On the Integrability of Non-Polynomial Scalar Evolution Equations

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We show the existence of infinitely many symmetries for λ-homogeneous equations when $\lambda = 0$. If the equation has one generalized symmetry, we prove that it has infinitely many and these can be produced by recursion operators. Identifying equations under homogeneous transformations, we find that the only integrable equations, in this class are Potential Burgers, Potential modified Korteweg-de Vries and Potential Kupershmidt. We can draw some conclusions from these results for the case $\lambda = -1$, which, although theoretically incomplete, seem to cover the known integrable systems for this case.

1. INTRODUCTION

The paper is devoted to the integrability of the k^{th}–order equation of the form

$$u_t = u_k + f(u, \cdots, u_{k-1}), \qquad u_i = \frac{\partial^i u}{\partial x^i} \tag{1}$$

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A system is said to be integrable if it has infinitely many independent symmetries. We sometimes call this symmetry-integrable, to avoid confusion with other possible definitions of integrability, but we have nothing to say about these others in this paper.

In most interesting integrable evolution equations, the right-hand side of equation is a homogeneous differential polynomial under a suitable weighting of its constituent monomials with coefficients in some ring of C^∞–functions of zero weight monomials. We introduce a weighting scheme by assigning a weight $\lambda + i$ to u_i. The weight of a monomial is the sum of the weights of its multiplicands. The differential equation (1) is said to be λ–homogeneous if all monomials in its right-hand side have the same weight. For example, the Korteweg–de Vries equation $u_t = u_3 + uu_1$ is 2–homogeneous of weight 3 (viewing the equation as the vectorfield $(u_3 + uu_1)\frac{\partial}{\partial u}$). The coefficient ring in this case is $\mathbb{R}$ or $\mathbb{C}$ since there are no monomials of zero weight.

If $m_1, \cdots, m_l$ are zero weight monomials, we denote the coefficient ring of C^∞–functions of them by $C^\infty(m_1, \cdots, m_l)$. The 0-homogeneous polynomials of weight 3 are generated by: u_3, u_1u_2 and u_1^3 with coefficient ring $C^\infty(u)$. At the final stage of the classification is does matter whether the functions are real or complex valued, since this determines whether some final constant can be scaled to 1 (in the complex case) or ± 1 (in the real case). Also we take the roots of functions quite formally and in concrete problems one should be careful to apply the results to the real case without going through the proof.

In [16], we produced the complete list of λ–homogeneous (symmetry) integrable equations with positive λ.

We now mention some equations which have appeared in the literature as integrable equations and which do not fall under the classification given in [16] since u's weight λ not strictly positive.

- The potential Burgers' equation $u_t = u_2 + u_1^2$, where $\lambda = 0$.
- The Krichever-Novikov equation (cf. [7])

$$u_t = u_3 - \frac{3}{2u_1}u_2^2, \qquad \lambda = -1. \tag{2}$$

- The Potential Kupershmidt equation (cf. [2] and [11], eq. [4.2.7])

$$u_t = u_5 + 5u_2u_3 - 5u_1^2u_3 - 5u_1u_2^2 + u_1^5, \qquad \lambda = 0. \tag{3}$$

- The following non-polynomial homogeneous equation with $\lambda = -1$ appeared in the list of integrable scalar 5^{th}-order equations in [11], equation

[4.2.13].

$$u_t = u_5 - \frac{15}{2}\frac{e^{5f}+2e^{2f}}{(e^{3f}-1)^2}u_2u_4 - \frac{45}{4}\frac{e^{2f}}{(e^{3f}-1)^2}u_3^2 + \frac{45}{4}\frac{(e^{10f}+22e^{7f}+13e^{4f}}{(e^{3f}-1)^4}u_2^2u_3 - \frac{3645}{16}\frac{2e^{12f}+4e^{9f}+e^{6f}}{(e^{3f}-1)^6}u_2^4, \tag{4}$$

where the $f \in C^\infty(u_1)$ is defined by the algebraic equation

$$2e^{3f} - 3u_1e^{2f} + 1 = 0.$$

In the present paper, we give the complete list of 0–homogeneous integrable systems. It turns out that such integrable equations are equivalent up to homogeneous transformations to the equations contained in hierarchies of:

- Heat equation/Potential Burgers: $u_t = u_2$.
- Potential modified Korteweg-de Vries: $u_t = u_3 + u_1^3$.
- Potential Kupershmidt equation (3).

We derive a list for $\lambda = -1$ from the results for $\lambda = 0$. This result is not complete since the coefficients only depend on u_1 (and not on uu_2, for example). However, it seems to cover all known such examples in the literature due to [8].

We first sketch the general method to classify integrable systems and then show the results for $\lambda > 0$ are valid for the case $\lambda = 0$. In section 3 we completely classify the symmetry-integrable 0–homogeneous equations. We use the computational results for $\lambda = 0$ to derive some integrable equations of $\lambda = -1$ in section 4, and we compare them with the literature, mainly [11]. Finally, we discuss the difficulties which one encounters in extending these results to the case $\lambda < 0$ in section 5. The main results rely on extensive computer algebra computations. We list the results as they come out from these and produce their recursion operators and Hamiltonians in Appendix 1.

2. CLASSIFICATION OF INTEGRABLE SYSTEMS

In [16] we have shown that the symmetry-integrability is determined by

- the existence of one nontrivial symmetry,
- the existence of approximate symmetries.

This led to the proof of a long-standing conjecture that "in all known cases the existence of one generalized symmetry implies the existence of infinitely many" [9] under fairly relaxed conditions.

To this end we formulated a theorem that is valid in the context of filtered Lie algebra modules. * We used it in the $\lambda > 0$ case to conclude that the proof of the integrability for an equation

- of order 2 needs a symmetry of order 3,
- of order 3 needs a symmetry of order 5,
- of order 5 needs a symmetry of order 7,
- of order 7 needs a symmetry of order 13.

This result reduced the classification of integrable equations to a computer algebra problem. The basic algorithm to find a symmetry of a given order is fairly straightforward:

- One writes out the general form of the equation and the symmetry. The implicit assumption here is that the search space can be defined in such a way that this is possible.
- One computes the Lie bracket of the equation and the candidate symmetry.
- One derives the equations for the coefficients which have to be satisfied to let the bracket vanish according to the definition of symmetry, cf. [12].
- Depending on whether the coefficients are constants or C^∞–functions, one uses (differential) Gröbner bases to find the generators of the (differential) ideal defined by these equations.
- Allowing for homogeneous transformations, the equation is then put into normal form. In the $\lambda > 0$ case the only transformations allowed are linear scalings of u, but for $\lambda = 0$ we allow for transformations of the type $u \to f(u)$.

The computation is described in somewhat more detail in [15].

In this paper we apply the theoretical results to the case $\lambda = 0$, in which case we have a filtered Lie algebra $\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n \supset \cdots$. Take a typical 0-homogeneous term in the differential equation, say $f(u)u_1u_2$. We will consider this to be an element of $\mathcal{F}^1$. In general, we say that $f(u)u_1^{k_1}\cdots u_m^{k_m} \in \mathcal{F}^{k_1+\cdots+k_m-1}$.
From the formula for the Lie bracket

$$[X, Y] = D_Y[X] - D_X[Y],$$

we see that if $X \in \mathcal{F}^k$ and $Y \in \mathcal{F}^l$, then $[X, Y] \in \mathcal{F}^{k+l}$. Therefore, we can speak of a filtered Lie algebra since $[\mathcal{F}^k, \mathcal{F}^l] \subset \mathcal{F}^{k+l}$. The filtering induces a grading given by $\mathcal{G}^n = \mathcal{F}^n/\mathcal{F}^{n+1}$. So, we can do all our calculations step by step in the graded spaces $\mathcal{G}^n$.

*The theorem is listed in Appendix 1 for convenience. The details can be found in [16].

The result in $\mathcal{G}^{k+l}$ does not contain any derivatives of the coefficients f when we compute $[\mathcal{G}^k, \mathcal{G}^l]$, in other words, the coefficients behave like constants. This guarantees that the conditions in Theorem 1.1 can be checked without difficulty, since these are formulated in terms of an approximate symmetry, the computation of which only depends on calculations, typically, in $\mathcal{G}^1$ and $\mathcal{G}^2$.

Let us give a concrete example here. We calculate the bracket of u_2 and fu_1u_2, where $f \in C^\infty(u)$. One has the Fréchet derivatives

$$D_{u_2} = D_x^2, \qquad D_{fu_1u_2} = fu_1D_x^2 + fu_2D_x + f'u_1u_2.$$

Therefore

$$\begin{aligned}
&[u_2, fu_1u_2] = D_{fu_1u_2}[u_2] - D_{u_2}[fu_1u_2] = \\
&= \left(fu_1D_x^2 + fu_2D_x + f'u_1u_2\right)u_2 - D_x^2 fu_1u_2 \\
&= fu_1u_4 + fu_2u_3 + f'u_1u_2^2 - D_x(f'u_1^2u_2 + fu_2^2 + fu_1u_3) \\
&= -2fu_2u_3 - 2f'u_1u_2^2 - 2f'u_1^2u_3 - f''u_1^3u_2 \\
&= -2fu_2u_3 \ mod\ \mathcal{F}^2.
\end{aligned}$$

For $\lambda = 0$ the space of monomials of a fixed positive weight is again finitely generated, as long as we consider it as an algebra with coefficients in $C^\infty(u)$. The equations and their symmetries will be polynomials in $u_1, u_2, \cdots$. Therefore, the results for positive λ in [16] are automatically valid for this case.

Theorem 2.1. *A nontrivial symmetry of a* 0*-homogeneous equation is part of a hierarchy starting at order* 3, 5 *or* 7 *in the odd case, and at order* 2 *in the even case.*

Proof. See the proof of Theorem 5.7 in [16]. ∎

In order to obtain the complete list of 0–homogeneous integrable equations, one only needs to consider the equations of order 2, 3, 5, and 7 and their candidate symmetries of order 3, 5, 7, and 13, respectively. One now computes the Lie bracket of the equation and the candidate symmetry and derives ODEs for the coefficients to make the Lie bracket vanish. The system of ODEs can be analyzed using the algorithm in [4] and [5] as implemented in the Maple package Diffalg. This leads to a system of ODEs that determines the integrable equations.

3. THE COMPLETE RESULTS FOR $\lambda = 0$

3.1. Symmetries of 2nd- and 3rd-order equations

In the sequel the $b_{ij}, c_{ij}, \phi, \psi$ and χ stand for elements in $C^\infty(u)$.
The 2nd-order equation:

$$u_t = u_2 + b_{11}u_1^2 \tag{5}$$

is known to be integrable [9]. The typical example is potential Burgers if $b_{11} = 1$. Note that it can be linearized, that is, $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2}$ if let $b_{11} = \phi''/\phi'$.
We consider the family of 3rd-order equations

$$u_t = u_3 + 3b_{11}u_1u_2 + b_{21}u_1^3. \tag{6}$$

One finds, cf. [15, 10], that the only obstruction equation for the existence of the 5th-order symmetry, i.e, the obstruction for integrability reads

$$\frac{\partial b_{21}}{\partial u} = \frac{\partial^2 b_{11}}{\partial u^2} + 2b_{11}b_{21} - 2b_{11}^3. \tag{7}$$

We can simplify this by putting $b_{11} = \psi'/\psi$ (here we start using the freedom to apply homogeneous transformations). Then the obstruction leads to

$$\frac{\partial b_{21}}{\partial u} = 2\frac{\psi'}{\psi}b_{21} + \frac{\psi'''}{\psi} - 3\frac{\psi'\psi''}{\psi^2} \tag{8}$$

Put $b_{21} = \psi^2\chi$. Then one finds

$$\chi' = \left(\frac{\psi''}{\psi^3}\right)'$$

or $b_{21} = \psi''/\psi + C\psi^2$, where C is constant. If we now put $\psi = \phi'$ the integrable equation is converted to

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^3} + C\left(\frac{\partial \phi}{\partial x}\right)^3.$$

We can therefore restrict our analysis to systems with $b_{11} = 0$ and $b_{21} = 0$, which is in the hierarchy of (5), or $b_{21} = 1$, which is known as the Potential modified Korteweg–de Vries equation (see [18] for its recursion operator).

3.2. Symmetries of 5th-order equations

We only need to look for 7th-order symmetries of 5th-order equation, since any symmetry of order 3 or 5 (mod 6) automatically gives rise to a symmetry of order 1 (mod 6).

Assume that the equations have nonzero quadratic terms. Otherwise the analysis reduces to 3^{rd}-order equations.

We consider the following family

$$u_t = u_5 + c_{11}u_1u_4 + c_{12}u_2u_3 + c_{21}u_1^2u_3 + c_{22}u_1u_2^2 + c_{31}u_1^3u_2 + c_{41}u_1^5,$$

and look for the condition that this equation possesses a 7^{th}-order symmetry in the same way that we did for 3^{rd}-order equation before. Although straightforward in principle, the calculation is quite large. The computational results are listed in Appendix 1. Here we analyze them for the different cases respectively.

Case I The quadratic terms are equal to $c_{12}(u_1u_4/2+u_2u_3)$ due to $c_{11} = c_{12}/2$. Its correspondent symbolic expression is

$$c_{12}(\xi_1\xi_2^4 + 2\xi_1^2\xi_2^3 + 2\xi_1^3\xi_2^2 + \xi_1^4\xi_2)/4 = c_{12}\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)/4.$$

Therefore, this equation has a 3^{rd}-order symmetry since $\xi_1^2 + \xi_1\xi_2 + \xi_2^2$ divides the quadratic terms. In other words, it is the image under the recursion operator of the 3^{rd}-order family (6).

Case II Once we know c_{11} and c_{12}, the other parameters in the equation are determined completely. We can simplify

$$\frac{\partial c_{11}}{\partial u} = \frac{1}{5}c_{11}^2 - \frac{1}{10}c_{11}c_{12} + \frac{1}{2}\frac{\partial c_{12}}{\partial u}$$

by putting $c_{11} = 5\frac{\psi'}{\psi}$. Then the obstruction leads to

$$\frac{\partial c_{12}}{\partial u} = \frac{\psi'}{\psi}c_{12} + 10\frac{\psi''}{\psi} - 20\frac{\psi'2}{\psi^2}$$

Put $c_{12} = \psi\chi$. Then one finds

$$\chi' = 10\left(\frac{\psi'}{\psi^2}\right)'$$

or $c_{12} = 10\psi'/\psi + C\psi$, where C is constant. If we now put $\psi = \phi'$ the corresponding integrable equation is converted to

$$\frac{\partial\phi}{\partial t} = \frac{\partial^5\phi}{\partial x^5} + C\frac{\partial^2\phi}{\partial x^2}\frac{\partial^3\phi}{\partial x^3} - \frac{C^2}{5}\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^3\phi}{\partial x^3} - \frac{C^2}{5}\frac{\partial\phi}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \frac{C^4}{625}\left(\frac{\partial\phi}{\partial x}\right)^5.$$

When $C = 5$, this is Potential Kupershmidt equation (3).

3.3. Reduction of 7^{th}-order λ-homogeneous equations

Suppose one can show that for a given 7^{th}-order equation to have a 13^{th}-order symmetry implies that the quadratic part of the equation in symbolic form divides through $\xi_1^2+\xi_1\xi_2+\xi_2^2$, then $\tilde{K}^1/(\xi_1^2+\xi_1\xi_2+\xi_2^2)$ is the quadratic part of a 5^{th}-order symmetry. We can now view the 7^{th}-order equation as a symmetry of a 5^{th}-order equation.

To this end we have analyzed the 7^{th}-order homogeneous equation with the following quadratic terms:

$$e_{11}u_1u_6 + e_{12}u_2u_5 + e_{13}u_3u_4 \tag{9}$$

Using Maple [6] and Form [17] the Lie bracket of the 7^{th}-order equation and the 13^{th}-order symmetry has been computed (cf. [19]) step by step until we find the necessary condition

$$e_{11} = 2e_{12} - e_{13}.$$

The quadratic term can be divided by $\xi_1^2+\xi_1\xi_2+\xi_2^2$ under this condition. This reduces the problem of 7^{th}-order equations to 5^{th}-order equations. By now, we have proved the following statement:

THEOREM 3.1. *The integrable nonlinear* 0*–homogeneous equations are equivalent by homogeneous transformations to equations contained in hierarchies of:*

• *Heat equation/Potential Burgers:*

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2}.$$

• *Potential modified Korteweg-de Vries:*

$$\frac{\partial\phi}{\partial t} = \frac{\partial^3\phi}{\partial x^3} + \left(\frac{\partial\phi}{\partial x}\right)^3.$$

• *Potential Kupershmidt equation:*

$$\frac{\partial\phi}{\partial t} = \frac{\partial^5\phi}{\partial x^5} + 5\frac{\partial^2\phi}{\partial x^2}\frac{\partial^3\phi}{\partial x^3} - 5\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^3\phi}{\partial x^3} - 5\frac{\partial\phi}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^5.$$

4. SOME CONSEQUENCES FOR $\lambda = -1$

In this section, we derive the symmetry–integrable equations for $\lambda = -1$ from the results we found for $\lambda = 0$. The method is to put $u = v_1$ and derive the corresponding equation for v, cf. [18].

4.1. Symmetries of 3^{th}-order equations

We put $u = v_1$ and derive the equation for v from the equation (6):

$$v_t \;=\; D_x^{-1}(v_4 + 3b_{11}v_2v_3 + b_{21}v_2^3) = v_3 + \frac{3}{2}b_{11}v_2^2 + D_x^{-1}((b_{21} - \frac{3}{2}\frac{\partial b_{11}}{\partial v_1})v_2^3).$$

To make this equation local, we require that $b_{21} = \frac{3}{2}\frac{\partial b_{11}}{\partial v_1}$. Substituting it into (7), we have the integrable condition

$$\frac{\partial^2 b_{11}}{\partial {v_1}^2} = 6b_{11}\frac{\partial b_{11}}{\partial v_1} - 4b_{11}^3 \tag{10}$$

for the 3^{rd}-order family

$$v_t = v_3 + \frac{3}{2}b_{11}(v_1)v_2^2. \tag{11}$$

If we now let $b_{11} = -\psi'/2\psi$, (10) becomes

$$\psi''' = 0,$$

i.e. $\psi = c_0 + c_1v_1 + \frac{c_2}{2}v_1^2$. So $b_{11} = -\frac{c_2v_1+c_1}{c_2v_1^2+2c_1v_1+2c_0}$. We make a distinction in

1. $c_2 = 0$. This implies $b_{11} = -\frac{1}{2}\frac{c_1}{c_1v_1+c_0}$. Since $c_1 = 0$ would make things trivial, we can assume $c_1 = 1$ to have $b_{11} = -\frac{1}{2}\frac{1}{v_1+c_0}$. Do transformation $\tilde{v} = v + c_0x$ to obtain [4.1.19] (numbers in [ ] refer to [11]), cf. [18].

2. $c_2 \neq 0$. We can assume $c_2 = 1$ to obtain $b_{11} = -\frac{v_1+c_1}{v_1^2+2c_1v_1+2c_0}$. Let $\tilde{v} = v + c_1x$. Then this leads to $\tilde{b}_{11} = -\frac{\tilde{v}_1}{\tilde{v}_1^2+2c_0-c_1^2}$. There are three subcases as follows:

(i) $c_0 = \frac{c_1^2}{2}$. So $b_{11} = -\frac{1}{v_1}$. This leads to the equation $v_t = v_3 - \frac{3}{2v_1}v_2^2$, which is [4.1.16], the Krichever–Novikov equation (2), cf. [18].

(ii) $c_0 > \frac{c_1^2}{2} \neq 0$. We put $\tilde{c}_0^2 = 2c_0 - c_1^2$, scale $\tilde{v} = \tilde{c}_0 w$ and obtain $w_t = w_3 - \frac{3}{2}\frac{w_1}{1+w_1^2}w_2^2$. This is the equation [4.1.14].

(iii) $c_0 < \frac{c_1^2}{2} \neq 0$. We put $\tilde{c}_0^2 = -2c_0 + c_1^2$, scale $\tilde{v} = \tilde{c}_0 w$ and obtain $w_t = w_3 - \frac{3}{2}\frac{w_1}{w_1^2-1}w_2^2$.

In the complex case the last two cases can be identified.

The integrable system (11) has a recursion operator

$$\mathfrak{R} = D_x^{-1}\left(D_x + b_{11}v_2 + v_2 D_x^{-1}(\frac{\partial b_{11}}{\partial v_1} - 2b_{11}^2)v_2\right)(D_x + b_{11}v_2)D_x. \quad (12)$$

REMARK 4.1. *When* $\frac{\partial b_{11}}{\partial v_1} - 2b_{11}^2 = 0$*, that is the case with* $c_2 = 0$*. Note that* $\mathfrak{R} = \mathcal{P}^2$ *where* $\mathcal{P} = D_x^{-1}(D_x + b_{11}v_2)D_x$*. But* $\mathcal{P}$ *itself is not a recursion operator. This reflects the equation derives from the* $3^{\rm rd}$*-order Potential Burgers.*

4.2. Symmetries of $5^{\rm th}$-order equations

Similarly, we put $u = v_1$ and derive the $5^{\rm th}$-order equation for v. This leads to

$$v_t = v_5 + \tilde{c}_{11}v_2v_4 + \tilde{c}_{12}v_3^2 + \tilde{c}_{22}v_2^2v_3 + \tilde{c}_{31}v_2^4,$$

where the coefficients are elements of $C^\infty(v_1)$ with the following relations

$$\begin{array}{ll} \tilde{c}_{11} = c_{11}, & \tilde{c}_{12} = \frac{1}{2}(c_{12} - c_{11}), \\ \tilde{c}_{22} = c_{21} - \frac{\partial c_{11}}{\partial v_1}, & \tilde{c}_{31} = \frac{1}{4}(c_{31} - \frac{\partial c_{21}}{\partial v_1} + \frac{\partial^2 c_{11}}{\partial {v_1}^2}), \\ c_{22} = \frac{1}{2}(\frac{\partial c_{12}}{\partial u} - 5\frac{\partial c_{11}}{\partial u} + 4c_{21}), & c_{41} = \frac{1}{4}(\frac{\partial c_{31}}{\partial u} - \frac{\partial^2 c_{21}}{\partial u^2} + \frac{\partial^3 c_{11}}{\partial u^3}). \end{array}$$

Using the Maple package Diffalg, we get the two cases corresponding to those when $\lambda = 0$ as follows:

1. (Case I) The relation $c_{11} = \frac{1}{2}c_{12}$ leads to $\tilde{c}_{11} = 2\tilde{c}_{12}$. So, the quadratic terms are equal to $\tilde{c}_{11}(u_2u_4+2u_3u_3)$. Its correspondent symbolic expression is

$$\tilde{c}_{12}(\xi_1^2\xi_2^4 + \xi_1^3\xi_2^3 + \xi_1^4\xi_2^2) = \tilde{c}_{12}\xi_1^2\xi_2^2(\xi_1^2 + \xi_1\xi_2 + \xi_2^2).$$

Hence, this equation has a $3^{\rm rd}$-order symmetry since $\xi_1^2 + \xi_1\xi_2 + \xi_2^2$ divides the quadratic terms. In other words, it is the image under the recursion operator (12) of the $3^{\rm rd}$-order family (11).

2. (Case II) There are two different cases:

(i) Let $\tilde{c}_{12} = 0$. Then $\frac{\partial \tilde{c}_{11}}{\partial v_1} = \frac{1}{5}\tilde{c}_{11}^2$, $\tilde{c}_{22} = \frac{1}{5}\tilde{c}_{11}^2$, $\tilde{c}_{31} = 0$. Solving the ordinary differential equation $\frac{\partial \tilde{c}_{11}}{\partial v_1} = \frac{1}{5}\tilde{c}_{11}^2$, we have $\tilde{c}_{11} = -\frac{5}{u_1+c}$. This leads to $u_t = u_5 - 5\frac{u_2u_4}{u_1} - 5\frac{u_2^2u_3}{u_1^2}$, the equation [4.2.11] in the list of [11] by taking all the parameters in it zero.

(ii) Let $\tilde{c}_{12} \neq 0$. The integrable equation is determined by the relations

$$\begin{aligned}
\tilde{c}_{11}\tilde{c}_{12} &= \frac{5}{4}\frac{\partial \tilde{c}_{12}}{\partial v_1} + \tilde{c}_{12}^2, \\
\tilde{c}_{22}\tilde{c}_{12}^2 &= \frac{1}{5}\tilde{c}_{12}^4 + \frac{7}{2}\tilde{c}_{12}^2\frac{\partial \tilde{c}_{12}}{\partial v_1} + \frac{5}{16}(\frac{\partial \tilde{c}_{12}}{\partial v_1})^2, \\
\tilde{c}_{31}\tilde{c}_{12} &= -\frac{2}{25}\tilde{c}_{12}^4 + \frac{3}{5}\tilde{c}_{12}^2\frac{\partial \tilde{c}_{12}}{\partial v_1} + \frac{9}{8}(\frac{\partial \tilde{c}_{12}}{\partial v_1})^2.
\end{aligned}$$

and solving the ordinary differential equation

$$\frac{\partial^2 \tilde{c}_{12}}{\partial {v_1}^2}\tilde{c}_{12} = -\frac{4}{25}\tilde{c}_{12}^4 + \frac{4}{5}\tilde{c}_{12}^2\frac{\partial \tilde{c}_{12}}{\partial v_1} + \frac{5}{4}(\frac{\partial \tilde{c}_{12}}{\partial v_1})^2. \tag{13}$$

Let $\frac{\partial \tilde{c}_{12}}{\partial v_1} = \chi(y)\tilde{c}_{12}^2$, where $\chi \in C^\infty(y)$ and $y = \ln \tilde{c}_{12}$. Then (13) becomes

$$\chi\chi' + \frac{3}{4}\left(\chi - \frac{4}{15}\right)\left(\chi - \frac{4}{5}\right) = 0. \tag{14}$$

We treat it as follows:

a. Let $\chi = \frac{4}{5}$. Using $\tilde{c}_{11}\tilde{c}_{12} = \frac{5}{4}\frac{\partial \tilde{c}_{12}}{\partial v_1} + \tilde{c}_{12}^2$, we have $\tilde{c}_{11} = 2\tilde{c}_{12}$. This reduced to Case I.

b. Let $\chi = \frac{4}{15}$. Solving the equation $\frac{\partial \tilde{c}_{12}}{\partial v_1} = \frac{4}{15}\tilde{c}_{12}^2$, we obtain $\tilde{c}_{12} = -\frac{15}{4v_1+\alpha}$, where α is constant. This leads to

$$v_t = v_5 - 5\frac{v_2v_4}{v_1} - \frac{15}{4}\frac{v_3^2}{v_1} - \frac{65}{4}\frac{v_2^2v_3}{v_1^2} - \frac{135}{16}\frac{v_2^4}{v_1^3},$$

the equation [4.2.12] in [11], taking all the parameters zero.

c. Let $\chi \neq \frac{4}{15}$ and $\chi \neq \frac{4}{5}$. Solving the equation (14), we obtain

$$\tilde{c}_{12} = \frac{\alpha(15\chi-4)^{\frac{2}{3}}}{(5\chi-4)^2} \quad v_1 = \frac{-100\chi}{\alpha(15\chi-4)^{\frac{2}{3}}} + \beta, \tag{15}$$

where $\alpha \neq 0, \beta$ are constant. We take $\beta = 0$ without losing generality. In fact, the transformation $v \to v + \beta x$ does the job.
Let $\chi = \frac{4}{5}v_1g^2$, where $g \in C^\infty(v_1)$ and $\alpha = -20$ (rescale v). Substituting them into the formula (15), we obtain:

$$4g^6 = (3v_1g^2+1)^2, \quad \tilde{c}_{12} = -5\frac{g^2}{(v_1g^2-1)^2}. \tag{16}$$

Analyzing them according to the values of g and v_1, we get

$$-2g^3 - 3v_1 g^2 + 1 = 0, \quad \tilde{c}_{12} = -\frac{45}{4}\frac{g^2}{(g^3+1)^2},$$

where we have absorbed the sign choice in g. To see what happens it helps to express χ in g:

$$\chi = \frac{4}{15}(1 - 2g^3).$$

Considering the limits for $g \to \infty$, $g \to 0$, $g \to -1$ and $g \to -\infty$, we obtain the following table.

g	∞	0	-1	$-\infty$
v_1	$-\infty$	∞	1	∞
χ	$-\infty$	$\frac{4}{15}$	$\frac{4}{5}$	∞
$\tilde{c}_{12}$	0	0	∞	0

From this table we obtain the following result. There are three solutions. For $0 \le g < \infty$, for which the v_1 domain is $(-\infty, \infty)$. One has $\lim_{v_1 \to \pm\infty} \tilde{c}_{12} = 0$. We plot $\tilde{c}_{12}$ against v_1 in Figure 1. Then there is a solution for $-1 \le g \le 0$, with v_1 domain $(1, \infty)$, plotted in Figure 2 and a solution with the same v_1 domain for $-\infty < g \le -1$, plotted in Figure 3. Both of these have $\lim_{v_1 \to 1} \tilde{c}_{12} = \infty$ and $\lim_{v_1 \to \infty} \tilde{c}_{12} = 0$.

Remark 4.2. *We would take $g = \pm e^f$ in (16), contrary to the choice in [11], we get*

$$\pm 2e^{3f} - 3v_1 e^{2f} + 1 = 0, \quad \tilde{c}_{12} = -\frac{45}{4}\frac{e^{2f}}{(\pm e^{3f} - 1)^2}.$$

This leads to equation (4) in the "+" case. Figure 1 coresponds the "-" case, Figure 3 corresponds the "+" case for $f \ge 0$ and Figure 2 corresponds the "+" case for $f \le 0$.

Remark 4.3. *A recursion operator of the integrable equations in case II can be obtained from $\mathfrak{R}(u)$ in (2.1), that is, $D_x^{-1}\mathfrak{R}(v_1)D_x$.*

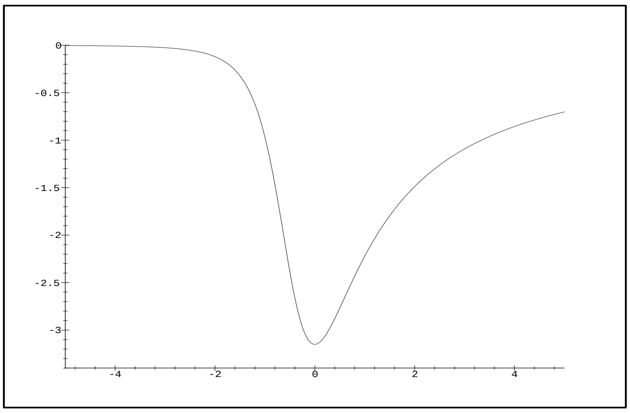

FIG. 1. Horizontal: v_1, vertical $\tilde{c}_{12}$

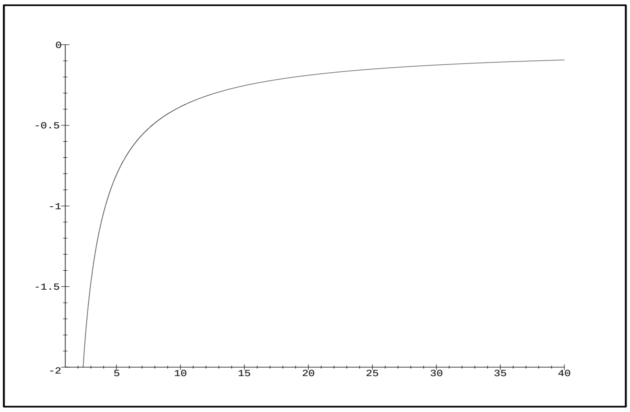

FIG. 2. Horizontal: v_1, vertical $\tilde{c}_{12}$

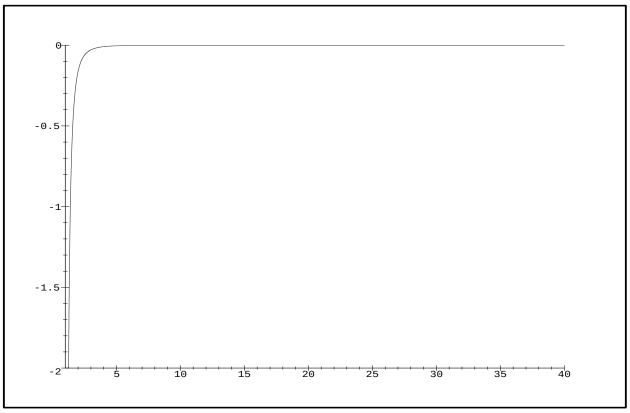

FIG. 3. Horizontal: v_1, vertical $\tilde{c}_{12}$

5. CONCLUDING REMARKS, OPEN PROBLEMS

One would expect the same methods to work for $\lambda < 0$. However, there are a few problems here.

• Relations among generators. When we have computed the Lie bracket, we need to have a basis for λ–homogeneous polynomials with weight zero coefficients. Therefore we need a Stanley decomposition (cf. [3]) of these polynomials. Preliminary calculations by K. Gatermann show that this is indeed possible for $\lambda = -\frac{3}{2}$, a test case chosen for its difficulty.

• Filtering. For $\lambda = 0$ the situation is simple: upon differentiating a function $f(u)$, one gets $\frac{\partial f}{\partial u} u_1$, and so the result is in a higher filtration space and can be ignored at the graded calculation form the moment. This is not so simple for $\lambda < 0$, since one can get out expressions with negative weight, which may recombine to give weight zero monomials.

• Diffalg, PDEs. The expressions one obtains are often too big for Diffalg.

• The noncommutative case (cf. [13, 14]) for $\lambda \leq 0$ seems to offer some interesting technical problems. It seems most promising to handle the cyclic case first.

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APPENDIX 1

An implicit function theorem

We repeat the main theorem in [16] that is used to prove Theorem 2.1.

In order to see the use of this theory, it helps if one thinks of K^0 as the linear part of the equation, K as the equation, S is a given symmetry, and $\bar{Q}$ is an approximation of a higher order symmetry for which the terms in graded space $\mathcal{G}^i$ have already been computed for $i \leq l$. Theorem 1.1 now states that under some technical conditions $\bar{Q}$ can be uniquely extended to a symmetry Q, commuting with K and S.

Consider a filtered Lie algebra $\mathcal{F} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n \supset \cdots$ and let $\mathcal{V}$ be a filtered $\mathcal{F}$-module $\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \cdots \supset \mathcal{V}^n \supset \cdots$ (with $\bigcap_{i=0}^{\infty} \mathcal{V}^j = 0$), where the action of $\mathcal{F}$ on $\mathcal{V}$ is such that if $X^i \in \mathcal{F}^i$ and $v^j \in \mathcal{V}^j$, then $X^i \cdot v^j \in \mathcal{V}^{i+j}$.

In the present paper $\mathcal{V}$ and $\mathcal{F}$ are to be identified, and the action is simply the one given by the Lie bracket. Nevertheless the reader should realize the potential of this theorem in the classification of cosymmetries, conservation laws, recursion operators, etc.

Definition 1.1. We call $S^0 \in \mathcal{F}^0$ relatively l-prime with respect to $K^0 \in \mathcal{F}^0$ if $S^0 \cdot v^j \in \operatorname{Im} K^0 \pmod{\mathcal{V}^{j+1}} \Rightarrow v^j \in \operatorname{Im} K^0|_{\mathcal{V}^j} \pmod{\mathcal{V}^{j+1}}$ for all $j \geq l$ and $v^j \in \mathcal{V}^j$.

In less precise wording this means that if $S^0 v^j \in \operatorname{Im} K^0$, then $v^j \in \operatorname{Im} K^0$. If S^0_j and K^0_j as symbols of the linear part of S^0 and K^0 acting on j-linear terms respectively, then this says that S^0_j and K^0_j are relatively prime for $j \geq l$. To show that S^0 is relatively $l+1$-prime with respect to K^0 is a nontrivial application of diophantine approximation theory for low l (cf. [1, 18]).

Definition 1.2. We call $K^0 \in \mathcal{F}^0$ nonlinear injective if for all $v^l \in \mathcal{V}^l$, $l > 0$, $K^0 \cdot v^l \in \mathcal{V}^{l+1} \Rightarrow v^l \in \mathcal{V}^{l+1}$.

In less precise wording this means that if $v^j \in \ker K^0$, then $j < l$. One usually sees only $l = 1$, since the scalar linear systems all commute. To check that K^0 is nonlinear injective is fairly straightforward in the scalar case.

Theorem 1.1. *Let $K^0, S^0 \in \mathcal{F}^0$ and $K^1, S^1 \in \mathcal{F}^1$. Put $K = K^0 + K^1$ and $S = S^0 + S^1$. Suppose there exists some $\bar{Q} \in \mathcal{V}^0$ such that*

- *$[K, S] = 0$,*
- *K^0 is nonlinear injective,*
- *S^0 is relatively $l+1$-prime with respect to K^0 (this implies that $K \neq S$),*

and there exists some $\bar{Q} \in \mathcal{V}^0$ such that

- *$K \cdot \bar{Q} \in \mathcal{V}^{l+1}$ and $S^0 \cdot \bar{Q} \in \mathcal{V}^1$.*

Then there exists a unique $Q = \bar{Q} + Q^{l+1}, Q^{l+1} \in \mathcal{V}^{l+1}$ such that

$$K \cdot Q = S \cdot Q = 0.$$

This means that once one has a symmetry $S \neq K$ of a system K, finding other invariants of an action of K is a matter of computing terms up till a fixed finite order. In other words, this is an implicit function theorem.

Although the formulation of the theorem and its proof are most naturally done in the filtered context, it is not difficult to derive explicit formulae in the graded case. The reader is advised to do so if the present approach seems too abstract.

APPENDIX 2

The computational results for $\lambda = 0$

We list the computational results for $\lambda = 0$ and give the recursion operators. The results for $\lambda = -1$ in section 4 are based on them.

2.1. RESULTS FOR 2^{ND}- AND 3^{RD}-ORDER EQUATIONS

There exists a 5^{th}-order (infinitively many) symmetry for the family

$$u_t = u_3 + 3b_{11}u_1u_2 + b_{21}u_1^3$$

if and only if

$$\frac{\partial b_{21}}{\partial u} = \frac{\partial^2 b_{11}}{\partial u^2} + 2b_{11}b_{21} - 2b_{11}^3.$$

The equation has a recursion operator

$$\mathfrak{R} = (D_x + b_{11}u_1 + 2u_1D_x^{-1}hu_1)(D_x + b_{11}u_1), \tag{2.1}$$

where $h = b_{21} - \frac{\partial b_{11}}{\partial u} - b_{11}^2 \in C^\infty(u)$.

The recursion operator can be split (if $h \neq 0$) as $\mathfrak{R} = \mathfrak{H}\mathfrak{I}$, where

$$\mathfrak{H} = (hD_x + hb_{11}u_1)^{-1}$$

and

$$\mathfrak{I} = h(D_x + b_{11}u_1)(D_x + b_{11}u_1 + 2u_1D_x^{-1}hu_1)(D_x + b_{11}u_1)$$

are the cosymplectic and the symplectic operator, respectively. The Hamiltonian function is given by

$$\frac{1}{2}h\left(u_2^2 - (\frac{2}{3}\frac{\partial b_{11}}{\partial u} + \frac{1}{3}b_{11}^2 + \frac{1}{2}h)u_1^4\right).$$

This is an example of a family of Hamiltonian systems parametrized by $h \in C^\infty(u)$. The only exceptional point is $h = 0$, which derives from the 2^{nd}-order equation (5) with a recursion operator $D_x + b_{11}u_1$.

2.2. RESULTS FOR 5^{TH}-ORDER EQUATIONS

The conditions of the existence of a 7^{th}-order (infinitively many) symmetry for

$$u_t = u_5 + c_{11}u_1u_4 + c_{12}u_2u_3 + c_{21}u_1^2u_3 + c_{22}u_1u_2^2 + c_{31}u_1^3u_2 + c_{41}u_1^5$$

are the following:

Case I

$$\begin{aligned}
\frac{\partial c_{22}}{\partial u} &= \frac{3}{2}\frac{\partial^2 c_{12}}{\partial u^2} + \frac{1}{5}c_{22}c_{12} - \frac{3}{100}c_{12}^3, \\
c_{11} &= \frac{1}{2}c_{12}, \\
c_{21} &= -\frac{1}{2}\frac{\partial c_{12}}{\partial u} + c_{22} - \frac{1}{20}c_{12}^2, \\
c_{31} &= -\frac{9}{20}c_{12}\frac{\partial c_{12}}{\partial u} + \frac{1}{2}c_{22}c_{12} - \frac{13}{200}c_{12}^3 + \frac{\partial^2 c_{12}}{\partial u^2}, \\
c_{41} &= \frac{1}{10}\frac{\partial^3 c_{12}}{\partial u^3} + \frac{1}{25}c_{12}\frac{\partial^2 c_{12}}{\partial u^2} + \frac{3}{200}(\frac{\partial c_{12}}{\partial u})^2 - \frac{2}{25}c_{22}\frac{\partial c_{12}}{\partial u} \\
&\quad - \frac{3}{250}c_{12}^2\frac{\partial c_{12}}{\partial u} + \frac{3}{50}c_{22}^2 + \frac{1}{500}c_{22}c_{12}^2 - \frac{31}{20000}c_{12}^4.
\end{aligned}$$

This is the image under the recursion operator (2.1) of the 3^{rd}-order family with the identifications $c_{12} = 10b_{11}$, $c_{22} = 15b_{21}$.

Case II

$$\begin{aligned}
\frac{\partial c_{11}}{\partial u} &= \frac{1}{5}c_{11}^2 - \frac{1}{10}c_{11}c_{12} + \frac{1}{2}\frac{\partial c_{12}}{\partial u},\\
c_{21} &= -\frac{1}{5}c_{12}^2 + \frac{4}{5}c_{12}c_{11} - \frac{2}{5}c_{11}^2 + \frac{\partial c_{12}}{\partial u},\\
c_{22} &= -\frac{1}{5}c_{12}^2 + \frac{11}{10}c_{12}c_{11} - \frac{4}{5}c_{11}^2 + \frac{3}{2}\frac{\partial c_{12}}{\partial u},\\
c_{31} &= \frac{4}{5}c_{12}c_{11}^2 - \frac{18}{25}c_{11}^3 + \frac{3}{5}c_{11}\frac{\partial c_{12}}{\partial u} + \frac{\partial^2 c_{12}}{\partial u^2} - \frac{1}{5}c_{11}c_{12}^2,\\
c_{41} &= \frac{1}{10}\frac{\partial^3 c_{12}}{\partial u^3} + \frac{1}{10}c_{11}\frac{\partial^2 c_{12}}{\partial u^2} - \frac{1}{100}c_{12}\frac{\partial^2 c_{12}}{\partial u^2} + \frac{3}{100}(\frac{\partial c_{12}}{\partial u})^2\\
&\quad -\frac{1}{25}c_{11}^2\frac{\partial c_{12}}{\partial u} + \frac{7}{100}c_{12}c_{11}\frac{\partial c_{12}}{\partial u} - \frac{19}{1000}c_{12}^2\frac{\partial c_{12}}{\partial u}\\
&\quad +\frac{1}{625}c_{12}^4 - \frac{8}{125}c_{11}^4 + \frac{7}{125}c_{11}^3c_{12} - \frac{9}{1000}c_{11}c_{12}^3.
\end{aligned}$$

First consider the special case $c_{11} = \frac{1}{2}c_{12}$. The equation can be identified with Case I. More specifically, it derives from the 2^{nd}-order equation $u_t = u_2 + \frac{1}{10}c_{12}u_1^2$. Otherwise, let $\gamma = c_{11} - \frac{1}{2}c_{12}$. Then

$$\begin{aligned}
\frac{\partial \gamma}{\partial u} &= \frac{1}{5}c_{11}\gamma,\\
c_{21} &= -\frac{4}{5}\gamma^2 - \frac{2}{5}c_{11}\gamma + \frac{2}{5}c_{11}^2 + 2\frac{\partial c_{11}}{\partial u},\\
c_{22} &= \frac{3}{5}c_{11}^2 - \frac{6}{5}c_{11}\gamma - \frac{4}{5}\gamma^2 + 3\frac{\partial c_{11}}{\partial u},\\
c_{31} &= \frac{2}{25}c_{11}^3 - \frac{8}{25}c_{11}^2\gamma - \frac{4}{5}c_{11}\gamma^2 + \frac{6}{5}c_{11}\frac{\partial c_{11}}{\partial u} + 2\frac{\partial^2 c_{11}}{\partial u^2} - \frac{2}{5}\gamma\frac{\partial c_{11}}{\partial u},\\
c_{41} &= \frac{1}{625}c_{11}^4 - \frac{2}{125}c_{11}^3\gamma - \frac{8}{125}c_{11}^2\gamma^2 + \frac{16}{625}\gamma^4 - \frac{2}{25}c_{11}\gamma\frac{\partial c_{11}}{\partial u}\\
&\quad - \frac{4}{25}\frac{\partial c_{11}}{\partial u}\gamma^2 + \frac{3}{25}(\frac{\partial c_{11}}{\partial u})^2 + \frac{6}{125}c_{11}^2\frac{\partial c_{11}}{\partial u} + \frac{4}{25}c_{11}\frac{\partial^2 c_{11}}{\partial u^2} + \frac{1}{5}\frac{\partial^3 c_{11}}{\partial u^3}.
\end{aligned}$$

The evolution equation in this case has a recursion operator

$$\begin{aligned}
&\mathfrak{R}(u) =\\
&= (D_x + \frac{2}{5}\gamma u_1 + \frac{1}{5}c_{11}u_1 - \frac{8}{25}u_1 D_x^{-1}\gamma^2 u_1)\cdot(D_x + \frac{2}{5}\gamma u_1 + \frac{1}{5}c_{11}u_1)\cdot\\
&(D_x + \frac{1}{5}c_{11}u_1)\cdot(D_x - \frac{2}{5}\gamma u_1 + \frac{1}{5}c_{11}u_1)\cdot\\
&(D_x - \frac{2}{5}\gamma u_1 + \frac{1}{5}c_{11}u_1 - \frac{8}{25}u_1 D_x^{-1}\gamma^2 u_1)\cdot(D_x + \frac{1}{5}c_{11}u_1).
\end{aligned} \tag{2.2}$$

It can be split (if $\gamma \neq 0$) as $\mathfrak{R} = \mathfrak{H}\mathfrak{I}$, where

$$\mathfrak{H} = (\gamma^2 D_x + \frac{1}{5}\gamma^2 c_{11}u_1)^{-1}$$

and $\mathfrak{I} = \mathfrak{H}^{-1}\mathfrak{R}$ are the cosymplectic and the symplectic operator, respectively. The Hamiltonian function is given by

$$-\gamma^2\cdot\Big(\frac{1}{2}u_3^2 + \{\frac{1}{3}\gamma - \frac{3}{10}c_{11}\}u_2^3 + \{\frac{1}{5}c_{11}\gamma + \frac{2}{5}\gamma^2 - \frac{3}{50}c_{11}^2 - \frac{9}{10}\frac{\partial c_{11}}{\partial u}\}u_1^2u_2^2$$

$$+\frac{1}{18750}\{80\gamma^4+525(\frac{\partial c_{11}}{\partial u})^2-40\gamma c_{11}^3+750\frac{\partial^3 c_{11}}{\partial u^3}+3c_{11}^4-180\gamma^2 c_{11}^2$$
$$-600\gamma^2\frac{\partial c_{11}}{\partial u}-300\gamma c_{11}\frac{\partial c_{11}}{\partial u}+120c_{11}^2\frac{\partial c_{11}}{\partial u}+450c_{11}\frac{\partial^2 c_{11}}{\partial u^2}\}u_1^6\Big)$$

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