varepsilon}{2} + o(1), \quad (4.3)$$

as $n \to \infty$. From the variation formula and (4.1) we then derive that

$$e^{-\tau_n}\mathcal{E}(c_n) + e^{\tau_n}\mathcal{W}(c_n) = -\varepsilon\left(e^{-\tau_n} - \frac{1}{2}e^{\tau_n/2}\right) + o(1)$$

as $n \to \infty$. On the other hand, $\mathcal{A}_\varepsilon(c_n, \tau_n) \to a_\varepsilon$ means that

$$e^{-\tau_n}\mathcal{E}(c_n) - e^{\tau_n}\mathcal{W}(c_n) = -\varepsilon\left(e^{-\tau_n} + e^{\tau_n/2}\right) + a_\varepsilon + o(1).$$

Combining these two estimates completes the proof. \qed
We obtain the following a priori bounds on $\tau_n$.

**Lemma 7.** Let $(c_n, \tau_n)$ be a Palais-Smale sequence. There are constants $T_0 < T_1$ (depending on $\varepsilon$) such that $T_0 \leq \tau_n \leq T_1$ for sufficiently large $n$.

**Proof.** From Equation (4.2) it follows that $\varepsilon(2e^{-\tau_n} + \frac{1}{2}e^{\tau_n}/2) \leq a_\varepsilon + 1$ for sufficiently large $n$, which proves the lemma.

From this we also obtain a priori bounds on the energy.

**Lemma 8.** Let $(c_n, \tau_n)$ be a Palais-Smale sequence. Then $\|\partial c_n\|^2_{L^2} = 2\mathcal{E}(c_n) \leq C(\varepsilon)$, independent of $n$.

**Proof.** The a priori bounds on $\tau_n$ can be used in Equation (4.2), which yields

$$2\mathcal{E}(c_n) = e^{\tau_n} \left\{ C_\varepsilon - \varepsilon(2e^{-\tau_n} + \frac{1}{2}e^{\tau_n}/2) + o(1) \right\} \leq C(\varepsilon),$$

which proves the lemma.

The following proposition establishes the Palais-Smale condition for the action $\mathcal{A}_\varepsilon$, $\varepsilon > 0$.

**Proposition 9.** Let $(c_n, \tau_n)$ be a Palais-Smale sequence for $\mathcal{A}_\varepsilon$. Then $(c_n, \tau_n)$ has an accumulation point $(c_\varepsilon, \tau_\varepsilon) \in \Lambda M \times \mathbb{R}$ that is a critical point, i.e. $d\mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) = 0$ and the action is bounded $0 < a_1 \leq \mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) = a_\varepsilon \leq a_2$.

**Proof.** From Lemmas 7 and 8 we have that $\mathcal{E}(c_n) \leq C$ and $\|\tau_n\| \leq C'$, with the constants $C, C' > 0$ depending only on $\varepsilon$. Fix $s_0 \in S^1$, then by Eq. (3.1) we have $d_M(c_n(s), c_n(s_0)) \leq |s - s_0|\sqrt{2C} \leq \sqrt{2C}$, and therefore $c_n(s) \in B_{\sqrt{2C}}(c_n(s_0))$, for all $s \in S^1$ and all $n$. We argue that $c_n(s_0)$ must remain in a compact set as $n \to \infty$. From this, and the energy bound on $c_n$, it follows that $c_n(s)$ remains in a compact set for all $s \in S^1$ as $n \to \infty$. The sequence $c_n$ is equicontinuous and the generalized Arzela-Ascoli Theorem shows that $c_n$ must have a convergent subsequence in the metric $d_{C_0}$. We then argue that this is actually an accumulation point for the metric $d_M$, which shows that $\mathcal{A}_\varepsilon$ satisfies the Palais-Smale condition.

We argue by contradiction. Suppose $d_M(c_n(s_0), K) \to \infty$ as $n \to \infty$. Since $K \subset M$ is compact, its diameter is finite and there exists a constant $C_K$ such that $d_M(q, \tilde{q}) \leq C_K$, for all $q, \tilde{q} \in K$. If $K \cap B_{\sqrt{2C}}(c_n(s_0)) \neq \emptyset$ for all $n$, then there exist points $q_n \in K \cap B_{\sqrt{2C}}(c_n(s_0))$, with $d_M(q, q_n) \leq C_K$ for all $q \in K$ and all $n$. This implies:

$$C_K + 2\sqrt{2C} \geq d_M(q_n, c_n(s_0)) \geq d_M(q, c_n(s_0)) \geq d_M(c_n(s_0), K) \to \infty,$$

as $n \to \infty$, which is a contradiction, and thus there exists an $N$ such that

$$K \cap B_{\sqrt{2C}}(c_n(s_0)) = \emptyset, \quad \text{for } n \geq N.$$
As a consequence $c_n(s) \in M \setminus K$, for all $s \in S^1$ and all $n \geq N$. Next we show it is impossible that $d_M(c_n(s_0), K) \to \infty$. Suppose there does not exist a constant $C''$ such that $d_M(c_n(s_0), K) \leq C''$, i.e., $d_M(c_n(s_0), K) \to \infty$ as $n \to \infty$. By the previous considerations there exists an $N$ such that $c_n(s) \in M \setminus K$, for all $s \in S^1$ and all $n \geq N$, and thus, using the asymptotic regularity, we have that

$$|\text{grad } V(c_n(s))|_{c_n(s)} \geq V_\infty > 0, \quad \text{for all } s \in S^1, \text{ and for all } n \geq N.$$ 

For $q \in M \setminus K$ we can define the smooth vector field on $M$ by

$$x(q) = -\frac{\text{grad } V(q)}{|\text{grad } V(q)|^2}.$$ 

For a loop $c \in C^\infty(S^1, M)$ the vector field along $c$ is given by

$$\xi(s) = x(c(s)) = -\frac{\text{grad } V(c(s))}{|\text{grad } V(c(s))|^2},$$

which is a smooth section in the pull-back bundle $C^\infty(c^*TM)$. We now investigate the $H^1$-norm of $\xi$ in $H^1(c^*TM)$. For a vector field $y$ on $TM$ we have that

$$|\nabla_y x(q)| \leq \left| -\frac{\nabla_y \text{grad } V(q)}{|\text{grad } V(q)|^2} - 2\left< \nabla_y \text{grad } V(q), \text{grad } V(q) \right> \frac{\text{grad } V(q)}{|\text{grad } V(q)|^4} \right| \text{grad } V(q) \right|$$

$$\leq 3 \left| \frac{\nabla_y \text{grad } V(q)}{|\text{grad } V(q)|^2} \right| \leq 3 \left| \frac{\text{Hess } V(q)}{|\text{grad } V(q)|^2} \right| |y|,$$

where we used the identity $|\nabla_y \text{grad } V(q)|^2 = \text{Hess } V(q)(y, \nabla_y \text{grad } V(q))$. Now set $\xi_n(s) = x(c_n(s))$. Using the asymptotic regularity of $V$, we obtain that

$$|\nabla_s \xi_n(s)| \leq 3 \left| \frac{\text{Hess } V(c_n(s))}{|\text{grad } V(c_n(s))|^2} \right| |\text{grad } c_n(s)| = o(1) |\text{grad } c_n(s)|,$$

and $|\xi_n(s)| \leq \frac{1}{V_\infty}$ as $d_M(c_n(s_0), K) \to \infty$. Thus the $H^1$ norm of $\xi_n$ is

$$|\xi_n|_{H^1} = \frac{1}{V_\infty} + o(1) |\text{grad } c_n|_{L^2}.$$

By the first variation formula in Lemma 5

$$d\mathcal{A}_\varepsilon(c_n, \tau_n)(\xi_n, 0) = e^{-\tau_n} \langle \partial c_n, \nabla \xi_n \rangle_{L^2} - e^{-\tau_n} \langle \text{grad } V \circ c_n, \xi_n \rangle_{L^2}$$

$$= o(1) \|\text{grad } c_n\|_{L^2}^2 + e^{-\tau_n} = o(1) + e^{-\tau_n}. \quad (4.4)$$

On the other hand, since $c_n(s)$ is a Palais-Smale sequence, we have

$$d\mathcal{A}_\varepsilon(c_n, \tau_n)(\xi_n, 0) = o(1) \|\xi_n\|_{H^1} = o(1) \left( \frac{1}{V_\infty} + C o(1) \right) = o(1),$$

which contradicts (4.4), since $|\tau_n|$ is bounded by Lemma 7. This now shows that $d_M(c_n(s_0), K) \leq C''$ and therefore there exists an $0 < R < \infty$ such that $c_n(s) \in B_R(K)$ for all $s \in S^1$ and all $n \geq N$. Since $(M, g)$ is complete, the Hopf-Rinow Theorem implies
that $B_R(K)$ is compact and thus $\{c_n(s)\} \subset M$ is pre-compact for any fixed $s \in S^1$. The sequence $\{c_n(s)\}$ is point wise relatively compact and equicontinuous by Eq. (3.1). Therefore, by the generalized version of the Arzela-Ascoli Theorem [16] there exists a subsequence $c_{n_k}$ converging in $C^0(S^1, M)$ (uniformly) to a continuous limit $c_\varepsilon \in C^0(S^1, M)$. It remains to show that $c_\varepsilon$ is an accumulation point in $\Lambda M$, thus in $H^1$ sense.

Due to the above convergence in $C^0(S^1, M)$, the sequence $\{c_n\}$ can be assumed to be contained in a fixed chart $(U(c_0), \exp^{-1})$, for a fixed $c_0 \in C^\infty(S^1, M)$. Following [13] it suffices to show that $\exp_{c_0}^{-1} c_n$ is a Cauchy sequence in $T_{c_0} \Lambda M = H^1(c_0^*TM)$. This final technical argument is identical to Theorem 1.4.7 in [13], which proves that $\{c_n\}$ has an accumulation point in $(c_\varepsilon, \tau_\varepsilon) \in \Lambda M \times \mathbb{R}$, proving the Palais-Smale condition for $\mathcal{A}_\varepsilon$. The limit points satisfy $d\mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) = 0$, and $\mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) = a_\varepsilon$.

For critical points of $\mathcal{A}_\varepsilon$ we prove additional a priori estimates on $\tau_\varepsilon$. The latter imply a priori estimates on $c_\varepsilon$. This allows us to pass to the limit as $\varepsilon \to 0$.

We start with pointing out that critical points of the penalized action $\mathcal{A}_\varepsilon$ satisfy the following Hamiltonian identity

$$H_\varepsilon(s) = \frac{e^{-2\tau_\varepsilon}}{2} |a_\varepsilon'(s)|^2 + V(c_\varepsilon(s)) \equiv \varepsilon\left( -e^{-2\tau_\varepsilon} + \frac{1}{2} e^{-\tau_\varepsilon/2} \right) = \varepsilon.$$  (4.5)

Thus the critical point $(c_\varepsilon, \tau_\varepsilon)$ corresponds to a closed characteristic on $\Sigma_\varepsilon$. Via the transformation $q_\varepsilon(t) = c_\varepsilon(te^{-\tau_\varepsilon})$ and the Legendre transform of $(q_\varepsilon, q_\varepsilon')$ to a curve $\gamma_\varepsilon$ on the cotangent bundle we see that the Hamiltonian action is

$$\mathcal{A}_\varepsilon^H(\gamma_\varepsilon, \tau_\varepsilon) = \int_{\gamma_\varepsilon} \Lambda + \varepsilon(e^{-\tau_\varepsilon} + e^{\tau_\varepsilon}/2).$$

The following a priori bounds are due to the uniform contact type of $\Sigma$, cf. Lemma 4.

**Lemma 10.** Let $(c_\varepsilon, \tau_\varepsilon)$ be critical points of $\mathcal{A}_\varepsilon$, with $0 < a_1 \leq \mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) \leq a_2$. Then there is a constant $T_2$, independent of $\varepsilon$, such that $\tau_\varepsilon \leq T_2$ for sufficiently small $\varepsilon$.

**Proof.** We start with the case $\tau_\varepsilon \geq 0$. The Hamiltonian action satisfies

$$\mathcal{A}_\varepsilon^H(\gamma_\varepsilon, \tau_\varepsilon) = \int_{\gamma_\varepsilon} \Lambda + \varepsilon(e^{-\tau_\varepsilon} + e^{\tau_\varepsilon}/2) \leq a_2,$$

and thus $\int_{\gamma_\varepsilon} \Lambda \leq a_2$. Since $\Sigma$ is of uniform contact type it holds for $\gamma_\varepsilon \subset \Sigma_\varepsilon$, with $\tilde{\varepsilon} \leq \varepsilon \leq \varepsilon_0$, that

$$a_2 \geq \int_{\gamma_\varepsilon} \Lambda = \int_{\gamma_\varepsilon} \Theta = \int_0^{\tau_\varepsilon} a_{\gamma_\varepsilon}(X_H) \geq a_n e^{\tau_\varepsilon}.$$

We conclude that always $\tau_\varepsilon \leq \max\{0, \log(a_2/a_n)\}$, which proves the lemma. $\square$

We can also establish a lower bound on $\tau_\varepsilon$ under the condition that $M$ is asymptotically flat. This is the only estimate that requires flat ends. All other estimates carry through under the weaker assumption of bounded geometry.
**Lemma 11.** Let \((c_{\varepsilon}, \tau_{\varepsilon})\) be critical points of \(\mathcal{A}_{\varepsilon}\), with \(0 < a_1 \leq \mathcal{A}_{\varepsilon}(c_{\varepsilon}, \tau_{\varepsilon}) \leq a_2\). If \((M, g)\) is asymptotically flat, then there is a constant \(T_3\), independent of \(\varepsilon\), such that \(\tau_{\varepsilon} \geq T_3\) for sufficiently small \(\varepsilon\).

**Proof.** Assume, by contradiction that \(\tau_{\varepsilon} \to -\infty\) as \(\varepsilon \to 0\). Then Equation (4.2) gives

\[
2\delta^{\varepsilon}(c_{\varepsilon}) = e^{\tau_{\varepsilon}a_{\varepsilon}} - 2\varepsilon - \frac{\varepsilon^3}{2}e^{3\tau_{\varepsilon}/2} \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{4.6}
\]

Fix \(s_0 \in S^1\). Then the previous equation implies, using Equation 3.1, that \(c_{\varepsilon}(s) \in B_{\delta'}(c_{\varepsilon}(s_0))\), where \(\delta' = \sqrt[4]{\varepsilon^{3/2}a_{\varepsilon} - 2\varepsilon - \varepsilon^{3/2}}\). We distinguish two cases:

(i) There exists an \(R > 0\) such that \(d_M(c_{\varepsilon}(s_0), K) \leq R\) for all \(\varepsilon\). Then \(c_{\varepsilon}(s) \in B_{\delta'+R}(K)\), and therefore \(|V(c_{\varepsilon}(s))| \leq C\) for all \(s \in S^1\) and all \(\varepsilon > 0\). This implies \(\int_0^1 e^{\tau_{\varepsilon}V(c_{\varepsilon}(s))}ds \to 0\), which contradicts (4.3), as \(a_{\varepsilon} > 0\), and thus \(\tau_{\varepsilon} \geq T_3\).

(ii) Now we assume no such \(R > 0\) exists, and assume thus that \(d_M(c_{\varepsilon}(s_0), K) \to \infty\) as \(\varepsilon \to 0\) to derive a contradiction. By bounded geometry of \(M\), every point \(q \in M\) has a normal charts \((U_q, \exp_q^{-1})\) and constants \(\rho_0, R_0 > 0\) such that \(B_{\rho_0}(q) \subset U_q\) and \(|\partial^k F_{ij}(q)| \leq R_0\). This implies that \(c_{\varepsilon}(s) \in U_{c_{\varepsilon}(s_0)}\) for sufficiently small \(\varepsilon\). We assume \(M\) has flat ends, and since \(d(c_{\varepsilon}(s_0), K) \to \infty\) the metric on the charts \(U_{c_{\varepsilon}(s_0)}\) is flat. We identify these charts with open subsets of \(\mathbb{R}^n\) henceforth. The differential equation \(c_{\varepsilon}\) satisfies is

\[
e^{-2\tau_{\varepsilon}} \nabla_{c_{\varepsilon}}' \gamma_{\varepsilon}(s) + \text{grad } V(c_{\varepsilon}(s)) = 0. \tag{4.7}
\]

Take the unique geodesic \(\gamma\) from \(c_{\varepsilon}(s_0)\) to \(c_{\varepsilon}(s)\) parameterized by arc length, i.e.

\[
\gamma(0) = c_{\varepsilon}(s_0), \quad \gamma(d_M(c_{\varepsilon}(s_0), c_{\varepsilon}(s))) = c_{\varepsilon}(s), \quad \text{and} \quad |\gamma'(t)| = 1.
\]

Then, by asymptotic regularity, \(\|\text{Hess } V(\gamma(t))\| \leq C\|\text{grad } V(\gamma(t))\|\) for some constant \(C > 0\), and

\[
\frac{d}{dt}\|\text{grad } V(\gamma(t))\|^2 = 2\text{Hess } V(\gamma(t))(\text{grad } V(\gamma(t)), \gamma'(t)) \leq 2\|\text{Hess } V(\gamma(t))\|\|\text{grad } V(\gamma(t))\| \leq 2C\|\text{grad } V(\gamma(t))\|^2.
\]

Gronwall’s inequality therefore implies that \(\|\text{grad } V(\gamma(t))\| \leq \|\text{grad } V(\gamma(0))\|e^{Ct}\). We identify \(U_{c_{\varepsilon}(s_0)}\) with an open subset of \(\mathbb{R}^n\), and we write

\[
\text{grad } V(\gamma(t)) = \text{grad } V(\gamma(0)) + \int_0^t \frac{d}{d\sigma}\text{grad } V(\gamma(\sigma))d\sigma.
\]

Hence

\[
\|\text{grad } V(\gamma(t)) - \text{grad } V(\gamma(0))\| \leq \int_0^t \|\text{Hess } V(\gamma(\sigma))\|d\sigma \leq C \int_0^t \|\text{grad } V(\gamma(\sigma))\|d\sigma \leq \|\text{grad } V(\gamma(0))\|(e^{Ct} - 1). \tag{4.8}
\]
For any solution to Equation (4.7), we compute
\[
\frac{d}{ds} e^{2\tau c} \langle \text{grad } V(c_\varepsilon(s_0)), c'_\varepsilon(s) \rangle = e^{2\tau c} \langle \text{grad } V(c_\varepsilon(s_0)), \nabla_s c'(s) \rangle \\
= -\langle \text{grad } V(c_\varepsilon(s_0)), \text{grad } V(c_\varepsilon(s)) \rangle \\
= -\langle \text{grad } V(c_\varepsilon(s_0)), \text{grad } V(c_\varepsilon(s_0)) \rangle \\
- \langle \text{grad } V(c_\varepsilon(s_0)), \text{grad } V(c_\varepsilon(s)) - \text{grad } V(c_\varepsilon(s_0)) \rangle
\]
By asymptotic regularity and Estimate (4.8), we find that
\[
\frac{d}{ds} e^{2\tau c} \langle \text{grad } V(c_\varepsilon(s_0)), c'_\varepsilon(s) \rangle \leq -V_\infty^2 + V_\infty^2 (e^{C_{dM}(c_\varepsilon(s_0), c_\varepsilon(s))} - 1).
\]
We see that as \( \varepsilon \to 0 \) that \( e^{2\tau c} \langle \text{grad } V(c_\varepsilon(s_0)), c'_\varepsilon(s) \rangle \) is monotonically decreasing in \( s \).
We conclude that \( c_\varepsilon \) cannot be periodic. This is a contradiction, therefore there exists a constant \( T_3 \) such that \( \tau_\varepsilon \geq T_3 \), for sufficiently small \( \varepsilon \).

**Proposition 12.** Let \( (c_\varepsilon, \tau_\varepsilon), \varepsilon \to 0 \) be a sequence satisfying \( d\mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) = 0 \), and \( 0 < a_1 \leq \mathcal{A}_\varepsilon(c_\varepsilon, \tau_\varepsilon) \leq a_2 \). If \( (M, g) \) has flat ends, then there exists a convergent subsequence \( (c_{\varepsilon'}, \tau_{\varepsilon'}) \to (c, \tau) \) in \( \Lambda M \times \mathbb{R} \), \( \varepsilon' \to 0 \). The limit satisfies \( d\mathcal{A}_\varepsilon(c, \tau) = 0 \), and \( 0 < a_1 \leq \mathcal{A}(c, \tau) \leq a_2 \).

**Proof.** From Lemmas 10 and 11 we obtain uniform bounds on \( \tau_\varepsilon \). We can now repeat the arguments of the proof of Proposition 9 on the sequence \( \{c_\varepsilon\} \), from which we draw the desired conclusion.

\[
\square
\]

5. Minimax Characterization

We prove that bounds of the action functional on certain linking homology classes give rise to critical values. Set \( E = \Lambda M \times \mathbb{R} \). For \( d \in \mathbb{R} \) define the sublevel sets
\[
\mathcal{A}^d = \{(c, \tau) \in E \mid \mathcal{A}(c, \tau) \leq d\}.
\]

**Lemma 13.** Suppose we have subsets \( A, B \subset E \) such that
(i) The morphism induced by inclusion \( i_{k+1} : H_{k+1}(A) \to H_{k+1}(E - B) \) is non-trivial.
(ii) The subset \( A \) is compact, and the action functional satisfies the bounds
\[
\mathcal{A}|_B \geq b > 0 \quad \mathcal{A}|_A \leq a < b.
\]
(iii) The homology of the loop space vanishes in the \( (k + 1) \)-st degree, and therefore
\[
H_{k+1}(E) = 0.
\]
Then there exist \( \bar{a} > 0 \) and \( \varepsilon^* > 0 \) such that for all \( 0 < \varepsilon \leq \varepsilon^* \), there exist a non-trivial \([y_\varepsilon]\) \in \( H_{k+2}(E, \mathcal{A}^0)\) and
\[
C_\varepsilon = \inf_{[y_\varepsilon] \in [y_\varepsilon]} \max \mathcal{A}_\varepsilon,
\]
satisfies the estimates \( \bar{a} < C_\varepsilon < C \) for a finite constant \( C \), independent of \( \varepsilon \).
Proof. Choose \( \bar{a} \) and \( \bar{b} \) such that \( a < \bar{a} < \bar{b} < b \), with \( \bar{a} > 0 \). Define \( \varepsilon^* = \frac{1}{2} \inf_{(c, \tau) \in A} \frac{\pi - a}{\varepsilon + \varepsilon^2} \), which is finite and positive because \( A \) is compact. For all \( 0 < \varepsilon \leq \varepsilon^* \), and \( (c, \tau) \in A \), the following estimate holds

\[
\mathcal{A}_\varepsilon(c, \tau) = \mathcal{A}(c, \tau) + \varepsilon \left( e^{-\tau} + e^{\frac{\pi}{2}} \right) < a + \frac{a - \bar{a}}{2} < \bar{a},
\]

For \( (c, \tau) \in B \) we obtain

\[
\mathcal{A}_\varepsilon(c, \tau) = \mathcal{A}(c, \tau) + \varepsilon \left( e^{-\tau} + e^{\frac{\pi}{2}} \right) > b > \bar{b}.
\]

For the remainder of this proof, we assume \( \varepsilon \leq \varepsilon^* \). One immediately verifies the following inclusions of sublevel sets

\[
A \longrightarrow \mathcal{A}_{\varepsilon^*} \longrightarrow \mathcal{A}_\varepsilon \longrightarrow \mathcal{A}_{\bar{b}} \longrightarrow E \setminus B.
\]

When we pass to homology, the sequence becomes

\[
H_{k+1}(A) \rightarrow H_{k+1}(\mathcal{A}_{\varepsilon^*}) \rightarrow H_{k+1}(\mathcal{A}_\varepsilon) \rightarrow H_{k+1}(E \setminus B), \tag{5.1}
\]

By assumption (i), the morphism induced by the inclusion is non-trivial, and it factors through this sequence. Therefore all the homology groups in Sequence (5.1) are non-trivial. Consider the following part of the long exact sequence of the pair \((E, \mathcal{A}_{\bar{b}})\)

\[
H_{k+2}(E, \mathcal{A}_{\bar{b}}) \rightarrow H_{k+1}(\mathcal{A}_{\bar{b}}) \rightarrow H_{k+1}(E).
\]

By assumption \( H_{k+1}(E) = 0 \), hence \( H_{k+2}(E, \mathcal{A}_{\bar{b}}) \rightarrow H_{k+1}(\mathcal{A}_{\bar{b}}) \) is surjective. Because \( H_{k+1}(\mathcal{A}_{\bar{b}}) \neq 0 \), we conclude \( H_{k+2}(E, \mathcal{A}_{\bar{b}}) \neq 0 \). The same argument shows that the homology groups \( H_{k+2}(E, \mathcal{A}_\varepsilon) \) are non-zero. Naturality of the boundary map with respect to the inclusions of pairs shows that the following diagram commutes

\[
\begin{array}{ccc}
H_{k+1}(A) & \longrightarrow & H_{k+1}(\mathcal{A}_{\varepsilon^*}) \longrightarrow H_{k+1}(\mathcal{A}_\varepsilon) \longrightarrow H_{k+1}(E \setminus B) \\
\uparrow & & \uparrow \\
H_{k+2}(E, \mathcal{A}_{\varepsilon^*}) & \longrightarrow & H_{k+2}(E, \mathcal{A}_\varepsilon) \longrightarrow H_{k+2}(E, \mathcal{A}_{\bar{b}})
\end{array} \tag{5.2}
\]

The vertical maps are the boundary homomorphisms of the long exact sequences of the pairs. Choose \([x] \in H_{k+1}(A)\) such that \( i_{k+1}([x]) \neq 0 \). Because the map \( i_{k+1} \) factors through Sequence (5.1), we get non-zero elements in those groups. We can lift the non-zero \([y] \in H_{k+1}(\mathcal{A}_{\varepsilon^*})\) to some non-zero \([y_{\varepsilon^*}] \in H_{k+2}(E, \mathcal{A}_{\varepsilon^*})\) by the surjectivity of the boundary homomorphism. The element \([x]\) is mapped by the inclusions in equation (5.1) as follows

\[
\begin{array}{cccccc}
[x] & \longmapsto & [x_{\varepsilon^*}] & \longmapsto & [x_{\varepsilon}] & \longmapsto & [x_{\bar{b}}] \\
& & \downarrow & & \downarrow & & \downarrow \\
& & [y_{\varepsilon^*}] & \longmapsto & [y_{\varepsilon}] & \longmapsto & [y_{\bar{b}}]
\end{array}
\]
This diagram defines the undefined elements. All elements in the above diagram are non-zero. Now define

\[ C_\varepsilon = \inf_{y_\varepsilon^\pi \in [y_\varepsilon^\pi]} \max_{|y_\varepsilon^\pi|} \mathcal{A}_\varepsilon. \]

Here we abuse notation and write \( y_\varepsilon^\pi \) for the representatives of \([y_\varepsilon^\pi]\). The infimum runs over all elements representing the class \([y_\varepsilon^\pi]\). The support, or image, of \( y_\varepsilon^\pi \) is denoted by \(|y_\varepsilon^\pi|\). The morphism induced by inclusion maps \( y_\varepsilon^\pi \mapsto y_\varepsilon^h \), an element in \( C_{k+2}(E, \mathcal{A}_\varepsilon^h) \), which represents the class \([y_\varepsilon^h]\). The support does not change, hence

\[ \max_{|y_\varepsilon^\pi|} \mathcal{A}_\varepsilon = \max_{|y_\varepsilon^h|} \mathcal{A}_\varepsilon > \bar{b}. \]

This follows from the following observation. If \( \max_{|y_\varepsilon^\pi|} \mathcal{A}_\varepsilon \leq \bar{b} \), then \(|y_\varepsilon^\pi| \subset \mathcal{A}_\varepsilon^h\), hence \([y_\varepsilon^h] = 0\) in \( H_{k+2}(E, \mathcal{A}_\varepsilon^h)\), which is a contradiction. We therefore conclude \( C_\varepsilon \geq \bar{b} > \bar{a} \).

If we now pick a representative \( y_\varepsilon^{\pi*} \in [y_\varepsilon^{\pi*}] \), which is mapped to \( y_\varepsilon^h \), we have that

\[ \inf_{y_\varepsilon^\pi \in [y_\varepsilon^\pi]} \max_{|y_\varepsilon^\pi|} \mathcal{A}_\varepsilon \leq \max_{|y_\varepsilon^{\pi*}|} \mathcal{A}_\varepsilon \leq \max_{|y_\varepsilon^{\pi*}|} \mathcal{A}_\varepsilon^{\pi*} =: C. \tag{5.3} \]

We conclude that we have, for each \( \varepsilon \leq \varepsilon^{\pi*} \), a non-trivial class \([y_\varepsilon^h] \in H_{k+2}(E, \mathcal{A}_\varepsilon^h)\), such that

\[ \bar{a} < C_\varepsilon < C. \tag{5.4} \]

The constants that bound \( C_\varepsilon \) are independent of \( \varepsilon \).

\[ \square \]

**Proposition 14.** Assume the hypotheses of Lemma 13 are met. There exists constants \( 0 < a_1 < a_2 < \infty \), such that, for all \( \varepsilon > 0 \) sufficiently small, \( \mathcal{A}_\varepsilon \) has a critical point \((c_\varepsilon, \tau_\varepsilon)\), satisfying \( a_1 < \mathcal{A}(c_\varepsilon, \tau_\varepsilon) < a_2 \).

**Proof.** The homology class \([y_\varepsilon^\pi]\) is homotopy invariant, and the minimax value \( C_\varepsilon \) over this class is finite and greater than zero, by Lemma 13. According to the Minimax principle, cf. [5], this gives rise to a Palais-Smale sequence \((c_n, \tau_n)\) for \( \mathcal{A}_\varepsilon \), with \( a_1 = \bar{a} \) and \( a_2 = C \). Proposition 9 states that \( \mathcal{A}_\varepsilon \) satisfies the Palais-Smale condition, hence this produces, for each \( \varepsilon > 0 \) sufficiently small, a critical point \((c_\varepsilon, \tau_\varepsilon)\) of \( \mathcal{A}_\varepsilon \).

\[ \square \]

6. **Some Remarks on the Geometry and Topology of the Loop Space**

The following observation is a simple relation between the homology of the loop space and the base manifold.

**Proposition 15.** There exists an isomorphism

\[ H_\#(\Lambda M) \cong H_\#(M) \oplus H_\#(\Lambda M, M). \tag{6.1} \]
structure on the normal bundle. By the zero section of the normal bundle is mapped diffeomorphically into that for inj. Thus Proposition 16. Thus Proposition 16.

Let c, thus c(s) = q for all s ∈ S. Elements ξ ∈ Nc(M) are characterized by the fact that ∫S1〈ξ(s), η0〉ds = 0, for all η0 ∈ TqM.

Proposition 16. Assume that M is of bounded geometry. Then there exists an open neighborhood V of ι(M) in ΛM and a diffeomorphism ϕ : NM → V, with the property that it maps ξ ∈ NM with ∥ξ∥H1 < inj M/2 to ϕ(ξ) ∈ ΛM with dH1(cq, ϕ(ξ)) = ∥ξ∥H1.

Proof. Let k : NM → NM be a smooth injective radial fiber-wise contraction such that

\[ k(ξ) = \begin{cases} ξ & \text{for } ∥ξ∥H1 < \frac{\text{inj } M}{2} \\ \frac{\text{inj } M}{1+∥ξ∥H1} ξ & \text{for } ∥ξ∥H1 \text{ large} \end{cases} \]

Thus ∥k(ξ)∥H1 < inj M for all ξ ∈ NM, and k is the identity on the disc bundle of radius inj M/2. Define the map ϕ : NM → ΛM by

\[ ϕ(ξ) = \exp_{cq} k(ξ), \]

for ξ ∈ Ncq M. This is a diffeomorphism onto an open subset V ⊂ ΛM. We use the fact that k is injective, and along with bounded geometry this shows that expcq is injective for ∥ξ∥H1 < inj M. We use bounded geometry to globalize this statement to the whole of M.

The inclusion of the zero section in the normal bundle is denoted by ζ : M → NM. The zero section of the normal bundle is mapped diffeomorphically into ι(M) ⊂ ΛM by ϕ. To form the required linking sets, we need a different, but equivalent, Riemannian structure on the normal bundle.
Proposition 17. On the normal bundle $NM$, the norm $\| \cdot \|_\perp$ defined by

$$\|\xi\|_\perp = \int_{S^1} \langle \nabla \xi(s), \nabla \xi(s) \rangle ds,$$  \hspace{1cm} (6.2)

is equivalent to the norm $\| \cdot \|_{H^1}$. To be precise the following estimate holds

$$\|\xi\|_\perp \leq \|\xi\|_{H^1} \leq \sqrt{2} \|\xi\|_\perp.$$  \hspace{1cm} (6.3)

Proof. It is obvious that $\|\xi\|_\perp \leq \|\xi\|_{H^1}$. We need to show the remaining inequality. We do this using Fourier expansions. For $\xi \in T_{c_q} \Lambda M$, a vector field along a constant loop $c_q$, we can write $\xi(s) = \sum_{k} \lambda_k e^{2\pi i ks}$, with $\lambda_k \in T_q M \otimes \mathbb{C}$. Then

$$\|\xi\|_{H^1}^2 = \sum_{k} \langle \lambda_k, \lambda_k \rangle + \sum_{k} 4\pi^2 k^2 \langle \lambda_k, \lambda_k \rangle.$$  

If $\xi \in NM$ then $\lambda_0 = 0$, by the characterization of the normal bundle given above, hence $\sum_{k} \langle \lambda_k, \lambda_k \rangle \leq \sum_{k} 4\pi^2 k^2 \langle \lambda_k, \lambda_k \rangle$, which gives that $\|\xi\|_{H^1} \leq \sqrt{2} \|\xi\|_\perp$. \hfill $\square$

We will use the next proposition to show that the linking sets we construct in the normal bundle persist in the loop space.

Proposition 18. Let $\mathcal{V}$ be the tubular neighborhood of $M$ inside $\Lambda M$, constructed in Proposition 16. Let $T$ be any subspace of $\mathcal{V}$ whose closure is contained in $\mathcal{V}$. Assume $H_{k+2}(\Lambda M) = 0$. Then the morphism

$$H_{k+1}(\mathcal{V} \setminus T) \to H_{k+1}(\Lambda M \setminus T),$$  \hspace{1cm} (6.4)

induced by inclusion is injective, and

$$H_{k+2}(\mathcal{V} \setminus T) \to H_{k+2}(\Lambda M \setminus T),$$  \hspace{1cm} (6.5)

is surjective.

Proof. We know from Proposition 15 that

$$H_{k+1}(\Lambda M) \cong H_{k+1}(\Lambda M, M) \oplus H_{k+1}(M).$$

We assume $H_{k+2}(\Lambda M) = 0$, thus $H_{k+2}(\Lambda M, M) = 0$. The tubular neighborhood $\mathcal{V}$ deformation retracts to $M$. If we apply the Five-Lemma to the long exact sequences of the pairs $(\Lambda M, M)$ and $(\Lambda M, \mathcal{V})$, where the vertical maps are induced by the deformation retraction,

$$
\begin{array}{cccccc}
H_{k+2}(\mathcal{V}) & \longrightarrow & H_{k+2}(\Lambda M) & \longrightarrow & H_{k+2}(\Lambda M, \mathcal{V}) & \longrightarrow & H_{k+1}(\mathcal{V}) & \longrightarrow & H_{k+1}(\Lambda M) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H_{k+2}(M) & \longrightarrow & H_{k+2}(\Lambda M) & \longrightarrow & H_{k+2}(\Lambda M, M) & \longrightarrow & H_{k+1}(M) & \longrightarrow & H_{k+1}(\Lambda M)
\end{array}
$$
we see that \( H_{k+2}(\Lambda M, V) \cong H_{k+2}(\Lambda M, M) \). Since \( T \) is contained in the closure of \( V \), we can excise \( T \). This gives an isomorphism \( H_{k+2}(\Lambda M \setminus T, V \setminus T) \cong H_{k+2}(\Lambda M, V) \cong 0 \). The long exact sequence of the pair \( (\Lambda M \setminus T, V \setminus T) \) is

\[
H_{k+2}(V \setminus T) \rightarrow H_{k+2}(\Lambda M \setminus T) \rightarrow H_{k+2}(\Lambda M \setminus T, V \setminus T) \rightarrow H_{k+1}(V \setminus T) \rightarrow H_{k+1}(\Lambda M \setminus T).
\]

The homology group \( H_{k+2}(\Lambda M \setminus T, V \setminus T) \) is zero by the preceding argument, thus the map on the right is injective and the map on the left is surjective.

\[\square\]

7. Relation of Topology of the Hypersurface to the Topology of its Projection

We investigate the relation between the topology of \( \Sigma \) and its projection \( N = \pi(\Sigma) \) to the base manifold. In the case \( N \) does not have a boundary, this relation is expressed by the classical Gysin sequence for sphere bundles. Recall that we assume \( H \) to be mechanical and that the hypersurface \( \Sigma = H^{-1}(0) \) is regular. Thus \( N \) and its boundary \( \partial N \) are given by

\[
N = \{ q \in M \mid V(q) \leq 0 \}, \quad \text{and} \quad \partial N = \{ q \in M \mid V(q) = 0 \},
\]

and \( \partial N \) is smooth. Let us consider a basic example, the harmonic oscillator on \( \mathbb{R} \). The potential is \( V(q) = \frac{k}{2} q^2 - V_0 \) for constants \( k, V_0 > 0 \). One directly verifies that \( N = \left[ -\sqrt{\frac{2V_0}{k}}, \sqrt{\frac{2V_0}{k}} \right] \) is an interval and \( \Sigma = \{ (q, \vartheta) \in \mathbb{R}^2 \mid \frac{1}{2} \vartheta^2 + \frac{k}{2} q^2 = V_0 \} \) is an ellipse. The fibers over the interior of \( N \) are copies of \( S^0 \), whose size is determined by the distance to the boundary \( \partial N \). Close to the boundary the fibers get smaller and at the boundary the fibers are collapsed to a point. Topologically the collapsing process can be understood by the gluing of an interval at the boundary of \( N \). The hypersurface \( \Sigma \) is thus homeomorphic
to $N \times S^0 \cup_{\partial N \times S^0} \partial N \times D^1$, which Figure 7.2 illustrates. This picture generalizes to arbitrary hypersurfaces $\Sigma$. We have the topological characterization

$$\Sigma \cong ST^*N \bigcup_{ST^*N|_{\partial N}} DT^*N|_{\partial N}. \quad (7.1)$$

The characterization is given in terms of the sphere bundle $ST^*N$ and the disc bundle $DT^*N$ in the cotangent bundle of $N$. The vertical bars denote the restriction of the bundles to the boundary. This topological characterization gives a relation between the homology of $\Sigma$ and $N$. In this section we identify $\Sigma$ with this characterization.

Recall that a map is proper if preimages of compact sets are compact. In the proof of the next proposition, compactly supported cohomology $H^*_c(M)$ is used, which is contravariant with respect to proper maps. In singular (co)homology, homotopic maps induce the same maps in (co)homology. For compactly supported cohomology, maps that are homotopic via a homotopy of proper maps, induce the same maps in cohomology. If $\partial N = \emptyset$ the following proposition directly follows from the Gysin sequence.

**Proposition 19.** There exist isomorphisms $H^i_c(\Sigma) \cong H^i_c(N)$ for all $0 \leq i \leq n - 2$.

**Proof.** Let $C$ be the closure of a collar of $\partial N$ in $N$, which always exists, cf. [9]. Thus $C$ deformation retracts via a proper homotopy onto $\partial N$. Now $\pi^{-1}(C)$ is the closure of a collar of $ST^*N|_{\partial N} = \partial ST^*N$ in $ST^*N$, and therefore it deformation retracts via a proper homotopy onto $ST^*N|_{\partial N}$. Define $D \subset \Sigma$ by

$$D = \pi^{-1}(C) \bigcup_{ST^*N|_{\partial N}} DT^*N|_{\partial N}.$$

This is a slight enlargement of the disc bundle of $M$ restricted to the boundary $\partial N$, which Figure 7.3 clarifies. By construction $D$ deformation retracts properly to $DT^*N|_{\partial N}$, which in turn deformation retracts properly to $\partial N$. This induces an isomorphism

$$H^*_c(D) \cong H^*_c(\partial N). \quad (7.2)$$

Let $S = ST^*N$. The intersection $D \cap S$ deformation retracts properly to $ST^*N|_{\partial N}$. Thus the isomorphism

$$H^*_c(D \cap S) \cong H^*_c(ST^*N|_{\partial N}). \quad (7.3)$$

holds. The inclusions in the diagram

$$\begin{alignedat}{2}
S \cap D &\to S &\to &\Sigma \\
&\downarrow i_1 &\downarrow j_1 &\downarrow i_2 \\
&\downarrow i_2 &\downarrow j_2 &D
\end{alignedat}$$
are proper maps, because the domains are all closed subspaces of the codomains. This gives rise to the contravariant Mayer-Vietoris sequence of compactly supported cohomology of the triad \((\Sigma, S, D)\)

\[
\cdots \longrightarrow H^i_c(\Sigma) \xrightarrow{\partial_i} H^i_c(S) \oplus H^i_c(D) \xrightarrow{\iota^i_1 + \iota^i_2} H^i_c(S \cap D) \longrightarrow H^{i+1}_c(\Sigma) \longrightarrow \cdots \tag{7.4}
\]

The map \(\iota^i_2\) is an isomorphism: this can be seen from the Gysin sequence for compactly supported cohomology as follows. Recall that from any vector bundle \(E \to B\) of rank \(n\) over a locally compact space \(B\), we can construct a sphere bundle \(SE \to B\). The Gysin sequence relates the cohomology of \(SE\) and \(B\),

\[
\cdots \longrightarrow H^i_c(\partial N) \xrightarrow{\pi^i} H^i_c(ST^*N |_{\partial N}) \longrightarrow 0 \quad \text{for} \quad 0 \leq i \leq n - 2. \tag{7.6}
\]

The map \(\varepsilon^i\) is the cup product with the Euler class of the sphere bundle. We apply this sequence to the sphere bundle in \(T^*N\) restricted to \(\partial N\). For dimensional reasons the sequence breaks down into short exact sequences

\[
0 \longrightarrow H^i_c(\partial N) \xrightarrow{\pi^i} H^i_c(ST^*N |_{\partial N}) \longrightarrow 0 \quad \text{for} \quad 0 \leq i \leq n - 2. \tag{7.6}
\]

The diagram

\[
\begin{array}{ccc}
H^i_c(D) & \xrightarrow{\iota^i_2} & H^i_c(S \cap D) \\
\varepsilon^i & \cong & \varepsilon^i \\
H^i_c(\partial N) & \xrightarrow{\pi^i} & H^i_c(ST^*N |_{\partial N})
\end{array}
\]

commutes. This shows that \(\iota^i_2\) is an isomorphism for \(0 \leq i \leq n - 2\). The map \(\iota^i_1 + \iota^i_2\) in the Mayer-Vietoris sequence, Equation (7.4), is surjective, and the sequence breaks down into short exact sequences

\[
0 \longrightarrow H^i_c(\Sigma) \longrightarrow H^i_c(S) \oplus H^i_c(D) \xrightarrow{\iota^i_1 + \iota^i_2} H^i_c(S \cap D) \longrightarrow 0 . \tag{7.7}
\]

More is true, since the sequence actually splits by the map \(p = (0, (\iota^i_2)^{-1})\). If we study the Gysin sequence for \(N\) and \(S\) we see that

\[
0 \longrightarrow H^i_c(N) \xrightarrow{\pi^i} H^i_c(S) \longrightarrow 0, \quad \text{for} \quad 0 \leq i \leq n - 2. \tag{7.8}
\]

The isomorphisms (7.8), (7.2), (7.3), and (7.6) can be applied to the sequence in Equation (7.7), which becomes

\[
0 \longrightarrow H^i_c(\Sigma) \longrightarrow H^i_c(N) \oplus H^i_c(\partial N) \xrightarrow{p} H^i_c(S \cap D) \longrightarrow 0, \quad \text{for} \quad 0 \leq i \leq n - 2.
\]

Because the sequence is split the stated isomorphism holds. \(\square\)
Proposition 20. For all $2 \leq i \leq n$ there is an isomorphism
\[ H_i(N, \partial N) \cong H_{i+n-1}(\Sigma). \]  

(7.9)

Proof. This is a double application of Poincaré duality for non-compact manifolds with boundary. The dimension of $N$ is $n$, and therefore Poincaré duality gives $H_i(N, \partial N) \cong H_{n-i}^c(N)$. The boundary of $\Sigma$ is empty, and its dimension equals $2n - 1$, thus $H_{n+i-1}(\Sigma) \cong H_{n-i}^c(\Sigma)$. By Proposition 19 we have $H_{n-i}^c(N) \cong H_{n-i}^c(\Sigma)$, for all $2 \leq i \leq n$. The isomorphism stated in the proposition is the composition of the isomorphisms.

We would also like the previous proposition to be true if $i = 1$. This is the case if the bundle $ST^*N$ is trivial, but in general this is not true. However, the following result is sufficient for our needs.

Proposition 21. If $H_n(\Sigma) \neq 0$ and $H_n(M) = 0$, then $H_1(N, \partial N) \neq 0$.

Proof. We will show that a non-zero element in $H_{n-1}^c(\Sigma)$ gives rise to a non-zero element in $H_{n-1}^c(N)$. A double application of Poincaré duality, as in the previous proposition, will give the desired result. We will use the same notation as in the proof of Proposition 19. The Gysin sequence, Equation (7.5), for the sphere bundle $ST^*N|_{\partial N}$ over $\partial N$ breaks down to the short exact sequence
\[ 0 \longrightarrow H_{n-1}^c(\partial N) \overset{\pi^*}{\longrightarrow} H_{n-1}^c(ST^*N|_{\partial N}) \overset{\delta}{\longrightarrow} H_{n}^0(\partial N) \longrightarrow 0. \]

(7.10)

Because $ST^*N|_{\partial N}$ is an $(n-1)$-dimensional sphere bundle over an $(n-1)$-dimensional manifold, it admits a section $\sigma: \partial N \to ST^*N|_{\partial N}$ and Equation (7.10) splits. We obtain

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7.3}
\caption{A sketch of the spaces $D$, and $S$. In the picture $N$ is a half-line, hence $\partial N$ is a point. The topology of the energy hypersurface can be recovered from its projection $N$.}
\end{figure}
the isomorphism
\[ H^{n-1}_c(S \cap D) \cong H^{n-1}_c(ST^*N|_{\partial N}) \cong H^{n-1}_c(\partial N) \oplus H^0_c(\partial N). \]
where the first isomorphism is induced by a homotopy equivalence. Now we look at the Mayer-Vietoris sequence for \( S, D \),
\[
0 \to H^{n-1}_c(\Sigma) \xrightarrow{\delta} H^{n-1}_c(S) \oplus H^{n-1}_c(D) \xrightarrow{\iota_1^{n-1} + \iota_2^{n-1}} H^{n-1}_c(S \cap D) \to H^{n-1}_c(\partial N) \to H^0_c(\partial N).
\]
We get a zero on the left of this sequence, because we have shown that in the previous step that the map \( \iota_1^{n-2} + \iota_2^{n-2} \) is surjective, cf. the argument before Equation (7.7). We claim that \( j_1^{n-1} \) is injective. Suppose otherwise, then there are \([x], [y] \in H^{n-1}_c(\Sigma)\), with \([x] \neq [y]\) such that \( j_1^{n-1}([x]) = j_1^{n-1}([y])\). Then \( j_1^{n-1}([x] - [y]) = 0 \). But since the map \((j_1^{n-1}, -j_2^{n-1})\) is injective, we realize that \( j_2^{n-1}([x] - [y]) = 0 \). But then \( j_2^{n-1} - j_2^{n-1}([x] - [y]) = 0 \) by the exactness of the sequence. Moreover
\[
\sigma^{n-1}_2 j_2^{n-1}([x] - [y]) = (\iota_2 \sigma)^{n-1}_2 j_2^{n-1}([x] - [y]).
\]
But, by the proper homotopy equivalence \( D \cong \partial N \), we realize that \((\iota_2 \sigma)^{n-1}_2 : H^{n-1}_c(D) \to H^{n-1}_c(\partial N) \) is an isomorphism, and \( (\iota_2 \sigma)^{n-1}_2([x] - [y]) \neq 0 \). This is a contradiction, hence \( j_1^{n-1} \) is injective. Recall that the Gysin sequence comes from the long exact sequence of the disc and sphere bundle, and the Thom isomorphism. From this we derive the following commutative diagram, which shows a naturality property of the Gysin sequence.

\[
\begin{array}{ccccccccc}
0 & \to & H^{n-1}_c(\partial N) & \to & H^{n-1}_c(ST^*N|_{\partial N}) & \to & H^0_c(\partial N) & \to &  \\
& & \mu & \to & \delta & & \to & \\
0 & \to & H^{n-1}_c(DT^*N|_{\partial N}) & \to & H^{n-1}_c(ST^*N|_{\partial N}) & \to & H^0_c(DT^*N|_{\partial N}, ST^*N|_{\partial N}) & \to & \\
& & \iota & \to & \iota^{n-1} & & \to & \\
0 & \to & H^{n-1}_c(DT^*N) & \to & H^{n-1}_c(ST^*N) & \to & H^0_c(DT^*N, ST^*N) & \to & \\
& & \cong & \to & \cong & & \to & \\
0 & \to & H^{n-1}_c(N) & \to & H^{n-1}_c(ST^*N) & \to & H^0_c(N) & \to & \\
& & \Phi & \cong & \Phi & & \cong & \\
\end{array}
\]
The top and bottom rows are the Gysin sequences of \((\partial N, ST^*N|_{\partial N})\) and \((N, ST^*N|_N)\) respectively. The vertical maps between the middle rows are the pullback maps of the inclusion of pairs \((DT^*N|_{\partial N}, ST^*N|_{\partial N}) \to (DT^*N, ST^*N|_N)\). The vertical maps \(\Phi\) are the Thom isomorphisms. The map \(\iota_1^{n-1}\) in the diagram is is the same as the map...
induced by \( \iota_1 : S \cap D \to S \), under the isomorphism induced by the homotopy equivalence \( S \cap D \cong ST^*N|_{\partial N} \), which we therefore denote by the same symbol.

We want to show that \( \delta j_1^{n-1}(y) = 0 \) for all \( y \in H_c^{n-1}(\Sigma) \). For this we argue as follows. Recall that \( H_0^c(N) \) consists of constant functions of compact support, and therefore is generated by the number of compact components of \( N \). If the vertical map in the third column, from \( H_0^c(N) \to H_0^c(\partial N) \) is not injective, then \( N \) has a compact component without boundary. This implies that \( M \) must have a compact component without boundary. But we assume that \( H_n(M) = 0 \), therefore \( M \) does not have orientable compact components and \( H_0^c(N) \to H_0^c(\partial N) \) is injective. Let \( [y] \in H_0^c(\Sigma) \) be non-zero. Obviously in \( H_c^{n-1}(ST^*N|_{\partial N}) \) we have the equality \( \iota_1^{n-1}j_1^{n-1}([y]) = \iota_2^{n-1}j_2^{n-1}([y]) \), and from the definition of the boundary map in the long exact sequence of the pair in the second row of the diagram, we obtain

\[
\delta j_1^{n-1}j_2^{n-1}([y]) = \delta j_2^{n-1}j_2^{n-1}([y]) = [p^{-1}d(\iota_2^{n-1})^{-1}\iota_2^{n-1}j_2^{n-1}y] = [p^{-1}d\iota_2^{n-1}y] = [p^{-1}j_2^{n-1}d\iota_2^{n-1}y] = 0.
\]

where \( p \) is the projection map in the defining short exact sequence. By the injectivity of the map \( H_0^c(N) \to H_0^c(\partial N) \), the commutativity of the diagram we must have that \( \delta j_1^{n-1}([y]) = 0 \in H_0^c(\Sigma) \). The exactness of the bottom row now shows that there must be an element in \( H_c^{n-1}(N) \) which is mapped to \( j_1^{n-1}([y]) \), because it \( j_1^{n-1}([y]) \) is in the kernel of \( \delta \). Poincaré duality for non-compact manifolds with boundary states that \( H_n(\Sigma) \cong H_c^{n-1}(\Sigma) \), and \( H_c^{n-1}(N) \cong H_1(\Sigma, \partial \Sigma) \). Thus, by the proceeding argument we get a non-zero class in \( H_1(N, \partial N) \).

\[\square\]

**Proposition 22.** Suppose that \( H_{k+4}(\Sigma) \not\cong 0 \) and \( H_{k+1}(M) = 0 \), for some \( 0 \leq k \leq n - 1 \). Then there exists a non-zero class in \( H_k(M \setminus N) \) which is mapped to zero in \( H_k(M) \) by the morphism induced by the inclusion.

**Proof.** Consider the long exact sequence of the pair \((M, M \setminus N)\)

\[
H_{k+1}(M) \to H_{k+1}(M, M \setminus N) \to H_k(M \setminus N) \to H_k(M).
\]

The homology group \( H_{k+1}(M) \) is zero by assumption, thus the middle map is injective. If we can find a non-zero element in \( H_{k+1}(M, M \setminus N) \), then we see it is mapped to a non-zero element of \( H_k(M \setminus N) \), which in turn is mapped to zero in \( H_k(M) \) by exactness of the sequence. Using excision, we will now show that \( H_{k+1}(M, M \setminus N) \) is isomorphic to \( H_{k+1}(N, \partial N) \). The latter group is non-zero, by Propositions 20 and 21. Take a neighborhood \( V \) of \( N \), which strongly deformation retracts to \( N \). This is possible because \( N \) is a submanifold with boundary. The complement of \( V \) in \( M \) is closed, and \( M \setminus V \subset M \setminus N \). The latter space is open, hence we can excise \( M - V \). Thus

\[
H_{k+1}(M, M \setminus N) \cong H_{k+1}(V, V \setminus N).
\]
The projection $N = \pi(\Sigma)$ is shrunk to $N_\nu$ using the gradient flow of the function $f$ defined in Equation (8.1). The boundary $\partial N$ is shown in red, and the boundary $\partial N_\nu$ is grey.

The pair $(V, V\setminus N)$ deformation retracts to the pair $(N, \partial N)$ by construction. By the Five Lemma applied to the long exact sequences of the pairs, we have $H_{k+1}(M, M\setminus N) \cong H_{k+1}(N, \partial N)$. Propositions 20 and 21 show that this homology group is non-zero. Thus there is a non-zero element of $H_k(M\setminus N)$ which is mapped to zero in $H_k(M)$ by the inclusion for $0 \leq k \leq n-1$. \hfill \square

Remark 3. In our setting, the assumption $H_{k+1}(M) = 0$ is automatically satisfied. This follows from the assumption $H_{k+1}(\Lambda M) = 0$ on the topology of the loop space, and Lemma 15.

8. The Link

8.1. The parameter $\nu$. For analytical reasons, we need to shrink the set $N = \pi(\Sigma) = \{q \in M \mid V(q) \leq 0\}$ to

$$N_\nu = \{q \in M \mid V(q) \leq -\nu \sqrt{1 + |\text{grad } V(q)|^2}\}. \quad (8.1)$$

For small $\nu$ this can be done diffeomorphically. On the modified set $N_\nu$ we estimate the potential $V$ uniformly. A sketch is given in Figure 8.1.

Lemma 23. There exist $\nu > 0$ sufficiently small, such that

- The spaces $N$ and $N_\nu$ are diffeomorphic, and $M\setminus N$ and $M\setminus N_\nu$ are diffeomorphic.
- If $H_{k+n}(\Sigma) \neq 0$ and $H_{k+1}(M) = 0$ for some $k$, there exists a non-zero class in $H_k(M - N_\nu)$ which is mapped to zero in $H_k(M)$ by the morphism induced by the inclusion.
- There exists a $\rho_\nu > 0$ such that, for all $q \in N_\nu$,

$$V(\tilde{q}) \leq -\frac{\nu}{2}, \quad \text{for all } \tilde{q} \in B_{\rho_\nu}(q). \quad (8.2)$$
Proof. Consider the function $f : M \to \mathbb{R}$ defined by

$$f(q) = \frac{V(q)}{\sqrt{1 + |\nabla V(q)|^2}}.$$ 

The gradient flow of this function induces the diffeomorphism. Because $N$ is non-compact, the standard Morse Lemma does not apply. We show that this function satisfies the Palais-Smale condition and that it has no critical values between 0 and $-\nu$. Because $M$ is assumed to be complete, a theorem of Palais [17, Theorem 10.2] shows that $N$ and $N_\nu$ are diffeomorphic.

We first show that $f$ satisfies the Palais-Smale condition. Let $q_n$ be a Palais-Smale sequence for $f$. That is, we assume there exists a constant $a$ such that the sequence satisfies

$$|f(q_n)| < a \quad \text{and} \quad |\nabla f(q_n)| \to 0 \quad \text{as} \quad n \to \infty.$$

Sequences for which $q_n$ stays in a compact set have a convergent subsequence, which must be a critical point of $f$. Thus we assume that $d(q_n, K) \to \infty$, and show that we get a contradiction. We compute the gradient of $f$

$$\nabla f(q) = \frac{\nabla V(q)}{\sqrt{1 + |\nabla V(q)|^2}} - \frac{1}{2} \frac{\nabla V(q)}{1 + |\nabla V(q)|^2},$$

and estimate

$$|\nabla f(q_n)| \geq \frac{|\nabla V(q_n)|}{\sqrt{1 + |\nabla V(q_n)|^2}} - \frac{1}{2} \frac{|\nabla f(q_n)|}{\sqrt{1 + |\nabla V(q_n)|^2}}.$$ 

The first term is bounded from below by $V_\infty/\sqrt{1 + V_\infty^2}$ for large $n$, by asymptotic regularity. In the second term, $|f(q_n)|$ is also bounded because $q_n$ is a Palais-Smale sequence, and a calculation reveals that $d|\nabla V(q_n)|^2 = 2 \text{Hess } V(q_n)(\nabla V(q_n), -)$. We find

$$|\nabla f(q_n)| \geq \frac{V_\infty}{\sqrt{1 + V_\infty^2}} - a \frac{\|\text{Hess } V(q_n)\|}{\|\nabla V(q_n)\|} \geq \frac{V_\infty}{2} \frac{V_\infty}{\sqrt{1 + V_\infty^2}},$$

for $n$ sufficiently large, since asymptotic regularity gives that $\frac{\|\text{Hess } V(q_n)\|}{\|\nabla V(q_n)\|} \to 0$ as $n \to \infty$. This is a contradiction for $n$ large. Hence $d(q_n, K)$ is bounded as $n \to \infty$. Since $M$ is complete, $q_n$ is contained in a compact set and therefore it contains a convergent subsequence. We conclude that $f$ satisfies the condition of Palais and Smale.

We use a similar argument to show that $f$ does not have critical values between 0 and $-\nu$, provided $\nu$ is small enough. Let $q \in N \setminus N_\nu$, then we have the following estimates with $C_1, C_2 > 0$, independent of $q$,

$$|\nabla V(q)| > C_1 \quad \text{and} \quad \|\text{Hess } V(q)\| < C_2 |\nabla V(q)|.$$

For $d(q, K) > 0$ this directly follows from asymptotic regularity. For $q \in K$ we argue as follows. By regularity of the energy surface $\nabla V \neq 0$ on $\partial N$. Hence the gradient
does not vanish on a tubular neighborhood of $\partial N$. For $\nu$ small the strip $(N \setminus N_\nu) \cap K$ lies inside this tubular neighborhood, by compactness. Thus $|\nabla V(q)|$ does not vanish on $(N \setminus N_\nu) \cap K$ for $\nu$ small. On the compact set $\{q \mid d(q, K) \leq R\}$, $\|\text{Hess} V\|$ is bounded, thus we can find a $C_2$ such that the second estimate holds.

Hence for all $q \in N \setminus N_\nu$, the norm of $\nabla f(q)$ is bounded from below

$$|\nabla f(q)| \geq \frac{\|\nabla V(q)\|}{\sqrt{1 + \|\nabla V(q)\|^2}} \cdot \frac{\|\nabla V(q)\|^2}{1 + \|\nabla V(q)\|^2},$$

$$\geq \frac{C_1}{\sqrt{1 + C_1^2}} - \nu \frac{\|\text{Hess} V(\nabla V(q), -)\|}{1 + \|\nabla V(q)\|^2},$$

$$\geq \frac{C_1}{\sqrt{1 + C_1^2}} - \nu C_2 > 0,$$  \hspace{1cm} (8.3)

provided $\nu > 0$ is chosen small enough. Thus $f$ does not have critical values between 0 and $-\nu$.

This shows that $N$ is diffeomorphic to $N_\nu$. By considering $-f$ we also see that $M \setminus N_\nu$ is diffeomorphic to $M \setminus N_\nu$. The following diagram commutes, by homotopy invariance

$$
\begin{array}{ccc}
H_k(M \setminus N) & \longrightarrow & H_k(M) \\
\downarrow & & \downarrow \\
H_k(M \setminus N_\nu) & & \\
\end{array}
$$

Proposition 22 therefore shows that there exists a non-zero class in $H_{k+1}(M \setminus N_\nu)$ that is mapped to zero in $H_k(M)$ by the morphism induced by the inclusion.

We now estimate $V$ uniformly on balls of radius $\rho_\nu$ around points of $N_\nu$. By continuity and compactness, there exists a $\rho_\nu > 0$ such that for all $q \in N_\nu$ with $d(q, K) < 1$, and all $\bar{q} \in B_{\rho_\nu}(q)$, the estimate $V(\bar{q}) \leq -\frac{\nu}{2}$ holds. Away from $K$, i.e. $q \in N_\nu$ with $d(q, K) \geq 1$, the following argument works. Take $\rho_\nu > 0$ small. By the reversed triangle inequality, then for all $\bar{q} \in B_{\rho_\nu}(q)$, $d(\bar{q}, K) > 0$. By asymptotic regularity there exists a $C > 0$, independent of $q$, such that

$$\|\text{Hess} V(\bar{q})\| \leq C |\nabla V(\bar{q})|.$$  \hspace{1cm} (8.5)

If $\rho_\nu$ is less than the injectivity radius $\text{inj} M$ of $M$, there exists a unique geodesic $c$ from $q$ to $\bar{q}$ parameterized by arc length, with length $\rho_\nu < \rho_\nu$. One then has:

$$
\frac{d}{ds}|\nabla V(c(s))|^2 = 2 |\text{Hess} V(\nabla V(c(s)), c'(s))|,
$$

$$\leq 2 |\text{Hess} V(c(s))| |\nabla V(c(s))| |c'(s)| \leq 2C |\nabla V(c(s))|^2.$$

The above estimate uses Estimate (8.5), and the fact that $c$ is parameterized by arc length. Gronwall’s inequality now implies

$$|\text{grad } V(c(s))|^2 \leq |\text{grad } V(c(0))|^2 e^{2Cs}.$$ 

Taking the square root of the previous equation gives the inequality

$$|\text{grad } V(c(s))| \leq |\text{grad } V(q)| e^{Cs}.$$ 

This allows us to estimate $V$ along the geodesic $c$. We compute

$$V(\bar{q}) = V(q) + \int_0^{\rho_\nu} \frac{d}{ds} V(c(s)) ds,$n
$$= V(q) + \int_0^{\rho_\nu} \langle \text{grad } V(c(s)), c'(s) \rangle ds,$n
$$\leq V(q) + |\text{grad } V(q)| \frac{e^{C\rho_\nu} - 1}{C},$$

$$\leq -\nu \sqrt{1 + |\text{grad } V(q)|^2} + |\text{grad } V(q)| \frac{e^{C\rho_\nu} - 1}{C}.$$ 

The function $x \mapsto -\nu \sqrt{1 + x^2} + \frac{e^{C\rho_\nu} - 1}{C} x$ has the maximum $-\nu^2 - \left(\frac{e^{2C\rho_\nu} - 1}{C}\right)^2$ for $\frac{e^{2C\rho_\nu} - 1}{C} \leq \nu$. We can find $\rho_\nu > 0$ small such that $V(\bar{q}) \leq -\frac{\nu}{2}$. This is independent of $q$, because $C$ is.
8.2. Constructing linking sets. We will use the topological assumptions in Theorem 2, to construct linking subspaces of the loop space. These are in turn used to find candidate critical values of the functional $\mathcal{A}$. The minimax characterization, cf. Section 5, shows that these candidate critical values contain critical points.

By Proposition 22 and Lemma 23 there exists a non-zero $[w] \in H_k(M - N_{\nu})$ such that $i_k([w]) = 0$ in $H_k(M)$. In this formula $i$ is the inclusion $i : M - N_{\nu} \to M$, and $i_k$ the induced map in homology of degree $k$. Because $i_k[w] = 0$, there exists a $u \in C_{k+1}(M)$ such that $\partial u = w$. We disregard any connected component of $u$ that does not intersect $w$. Set $W = |w|$ and $U = |u|$ where $|\cdot|$ denotes the support of a cycle. Both are compact subspaces of $M$. The inclusion $H_k(W) \to H_k(M - N_{\nu})$ is non-trivial by construction. We say that $W$ ($k$)-links $N_{\nu}$ in $M$. The linking sets discussed above will be used to construct linking sets in the loop space, satisfying appropriate bounds, cf. Proposition 27. A major part of this construction is carried out by the “hedgehog” function. This function is constructed in section 11, and is a continuous map $h : [0,1] \times U \to \Lambda M$ with the following properties

(i) $h_0(U) \subset \mathcal{V}$, with the tubular neighborhood $\mathcal{V}$ defined in Proposition 16.
(ii) The restriction $h_t|_W$ is the inclusion of $W$ in the constant loops in $\Lambda M$.
(iii) Only $W$ is mapped to constant loops. Thus $h_t(q) \in i(M)$ if and only if $q \in W$.
(iv) $\int_0^1 V(h_1(q)(s))ds > 0$ for all $q \in U$.

The idea is that, for $t = 0$, a point $q \in U$ is mapped to a loop close (in $H^1$ sense) to the constant loop $c_q(s) = q$. Points on the boundary $W$ are mapped to constant loops, but other points are never mapped to a constant loop. This ensures the first three properties (for $t = 0$). The construction shows that there are a finite number of points, such that the loops stay at these points for almost all time. These points are then homotoped to points where the potential is positive. This ensures the last property, using compactness...
The projected normal bundle $\hat{\pi}(NM) = M \times \mathbb{R}$. The set $Z$ and $N_\nu \times \{\rho\}$ $(k+1)$ link in $M \times \mathbb{R}$.

of $U$. Properties (i) and (ii) are used to lift the link of $M$ to a link in $\Lambda M$. The remaining properties are used to deform the link to sets where the functional satisfies appropriate bounds, and show that the link is not destroyed during the homotopy.

Because $U$ is compact, and $N_\nu$ is closed, $\iota(U \cap N_\nu)$ is compact. Moreover, it does not intersect $h_t(U)$ for any $t$, by property (iii). Hence $d_{\Lambda M}(h_{[0,1]}(U), \iota(U \cap N_\nu)) > 0$. Set $0 < \rho < \min\left(\frac{\inj M}{2}, \frac{\rho_s}{2}\right)$ such that

$$\rho < \frac{1}{2} d_{\Lambda M}(h_{[0,1]}(U), \iota(U \cap N_\nu)). \quad (8.1)$$

Define $f : U \to NM$ by the equation $f(q) = \varphi^{-1}h_0(q)$, where $\varphi : NM \to \mathcal{V}$ is defined in Proposition 16. The restriction of $f$ to $W$ is the inclusion of $W$ into the zero section of $NM$ by Property (ii), see also Figure 8.1. Recall that the normal bundle comes equipped with the equivalent norm $\| \cdot \|_\perp$, cf. Proposition 17. Define the map $\hat{\pi} : NM \to M \times \mathbb{R}$ by

$$\hat{\pi}(q, \xi) = (q, \|\xi\|_\perp).$$

Define $S = \hat{\pi}^{-1}(N_\nu \times \{\rho\})$. This is a sphere sub-bundle of radius $\rho$ in the normal bundle over $N_\nu$. Recall that the inclusion of $M$ as the zero section in $NM$ is denoted by $\zeta : M \to NM$. Set

$$Z = \hat{\pi}(\zeta(U) \cup f(U)) = U \times \{0\} \cup \hat{\pi}(f(U)).$$

The sets are depicted in Figure 8.2. Because $W$ $(k)$-links $N_\nu$ in $M$, the set $Z$ $(k+1)$-links $\hat{\pi}(S) = N_\nu \times \{\rho\}$ in $M \times \mathbb{R}$, as we prove below.

**Lemma 24.** Suppose $H_k(W) \to H_k(M \setminus N_\nu)$ is non-trivial. Then

$$H_{k+1}(Z) \to H_{k+1}(M \times \mathbb{R} \setminus N_\nu \times \{\rho\}),$$

is non-trivial.

**Proof.** Recall that $W = |w|$ and $w = \partial u$ with $u \in C_{k+1}(M)$ a $(k+1)$-cycle. Define the cycle $x \in C_{k+1}(Z)$ by

$$x = \hat{\pi}_{k+1} \zeta_{k+1}(u) - \hat{\pi}_{k+1} f_{k+1}(u).$$
This cycle is closed, because
\[
\partial \mathbf{x} = \hat{\pi}_k \zeta_k (\partial \mathbf{u}) - \hat{\pi}_k f_k (\partial \mathbf{u}) \\
= \hat{\pi}_k \zeta_k (\mathbf{w}) - \hat{\pi}_k f_k (\mathbf{w}) = 0.
\]

In the last step we used that \( f \vert_W = \zeta \vert_W \). Hence \( [\mathbf{x}] \in H_{k+1} (Z) \). We show that this class is mapped to a non-trivial element in \( H_{k+1} (M \times \mathbb{R} \setminus N_\nu \times \{ \rho \}) \).

For technical reasons we need to modify \( Z \) and \( N_\nu \times \{ \rho \} \). Define the set \( \tilde{Z} \), which is depicted in Figure 8.3, by
\[
\tilde{Z} = U \times \{ 0 \} \cup W \times [0, \rho] \cup T_\rho (\hat{\pi} f (U)),
\]
where \( T_\rho : M \times \mathbb{R} \rightarrow M \times \mathbb{R} \) is the translation over \( \rho \) in the \( \mathbb{R} \) direction, i.e. \( T_\rho(q, r) = (q, r + \rho) \). Denote by \( I_\rho \) the interval \( (\frac{\rho}{2}, \frac{\rho}{3}) \). There exists a homotopy \( m_t : M \times \mathbb{R} \rightarrow M \times \mathbb{R} \), with the following properties:

(i) \( m_0 = \text{id} \),
(ii) \( m_t (\tilde{Z}) \cap m_t (N_\nu \times I_\rho) = \emptyset \), for all \( t \),
(iii) \( m_1 (\tilde{Z}) = Z \),
(iv) \( m_1 (N_\nu \times I_\rho) = N_\nu \times \{ \rho \} \).

These properties ensure that \( Z \) \((k + 1)\)-links \( N_\nu \times \{ \rho \} \) if and only if \( \tilde{Z} \) \((k + 1)\)-links \( N_\nu \times I_\rho \). Define \([\tilde{x}] = (m_1)_{k+1}^{-1} \mathbf{x} \in H_{k+1} (\tilde{Z}) \). This is well defined because \((m_1)_{k+1} \) is an isomorphism. We will reason that this class includes non-trivially in \( H_{k+1} (M \times \mathbb{R} \setminus N_\nu \times I_\rho) \). For this we apply Mayer-Vietoris to the triad \((\tilde{Z}, U_1, U_2)\), with
\[
U_1 = U \times \{ 0 \} \cup W \times [0, \frac{2\rho}{3}],
\]
\[
U_2 = W \times (\frac{\rho}{3}, \rho] \cup T_\rho (\hat{\pi} f (U)).
\]
Note that $U_1 \cap U_2 = W \times I_\rho$. From the Mayer-Vietoris sequence for the triad we get the boundary map

$$H_{k+1}(\tilde{Z}) \xrightarrow{\delta} H_k(W \times I_\rho).$$

By definition of the boundary map $\delta$ in the Mayer-Vietoris sequence, we have that $\delta[u] = (m_1)^{-1}_k \pi_k \zeta_k[w]$. Now we consider a second Mayer-Vietoris sequence, the Mayer-Vietoris sequence of the triad

$$(M \times \mathbb{R} \setminus N_\nu \times \{I_\rho\}, M \times \mathbb{R} \setminus \frac{1}{2} N_\nu \times \{I_\rho\}, M \times \mathbb{R} \setminus \frac{2}{3} N_\nu \times \{I_\rho\}).$$

By naturality of Mayer-Vietoris sequences, the following diagram commutes

$$
\begin{array}{c}
H_{k+1}(\tilde{Z}) \\
\downarrow i_{k+1}
\end{array}
\xrightarrow{\delta}

\begin{array}{c}
H_k(W \times I_\rho) \\
\downarrow i_k
\end{array}
\xrightarrow{\delta}

\begin{array}{c}
H_{k+1}(M \times \mathbb{R} \setminus N_\nu \times \{I_\rho\}) \\
\downarrow i_{k+1}
\end{array}.
$$

We argued that $\delta[u] = (m_1)^{-1}_k \pi_k \zeta_k[w]$. We have that $i_k(m_1)^{-1}_k \pi_k \zeta_k[w] \neq 0$ by assumption. By the commutativity of the above diagram we conclude that $i_{k+1}[\tilde{x}] \neq 0$. Thus $\tilde{Z}$ $(k + 1)$-links $N_\nu \times \{I_\rho\}$ in $M \times \mathbb{R}$, which implies that $Z$ $(k + 1)$-links $N_\nu \times \{\rho\}$ in $M \times \mathbb{R}$.

The previous lemma lifted the link in the base manifold to a link in $M \times \mathbb{R}$. We now lift this link to the full normal bundle.

**Lemma 25.** The fact that $H_{k+1}(Z) \rightarrow H_{k+1}(M \times \mathbb{R} \setminus N_\nu \times \{\rho\})$ is non-trivial implies that $H_{k+1}(\zeta(U) \cup f(U)) \rightarrow H_{k+1}(NM\setminus S)$ is non-trivial.

**Proof.** The following diagram commutes

$$
\begin{array}{c}
H_{k+1}(\zeta(U) \cup f(U)) \\
\downarrow i_{k+1}
\end{array}
\xrightarrow{\hat{\pi}_{k+1}}

\begin{array}{c}
H_{k+1}(Z) \\
\downarrow i_{k+1}
\end{array}
\xrightarrow{\hat{\pi}_{k+1}}

\begin{array}{c}
H_{k+1}(NM\setminus S) \\
\downarrow i_{k+1}
\end{array}.
$$

Define $[y] = \zeta_{k+1}[u] - f_{k+1}[u]$. Recall that $[x] = \hat{\pi}_{k+1}[y]$ includes non-trivially in $H_{k+1}(M \times \mathbb{R} \setminus N_\nu \times \{\rho\})$ by the construction in lemma 24. By the commutativity of the above diagram $\pi_k i_{k+1}[y] = i_{k+1} \pi_k [y] \neq 0$. Thus $i_{k+1}[y] \neq 0$. The inclusion $H_{k+1}(\zeta(U) \cup f(U)) \rightarrow H_{k+1}(NM\setminus S)$ is non-trivial.

The domain of $\mathcal{A}$ is not the free loop space $\Lambda M$, but $\Lambda M \times \mathbb{R}$. The extra parameter keeps track of the period of the candidate periodic solutions. Thus we need once more to lift the link to a bigger space. In this process we also globalize the link, moving it from
the normal bundle to the full free loop space. Recall that we write \( E = \Lambda M \times \mathbb{R} \). The subsets \( A' = A_I \cup A_{II} \cup A_{III} \) are defined by
\[
A_I = \varphi(\zeta(U)) \times \{\sigma_1\} \\
A_{II} = \varphi(\zeta(W)) \times [\sigma_1, \sigma_2] \\
A_{III} = h_t(U) \times \{\sigma_2\}
\]
The constants \( \sigma_1 < \sigma_2 \) will be specified in Proposition 27. Finally we define the sets \( A, B \subset E \) by
\[
A = A^1 \quad \text{and} \quad B = \varphi(S) \times \mathbb{R}. \quad (8.2)
\]
Figure 8.4 depicts the sets \( A, B \).

**Lemma 26.** Assume that \( H_{k+2}(\Lambda M) = 0 \). The fact that \( H_{k+1}(\zeta(U) \cup f(U)) \rightarrow H_{k+1}(\Lambda M \setminus S) \) is non-trivial implies that the inclusion \( H_{k+1}(A) \rightarrow H_{k+1}(E \setminus B) \) is non-trivial.

*Proof.* By Lemma 25 the morphism induced by the inclusion \( H_{k+1}(\zeta(U) \cup f(U)) \rightarrow H_{k+1}(\Lambda M \setminus S) \) is non-trivial. By applying the diffeomorphism \( \varphi \), we see therefore that
\[
H_{k+1}(\varphi(\zeta(U)) \cup \varphi(f(U))) \rightarrow H_{k+1}(\mathcal{V} \setminus \varphi(S)),
\]
is non-trivial. Proposition 18 shows that the map
\[
H_{k+1}(\mathcal{V} \setminus \varphi(S)) \rightarrow H_{k+1}(\Lambda M \setminus \varphi(S)),
\]
is injective, because we assumed \( H_{k+2}(\Lambda M) = 0 \), and \( \varphi(S) \) is closed in \( \mathcal{V} \). It follows that
\[
H_{k+1}(\varphi(\zeta(U) \cup f(U))) \rightarrow H_{k+1}(\Lambda M \setminus \varphi(S)),
\]
is non-trivial. Let \( \pi_1 : \Lambda M \times \mathbb{R} \rightarrow \Lambda M \) be the projection to the first factor. Because of the choice of \( \rho \), cf. Equation (8.1) the set \( \pi_1(A^t) \) never intersects \( \pi_1(B) \). By the construction of the sets \( A^t \) and \( B \), the map \( \pi_1 \) induces a homotopy equivalence between \( A^t \) and \( \pi_1(A^t) \) and between \( E \setminus B \) and \( \Lambda M \setminus \pi_1(B) \), so that the diagram
\[
\begin{array}{ccc}
H_{k+1}(A^t) & \longrightarrow & H_{k+1}(E \setminus B) \\
\quad \downarrow_{\pi_1)_{k+1}} & & \downarrow_{(\pi_1)_{k+1}} \\
H_{k+1}(\pi_1(A^t)) & \longrightarrow & H_{k+1}(\Lambda M \setminus \pi_1(B)),
\end{array}
\]
commutes. We see that \( H_{k+1}(A^t) \rightarrow H_{k+1}(E \setminus B) \) is non-trivial if and only if \( H_{k+1}(\pi_1(A^t)) \rightarrow H_{k+1}(\Lambda M \setminus \pi_1(B)) \) is non-trivial. For all \( t \in [0, 1] \) the induced maps are the same, because of homotopy invariance. For \( t = 0 \) we have that \( \pi_1(A^0) = \varphi(\zeta(U) \cup f(U)) \), and \( \Lambda M \setminus \pi_1(B) = \Lambda M \setminus \varphi(S) \). We conclude that \( A \) \((k + 1)\)-links \( B \) in \( \Lambda M \). \( \square \)
9. Estimates

We need to estimate $\mathcal{A}$ on the sets $A, B \subset E$, defined in Equation (8.2).

**Proposition 27.** If $\nu$ and $\rho$ are sufficiently small, then there exist constants $\sigma_1 < \sigma_2$ and $0 < a < b$, such that

$$
\mathcal{A}|_A \leq a \quad \text{and} \quad \mathcal{A}|_B > b.
$$

**Proof.** We first estimate $\mathcal{A}$ on $B = \varphi(S) \times \mathbb{R}$. Let $(c_1, \tau) \in \varphi(S) \times \mathbb{R}$. Then $c_1 = \varphi(\xi) = \exp_{c_0}(\xi)$ where $\xi$ is a vector field along a constant loop $c_0$ at $q \in N_\nu$, for which $\|\xi\|_{L^2} = \|\nabla \xi\|_{L^2} = \rho$. From the Gauss lemma, and the following estimate, cf. Equations (3.3) and (6.3),

$$
\|\xi\|_{C^0} \leq \sqrt{2}\|\xi\|_{H^1} \leq 2\|\xi\|_{L^2}
$$

we see that $\sup_{s \in S^1} d_M(c_0(s), c_1(s)) \leq \rho$. Recall that we assumed $\rho \leq \frac{\rho_0}{2}$. Hence for all $s \in S^1$, we have $c_1(s) \in B_{\rho_0}(q)$ and therefore $V(c_1(s)) \leq -\frac{\nu}{2}$, by Lemma 23. We use this to estimate the second term of

$$
\mathcal{A}(c_1, \tau) = \frac{e^{-\tau}}{2} \int_0^1 |\partial c_1(s)|^2 ds - e^{\tau} \int_0^1 V(c_1(s)) ds.
$$

Let us now concentrate on the first term. We construct the geodesic from $c_0$ to $c_1$ in the loop space, namely

$$
c_t(s) = \exp_{c_0}(t \xi(s)).$$
This can also be seen as a singular surface in $M$, cf. [14]. Now we apply Taylor’s formula with remainder to $t \mapsto E(c_t)$. There exists a $0 \leq \tilde{t} \leq 1$ such that
\[
E(c_1) = E(c_0) + \frac{d}{dt} E(c_t) \Big|_{t=0} + \frac{1}{2} \frac{d^2}{dt^2} E(c_t) \Big|_{t=0} + \frac{1}{6} \frac{d^3}{dt^3} E(c_t) \Big|_{t=\tilde{t}}. \tag{9.3}
\]
It is obvious that $E(c_0) = 0$, since $c_0$ is a constant loop. Because $t \mapsto c_t$ is a geodesic $\frac{d}{dt} E(c_t) = 0$. The second order neighborhood of a closed geodesic is well studied [14, Lemma 2.5.1]. We see that $c_0$ is a (constant) closed geodesic, therefore
\[
\frac{d^2}{dt^2} E(c_t) = D^2 E(c_0)(\xi, \xi) = \|\xi\|^2 = \rho^2.
\]
The curvature term in the second variation vanishes at $t = 0$, because $c_0$ is a constant loop. The third derivative of the energy functional can be bounded in terms of the curvature tensor and its first covariant derivative times a third power of $\|\xi\|$. By the assumption of bounded geometry, we can therefore uniformly bound $E(c_t)$. The main point is that for $\rho$ sufficiently small, $E(c_1) \geq C\rho^2$, for some constant $C > 0$. We now can estimate $\mathcal{A}$ on $B$.
\[
\mathcal{A}(c, \tau) \geq \frac{e^{-\tau}}{2} CP^2 + \frac{e^{\tau}}{2} \nu \geq \sqrt{C} \nu \rho \tag{9.4}
\]
Set $b = \sqrt{C} \nu \rho$, then $\mathcal{A}|_B > b$. It remains to estimate $\mathcal{A}$ on the set $A = A_I \cup A_{II} \cup A_{III}$. Let $(c, \sigma_1) \in A_I = \varphi(\zeta(U)) \times \{\sigma_1\}$. Recall that $U$ is compact, hence $V_{\max} = \sup_{q \in U} -V(q) < \infty$. Because $c$ is a constant loop, we find
\[
\mathcal{A}(c, \sigma_1) = -e^{\sigma_1} \int_0^1 V(c(s))ds \leq e^{\sigma_1} V_{\max}. \tag{9.5}
\]
By choosing $\sigma_1 \leq \log(\frac{b}{2V_{\max}})$ we get $\mathcal{A}|_{A_I} \leq b/2$. On $A_{II} = \varphi(W) \times [\sigma_1, \sigma_2]$ all the loops are constants as well, moreover their image is contained in $W$. The potential is positive on $W$ hence $\mathcal{A}|_{A_{II}} < 0 < b/2$. It remains to estimate $\mathcal{A}$ on $A_{III} = h_1(U) \times \{\sigma_2\}$. Recall that we constructed $h$ in such a way that for any $q \in U$ we have $\int_0^1 V(h_1(q)(s))ds > 0$. This gives
\[
\mathcal{A}(c, \sigma_2) = \frac{e^{-\sigma_2}}{2} \int_0^1 |\partial h_1(q)(s)|^2 ds - e^{\sigma_2} \int_0^1 V(h_1(q)(s))ds \\leq \frac{e^{-\sigma_2}}{2} \int_0^1 |\partial h_1(q)(s)|^2 ds. \tag{9.6}
\]
Because $h$ is continuous and $U$ is compact, $E_{\max} = \sup_{q \in U} \mathcal{E}(h_1(q)) < \infty$. And therefore
\[
\mathcal{A}(c, \sigma_2) \leq \frac{e^{-\sigma_2}}{2} E_{\max}. \tag{9.7}
\]
By setting $\sigma_2 > \max(\log(\frac{E_{\max}}{b}), \sigma_1)$ we get $\mathcal{A}|_{A_{III}} \leq \frac{b}{2}$. Now set $a = b/2$, and we see that $\mathcal{A}|_A < a < b$. \qed
10. Proof of the Main Theorem

Proof of Theorem 2. Recall that we assume, see Subsection 1.5, that we have chosen an admissible asymptotically regular embedding \( \iota : \Sigma \to T^*M \). From the assumptions \( H_{k+1}(\Sigma) \neq 0 \) and \( H_{k+1}(\Lambda M) = H_{k+2}(\Lambda M) = 0 \), we are able to construct linking sets \( A \) and \( B \) in the loop space, cf. Lemma 26. We estimate \( \mathcal{A} \) on \( A \) and \( B \) in Proposition 27. This gives rise to estimates for the penalized functionals \( \mathcal{A}_\varepsilon \), and this in turn gives the existence of critical points, cf. Proposition 14. Under the assumption of flat ends, Proposition 12 produces a critical point of \( \mathcal{A} \), by taking the limit \( \varepsilon \to 0 \). The critical point of \( \mathcal{A} \) corresponds to a closed characteristic on \( \Sigma \).

11. Appendix: Construction of the Hedgehog

In this section we construct the hedgehog function \( h : [0, 1] \times U \to \Lambda M \). Recall that this function needs to satisfy the following properties:

(i) \( h_0(U) \subset V \), with the tubular neighborhood \( V \) defined in Proposition 16.
(ii) The restriction \( h_t|_W \) is the inclusion of \( W \) in the constant loops in \( \Lambda M \),
(iii) Only \( W \) is mapped to constant loops. Thus \( h_t(q) \in \iota(M) \) if and only if \( q \in W \).
(iv) \( \int_0^1 V(h_1(q)(s))ds > 0 \) for all \( q \in U \).

The precise construction of the hedgehog is technically involved, but the idea behind it is simple. It is based on a similar construction in [3], but we have to construct the map locally, which adds some subtleties. Points in \( W \) are mapped to constant loops, and, for \( t = 1 \), points in \( U \setminus W \) are mapped to non-constant loops which remain for most time at points where the potential is positive.

When a space admits a triangulation, simplicial and singular homology coincide. We will exploit this fact by constructing our map simplex by simplex in simplicial homology. It is possible to find a smooth triangulation of \( M \), which restricts to a smooth triangulation of \( N_r \) [15, Theorem 10.4], and by subdivision we can make this triangulation as fine as we want, i.e. we assume that each simplex is contained in a single chart, and that the diameter (measured with respect to the Riemannian distance) of each simplex is bounded uniformly, by \( 0 < \frac{\xi}{2} < \text{inj } M \). Recall that \( U \) and \( W \) are images of cycles, and as such are triangulated and compact.

The hedgehog is constructed by the following procedure. First we construct \( h_0 \). Each point in \( U \) is mapped to a loop whose image is in the simplex it is contained in, or in small spines emanating from the corners. This is done in such a manner for most times \( s \), the loop is locally constant with values at the corner points and the spines. By starting with a fine triangulation this ensures that this map satisfies property (i). The points on \( W \) are mapped to constant loops, which ensures property (ii). We also ensure that these are the only points which are sent to constant loops, which enforces (iii). Finally we apply a homotopy \( f_t : M \to M \), which is the identity outside a neighborhood of \( U \), and maps
the 0-simplices in the triangulation that are not in $W$ to points in $M \setminus (N \cup W)$. The potential $V$ is positive on this set. We then define $h_t = f_t \circ h_0$, and this, together with the property that the loops are for most times at these corner points, makes sure we satisfy property (iv).

For the construction we need some spaces which closely resemble the standard simplex.

**Definition 5.** The standard $m$-simplex $\Delta_m$ is defined by

$$\Delta_m = \left\{ y = (y_0, \ldots, y_m) \in \mathbb{R}^{m+1}_{\geq 0} \mid \sum_{i=0}^{n} y_i = 1 \right\} , \quad (11.1)$$

the extended $m$-simplex $\overline{\Delta}_m$ by

$$\overline{\Delta}_m = \left\{ y = (y_0, \ldots, y_m) \in \mathbb{R}^{m+1}_{\geq 0} \mid 1 \leq \sum_{i=0}^{n} y_i \leq 2 \right\} , \quad (11.2)$$

and the $m$-simplex with spines by

$$\underline{\Delta}_m = \Delta_m \cup \left\{ y \in \mathbb{R}^{m+1}_{\geq 0} \mid 1 \leq y_j \leq 2 \quad \text{and} \quad y_{i+j} = 0 \right\} . \quad (11.3)$$

These simplices are sketched in figure 11.5. We can now make the previous discussion more precise. Denote by $\mathcal{T}_U = \{ l_m : \Delta_m \to M \}$ the triangulation of $U$. Because $U$ is a $(k+1)$-cycle, $\mathcal{T}_U$ only contains simplices of dimension less or equal to $k+1$. To make consistent choices in the interpolations to come, we need to fix a total order $<$ on the zero simplices. All the other simplices respect this order, in the sense that for any simplex $l_k : \Delta_k \to M$ we have

$$l_k(1,0,\ldots,0) < l_k(0,1,0,\ldots,0) < \ldots < l_k(0,\ldots,0,1).$$

Now we start with some fixed top-dimensional, i.e. $(k+1)$-dimensional, simplex in the triangulation $\mathcal{T}_U$, say $l_{k+1} : \Delta_m \to M$. We define the map $\tilde{h}_{k+1}$, which constructs initial non-constant loops on the vertices not in $W$, and interpolates between them. The images of these loops lie in the extended simplex. The loops produced in this manner
are not smooth, only of class $L^2$. Therefore we need to smooth these loops, by taking
the convolution $\ast$ with a standard mollifier $\xi_{\varepsilon}$, which lands us in $C^\infty(S^1, \{\Delta_m\})$. This is
possible because the extended simplex is a convex set. The map $\Phi$ contracts the extended
simplex to the simplex with spines. By post-composing the loops with this contraction,
denoted $\Phi \cdot$, we get loops in the simplex with spines. This simplex with spines is mapped
to the free loop space of $M$ using $\tilde{I}_{k+1}$. This is a slight modification of the simplex. The
spines are mapped to paths close-by the corner points. This finalizes the construction of
$\Lambda_0$ restricted to the simplex. After this we homotope $M$ moving all the 0-simplexes of
the triangulation to points with positive potential, and not in $W$. The construction is such that
on intersection of simplices in the triangulation of $U$ the maps coincide. Therefore these
maps can be patched together to make a continuous map from $U \to \Lambda M$. For reference,
the points in the simplex $l_{k+1}$ will eventually be mapped to $\Lambda M$ by the following sequence

$$
\begin{align*}
    l_{k+1}(\Delta_{k+1}) & \xrightarrow{\tilde{I}_{k+1}} \Delta_{k+1} \xrightarrow{\tilde{h}_{k+1}} L^2(S^1, \{\Delta_{k+1}\}) \xrightarrow{\xi_{\varepsilon} \ast} C^\infty(S^1, \{\Delta_{k+1}\}) \\
    & \xrightarrow{\Phi} H^1(S^1, \{\Delta_{k+1}\}) \xrightarrow{\tilde{I}_{k+1}} \Lambda M \xrightarrow{f_\mu} \Lambda M.
\end{align*}
$$

The precise details on the construction of the maps in this diagram follows.

**The interpolation operator $I_{\Delta_m}$**. We will construct the loops inductively. We initially set
loops at the vertices in the next paragraph. We want to interpolate them to the 1 dimen-
sional faces. These are then in turn interpolated to the 2 dimensional faces, etc. Thus
we need to construct a continuous interpolation operator $I_{\Delta_m} : C(\partial \Delta_m, L^2(S^1, \{\Delta_m\}) \to
C(\Delta_m, L^2(S^1, \{\Delta_m\}))$. We define the loop at the center of the simplex to be the concatena-
tion of loops at the corner points. Each interior point on the simplex lies on a unique line
segment from the barycenter to a boundary point. We can parameterize this line segment
by a parameter $0 \leq \lambda \leq 1$. A point on the line segment corresponding to the parameter $\lambda$
is mapped to the loop which follows the boundary loop up to time $s = \lambda$ and then follows
the loop of the barycenter of the simplex, for time $\lambda < s < 1$.

Fix $m$ and consider the barycenter $M = \left(\frac{1}{m+1}, \ldots, \frac{1}{m+1}\right)$ of the $m$-simplex. Set, for
$y = (y_1, \ldots, y_m) \in \Delta_m - M$,

$$
\lambda_y = \min_{0 \leq i \leq m, \ y_i < 1/(m+1), \ y_i < 1/(m+1)} \frac{1}{1 - (m+1) y_i}.
$$

(11.4)

This is the smallest positive value $\lambda_y$ such that $M + \lambda_y(y - M)$ is on the boundary,
cf. Figure 11.6. Now assume we have a continuous map $k : \partial \Delta_m \to L^2(S^1, \{\Delta_m\})$. 

...
Interpolate this map to the barycenter by
\[
(I_{\triangle_m} k)(\mathcal{M})(s) = \begin{cases} 
  k(1,0,\ldots,0) & \text{if } 0 \leq s \leq \frac{1}{m}, \\
  k(0,1,0\ldots,0) & \frac{1}{m} \leq s \leq \frac{2}{m}, \\
  \vdots & \vdots \\
  k(0,\ldots,0,1) & \frac{m-1}{m} \leq s \leq 1
\end{cases}
\]
and interpolate it to the full simplex \( \triangle_m \) by:
\[
(I_{\triangle_m}(k))(y)(s) = \begin{cases} 
  k(M + \lambda_y(y - \mathcal{M}))(s) & \text{if } 0 \leq s \leq 1/\lambda_y, \\
  k(M)(s) & 1/\lambda_y < s \leq 1,
\end{cases}
\]
for all \( y \) in the interior of \( \triangle_m \) but not equal to \( \mathcal{M} \). If \( y \) is on the boundary, then \( I_{\triangle_m}(k)(y)(s) = k(y)(s) \). If \( k \) was continuous, we have that \( I_{\triangle_m}(k) \in C(\triangle_m, L^2(S^1, \Delta_m)) \).

The map \( \bar{h}_{k+1} \). Recall that we have chosen a fixed top dimensional, that is \((k+1)\)-simplex \( l_{k+1} : \Delta_{k+1} \rightarrow M \) in \( \mathcal{T}_W \). For each lower dimensional face of \( \Delta_{k+1} \) the image of its interior is either disjoint from \( W \) or it is contained in \( W \). We say that the face is in \( W \) in the latter case. To construct \( \bar{h}_{k+1} \) we assign non-constant initial loops to the 0-dimensional faces not in \( W \). The 0-dimensional faces in \( W \) are mapped to constant loops. Then we interpolate, using \( I_{\Delta_1} \), on all the 1-dimensional faces not in \( W \). On the 1-dimensional faces in \( W \) we will have constant loops. This process can be done on higher dimensional faces by induction. The resulting map is \( \bar{h}_{k+1} : \Delta_{k+1} \rightarrow L^2(S^1, \Delta_{m+1}) \). Let us now formalize the discussion.
To start, let us define \( \tilde{h}_{k+1} \) on the vertices in \( \triangle_{k+1} \). For a vertex \( y = (0, \ldots, 1, \ldots, 0) \) set

\[
\tilde{h}_{k+1}(y)(s) = \begin{cases} 
y & \text{if } y \text{ is in } W \text{ or } 0 \leq s \leq \frac{1}{2} \\
2y & \text{if } y \text{ is not in } W \text{ and } \frac{1}{2} \leq s \leq 1 
\end{cases}.
\] (11.7)

Now suppose \( \tilde{h}_{k+1} \) is defined for all \( m \) dimensional faces with \( m < k + 1 \). We now define \( \tilde{h}_{k+1} \) on the \((m+1)\)-dimensional faces. If the \((m+1)\)-dimensional face is in \( W \) set \( \tilde{h}_{k+1}(y)(s) = y \) for \( y \) in this simplex. For each \((m+1)\)-dimensional face not in \( W \) we will use the interpolation operator \( I_{\triangle_{m+1}} \) in the following manner.

We have standard projections \( \pi : \mathbb{R}^{(k+1)+1} \rightarrow \mathbb{R}^{(m+1)+1} \), which drops \( k - m \) variables, and inclusions \( \iota : \mathbb{R}^{(m+1)+1} \rightarrow \mathbb{R}^{(k+1)+1} \), which set \( k - m \) coordinates to zero. There are many such maps, and we need them all, but we denote them by the same symbol. These maps are linear maps that respect the total order \( \prec \). Now if \( \tilde{h} \) has been defined on the \( m \)-dimensional faces of \( \triangle_{k+1} \), then for an \((m+1)\)-dimensional face

\[
\pi \circ \tilde{h} \circ \iota|_{\partial \triangle_{m+1}} \in C(\partial \triangle_{m+1}, L^2(\mathbb{S}^1, [\triangle_{m+1}])),
\] (11.8)

is a map we can apply \( I_{\triangle_{m+1}} \) to. This gives a continuous map in \( C(\triangle_{m+1}, L^2(\mathbb{S}^1, [\triangle_{m+1}])) \). By post composing with \( \iota \) we can map this back to \( \triangle_{k+1} \). Doing this to all \((m+1)\)-dimensional faces, gives us a continuous map on the \((m+1)\)-dimensional faces. Because the interpolation is a continuous operator, we can proceed by induction to obtain the full map \( \tilde{h}_{k+1} : \triangle_{k+1} \rightarrow L^2(\mathbb{S}^1, [\triangle_{k+1}]) \).

The loops constructed in this way are locally constant a.e. with values at the special points \((0, \ldots, 0, 1, 0, \ldots, 0)\) and \((0, \ldots, 0, 2, 0, \ldots, 0)\), or points that are mapped to \( W \). Furthermore, if \( y \in \triangle_{k+1} \) is not in \( W \), then \( \tilde{h}_{k+1}(y) \) attains at least two different values on sets with non-zero measure.

**The convolution \( \xi_{\hat{\varepsilon}} \).** The set \([\triangle]_{k+1}\) is a convex subset of \( \mathbb{R}^{n+1} \) so we can apply to each loop we got from \( \tilde{h}_{k+1} \) the convolution with a standard mollifier \( \xi_{\hat{\varepsilon}} \) with parameter \( \hat{\varepsilon} \) to obtain smooth loops in \([\triangle]_{k+1}\). The smooth loops obtained in this manner are constant at the corner points, on a set of measure at least \( 1 - 2(k + 1)\hat{\varepsilon} \). The loops that were constant before mollifying, i.e. those loops that arose from points in \( W \), are left unchanged by this process. No non-constant loop is mapped to a constant loop, if \( \hat{\varepsilon} \) is chosen small enough.

**The retraction \( \Phi \).** We now construct a retraction from \([\triangle]_{k+1}\) to \([\triangle]_{k+1}\). A sketch of \( \Phi \) is depicted in Figure 11.7. Let \( y \in [\triangle]_{k+1} \). Denote by \( 2\text{ndmax}(y) \) the second largest component of \( y \). Define

\[
\mu_y = \min(2\text{ndmax}(y), \frac{1}{k + 2} \left( \sum_{k=0}^{k+1} y_i - 1 \right)).
\]
Then the map $\Phi : \triangle_{k+1} \to \triangle_{k+1}$ is defined component-wise by

$$\Phi^i(y) = \max(0, y_i - \mu y).$$

Note, that if $y \in \triangle_{k+1}$ then $\Phi(y) \in \triangle_{k+1}$. Post composing an $H^1$-loop in $\triangle_{k+1}$ with $\Phi$ produces an $H^1$ loop with image in $\triangle_{k+1}$.

The spine maps $\tilde{l}_{k+1}$. For each 0-simplex $i_0^1$ in $\mathcal{T}_U - \mathcal{T}_W$, with image $x_i$ choose a point $\bar{x}_i$ close by. To be precise assume that $d_M(x_i, \bar{x}_i) < \frac{\mu}{2}$. For each $i$ there exists a unique geodesic $\gamma_{x_i}$ in the simplex with $\gamma_{x_i}(0) = x_i$ and $\gamma_{x_i}(1) = \bar{x}_i$. Without loss of generality we can choose $\bar{x}_i$ such that the geodesics $\gamma_{x_i}$ don’t intersect one another. For the 0-simplices in $W$, i.e. $i_0^1 \in \mathcal{T}_W$, with images $x_i$ we choose the points $\bar{x}_i = x_i$ and $\gamma_{x_i}(s) = x_i$ is the constant geodesic at $x_i$.

We have fixed a top dimensional simplex $l_{k+1} : \triangle_{k+1} \to M$ in $\mathcal{T}_U - \mathcal{T}_W$. The map $\tilde{l}_{k+1}$ is defined by

$$\tilde{l}_{k+1}(y) = \begin{cases} 
l_{k+1}(y) & \text{if } y \in \triangle_{k+1} \\
\gamma_{l_{k+1}(0,\ldots,1,\ldots,0)}(y_i - 1) & \text{if } y = (0,\ldots,y_i,\ldots,0) \end{cases} \quad (11.9)$$

On the simplex itself $\tilde{l}_{k+1}$ is just $l_{k+1}$, but on the spines it follows the geodesics $\gamma_{l_{k+1}(0,\ldots,1,\ldots,0)}$ emanating form the images of the corners of the $(k+1)$-simplex constructed above. This map is smooth in the following sense. The restriction to the interior, or the restriction to the interior of $\triangle_{k+1}$ is smooth, because we started with a smooth triangulation. The restriction to the spines is also smooth, because the geodesics are.

Patching. If we apply the above maps after each other we obtain a map $l_{k+1}(\triangle_{k+1}) \to \Lambda M$. If we have another simplex $l'_{k+1}$, the construction coincides on common lower dimensional faces, for which we use the total order $\prec$. The simplices are closed, hence the maps patch together to a continuous map $h_0 : U \to \Lambda M$. The image of a point $q \in U$ is in $H^1$, because $\Phi$ is the only $H^1$ map and all other maps in the construction are piecewise smooth.
We see that for small enough loops $\tilde{c}$.

Proof. Let $\gamma_0$ be a fixed simplex $\gamma_0$, and a point $q$ in its interior, and denote by $c_0$ the constant loop at $q$. Because the image of $c_1$ is contained in the simplex $l_{k+1}$ and the geodesics $\gamma_{x_i}$ emanating from its corners points (which are maximally $\epsilon_1 = \frac{a_1}{2} + \frac{a_2}{2}$ away from $q$), we can estimate that $d(\gamma_0(c_0, c_1) \leq \epsilon_1$. Because the loops are only non-constant on a set of measure at most $2(k+1)\epsilon$, and the derivate of the loop can be estimated in terms of of $\epsilon_1$ we argue that the energy of $c_1$ is also bounded from above, by $\epsilon_2$, which is small if $\epsilon_1$ is small. The following lemma shows that the image of $h_0$ is contained in $V$.

**Lemma 28.** Suppose $c_0$ is a constant loop and $c_1 \in \Lambda M$ is such that $d(\gamma_0(c_0, c_1) < \epsilon_1 < \text{inj } M$ and $\delta'(c_1) < \epsilon_2$, then $\|\xi\|_{H^1} \leq \sqrt{\epsilon_1^2 + 2\epsilon_2}$.

**Proof.** Because $d(\gamma_0(c_0, c_1) < \epsilon_1 < \text{inj } M$ we can write $c_1(s) = \exp_{c_0} \xi(s)$ with $\xi \in T_{c_0} \Lambda M$. We expand $\delta'(c_1)$ along the geodesic $c_1 = \exp_{c_0}(t\xi)$, cf. Equation (9.3). Because $c_0$ is constant, $\delta'(c_0) = 0$, and because $c_1$ is a geodesic on the loop space $\frac{d}{dt}\big|_{t=0} \delta'(c_t) = 0$. Thus

$$\delta'(c_1) = D^2E(\xi, \xi) + o(\|\xi\|_{H^1}).$$

We see that for small enough loops $D^2E(\xi, \xi) \leq 2\epsilon_2$. But at a constant loop $D^2E(\xi, \xi) = \|\nabla \xi\|_{L^2}$, Thus, by the inequality $\|\xi\|_{L^2} \leq \|\xi\|_{C^0}$, Equation (3.3), we have that

$$\|\xi\|_{H^1}^2 = \|\xi\|_{L^2}^2 + \|\nabla \xi\|_{L^2}^2 \leq \epsilon_1^2 + 2\epsilon_2.$$

Because we control the size of $\epsilon_1$, and therefore of $\epsilon_2$ we can ensure $h_0$ lands in $V$. 

**Figure 11.8.** The spine map $\tilde{c}_{k+1}$. The spines of $\bigtriangleup_{k+1}$ are mapped to geodesics in $U$. 

**Tubular neighborhood.** We need to argue that $h_0$ lies in the tubular neighborhood $V$ of $M$ in $\Lambda M$. For this it is sufficient to show that the $H^1$ distance of a loop $c_1 := h_0(q)$ and a constant loop $c_0 \in \iota(M)$ is as small as we want, see proposition 16. We consider the fixed simplex $l_{k+1}$, and a point $q$ in its interior, and denote by $c_0$ the constant loop at $q$. Because we control the size of $\epsilon_1$, and a point $q$ in its interior, and denote by $c_0$ the constant loop at $q$. We need to argue that $h_0$ lands in $V$. 

**Lemma 28.** Suppose $c_0$ is a constant loop and $c_1 \in \Lambda M$ is such that $d(\gamma_0(c_0, c_1) < \epsilon_1 < \text{inj } M$ and $\delta'(c_1) < \epsilon_2$, then $\|\xi\|_{H^1} \leq \sqrt{\epsilon_1^2 + 2\epsilon_2}$.
The homotopy $f_t$. For each 0-simplex $l_0^i$ in $T_U \setminus T_W$ construct paths starting at $x_i$ and $\tilde{x}_i$, and ending at points in $M \setminus (N \cup W)$, making sure that these paths do not intersect. Define $f_t : M \rightarrow M$, to be the identity outside small enough balls around the $x_i$'s, and $\tilde{x}_i$'s. For all points inside these balls, trace a small tube along the path constructed, cf. Figure 11.9, making sure that $f_1(x_i) \not= f_1(\tilde{x}_i)$ and $f_1(x_i), f_1(\tilde{x}_i) \in M \setminus N \cup W$. There are no topological obstructions to this construction. To see this, recall that we assumed that all connected components of $U$ intersect $W$, and we could construct the paths lying in $U$.

Estimates. Define $h_t = f_t \circ h_0$. The set $U$ is compact, $h_1$ is a continuous function hence there exists a constant such that $V(g_1(u)(s)) > -\tilde{C}$ for some $\tilde{C} > 0$. Define $\tilde{C}$ such that $V(f_1(x_i)), V(f_1(\tilde{x}_i)) > \tilde{C} > 0$. If the mollifying parameter $\tilde{\varepsilon} > 0$ has been chosen sufficiently small the loop $g_1(u)$ is locally constant on a set of measure at least $1 - 2(k + 1)\tilde{\varepsilon}$, with values in $\bigcup_i V(f_1(x_i)) \cup V(f_1(\tilde{x}_i))$. Thus we can bound

$$
\int_0^1 V(g_1(u)(s))ds \geq (1 - 2(k + 1)\tilde{\varepsilon})\tilde{C} - 2(k + 1)\tilde{\varepsilon} \tilde{C} > 0,
$$

(11.10)

if $\tilde{\varepsilon}$ is chosen sufficiently small. If $u \in T_W$ then $g_1(u)(s) = u \in W$ and $V|_W > 0$. Therefore $h_t$ satisfies property (iv). Remark that $h_t(u)$ attains at least two different values if $u \notin T_W$, and is thus never constant, so that we satisfy (iii). This completes the construction of the hedgehog function.
12. Appendix: Local computations

In this section we show that equation (2.2) is valid. The computation is done in local coordinates. We use Einstein’s summation convention, and we denote by \( \partial_i \) the \( i \)-th partial derivative with respect to the coordinates on the base manifold. Similarly the the metric tensor \( g^{ij} \) and the Christoffel symbols \( \Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{kj} + \partial_j g_{ik} - \partial_l g_{ij}) \) refer to those on the base manifold \( M \). We omit the subscript \( q \) in \( \vartheta_q \) and write \((q, \vartheta) \in T_q M\). In local coordinates the Hamiltonian is expressed as

\[
H(q, \vartheta) = \frac{1}{2} g^{ij} \partial_i \vartheta_j + V(q),
\]

and the Hamiltonian vector field is

\[
X_H = \left( -\partial_i + \Gamma_i^k g^{lj} \partial_k \vartheta_j \right) \frac{\partial}{\partial \vartheta_i} + g^{ij} \partial_j \frac{\partial}{\partial \vartheta^i}.
\]

To relieve notational burden we introduce the function \( h = \frac{1}{1 + |\text{grad } V|^2} \) so

\[
f = \vartheta_i v^i = h \partial_i g^{ij} \partial_j V.
\]

Then

\[
X_H(f) = h g^{ij} \partial_j V \left( -\partial_i V + \Gamma_i^k g^{lj} \partial_k \vartheta_j \right) + \partial_i g^{jk} \partial_j \partial_k \left( h g^{lj} \partial_l V \right).
\]

If we rewrite this, relabeling dummy indices, we arrive at

\[
X_H(f) = -h g^{ij} \partial_i \partial_j V + h\vartheta_i \vartheta_j g^{ij} \left( \Gamma_i^k g^{km} \partial_m V + \partial_l (g^{ik} \partial_k V) \right) + \vartheta_i \vartheta_j g^{il} \partial_l V g^{jk} \partial_k h.
\]

The first term, Equation (12.2) is equal to \( \frac{|\text{grad } V|^2}{1 + |\text{grad } V|^2} \), while the third term, Equation (12.4) can be expressed as follows

\[
\vartheta_i \vartheta_j g^{il} \partial_l V g^{jk} \partial_k h = \vartheta(\text{grad } V) \vartheta(\text{grad } h)
\]

\[
= \vartheta(\text{grad } V) \vartheta(h^2 |\text{grad } V|^2)
\]

\[
= \frac{2\vartheta(\text{grad } V) \text{Hess } V(\text{grad } V, \# \vartheta)}{(1 + |\text{grad } V|^2)^2}.
\]

We now focus on the second term, Equation (12.3). If we write out the derivative, and we relabel some dummy indices, we obtain

\[
h\vartheta_i \vartheta_j g^{ij} \left( \Gamma_i^k g^{km} \partial_m V + \partial_l (g^{ik} \partial_k V) \right) = h g^{ik} \partial_i g^{lj} (\partial_l \partial_k V - \Gamma_i^m \partial_m V)
\]

\[
= h \text{Hess } V(\# \vartheta, \# \vartheta) = \frac{\text{Hess } V(\# \vartheta, \# \vartheta)}{1 + |\text{grad } V|^2}.
\]

Combining all the terms gives Equation (2.2).
REFERENCES


