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On Integrability of Evolution Equations and Representation Theory

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Abstract. In this paper we observe that the existence of an $\mathfrak{sl}(2,\mathbb{R})$ can help us in organizing the analysis and computation of time-dependent symmetries of nonlinear evolution equations. We apply this idea to the Burgers and Ibragimov–Shabat equations. We then use $\mathfrak{sl}(2,\mathbb{R})$ to compute their Bäcklund transformations.

This leads to unexpected consequences for the Kadomtsev–Petviashvili equation: we show that the Jacobi identity does not hold, due to nonlocal terms in the equation. Nevertheless we are able to compute time–dependent symmetries for this equation too.

1. Introduction

If one considers nonlinear integrable systems, let us say evolution equations with one or two spatial variables, one notices that the biHamiltonian systems cover a high percentage of the known integrable systems. But to really understand what integrability is, one also needs to have an explanation for those systems that fall outside this category. Well known exceptional cases are Burgers equation and the Ibragimov-Shabat equation.

In this paper we describe how the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ can help us to systematically find the (time-dependent) symmetries of these equations and we observe that the existence of an $\mathfrak{sl}(2,\mathbb{R})$ containing the usual scaling symmetry appears to be connected with the exceptional character of the equation. Why this is so remains completely mysterious to us (in the sense that both the biHamiltonian systems and $\mathfrak{sl}(2,\mathbb{R})$ rely on certain duality properties. For instance in the finite dimensional context one needs a nondegenerate Killing form in order to prove the existence of an $\mathfrak{sl}(2,\mathbb{R})$ from the existence of a nilpotent element), but the observation in itself seems interesting enough to draw attention to it.

And if no deeper explanation of these facts follows, what we present is a good way to organize the time-dependent symmetries, since they can now be considered as

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part of an $\mathfrak{sl}(2,\mathbb{R})$-module, which makes their structure much more simple than the usual approach leading to infinite dimensional Lie algebras.
In section 2 we motivate the use of $\mathfrak{sl}(2,\mathbb{R})$ as an organizing element in our symmetry calculation. We start with the problems where the equation, the $\mathfrak{sl}(2,\mathbb{R})$ and the symmetries are all local. In section 3, we construct t–dependent symmetries for the Burgers equation and we use known classification results to show that we found them all. Next we consider the Ibragimov–Shabat equation, where such results are not known to us. We can only conjecture that we found all time-dependent symmetries by our analysis. Then we turn to an equation with nonlocal terms: the Kadomtsev–Petviashvili equation. This immediately presents us with a problem, since rather elementary computations conflict with the Jacobi identity. However, there exist $\mathfrak{sl}(2,\mathbb{R})$'s, where we do not encounter these difficulties and we present the results of our analysis, referring to the solution of the Jacobi identity problem to [**OSW01**].

2. Symmetries of evolution equations

Let us consider an evolution equation

$$u_t = K(x, u, u_1, \cdots, u_n); \quad u_t = \frac{\partial u}{\partial t}, \quad u_n = \frac{\partial^n u}{\partial x^n}.$$

A symmetry of this equation is an $S(t, x, u, \cdots, u_m)$ such that

$$L_K S = \frac{\partial S}{\partial t} + ad_K S = \frac{\partial S}{\partial t} + D_S[K] - D_K[S] = 0.$$

Let $\psi_0(x, u, \cdots, u_k)$ be such that the formal expression

$$\psi = exp(-t\ ad_K)\psi_0 \tag{1}$$

converges in whatever topology we have defined. In particular we are interested in the case where for some $m \in \mathbb{N}$ one has $ad_K^m \psi_0 = 0$, in which case the sum defining the exp is finite and the topology is not relevant. Then

$$L_K \psi = -ad_K \psi + ad_K \psi = 0.$$

Thus ψ is a symmetry of K. Moreover $\frac{\partial^r \psi}{\partial t^r}$ is also a symmetry of K:

$$L_K \frac{\partial^r \psi}{\partial t^r} = \frac{\partial^{r+1} \psi}{\partial^{r+1} t} + ad_K \frac{\partial^r \psi}{\partial t^r} = \frac{\partial^r L_K \psi}{\partial t^r} = 0.$$

In particular, this implies that if ψ is a **polynomial of degree** p in t, then $\frac{\partial^p \psi}{\partial t^p}$ is a time independent symmetry of K. Notice that it is essential for this argument to work that both K and ψ have no explicit time-dependence.
Does this elementary observation (Cf. [**Fuc83**]) trivialize the integrability theory of evolution equations? Not quite. For a given K to find ψ_0 for which the whole construction makes sense is still a nontrivial problem.
However, there is a general way to consider this problem. The sufficient condition that for some m one has $ad_K^m \psi_0 = 0$ reminds one of the representation theory of $\mathfrak{sl}(2,\mathbb{R})$. Could it be that the $ad_K^j \psi_0, j = 0, \cdots, m-1$ form a representation of some (low-dimensional) Lie algebra, for instance $\mathfrak{sl}(2,\mathbb{R})$?
Let us first try to construct an $\mathfrak{sl}(2,\mathbb{R})$ from K. The usual assumption is the existence of a scaling symmetry H such that $ad_H M = \lambda M$, where typically $M = K$.

Rescaling H, if necessary, such that $\lambda = 2$, we see that we have here 'half' of $\mathfrak{sl}(2,\mathbb{R})$. Suppose now that we can complete this to an $\mathfrak{sl}(2,\mathbb{R})$ by finding an N such that

$$\begin{aligned} [H,\ M] &= 2M \\ [H,\ N] &= -2N \\ [M,\ N] &= H \end{aligned}$$

In the finite dimensional case it follows from the Jacobson-Morozov lemma that in a *reductive* Lie algebra a nilpotent M can always be imbedded in an $\mathfrak{sl}(2,\mathbb{R})$ in this manner. The problem of computing such an extension is a linear problem, and so we can just try to do this in our concrete case, following the algorithm given in [**Hel78**].

We show the results of this construction for the exceptional equations mentioned in the introduction, and proceed to use this construction to explicitly compute the time-dependent symmetries of these equations.

3. Burgers equation

Consider Burgers equation

$$u_t = K = u_2 + uu_1 \ .$$

One can construct two different $\mathfrak{sl}(2,\mathbb{R})$'s around the scaling $xu_1 + u$:

$$M = K,\ N = -x,\ H = xu_1 + u; \tag{2}$$

$$M_2 = 1,\ N_2 = -(4u_1 + 2xu_2 + u^2 + 2xuu_1),\ H_2 = -2H. \tag{3}$$

In the sequel we will write Xv for $ad_X v = [X, v]$, $XYv = [X, [Y, v]]$, and $(XY)v$ for $ad_{[X,Y]}v$ without leading to any confusion. E.g., we write $M_2K = u_1$ instead of $[M_2,\ K] = u_1$. And the Jacobi identity looks like

$$(XY)Z = X(YZ) + Y(ZX).$$

Remark 1. *We can also take another N_2 as $N_2 + 4u_1$ instead. We choose the one as in (3) so that we have $NN_2K = -3N_2$.*

Let us remind the reader that if there exists a v_0 such that $Mv_0 = 0$ and $Hv_0 = \mu v_0, \mu \in \mathbb{N}$, then one constructs an irreducible $\mathfrak{sl}(2,\mathbb{R})$-representation V_μ of dimension $\mu + 1$ spanned by $v_j = N^j v_0$ $(j = 0, 1, \cdots, \mu)$ as follows:

$$\begin{aligned} Hv_j &= (\mu - 2j)v_j \\ Nv_j &= (\mu - j)v_{j+1} \\ Mv_j &= jv_{j-1}, \end{aligned}$$

where we take v_{-1} and $v_{\mu+1}$ to be zero, by convention. If we now take for v_0 the equation itself, we see that $Mv_0 = 0$ and $Hv_0 = 2v_0$.

Notation 1. *We write $Q \sim P$ when $Q = \kappa P$, where κ is a nonzero constant. If in a diagram we have $f \stackrel{N}{\to} g$, for instance, this means that $Nf \sim g$.*

Notice $M_2K \sim u_1$ and $H_2u_1 = -2u_1$. Starting from M_2K, we can construct an infinite–dimensional representation of $\mathfrak{sl}(2,\mathbb{R}) = \{M_2, H_2, N_2\}$.

It is easy to see that $Hu_1 = u_1$ and $[K, u_1] = 0$. Therefore the space spanned by M_2K and $NM_2K \sim M_2$ is a 2–dimensional irreducible representation of $\{M, H, N\}$. Moreover, we have the following

Proposition 1. *Fix $n \in \mathbb{N}$. The space spanned by $N^j N_2^n K$, $j = 0, 1, \cdots, n+2$, is an $n+3$–dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R}) = \{M, H, N\}$.*

Before we give the proof, we recall Proposition 3.8 in [**SW98**]: the x, t–independent polynomial symmetries of the equation $u_t = u_j$ $(j \neq 1)$ are u_i for all $i \in \mathbb{N}$. This is equivalent to

Proposition 2. *Let F, G be x, t-independent polynomial scalar vectorfields. Then $0 \neq \pounds F \neq \kappa u_1$ and $[F, G] = 0$ implies that $\pounds G \neq 0$ or $G = 0$, where we denote by $\pounds$ the projection of a vectorfield on its linear part, as in $\pounds K = u_2$.*

Proposition 3. *We have $\pounds N_2^n K = (-2)^n (n+1)!\ u_{2+n}$.*

Proof. We prove the statement by induction to n. For $n = 0$ it is trivial.

$$\begin{aligned} & \pounds N_2^{n+1} K = [-2xu_2,\ \pounds N_2^n K] = [-2xu_2,\ (-2)^n (n+1)!\ u_{n+2}] \\ = & \ (-2)^{n+1}(n+2)!\ u_{2+n+1}. \end{aligned}$$

Thus we proved the statement. □

Proposition 4. *Assume $N_2 K \in \ker K$. Then*

$$N_2^n K \in \ker K \cap \ker M_2 K\ .$$

Remark 2. *The assumption $N_2 K \in \ker K$ is equivalent to the definition of* **mastersymmetry** *in* [**CLL83, Fuc83**] *in order to produce the infinitely many symmetries (the first vertical line in our diagram) for the integrable equations.*

Proof. Notice $M_2 K = u_1$. Thus $(M_2 K) K = 0$. We prove the statement by induction to n. For $n = 0$ it is trivial.

$$KN_2^{n+1} K = (KN_2) N_2^n K = [KN_2,\ N_2^n K] \tag{4}$$

By assumption and the induction hypothesis, one has

$$[K,\ [KN_2,\ N_2^n K]] = -[N_2^n K, [K, KN_2]] - [KN_2, [N_2^n K, K] = 0$$

Clearly $\pounds [KN_2, N_2^n K] = 0$ due to Proposition 3. Note that KN_2 and $N_2^n K$ are both x, t–independent since they are in $\ker M_2 K$. Using Proposition 2 we conclude that $[KN_2,\ N_2^n K] = 0$ and therefore $KN_2^{n+1} K = 0$, i.e., $N_2^{n+1} K \in \ker K$. Observe that $H_2 K = -4K$ and $N_2 M_2 K = -2K$. So we have

$$M_2 N_2^n K = -(n^2 + 3n + 2) N_2^{n-1} K,$$

which can easily be proved by induction. Thus

$$(M_2 K) N_2^{n+1} K = K M_2 N_2^{n+1} K + M_2 K N_2^{n+1} K = -(n^2 + 5n + 6) K N_2^n K = 0$$

and the statement is proved. □

Proof. [1] By Proposition 4, we have $KN_2^n K = 0$. So we only need to prove that $HN_2^n K = (n+2) N_2^n K$ by induction to n. When $n = 0$, it is true, namely $HK = 2K$.

$$\begin{aligned} & HN_2^{n+1} K = (HN_2) N_2^n K + N_2 H N_2^n K \\ = & -\frac{1}{2}(H_2 N_2) N_2^n K + (n+2) N_2^{n+1} K = N_2^{n+1} K + (n+2) N_2^{n+1} K \\ = & \ (n+3) N_2^{n+1} K \end{aligned}$$

and thus the statement is proved. □

[1] Of Proposition 1

Proposition 1 implies that $N^l N_2^n K \in \ker K^l$. This leads to the following theorem.

THEOREM 1. *The expressions* $exp(-tad_K)N^l N_2^n K$, *where* $0 \le l \le n+2$ *and* $n \in \mathbb{N}$, *are symmetries of Burgers equation.*

Before giving the proof we give an alternative formulation, more in the spirit of section 2.

REMARK 3. *The expressions* $\frac{\partial^l}{\partial t^l} exp(-tad_K)N^{n+2} N_2^n K$, *where* $0 \le l \le n+2$ *and* $n \in \mathbb{N}$, *are symmetries of Burgers equation.*

PROOF. First we check that $N_2 K \in \ker K$. Then, since $N^l N_2^n K \in \ker K^l$, $exp(-tad_K)N^l N_2^n K$ is well defined. It follows that it is a symmetry of K, together with its explicit time derivatives. □

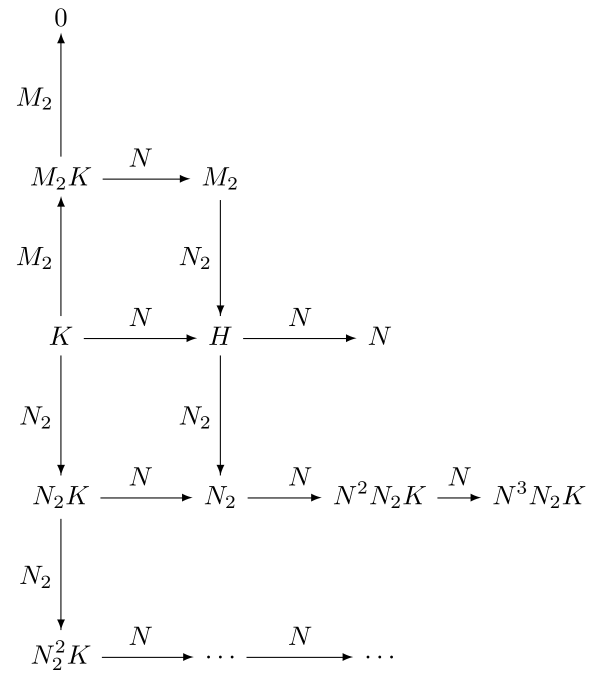
We list a few of them here:

$$t^m u_1 + m t^{m-1}, \quad m = 0, 1$$

$$t^m(u_2 + uu_1) + \frac{m}{2} t^{m-1}(xu_1 + u) + \binom{m}{2} t^{m-2} x, \quad m = 0, 1, 2$$

$$t^m \left(u_3 + \frac{3}{2} uu_2 + \frac{3}{2} u_1^2 + \frac{3}{4} u^2 u_x \right)$$
$$+ \frac{1}{4} m t^{m-1} \left(2xu_2 + 2xuu_1 + 4u_1 + u^2 \right) + \frac{1}{4} \binom{m}{2} t^{m-2} (x^2 u_1 + 2xu + 2)$$
$$+ \frac{3}{4} \binom{m}{3} x^2, \quad m = 0, 1, 2, 3$$

It is proved (Cf. [**BCD+99**, p.172]) that for any $k > 0$, the equation possesses $k+1$ symmetries of order k in the form $t^m u_k$, $m = 0, 1, \cdots, k$. Therefore, Theorem 1 gives all symmetries of the Burgers equation. We give the following diagram to show how they are produced.



Notice that there are many $\mathfrak{sl}(2,\mathbb{R})$'s around H. The choice is dictated by our wish to incorporate the elements on the boundary (like M_2 and N) in our analysis.

4. Ibragimov–Shabat equation

Consider the Ibragimov–Shabat equation ([**IS80**]):

$$u_t = K = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1.$$

Around the scaling $2xu_1 + u$ one can construct $\mathfrak{sl}(2,\mathbb{R})$:

$$M = u_1,\ N = -(xu + x^2u_1),\ H = 2xu_1 + u.$$

We use the similar method as in section 3 to prove the following statement.

PROPOSITION 5. *The space spanned by* $v_j = N^j(KN)^nK$, *where* $n \in \mathbb{N}$ *and* $j = 0, 1, \cdots, 4n+6$ *is an irreducible* $\mathfrak{sl}(2,\mathbb{R})$*-representation of dimension* $4n+7$.

Before we give the proof, we compute

$$\begin{aligned} & £KNu_{2k+1} = -£K[x^2u_1 + xu,\ u_{2k+1}] \\ = \ & -(2k+1)£[K,\ 2xu_{2k+1} + (2k+1)u_{2k}] \\ = \ & 6(2k+1)u_{2k+3}. \end{aligned}$$

It is easy to see that $£(KN)^nK \neq 0$.

PROPOSITION 6. *Assume* $K^2N \in \ker K$. *Then*

$$(KN)^nK \in \ker K \cap \ker M\ .$$

PROOF. We prove the statement by induction to n. For $n = 0$ it is trivial.

$$\begin{aligned} K(KN)^{n+1}K &= K(KN)(KN)^nK = K(KN)(KN)^nK \\ &= (K^2N)(KN)^nK = [K^2N, (KN)^nK]\ . \end{aligned}$$

By assumption and the induction hypothesis, one has

$$[K, [K^2N, (KN)^nK]] = -[(KN)^nK, [K, K^2N]] - [K^2N, [(KN)^nK, K] = 0$$

By the same reasoning to prove that the formula (4) equals to zero, we conclude that $[K^2N, (KN)^nK] = 0$ and therefore $K(KN)^{n+1}K = 0$, i.e., $(KN)^{n+1}K \in \ker K$. Now we prove that $(KN)^{n+1}K \in \ker M$. First we observe that

$$MKN = -MNK = KMN + NKM = KH = -6K.$$

Then

$$\begin{aligned} M(KN)^{n+1}K &= MKN(KN)^nK = M(KN)(KN)^nK \\ &= (MKN)(KN)^nK = [MKN, (KN)^nK] \\ &= -6[K, (KN)^nK] = 0\ . \end{aligned}$$

Thus we proved the statement. □

PROPOSITION 7. *We have* $£K^nN^nu_{2k+1} = 3^n\binom{4k+2}{n}n!^2u_{2k+2n+1}$.

PROOF. Notice that the highest order term in N^nu_{2k+1} is

$$(-1)^n\binom{4k+2}{n}n!\ x^nu_{2k+1}$$

and $£K^ix^nu_{2k+1} = (-3)^i\binom{n}{i}i!x^{n-i}u_{2k+2i+1}$ for $i \leq n$. Combining them, we prove the formula. □

PROPOSITION 8. *Assume $K^2N \in \ker K$. Then*

$$K^r N^r (KN)^n K \in \ker K \cap \ker M \ .$$

PROOF. We prove the statement by induction to r. For $r = 0$, it is the result of Proposition 6. We prove the same statement for $n+1$, assuming it has been proved for all lower values. We observe that

$$\begin{aligned} K^m N \cdot &= NK^m \cdot + \sum_{i=1}^{m} \binom{m}{i} (K^i N) K^{m-i} \cdot \\ &= NK^m \cdot + \sum_{i=1}^{2} \binom{m}{i} (K^i N) K^{m-i} \cdot, \end{aligned}$$

where $\cdot$ is any element in Lie algebra. Then

$$\begin{aligned} & KK^{r+1}N^{r+1}(KN)^n K = \\ = \ & NK^2K^rN^r(KN)^nK + \sum_{i=1}^{2} \binom{r+2}{i} (K^iN)K^{2-i}K^rN^r(KN)^nK \\ = \ & \binom{r+2}{2} [K^2N, K^rN^r(KN)^nK] = 0 \end{aligned}$$

by the same arguments as we proved the formula (4). Compute

$$\begin{aligned} & MK^{r+1}N^{r+1}(KN)^nK = K^{r+1}MN^{r+1}(KN)^nK \quad (MK = 0) \\ = \ & K^{r+1}(MN)N^r(KN)^nK + K^{r+1}NMN^r(KN)^nK \\ = \ & K^{r+1}HN^r(KN)^nK + K^{r+1}NMN^r(KN)^nK \\ = \ & K^{r+1}N^{r+1}M(KN)^nK = 0 \ . \end{aligned}$$

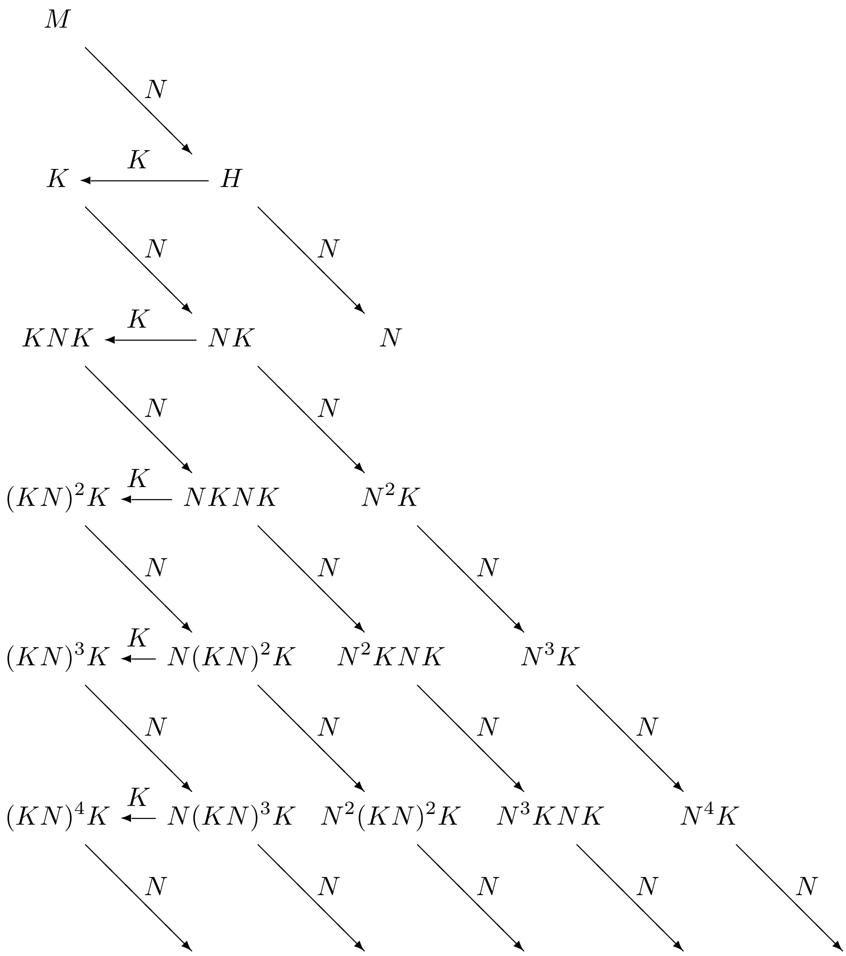
This concludes the proof of the proposition. □

PROOF. [2] We only need to show $H(KN)^nK = (4n+6)(KN)^nK$, which can be proved by induction to n. First note that $HK = 6K$. Using induction hypothesis, we have

$$\begin{aligned} & H(KN)^nK = (HK)N(KN)^{n-1}K + KHN(KN)^{n-1}K \\ = \ & 6(KN)^nK + K(HN)(KN)^{n-1}K + KNH(KN)^{n-1}K \\ = \ & 6(KN)^nK - 2(KN)^nK + (4(n-1)+6)(KN)^nK \\ = \ & (4n+6)(KN)^nK \ . \end{aligned}$$

Thus we proved the statement. □

[2] Of Proposition 5



THEOREM 2. *The expressions in the form of* $exp(-tad_K)N^r(KN)^jK$*, where* $0 \leq r \leq 4j+6$ *and* $j \geq 0$ *are symmetries of Ibragimov–Shabat equation.*

Note that there is a unique independent symmetry given the degree in t and the order of polynomial. E.g., the symmetry $\frac{\partial}{\partial t}\left(exp(-tad_K)N^4K\right)$ is dependent with the symmetry $exp(-tad_K)N^3KNK$ in the sense that the difference between these two can be expressed as the sum of low order symmetries. Therefore we only need the elements on the contour of the diagram to generate the independent symmetries.

We list the low order symmetries:

$$
\begin{aligned}
&u_1,\\
&t^m(u_3+3u^2u_2+9uu_1^2+3u^4u_1)+\frac{m}{6}(2xu_1+u), \quad m=0,1\\
&t^m\left(u_5+5u^2u_4+40uu_1u_3+10u^4u_3+25uu_2^2+50u_1^2u_2+120u^3u_1u_2\right.\\
&\left.+10u^6u_2+140u^2u_1^3+70u^5u_1^2+5u^8u_1\right)\\
&+mt^{m-1}\left(\frac{1}{3}xu_3+xu^2u_2+\frac{u_2}{2}+3xuu_1^2+\frac{5}{3}u^2u_1+xu^4u_1+\frac{1}{6}u^5\right)\\
&+\frac{1}{18}\binom{m}{2}t^{m-2}(x^2u_1+xu), \quad m=0,1,2\\
&t^m\left(u_7+7u^2u_6+70uu_1u_5+21u^4u_5+140uu_2u_4+140u_1^2u_4\right.\\
&+350u^3u_1u_4+35u^6u_4+91uu_3^2+518u_1u_2u_3+588u^3u_2u_3\\
&+1848u^2u_1^2u_3+728u^5u_1u_3+35u^8u_3+126u_2^3+2422u^2u_1u_2^2\\
&+483u^5u_2^2+3164uu_1^3u_2+4914u^4u_1^2u_2+784u^7u_1u_2+21u^{10}u_2\\
&\left.+280u_1^5+3360u^3u_1^4+1876u^6u_1^3+231u^9u_1^2+7u^{12}u_1\right)\\
&+mt^{m-1}\left(\frac{1}{3}xu_5+\frac{5}{3}xu^2u_4+\frac{5}{6}u_4+\frac{40}{3}uu_1u_3+\frac{10}{3}xu^4u_3+4u^2u_3\right.\\
&+\frac{25}{3}xuu_2^2+\frac{50}{3}xu_1^2u_2+40xu^3u_1u_2+22uu_1u_2+\frac{10}{3}xu^6u_2+7u^4u_2\\
&\left.+\frac{140}{3}xu^2u_1^3+\frac{20}{3}u_1^3+\frac{70}{3}xu^5u_1^2+30u^3u_1^2+6u^6u_1+\frac{5}{3}xu^8u_1+\frac{1}{6}u^9\right)\\
&+\frac{1}{18}\binom{m}{2}t^{m-2}\left(x^2u_3+3x^2u^2u_2+3xu_2+9x^2uu_1^2+3x^2u^4u_1\right.\\
&\left.+10xu^2u_1+\frac{6}{5}u_1+xu^5+u^3\right), \quad m=0,1,2
\end{aligned}
$$

5. Bäcklund transformations and $\mathfrak{sl}(2,\mathbb{R})$

The reader may have noticed that the equations which we could successfully analyze using $\mathfrak{sl}(2,\mathbb{R})$ are also the equations which can be linearized using Bäcklund transformations. In this section we illustrate how the Bäcklund transformation can be found in the case of the Burgers and Ibragimov-Shabat equation using $\mathfrak{sl}(2,\mathbb{R})$. The Bäcklund transformation is defined to be an invariant of a pair of equations. This pair consists of the equation itself and another equation at ones choice.

5.1. The Burgers equation. For this equation we choose the pair

$$
\begin{aligned}
u_t &= u_2+uu_1\\
v_t &= v_2.
\end{aligned}
$$

Since the Bäcklund transformation is a Lie algebra homomorphism, it should also transform the $\mathfrak{sl}(2,\mathbb{R})$'s into one another. This leads to the problem of finding a Bäcklund transformation for the equations

$$
\begin{aligned}
u_t &= 1\\
v_t &= \frac{1}{2}xv,
\end{aligned}
$$

where $\frac{1}{2}xv$ is part of the $\mathfrak{sl}(2,\mathbb{R})$

$$\langle \frac{1}{2}xv, xv_1 + \frac{1}{2}v, \cdots \rangle.$$

Solving this we obtain

$$u = t + f(x) \qquad log(v) = \frac{1}{2}xt + g(x)$$

and it follows that $\frac{v_1}{v} = \frac{1}{2}(u - f(x)) + g'(x)$. Taking $2g'(x) = f(x)$ this leads to the Bäcklund generating function

$$B(u,v) = uv - 2v_1,$$

which gives us the well known Miura transformation $u = \frac{2v_1}{v}$.
A similar approach appeared in [**Olv93**], where the author used the similarity between the symmetry algebras of Burgers equation and heat equation.

5.2. The Ibragimov-Shabat equation. For this equation we choose the pair

$$\begin{aligned} u_t &= u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1 \\ v_t &= v_3. \end{aligned}$$

Following the same method as in the case of the Burgers equation, we have to find a Bäcklund transformation for the equations

$$\begin{aligned} u_t &= x^2u_1 + xu \\ v_t &= x^2v_1 + axv, \end{aligned}$$

where $x^2v_1 + axv$ is part of the $\mathfrak{sl}(2,\mathbb{R})$

$$\langle v_1, 2xv_1 + av, x^2v_1 + axv \rangle$$

and the free parameter a reflects the freedom of scaling we have in the linear equation $v_t = v_3$. This equation has as a general solution

$$u = \frac{1}{x}f(t - \frac{1}{x}) \qquad v = \frac{1}{x^a}f(t - \frac{1}{x})$$

This implies that

$$\frac{v}{u^a} = F(t - \frac{1}{x}).$$

Furthermore, $\int u^2 dx = \int \frac{1}{x^2}f^2(t - \frac{1}{x})dx = \int f^2(\tau)d\tau = G(t - \frac{1}{x})$. This allows us to write

$$v = u^a g(D_x^{-1}u^2)$$

and this leads to the Bäcklund generating function

$$B(u,v) = v - u^a g(D_x^{-1}u^2),$$

where g is a completely arbitrary differentiable function. We now proceed to determine g by using this Bäcklund generating function for the original pair[3]. First

[3] One might first want to try to use the correspondence between the other $\mathfrak{sl}(2,\mathbb{R})$ elements, but it turns out that this gives no restriction on a nor f, due to the fact that u_1 is a trivial symmetry.

we obtain, denoting the k-th derivative of g by g_k,

$$\begin{array}{rcl} v & = & u^a g \\ v_1 & = & a u_1 u^{a-1} g + u^{a+2} g_1 \\ v_2 & = & a u_2 u^{a-1} g + a(a-1) u_1^2 u^{a-2} g + 2(a+1) u^{a+1} u_1 g_1 + u^{a+4} g_2 \\ v_3 & = & a u_3 u^{a-1} g + (3a+2) u^{a+1} u_2 g_1 + 3a(a-1) u_2 u_1 u^{a-2} g \\ & & + a(a-1)(a-2) u_1^3 u^{a-3} g + (3a^2+3a+2) u_1^2 u^a g_1 \\ & & + 3(a+2) u_1 u^{a+3} g_2 + u^{a+6} g_3. \end{array}$$

If we now differentiate the generating function, we obtain

$$\begin{array}{rcl} 0 & = & v_t - a u_t u^{a-1} g - 2u^a D_x^{-1}(u u_t) g_1 \\ & = & v_3 - a(u_3 + 3u^2 u_2 + 9 u u_1^2 + 3u^4 u_1) u^{a-1} g \\ & & -2u^a D_x^{-1}(u u_3 + 3u^3 u_2 + 9u^2 u_1^2 + 3u^5 u_1) g_1 \\ & = & u^{a-1}(a u_3 g + (3a+2) u^2 u_2 g_1 + 3a(a-1) u_2 u_1 u^{-1} g \\ & & + a(a-1)(a-2) u_1^3 u^{-2} g + (3a^2+3a+2) u_1^2 u g_1 \\ & & + 3(a+2) u_1 u^4 g_2 + u^7 g_3 \\ & & - a(u_3 + 3u^2 u_2 + 9 u u_1^2 + 3u^4 u_1) g \\ & & -2u(u u_2 - \frac{1}{2} u_1^2 + 3u^3 u_1 + \frac{1}{2} u^6) g_1) \\ & = & u^{a-1}(3a u^2 u_2 (g_1 - g) + 3a(a-1) u_2 u_1 u^{-1} g \\ & & + a(a-1)(a-2) u_1^3 u^{-2} g + 3u_1^2 u((a^2+a+1) g_1 - 3ag) \\ & & + u_1 u^4 (3(a+2) g_2 - 3ag - 6g_1) + u^7 (g_3 - g_1)). \end{array}$$

The conclusion is that $a = 1$ and $g_1 = g$ and therefore the Bäcklund transformation is given by

$$v = u e^{D_x^{-1} u^2},$$

a well known result [**SS84**].

6. Kadomtsev–Petviashvili equation

We now turn our attention to problems involving nonlocal terms, either in the equation itself or in the $\mathfrak{sl}(2, \mathbb{R})$. Consider the Kadomtsev–Petviashvili equation (KP):

$$u_t = K = D_x^{-1} u_{yy} - 6 u u_x - u_{xxx}.$$

This equation is the main example of an integrable equation in two spatial variables. Ignoring the y dependence, it reduces to KdV. It is however this extra degree of freedom which enables us to do what we can not do for KdV, namely construct a local $\mathfrak{sl}(2, \mathbb{R})$. Ignoring the y-dependence in the $\mathfrak{sl}(2, \mathbb{R})$ leads to a contraction of $\mathfrak{sl}(2, \mathbb{R})$ and does not give rise to anything interesting.

6.1. A naive approach. One can construct, using the usual computation rules as given in the literature, two different "$\mathfrak{sl}(2, \mathbb{R})$"'s around the scaling:

$$M = \frac{1}{12} K_2, \; N = y, \; H = y u_y + \frac{1}{2} x u_x + u,$$
$$M_2 = y u_x, \; N_2 = -(3yK + 2x u_y + 4 D_x^{-1} u_y), \; H_2 = -4H,$$

where $K_2 = D_x^{-2}u_{yyy} - 3u_{xxy} - 12uu_y - 6u_x D_x^{-1}u_y$. Compute

$$HM_2M = (HM_2)M + M_2HM = -\frac{1}{4}(H_2M_2)M + 2M_2M = \frac{3}{2}M_2M$$

Note that $M_2^3K \sim u_x$ and $H_2u_x = -2u_x$. So $Hu_x = -\frac{1}{4}H_2u_x = \frac{1}{2}u_x$. However, we have $Mu_x = Nu_x = 0$. This implies that the space spanned by u_x is a 1–dimensional representation of $\mathfrak{sl}(2,\mathbb{R}) = \{M,\ H,\ N\}$. But this would imply that $Hu_x = 0$ by the theory of finite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$. This implies that the Jacobi identity does not hold. This is caused by the occurrence of nonlocal terms in the expressions we study. The source of trouble is $\ker D_x$. The calculations with terms in $\ker D_x$ need to be done with extreme care. For instance, one cannot say that $D_x1 = 0$, since $D_x^{-1}D_x1 = 1$. The framework for this is the subject of a forthcoming paper with Peter Olver [**OSW01**] and is still work in progress at the moment of writing the present paper.
At this point it does not make sense to continue our present analysis, since we could obtain any result we wanted using the contradiction sketched above. We list some of the infinitely many t–dependent symmetries in [**CLL83, CL87**]:

$$t^m u_x - \frac{m}{6}t^{m-1}, \quad m = 0,1$$

$$t^m u_y + \frac{m}{2}t^{m-1}yu_x - \frac{1}{6}\binom{m}{2}t^{m-2}y, \quad m = 0,1,2$$

$$t^m K + \frac{m}{3}t^{m-1}\left(2yu_y + xu_x + 2u\right) + \frac{1}{9}\binom{m}{2}t^{m-2}(3y^2u_x - x)$$
$$-\frac{1}{6}\binom{m}{3}t^{m-3}y^2, \quad m = 0,1,2,3$$

$$t^m K_2 + \frac{1}{4}mt^{m-1}\left(3yK + 2xu_y + 4D_x^{-1}u_y\right)$$
$$+\frac{1}{2}\binom{m}{2}t^{m-2}\left(y^2u_y + xyu_x + 2yu\right) + \frac{1}{4}\binom{m}{3}t^{m-3}(y^3u_x - xy)$$
$$-\frac{1}{6}\binom{m}{4}t^{m-4}y^3, \quad m = 0,1,2,3,4.$$

This list suggested to us the choice of "$\mathfrak{sl}(2,\mathbb{R})$"'s. From the naive point of view, and ignoring the difficulty with the representation theory, the "$\mathfrak{sl}(2,\mathbb{R})$"'s explain the hierarchy of the time–dependent symmetries quite nicely, following the scheme of section 2.

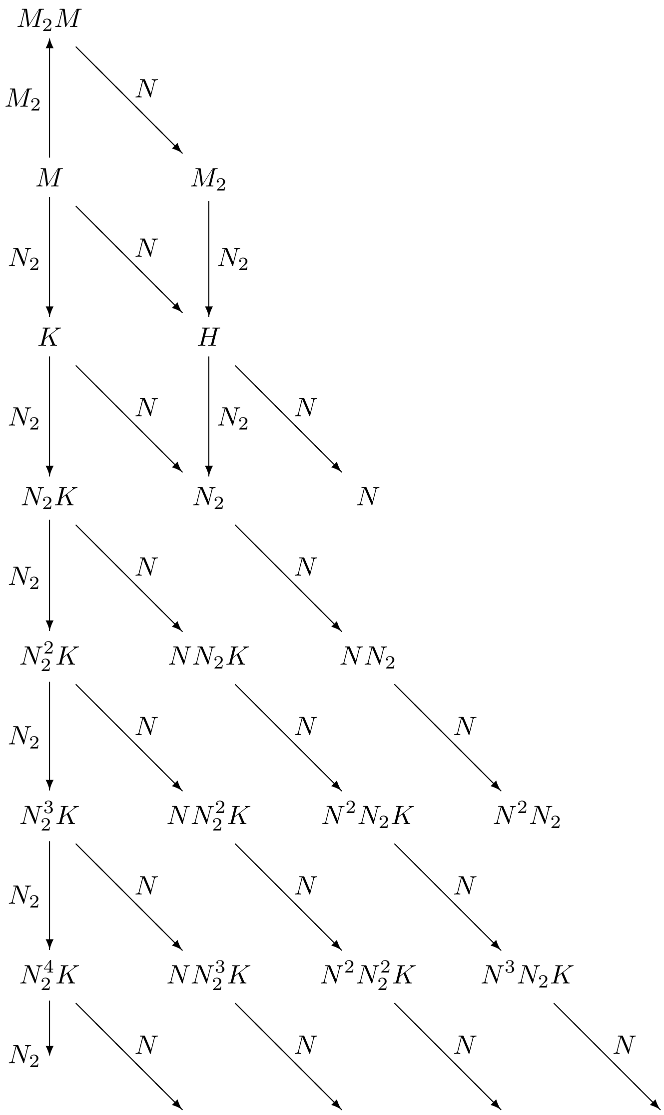
6.2. Avoiding ghosts.

As a result of repairing the Jacobi identity for nonlocal vectorfields, it turned out that one of the two "$\mathfrak{sl}(2,\mathbb{R})$"'s was not an $\mathfrak{sl}(2,\mathbb{R})$. With this in mind we change the first one to

$$M = u_y,\ N = -(y^2u_y + xyu_x + 2yu),\ H = 2yu_y + xu_x + 2u.$$

The elements in these two $\mathfrak{sl}(2,\mathbb{R})$'s have no terms in $\ker D_x$, and the ghost problems will not appear in the following calculations. If we ignore the y-dependence, only H is left, so the results are not likely to shed any light on the KdV hierarchy.

Notice $M_2M \sim u_x$ and $H_2u_x = -2u_x$. Starting from M_2M, we can construct an infinite–dimensional representation of $\mathfrak{sl}(2,\mathbb{R}) = \{M_2, H_2, N_2\}$ as in the diagram:



It is easy to see that $Hu_x = u_x$ and $Mu_x = 0$. Therefore the space spanned by M_2M and $NM_2M \sim M_2$ is a 2–dimensional irreducible representation of $\{M, H, N\}$. Moreover, we have the following

PROPOSITION 9. *Fix $n \in \mathbb{N}$. The space spanned by $N^jN_2^nK$, $j = 0, 1, \cdots, n+3$, is an $n+4$–dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R}) = \{M, H, N\}$.*

PROOF. This result is obtained by computing the H-eigenvalue of $N_2^nK \in \ker M$. We see by inspection that $HK = 3K$. The action of N_2 lowers the H_2-eigenvalue by 2 and therefore raises the H-eigenvalue by 1. Thus we find that the

H-eigenvalue of $N_2^n K$ equals $3+n$. So this implies that we have an $n+4$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. □

We notice $N_2 K \in \ker K$. Using the same method as before, we can prove

THEOREM 3. *The expressions* $exp(-tad_K)N^r N_2^j K$, *where* $0 \leq r \leq j+2$ *and* $j \geq 0$ *are symmetries of the Kadomtsev–Petviashvili equation.*

Here we list the first few.

$$
\begin{aligned}
& u_x \\
& t^m u_y + \frac{m}{2} t^{m-1} y u_x, \quad m = 0, 1 \\
& t^m K + \frac{m}{3} t^{m-1} \left(2yu_y + xu_x + 2u\right) \quad m = 0, 1 \\
& t^m K_2 + \frac{1}{4} m t^{m-1} \left(3yK + 2xu_y + 4D_x^{-1} u_y\right) \\
& \quad + \frac{1}{2} \binom{m}{2} t^{m-2} \left(y^2 u_y + xyu_x + 2yu\right) \quad m = 0, 1, 2,
\end{aligned}
$$

where $K_2 = D_x^{-2} u_{yyy} - 3u_{xxy} - 12uu_y - 6u_x D_x^{-1} u_y$.
Notice that we have lost some of the "symmetries" in the list in [**CLL83, CL87**], namely those containing terms in $\ker D_x^i$. In ghost language ([**OSW01**]), what happens is that the commutator of the equation and the "symmetry" is not zero, but a ghost.

7. Discussion

One may notice that N_2 in section 3 and NK in section 4 are the mastersymmetries [**Olv93, Fuc83**] of the equations we considered. The theory of the mastersymmetry has not been well-developed yet. According to [**Dor93**, Theorem 7.1] the mastersymmetry does produce (under suitable conditions) commutating symmetries (more references on the concept of mastersymmetry are cited in this book). However, the conditions do not hold for the Ibragimov–Shabat equation since there is no element above H, i.e., in the notation of [**Dor93**], the τ_{-1} in the theorem.
The $\mathfrak{sl}(2,\mathbb{R})$ approach also works for systems, as is illustrated by the Diffusion system

$$
\left\{
\begin{array}{lcl}
u_t & = & u_2 + v^2 \\
v_t & = & v_2,
\end{array}
\right.
$$

which does not have a symplectic operator. It seems that the occurrence of a local $\mathfrak{sl}(2,\mathbb{R})$ for a nonlinear equation is complementary to the existence of nontrivial conservation laws. This is confirmed in the scalar $1+1$–dimensional case by the results in [**Ser00**]. The time-dependent symmetries of the KdV equation are all characterized in [**MS81**]. The nonexistence of higher order polynomial symmetries in t is consistent with the fact that we could not find a local $\mathfrak{sl}(2,\mathbb{R})$.
Finally we make a remark that was inspired by conversations on the realization of abstract Lie algebras with Jan Draisma of the Eindhoven University of Technology. The search for $\mathfrak{sl}(2,\mathbb{R})$ was inspired by the application in normal form theory, cf. [**CS86**], and the present results make it possible to effectively define normal forms for those equations where one can find an $\mathfrak{sl}(2,\mathbb{R})$ containing the equation itself, as in the case of Burgers equation. The fact that the representation space is infinite dimensional does not prohibit the application of the splitting algorithm, as

in [**San94**], as long as one of the elements in $\mathfrak{sl}(2,\mathbb{R})$ acts nilpotently on polynomial vectorfields.

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References

[BCD+99] A. V. Bocharov, V. N. Chetverikov, S. V. Duzhin, N. G. Khor'kova, I. S. Krasil'shchik, A. V. Samokhin, Yu. N. Torkhov, A. M. Verbovetsky, and A. M. Vinogradov. *Symmetries and conservation laws for differential equations of mathematical physics.* American Mathematical Society, Providence, RI, 1999. Edited and with a preface by Krasil' shchik and Vinogradov, Translated from the 1997 Russian original by Verbovetsky [A. M. Verbovetskiĭ] and Krasil' shchik.

[CL87] H. H. Chen and J. E. Lin. On the infinite hierarchies of symmetries and constants of motion for the Kadomtsev-Petviashvili equation. *Phys. D*, 26(1-3):171–180, 1987.

[CLL83] H. H. Chen, Y. C. Lee, and Jeng Eng Lin. On a new hierarchy of symmetries for the Kadomtsev-Petviashvili equation. *Phys. D*, 9(3):439–445, 1983.

[CS86] Richard Cushman and Jan A. Sanders. Nilpotent normal forms and representation theory of sl$(2,\mathbb{R})$. In *Multiparameter bifurcation theory (Arcata, Calif., 1985)*, volume 56 of *Contemp. Math.*, pages 31–51. Amer. Math. Soc., Providence, R.I., 1986.

[Dor93] Irene Dorfman. *Dirac structures and integrability of nonlinear evolution equations.* John Wiley & Sons Ltd., Chichester, 1993.

[Fuc83] Benno Fuchssteiner. Mastersymmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations. *Progr. Theoret. Phys.*, 70(6):1508–1522, 1983.

[Hel78] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.

[IS80] N. H. Ibragimov and A. B. Shabat. Infinite Lie-Bäcklund algebras. *Funktsional. Anal. i Prilozhen.*, 14(4):79–80, 1980.

[MS81] B. A. Magadeev and V. V. Sokolov. Complete Lie–Bäcklund algebra for the Korteweg–de Vries equation. *Dinamika Sploshn. Sredy*, 52:48–55, 1981.

[Olv93] Peter J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1993.

[OSW01] Peter J. Olver, Jan A. Sanders, and Jing Ping Wang. Nonlocal symmetries and ghosts. Technical report, Vrije Universiteit Amsterdam, 2001. In preparation.

[San94] J. A. Sanders. Versal normal form computations and representation theory. In E. Tournier, editor, *Computer algebra and differential equations*, volume 193 of *London Mathematical Society Lecture Note Series*, pages 185–210. Cambridge University Press, Cambridge, 1994. Papers from the conference (CADE-92) held in June 1992.

[Ser00] Artur Sergyeyev. On time-dependent local symmetries of integrable evolution equations. In *Symmetry in Nonlinear Mathematical Physics,* Proceedings of Institute of Mathematics of NAS of Ukraine, volume 30, pages 196–203, 2000. *Proc. 3rd Int. Conference "Symmetry in Nonlinear Mathematical Physics" (Kyiv, July 13-18, 1999).*

[SS84] V. V. Sokolov and A. B. Shabat. Classification of integrable evolution equations. In *Mathematical physics reviews, Vol. 4*, volume 4 of *Soviet Sci. Rev. Sect. C: Math. Phys. Rev.*, pages 221–280. Harwood Academic Publ., Chur, 1984.

[SW98] Jan A. Sanders and Jing Ping Wang. On the integrability of homogeneous scalar evolution equations. *J. Differential Equations*, 147(2):410–434, 1998.

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