

The symbolic method and cosymmetry integrability of evolution equations

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ABSTRACT. We determine the existence of cosymmetries for the scalar evolution equations

$$u_t = u_k + f(u, \dots, u_{k-1})$$

by using the symbolic method and emphasize the role played by the permutation group in answering divisibility questions. In some special cases, we conclude that there is only one possible cosymmetry. The method can also be used to classify the evolution equations by generating functions. As an example, we give the complete list for 7th-order KdV-like equations with nontrivial conservation laws.

1. Introduction

In the history of integrable evolution equations a central question has always been: how to determine the number of conservation laws for a given equation? A lot of papers were dedicated to this subject, and several computer programs were developed for this purpose, [GH97, Gök96].

In this paper, we are devoted to answering this question using the symbolic method, [GD75] for an equation of the form

$$(1) \quad u_t = K([u]) = u_k + f(u, \dots, u_{k-1})$$

with f a formal power series starting with terms that are at least quadratic. A *local conservation law* of the equation (1) is a relation of the form $D_t \rho = D_x \sigma$, where ρ and σ are functions of x , t and x -derivatives of u . The function ρ is called the *conserved density*, while σ is called the *conserved flow*. The variational derivative of the conserved density ρ of (1) denoted by Q satisfies

$$(2) \quad D_Q[K] + D_K^*[Q] = 0,$$

where D_Q is the Fréchet derivative of Q and D_K^* is the dual operator of D_K . Note that (2) can be viewed as $L_K Q = 0$, i.e. Q is invariant under the Lie derivative with respect to K . The quantity Q is called a *cosymmetry* of equation (1).

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An equation is called *(co)symmetry-integrable* if it has at infinitely many higher order local (co)symmetries. It is called *almost (co)symmetry-integrable (of depth n)* if it has $n < \infty$ many higher order local (co)symmetries.

In [SW98b] we formulated a theorem stating that if there is a nontrivial symmetry S of an equation $u_t = K$, then one only has to solve the equation

$$L_K Q = 0$$

up till a certain fixed (depending only on K) degree to conclude that the equation can be solved in the filtration topology. We then used this theorem to classify equations with respect to the number of their symmetries. In the scalar case we proved for homogeneous equations of the form (1) that almost symmetry-integrable of depth 1 implies symmetry-integrable. For systems this is no longer the case, since there exist such systems which are not symmetry-integrable, cf [BSW98].

The classification of scalar equations with respect to the number of their cosymmetries is more complicated, since one also encounters equations (usually without a generalized symmetry) with only a finite number of cosymmetries.

In most interesting integrable evolution equations, the right-hand side of the equation is a homogeneous differential polynomial under a suitable weighting scheme. The differential equation (1) is said to be λ -homogeneous of weight μ if it admits the one-parameter group of scaling symmetries

$$(x, t, u) \longmapsto (a^{-1}x, a^{-\mu}t, a^\lambda u), \quad a \in \mathbb{R}^+.$$

For example, the Korteweg–de Vries equation $u_t = u_3 + uu_1$ is homogeneous of weight 3 for $\lambda = 2$.

Our work is concentrated on the λ -homogeneous evolution equations when $\lambda > 0$. We give in some detail the necessary results from the symbolic method in section 2.1 and discuss in section 2.2 the role of the permutation group \mathbb{S}^n . This is illustrated by some results for specific types of arbitrary odd order systems. After these theoretical developments, the classification of homogeneous equations with respect to (almost) cosymmetry-integrability becomes a matter of computer algebra. The computational details are to be found in [SW98a]. We illustrate the power of the symbolic method by giving in section 3 a complete classification of 7th-order 2-homogeneous equations (the KdV-like equations). The 5th-order case was given in [SW98a]. In [SW97] one finds results of a more general nature about conservation laws of KdV-like equations of arbitrary order and results about t -dependent conservation laws.

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2. Cosymmetries of λ -homogeneous equations

2.1. The symbolic method.

First we recall the basic idea of the symbolic method, cf. [GD75] and [Olv99] for full details: simply replacing u_i , where i is an index — in our case counting the number of derivatives — by ξ^i , where ξ is now a symbol. We see that the operation of differentiation, i.e. replacing u_i by u_{i+1} , is now replaced by multiplication with ξ , as is the case in Fourier transform theory. For higher degree terms with multiple u 's, one uses different symbols to denote differentiation; for example, the binomial $u_i u_j$

has symbolic form $\frac{1}{2} \left(\xi_1^i \xi_2^j + \xi_1^j \xi_2^i \right) \cdot u^2$. One needs to average over permutations of the differentiation symbols so that $u_i u_j$ and $u_j u_i$ have the same symbolic form.

A differential monomial takes the form $u_I = u_{i_1} u_{i_2} \cdots u_{i_k}$. We call k the *degree* of the monomial, $\#I = i_1 + \cdots + i_k$ the *index* and $\max(i_j, j = 1, \dots, k)$ the *order*. We let \mathcal{U}_n^k denote the set of differential polynomials of degree $k+1$ and index n . Let $\mathcal{U}^k = \bigoplus_n \mathcal{U}_n^k$, and $\mathcal{U} = \bigoplus_{n,k \geq 0} \mathcal{U}_n^k$, the algebra of all differential polynomials. The *order* of a differential polynomial is the maximum of the orders of its constituent monomials, that is, the order of $f(u, \dots, u_n)$ is n .

The *transform* or *symbolic form* defines a linear isomorphism between the space \mathcal{U}^k of differential polynomials of degree $k+1$ and the space $\mathcal{A}^k = \mathbb{R}[\xi_1, \dots, \xi_{k+1}]$ of algebraic polynomials in $k+1$ variables. Particularly, let $\tilde{\mathcal{A}}^k$ be the set of its symmetrized elements $\tilde{f} \stackrel{\text{def}}{=} \langle f \rangle$. The transform is uniquely defined by its action on monomials.

REMARK 1. For all u^k , their symbolic forms are the same: constant 1. To distinguish them, we put u^k in front of the symbolic form to indicate its degree, cf. [SW98b]. We also remark that by our conventions $1 \notin \mathcal{U}$.

DEFINITION 2. The symbolic form of a differential monomial is defined as

$$u_{i_1} u_{i_2} \cdots u_{i_k} \longmapsto \sum_{\pi \in \mathbb{S}^k} \xi_{\pi(1)}^{i_1} \xi_{\pi(2)}^{i_2} \cdots \xi_{\pi(k)}^{i_k} \cdot u^k.$$

PROPOSITION 3. Let $K \in \mathcal{U}^m$ and $Q \in \mathcal{U}^n$. Then $D_K(Q) \in \mathcal{U}^{m+n}$, where D_K is the Fréchet derivative of K , and

$$\widehat{D_K[Q]} = (m+1) \langle \widehat{K} \left(\xi_1, \dots, \xi_m, \sum_{i=0}^n \xi_{m+1+i} \right) \widehat{Q}(\xi_{m+1}, \dots, \xi_{m+n+1}) \rangle;$$

PROPOSITION 4. Let $K \in \mathcal{U}^m$ and $Q \in \mathcal{U}^n$, and define ξ_0 by

$$\xi_0 + \xi_1 + \cdots + \xi_{n+m+1} = 0.$$

Then $D_K^*[Q] \in \mathcal{U}^{m+n+1}$, where D_K^* is the dual operator of D_K and

$$\widehat{D_K^*[Q]} = (m+1) \langle \widehat{K}(\xi_1, \dots, \xi_m, \xi_0) \widehat{Q}(\xi_{m+1}, \dots, \xi_{m+n+1}) \rangle;$$

The following polynomials play a critical role in the analysis.

DEFINITION 5. The *cosymmetry-G-functions* are the polynomials

$$G_k^{(m)} = \xi_0^k + \xi_1^k + \cdots + \xi_{m+1}^k, \quad \text{where} \quad \xi_0 + \xi_1 + \cdots + \xi_{m+1} = 0.$$

The key fact is the formula for the Lie derivative of a linear differential polynomial:

$$(3) \quad \widehat{L_{u_k} Q} = G_k^{(m)} \widehat{Q}, \quad \text{whenever} \quad Q \in \mathcal{U}^m.$$

This follows directly from Proposition 3, Proposition 4 and the fact that u_k has symbolic form $\widehat{u_k} = \xi_1^k$. An immediate application is the following result:

PROPOSITION 6. Consider linear evolution equation $u_t = \sum_{j=1}^p \lambda_j u_j$, where λ_j are constants and $\lambda_p \neq 0$. The space of the cosymmetries of $u_t = f$ in \mathcal{U} is

- \mathcal{U} iff $p = 1$;
- \emptyset iff $p > 1$ and at least one of j is even.
- \mathcal{U}^0 iff $p > 1$ and all j are odd.

If one of the j is even, the equation $L_{K^1}Q = 0$ has no solution. This means the absence of local conservation laws for the even order scalar evolution equations, cf. [AG79]. Since we are interested in conservation laws, a cosymmetry is the variational derivative of a conserved density, we only need to consider the even order cosymmetries for the odd order equations.

Any λ -homogeneous evolution equation can be broken up into its homogeneous components, and so takes the form

$$(4) \quad u_t = K = K^0 + K^1 + K^2 + \cdots, \quad \text{where} \quad K^i \in \mathcal{U}^i.$$

We assume that $K^0 = u_{2n+1}$ and $0 < \lambda \in \mathbb{Q}$. Note that the order of K^i is $2n+1 - i\lambda \geq 0$. When $i\lambda \notin \mathbb{N}$, $K^i = 0$. This reduces the number of relevant λ to a finite set.

Let $Q \in \mathcal{U} = \bigoplus_{i=0}^{\infty} \mathcal{U}^i$ be a $2m^{\text{th}}$ -order cosymmetry of the equation (4). We break up the formula (2) into its homogeneous summands, leading to the series of successive identities

$$(5) \quad \sum_{i+j=r} L_{K^i} Q^j = \sum_{i+j=r} (D_{K^i}^*[Q^j] + D_{Q^j}[K^i]) = 0, \quad r = 0, 1, \dots$$

Note that Q must have a nontrivial linear term from Proposition 6, i.e., $Q^0 \neq 0$. And we can set $Q^0 = u_{2m}$ without loss of generality. The first equation to be solved is

$$L_{K^1} u_{2m} + L_{u_{2n+1}} Q^1 = 0.$$

This is trivially satisfied if K has no quadratic terms: $K^1 = 0$.

If $K^i = 0$ for $i = 1, \dots, j-1$, we first need solve the equation $L_{K^j} u_{2m} + L_{u_{2n+1}} Q^j = 0$. By Proposition 3 and 4, we obtain,

$$(6) \quad \tilde{Q}^j = - \frac{\sum_{i=0}^{j+1} \tilde{K}^j(\xi_{i+1}, \dots, \xi_{j+1}, \xi_0, \dots, \xi_{i-1}) \xi_i^{2m}}{G_{2n+1}^{(j)}},$$

where $G_{2n+1}^{(j)} = \xi_0^{2n+1} + \xi_1^{2n+1} + \cdots + \xi_{j+1}^{2n+1}$ and $\xi_0 + \xi_1 + \cdots + \xi_{j+1} = 0$. The necessary condition for equation (4) to possess a $2m^{\text{th}}$ -order cosymmetry is that the right hand side of (6) is a polynomial.

2.2. The role of the permutation group \mathbb{S}^n .

In this section we reformulate the problem using the symmetry groups. Let the $f_i = \sum_{1 \leq j_1 \leq \dots \leq j_i \leq n} \xi_{j_1} \cdots \xi_{j_i}$ span the \mathbb{S}^n symmetric functions of degree $1, \dots, n$. And let the $g_i = \sum_{0 \leq j_1 \leq \dots \leq j_i \leq n} \xi_{j_1} \cdots \xi_{j_i}$, $i = 2, \dots, n+1$, with $\sum_{i=0, \dots, n} \xi_i = 0$, span the \mathbb{S}^{n+1} symmetric functions. Recall that

$$\tilde{\mathcal{A}}^{n-1} = k[\xi_1, \dots, \xi_n]^{\mathbb{S}^n} = k[f_1, \dots, f_n].$$

So $\tilde{\mathcal{A}}^{n-1}$ is a polynomial algebra in the $f_i, i = 1, \dots, n$, with Poincaré series

$$P_{\tilde{\mathcal{A}}^{n-1}}(\tau) = \prod_{i=1}^n \frac{1}{1 - \tau^i}.$$

We define the subalgebra

$$\mathcal{R}^{n-1} = k[\xi_0, \xi_1, \dots, \xi_n]^{\mathbb{S}^{n+1}} / (\xi_0 + \cdots + \xi_n) = k[g_2, \dots, g_{n+1}]$$

with Poincaré series $P_{\mathcal{R}^{n-1}}(\tau) = \prod_{i=2}^{n+1} \frac{1}{1 - \tau^i}$. Note that $\tilde{Q}^j \in \mathcal{R}^j$.

PROPOSITION 7. $\tilde{\mathcal{A}}^{n-1}$ is a free \mathcal{R}^{n-1} -module with generators $f_0 = 1, f_1, \dots, f_n$.

PROOF. (**A.M. Cohen**) Let $M = \sum_{i=0}^n \mathcal{R}^{n-1} f_i$. We need to prove that M is a subalgebra of $\tilde{\mathcal{A}}^{n-1}$. Because M contains the generators of $\tilde{\mathcal{A}}^{n-1}$ it follows that $\tilde{\mathcal{A}}^{n-1} = M$.

We write the $g_i = \xi_0 \sum_{1 \leq j_2 \leq \dots \leq j_i \leq n} \xi_{j_2} \cdots \xi_{j_i} + f_i$ with $f_{n+1} = 0$, that is

$$g_i = f_i - f_1 f_{i-1}.$$

This implies that $f_1 M \subset M$, since

$$(7) \quad f_1 f_i = f_{i+1} + g_{i+1} f_0, \quad \text{for } i = 2, \dots, n.$$

We now show by induction that $f_j f_i \in M$. Assume $f_j f_i \in M$ for all $j < J$. Then one has, using (7) for both f_i and f_J ,

$$f_J f_i = g_J g_i f_0 - g_i f_1 f_{J-1} - g_J f_1 f_{i-1} + f_1^2 f_{J-1} f_{i-1}$$

and it follows that $f_J f_i \in M$. This formula can be used to put any element of $\tilde{\mathcal{A}}^{n-1}$ in the desired form.

To show that $\tilde{\mathcal{A}}^{n-1}$ a free \mathcal{R}^{n-1} -module is, we observe that

$$\frac{P_{\tilde{\mathcal{A}}^{n-1}}(\tau)}{P_{\mathcal{R}^{n-1}}(\tau)} = \frac{\prod_{i=2}^{n+1} (1 - \tau^i)}{\prod_{i=1}^n (1 - \tau^i)} = \frac{1 - \tau^{n+1}}{1 - \tau} = \sum_{i=0}^n \tau^i$$

This implies that

$$\tilde{\mathcal{A}}^{n-1} = \bigoplus_{i=0}^n \mathcal{R}^{n-1} \langle f_i \rangle$$

and concludes the proof. \square

Now we reconsider the formula (6). If $\tilde{K}^j(\xi_1, \dots, \xi_{j+1}) = (\xi_1 + \dots + \xi_{j+1}) \tilde{K}_{\mathcal{R}}^j$, where $\tilde{K}_{\mathcal{R}}^j \in \mathcal{R}^j$, then (6) becomes $\tilde{Q}^j = \frac{\tilde{K}_{\mathcal{R}}^j G_{2m+1}^{(j)}}{G_{2n+1}^{(j)}}$. Using the divisibility properties of the G -functions in [Beu97, SW98b], we have the following:

PROPOSITION 8. Assume that $K^i = 0$ for $i = 1, \dots, j-1$ in the equation (4) when $\lambda > 0$, Then

- If $\tilde{K}^j(\xi_1, \dots, \xi_{j+1}) = (\xi_1 + \dots + \xi_{j+1}) \tilde{K}_{\mathcal{R}}^j$, where $\tilde{K}_{\mathcal{R}}^j \in \mathcal{R}^j$, the equation has only one possible cosymmetry of order $2n$ when $j \geq 3$.
- If $\tilde{K}^j(\xi_1, \dots, \xi_{j+1}) = (\xi_1 \cdots \xi_{j+1}) \tilde{K}_{\mathcal{R}}^j$, where $\tilde{K}_{\mathcal{R}}^j \in \mathcal{R}^j$, then (6) becomes $\tilde{Q}^j = -\frac{\tilde{K}_{\mathcal{R}}^j g_{j+2} G_{2m-1}^{(j)}}{G_{2n+1}^{(j)}}$, and the equation (4) has only one possible cosymmetry of order $2n+2$ when $j \geq 3$.

COROLLARY 9. The equation $u_t = u_{2n+1} + \gamma u^l u_1$ has a cosymmetry $u_{2n} + \frac{\gamma}{l+1} u^{l+1}$, i.e., the conserved density $\frac{(-1)^n}{2} u_n^2 + \frac{\gamma}{(l+1)(l+2)} u^{l+2}$.

The equation $u_t = u_{2n+1} + \gamma u_1^l$ has a cosymmetry $u_{2n+2} + \gamma l u_1^{l-1} u_2$, i.e., the conserved density $\frac{(-1)^{n+1}}{2} u_{n+1}^2 - \frac{\gamma}{(l+1)} u_1^{l+1}$.

3. Application: 7th-order KdV-like equations

First we give a simple example to show how the method is being used.

EXAMPLE 10. The Ibragimov-Shabat equation $u_t = u_3 + 3u^2 u_2 + 9u u_1^2 + 3u^4 u_1$ has only one cosymmetry u , i.e., only one conservation law $\frac{u^2}{2}$. [Kap82].

PROOF. The cubic term of the equation $\tilde{K}^2 = 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3) - g_2^{(2)}$. Notice that $G_3^{(2)} = 3g_3^{(2)}$. If the equation has a $2m^{\text{th}}$ -order cosymmetry, then

$$\tilde{Q}^2 = \frac{g_2^{(2)}G_{2m}^{(2)} - \sum_{i=0}^3 f(\xi_{i+1}, \dots, \xi_3, \xi_0, \dots, \xi_{i-1}) \xi_i^{2m}}{3g_3^{(2)}},$$

where $f(\xi_1, \xi_2, \xi_3) = 2(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3)$ and $\xi_0 + \xi_1 + \xi_2 + \xi_3 = 0$. Now we prove that it is not a polynomial except $m = 0$ by generating functions. First

$$\sum_{i=0}^{\infty} g_2^{(2)}G_{2i}^{(2)}\tau^{2i} = \frac{g_2^{(2)} \left(4 + 6g_2^{(2)}\tau^2 + (2g_2^{(2)2} + 4g_4^{(2)})\tau^4 + (2g_2^{(2)}g_4^{(2)} - g_3^{(2)2})\tau^6 \right)}{1 + 2g_2^{(2)}\tau^2 + (g_2^{(2)2} + 2g_4^{(2)})\tau^4 + (2g_2^{(2)}g_4^{(2)} - g_3^{(2)2})\tau^6 + g_4^{(2)2}\tau^8}.$$

In the same way, we compute the second part. If we put $g_3^{(2)} = 0$, the generating function, denoted by $[\tilde{Q}^2]_{\tau}$ equals $\frac{2(g_2^{(2)2} - 4g_4^{(2)})\tau^2}{1 + g_2^{(2)}\tau^2 + g_4^{(2)}\tau^4}$. And

$$\lim_{g_4^{(2)} \rightarrow 0} [\tilde{Q}^2]_{\tau} = \frac{2g_2^{(2)2}\tau^2}{1 + g_2^{(2)}\tau^2} = \sum_{i=1}^{\infty} 2(-1)^{i-1}g_2^{(2)i+1}\tau^{2i}.$$

This implies that for any $m > 0$, there is no $2m^{\text{th}}$ -order cosymmetry. Only when $m = 0$ is this possible. Further computation shows that indeed u is a cosymmetry of the equation. \square

Consider the following 7^{th} -order KdV-like equation:

$$(8) \quad u_t = u_7 + \alpha_0uu_5 + \alpha_1u_1u_4 + \alpha_2u_2u_3 + \beta_0u^2u_3 + \beta_1uu_1u_2 + \beta_2u_1^3 + \gamma u^3u_1.$$

We write out its homogeneous summands, the \mathbb{S}^{j+1} invariants as a freely generated \mathcal{R}^j -module with generators $f_0^{(j)} = 1, f_1^{(j)}, \dots, f_{j+1}^{(j)}$:

$$\begin{aligned} \tilde{K}_5^1 &= \frac{1}{2}(\alpha_1 - 3\alpha_0)g_2^{(1)}g_3^{(1)} + \frac{1}{2}\alpha_0g_2^{(1)2}(\xi_1 + \xi_2) - \frac{1}{2}(\alpha_0 + \alpha_2 - 2\alpha_1)g_3^{(1)}\xi_1\xi_2; \\ \tilde{K}_3^2 &= \frac{1}{6}(4\beta_0 - \beta_1)g_3^{(2)} - \frac{1}{3}\beta_0g_2^{(2)}(\xi_1 + \xi_2 + \xi_3) + \frac{1}{3}(\beta_0 + 3\beta_2 - \beta_1)\xi_1\xi_2\xi_3; \\ \tilde{K}_1^3 &= \frac{\gamma}{4}(\xi_1 + \xi_2 + \xi_3 + \xi_4). \end{aligned}$$

This decomposition is meaningful, since one can formulate the existence questions of cosymmetries in terms of divisibility between the elements in \mathcal{R}^j .

If the equation (8) has a $2m^{\text{th}}$ -order cosymmetry, by the formula (6),

$$\tilde{Q}^1 = -\frac{1}{2} \frac{(\alpha_1 - 3\alpha_0)g_2^{(1)}g_3^{(1)}G_{2m}^{(1)} - \alpha_0g_2^{(1)2}G_{2m+1}^{(1)} + (\alpha_0 + \alpha_2 - 2\alpha_1)g_3^{(1)2}G_{2m-1}^{(1)}}{G_{2m+1}^{(1)}}.$$

So there exists the same period 6 in cosymmetries for the equation (8), as in symmetries, [SW98b]. We carry out the procedures for generating functions produced by all possible cosymmetries with order $6p + 2s$, where $p \geq 0$ and $s = 0, 1, 2$, so that we can handle the whole hierarchy in one computation, cf. [SW98a] for the details.

THEOREM 11. *The following cases completely describe the even order x, t -independent cosymmetries for equation (8).*

- When $\beta_1 = 2(\beta_0 + \beta_2)$, it has cosymmetry 1.

- When $\alpha_1 = 3\alpha_0$, $\alpha_2 = 5\alpha_0$, $\beta_0 = \frac{5}{14}\alpha_0^2$, $\beta_1 = \frac{10}{7}\alpha_0^2$, $\beta_2 = \frac{5}{14}\alpha_0^2$, $\gamma = \frac{5}{98}\alpha_0^3$, it has a cosymmetry for every even order, i.e. it is cosymmetry integrable. The equation is 7th-order symmetry of the KdV equation: $u_t = u_3 + \frac{3}{7}\alpha_0 u u_1$.
- When $\alpha_1 = 2\alpha_0$, $\alpha_2 = 3\alpha_0$, $\beta_0 = \frac{2}{7}\alpha_0^2$, $\beta_1 = \frac{6}{7}\alpha_0^2$, $\beta_2 = \frac{1}{7}\alpha_0^2$, $\gamma = \frac{4}{147}\alpha_0^3$, or $\alpha_1 = \frac{7}{2}\alpha_0$, $\alpha_2 = 6\alpha_0$, $\beta_0 = \frac{2}{7}\alpha_0^2$, $\beta_1 = \frac{9}{7}\alpha_0^2$, $\beta_2 = \frac{5}{14}\alpha_0^2$, $\gamma = \frac{4}{147}\alpha_0^3$, it has order $6m + 2$ and $6m + 4$ ($m \geq 0$) cosymmetries, i.e. it is cosymmetry integrable. The equation is 7th-order symmetry of the Sawada equation:

$$u_t = u_5 + \frac{5}{7}\alpha_0 u u_3 + \frac{5}{7}\alpha_0 u_1 u_2 + \frac{5}{49}\alpha_0^2 u^2 u_1$$

or the Kaup–Kupershmidt equation:

$$u_t = u_5 + \frac{5}{7}\alpha_0 u u_3 + \frac{25}{14}\alpha_0 u_1 u_2 + \frac{5}{49}\alpha_0^2 u^2 u_1.$$

- There is one cosymmetry $u_2 + \frac{1}{14}(2\alpha_1 - 3\alpha_0)u^2$, when

$$\begin{aligned} \beta_1 &= \frac{45}{14}\alpha_0\alpha_1 - \frac{45}{14}\alpha_0^2 - \frac{5}{7}\alpha_1^2 + \frac{1}{7}\alpha_1\alpha_2 - \frac{3}{14}\alpha_0\alpha_2 + \beta_0; \\ \gamma &= \frac{3}{7}\beta_0\alpha_1 - \frac{9}{14}\beta_0\alpha_0 - \frac{30}{49}\alpha_0\alpha_1^2 + \frac{225}{196}\alpha_1\alpha_0^2 + \frac{5}{49}\alpha_1^3 - \frac{1}{49}\alpha_2\alpha_1^2 \\ &+ \frac{3}{49}\alpha_1\alpha_2\alpha_0 - \frac{135}{196}\alpha_0^3 - \frac{9}{196}\alpha_2\alpha_0^2 + \frac{2}{21}\beta_2\alpha_1 - \frac{1}{7}\beta_2\alpha_0. \end{aligned}$$
- There is one 4th-order cosymmetry
 1. $u_4 + \frac{1}{3}\beta_1 u^3$ when $\alpha_1 = \frac{\alpha_0}{2}$, $\alpha_2 = 0$, $\beta_1 = 6\beta_0$, $\beta_2 = 2\beta_0$, $\gamma = \frac{7}{6}\alpha_0\beta_0$;
 2. u_4 when $\alpha_1 = \frac{1}{2}\alpha_0$, $\alpha_2 = 0$, $\beta_1 = -\beta_0$, $\beta_2 = \frac{4}{9}\beta_0$, $\gamma = 0$;
 3. $u_4 + \frac{1}{7}\alpha_2 u u_2 + \frac{1}{14}\alpha_2 u_1^2 + \left(\frac{2}{147}(4\alpha_0^2 - 9\alpha_0\alpha_1 + 2\alpha_1^2) + \frac{\beta_0}{3}\right)u^3$, when $\alpha_2 = 2\alpha_1 - \alpha_0$, and

$$\begin{aligned} \beta_1 &= 6\beta_0 + \frac{1}{7}(2\alpha_1^2 - 9\alpha_0\alpha_1 + 4\alpha_0^2), \\ \beta_2 &= 2\beta_0 + \frac{1}{14}(4\alpha_0^2 - 9\alpha_0\alpha_1 + 2\alpha_1^2), \\ \gamma &= \frac{2}{147}(17\alpha_0\alpha_1^2 - 40\alpha_0^2\alpha_1 + 16\alpha_0^3 - 2\alpha_1^3) + \frac{1}{3}(4\beta_0\alpha_0 - \beta_0\alpha_1). \end{aligned}$$
- There is one 6th-order cosymmetry
 1. u_6 when $\alpha_1 = -\frac{\alpha_0}{2}$, $\alpha_2 = \frac{\alpha_0}{10}$, $\beta_0 = \beta_1 = \beta_2 = \gamma = 0$;
 2. $u_6 + \alpha_0 u u_4 + 2\alpha_0 u_1 u_3 + \frac{3\alpha_0}{2}u_2^2 + \beta_0 u^2 u_2 + \beta_0 u u_1^2 + \frac{7}{4}u^3$ when $\alpha_1 = 3\alpha_0$, $\alpha_2 = 5\alpha_0$, $\beta_1 = 4\beta_0$, $\beta_2 = \beta_0$.
- There is one 8th-order cosymmetry:
 1. u_8 when $\alpha_1 = -\frac{3}{2}\alpha_0$, $\alpha_2 = \frac{5}{7}\alpha_0$, $\beta_0 = \beta_1 = \beta_2 = \gamma = 0$;
 2. $u_8 + 3\beta_2 u_1^2 u_2 + \alpha_2 u_3^2 + \alpha_2 u_2 u_4$ when $\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = \gamma = 0$;
 - 3.

$$\begin{aligned} &u_8 + \frac{10}{7}\alpha_0 u u_6 + \frac{30}{7}\alpha_0 u_1 u_5 + \frac{325}{42}\alpha_0 u_2 u_4 + \frac{25}{6}\alpha_0 u_3^2 \\ &+ \frac{125}{196}\alpha_0^2 u^2 u_4 + \frac{125}{49}\alpha_0^2 u u_1 u_3 + \frac{375}{196}\alpha_0^2 u u_2^2 + \frac{275}{98}\alpha_0^2 u_1^2 u_2 \\ &+ \frac{75}{686}\alpha_0^3 u^3 u_2 + \frac{225}{1372}\alpha_0^3 u^2 u_1^2 + \frac{45}{38416}\alpha_0^4 u^5 \end{aligned}$$

when $\alpha_1 = \frac{7}{2}\alpha_0$, $\alpha_2 = \frac{25}{6}\alpha_0$, $\beta_0 = \frac{5}{28}\alpha_0^2$, $\beta_1 = \frac{10}{7}\alpha_0^2$, $\beta_2 = \frac{15}{28}\alpha_0^2$ and $\gamma = 0$.

REMARK 12. For a given equation in the form of (8), we can answer how many cosymmetries (conservation laws) it has by checking the relations between its coefficients according to the above theorem. For example, the equation

$$u_t = u_7 + \alpha_2 u_2 u_3 + \beta_2 u_1^3,$$

where $\alpha_2 \beta_2 \neq 0$, is almost cosymmetry integrable of depth 2, that is, it has two cosymmetries u_2 and $u_8 + 3\beta_2 u_1^2 u_2 + \alpha_2 u_3^2 + \alpha_2 u_2 u_4$, i.e. conserved density $\frac{u_1^2}{2}$ and $\frac{u_4^2}{2} + \frac{1}{6}\alpha_2 u_2^3 - \frac{1}{4}\beta_2 u_1^4$.

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