# On the Integrability of Systems of second order Evolution Equations with two Components

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## 1. INTRODUCTION

In this paper we consider two component evolution equations with two independent variables – time and space. We restrict ourselves to the case that the system is homogeneous polynomial in the dependent variables and their derivatives and it has a diagonal linear part of order two.
An equation that has a generalized symmetry on infinitely many orders is called *(symmetry) integrable.* The purpose of this paper is to classify all (symmetry) integrable equations within the class specified above. A typical example, with free parameters $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{C}$, looks like

$$\begin{cases} u_t = \sigma_1 u_2 + \sigma_2 u_1 v + \frac{\sigma_2(\sigma_1 - 1)}{2\sigma_1} u v_1 + \frac{\sigma_2(\sigma_2 - \sigma_3)}{4\sigma_1} u v^2 \\ v_t = v_2 + \sigma_3 v_1 v. \end{cases}$$

This is a much more restricted classification than the program reviewed in [MSS91], but it has the advantage that it is easy to check whether a given equation is in the list of integrable equations or not. Further research will be needed to gradually extend the transformation pseudogroup until a classification is reached that both effective and simple to state.
This paper is part of a series that started with [SW98] where the one component equations with positive weight were classified, followed by [SW00], handling zero weight case, and [BSW98] proving that the Bakirov example only has one symmetry, and [BSW01] classifying systems of Bakirov type. The last paper is closest to what we do here in its technical details.
Let us sketch the methods by which we prove the completeness of the classification. First we compute all possible weights and write out the homoge-

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neous equations with these weights. We then translate the equations in the derivatives of the field variables into polynomial expressions using the symbolic method. Once the linear part of the system is diagonal, we know that its $ad$-action on the symmetry will be semisimple, where $ad(X)Y = [X, Y]$. This allows us to derive a number of equations for the linearly independent components when we compute the symmetry. We then proceed to solve these.

Once we have found all the relations on the parameters, we can use the fact that these relations have to be true for infinitely many orders, since the system is assumed to be integrable. We are now in the position to apply the Lech-Mahler Theorem A.4. This forces some ratios to be roots of unity and enables us, by repeated application of the above, to explicitly compute the spectrum and the order of the possible symmetries. Having obtained these necessary conditions we then set out to find the lowest possible order symmetry. After we have found such a symmetry and checked some technical conditions, we can conclude integrability from our implicit function Theorem A.3 for filtered Lie algebras, cf. [SW98]. We remark here that the knowledge of the spectrum is essential to get through the necessary calculations. If we write out the Lie bracket of an arbitrary equation and a candidate symmetry, this leads in general to a big system of polynomial equations, the solution of which is not within the reach of present day computer algebra packages.

Another technique, which is especially useful in the case that the quadratic terms are zero, is to look at the resultant of two polynomials we derived in order to find the symmetry. This quickly leads to conditions on the spectrum, and it is useful in more component systems too.

The organization of this paper is as follows. We start with an introduction to the symbolic method in section 2. Then we set up the equations that have to be solved in order to get a symmetry for a given system in section 3. In section 4 we assume that there are no quadratic and cubic terms in the equation, and therefore also none in the symmetry. We use purely geometric methods to show that such equations can never be integrable, following an idea of Frits Beukers [Wang98, Appendix A]. Then in section 5 we assume the quadratic terms to be zero, but the cubic terms not. Here we use resultant calculations to show that integrability implies that the ratio of the eigenvalues of the linear part of the system has to be $-1$. And finally in section 6 we complete the analysis and give the lists of integrable systems. The results depend heavily on the use of the Lech-Mahler theorem. We then compare in section 7 our results with those in the literature as far as applicable. In appendix A we give the implicit function theorem which allows us draw conclusions from computations on lower degrees (of the dependent variables), and the Lech-Mahler Theorem.

## 2. THE SYMBOLIC METHOD

The symbolic method, first introduced in [GD75], is based on the simple idea of replacing $u_k = \frac{\partial u}{\partial k}$ by $u\xi_1^k$, following similar procedures in classical invariant theory [Olv99]. If we have a monomial, for instance $u_1u_2$, we use a symbol for each $u$, so that we get $\xi_1\xi_2^2u^2$. Since $u_1u_2 = u_2u_1$ in the commutative case, the expression should be symmetric in $\xi_1$ and $\xi_2$. To this end, we average over the permutation group $\Sigma_2$ to obtain $\frac{1}{2}\xi_1\xi_2(\xi_1+\xi_2)u^2$. The necessary formalism in one dependent variable has been derived in [SW98]. We remark that the methods work equally well in the noncommutative case, as illustrated in [OS98, OW00, OSW99].

If we have several variables, like $u$ and $v$, we introduce a symbol for each of them, for instance $\xi$ and $\eta$. Thus the symbolic expression for $u_1u_2v_3$ is $\frac{1}{2}\xi_1\xi_2\eta_1^3(\xi_1+\xi_2)u^2v$. If we would not carry along the $u$'s and $v$'s, information would be lost: consider the expressions $uv$ and $u^2$. The alternative would be to keep the zeroth power of any symbol so that $uv$ would go to $\xi_1^0\eta_1^0$. However, this is very awkward in actual polynomial computations.

The main motivation for the introduction of the symbolic notation lies in the fact that all operations that one is used to be doing in differential algebra translate naturally. For instance, taking the derivative of an expression translates into multiplication with the sum of all symbols: $(\xi_1+\xi_2+\eta_1)u^2v$ corresponding to $2uu_1v + u^2v_1 = D_xu^2v$.

Here we don't give the general formula for the Lie bracket since it is quite complicated. We only give the one we need later on.

$$\begin{aligned}&[v^2f(y_1,y_2)\tfrac{\partial}{\partial u}, u^2g(x_1,x_2)\tfrac{\partial}{\partial v}]\\ &= -2u^2vf(y_1,x_1+x_2)g(x_1,x_2)\tfrac{\partial}{\partial u} + 2uv^2g(x_1,y_1+y_2)f(y_1,y_2)\tfrac{\partial}{\partial v}.\end{aligned} \quad (1)$$

To illustrate how the symbolic method works, we give the symbolic calculation for the fifth order symmetry of the Korteweg–de Vries equation. When one computes a symmetry, the natural approach is to do this degree by degree. So for instance, if we have as the equation

$$u_t = K = K^0 + K^1 = u_3 + uu_1 \qquad \text{(KdV)}$$

then we compute as a symmetry

$$S = S^0 + S^1 + \cdots = u_5 + a_1uu_3 + a_2u_1u_2 + \cdots$$

Since $[K^0, S^0]$ is automatically zero, the first equation we have to solve is $[K^0, S^1] + [K^1, S^0] = 0$, i.e.,

$$\begin{aligned}&D_x^3S^1 + uD_xS^0 + u_1S^0 =\\ &= D_x^5K^1 + a_1uD_x^3K^0 + a_1u_3K^0 + a_2u_1D_x^2K^0 + a_2u_2D_xK^0.\end{aligned}$$

If we translate this into the symbolic method we obtain

$$(\xi_1+\xi_2)^3\hat{S}^1+(\xi_1^5+\xi_2^5)\hat{K}^1=(\xi_1+\xi_2)^5\hat{K}^1+(\xi_1^3+\xi_2^3)\hat{S}^1.$$

We can formally solve

$$\hat{S}^1=\frac{(\xi_1+\xi_2)^5-\xi_1^5-\xi_2^5}{(\xi_1+\xi_2)^3-\xi_1^3-\xi_2^3}\hat{K}^1,$$

and this is a real solution if $\hat{S}^1$ turns out to be a polynomial. So we have translated our problem into the following question. If we let

$$G^n=(\xi_1+\xi_2)^n-\xi_1^n-\xi_2^n,$$

which factors do $G^n$ and $G^m$ have in common? In this case, the answer is simple, that is,

$$\hat{S}^1=\frac{5}{3}(\xi_1^2+\xi_1\xi_2+\xi_2^2)\hat{K}^1=\frac{5}{6}(\xi_1^3+2\xi_1^2\xi_2+2\xi_1\xi_2^2+\xi_2^3)u^2.$$

We now compute $S^2$ by solving $[S^1,K^1]+[S^2,K^0]=0$. This leads to

$$\hat{S}^2=\frac{5}{6}\frac{(\xi_1+\xi_2)(\xi_2+\xi_3)(\xi_1+\xi_3)(\xi_1+\xi_2+\xi_3)}{(\xi_1+\xi_2+\xi_3)^3-\xi_1^3-\xi_2^3-\xi_3^3}u^3=\frac{5}{18}(\xi_1+\xi_2+\xi_3)u^3.$$

Note that $[S^2,K^1]=0$ on the next degree. Therefore, the fifth order symmetry is

$$S=S^0+S^1+S^2=u_5+\frac{5}{3}uu_3+\frac{10}{3}u_1u_2+\frac{5}{6}u^2u_1,$$

the well-known Lax equation.

## 3. SYMMETRIES OF HOMOGENEOUS SYSTEMS

We consider the systems with 2 components $u$ and $v$ and assume that the weights are $w(u)=\lambda_1$ and $w(v)=\lambda_2$ respectively.

Assumption 3.1. *We assume* $0<\lambda_1\leq\lambda_2$ *and* $\lambda_2-\lambda_1\notin\mathbb{N}_{>0}$.

There is one subtle point in this assumption. In the sequel we want to scale (by changing the time) the last eigenvalue to 1. If it happens to be zero, we would like to permute our variables so that a nonzero eigenvalue appears at the last position. But this permutation on the variables is only allowed when $\lambda_1=\lambda_2$. We therefore will assume both eigenvalues to be

nonzero, but some of the results will still be valid for one zero eigenvalue. The reader should be careful in the application of the results in this paper to this case.
The $n^{\mathrm{th}}$-order system can be written as:

$$u_t\frac{\partial}{\partial u}+v_t\frac{\partial}{\partial v}=\sum_{i,j}K^{(i,j)}_{n-i\lambda_1-j\lambda_2},\quad i,j\geq -1,\ i+j\geq 0. \tag{2}$$

Here $K^{(i,j)}_m$ indicates a term with $m$ $x$-derivatives altogether, and degree $i$ in $u$ and $j$ in $v$. This degree can be $-1$: $K^{(-1,0)}_0=\frac{\partial}{\partial u}$. Only when $n-i\lambda_1-j\lambda_2\in\mathbb{N}$ does the term $K^{(i,j)}$ make sense and can appear in the system. The linear part of the system can be written as $K^{(0,0)}_n+K^{(-1,1)}_{n-\lambda_2+\lambda_1}$, where $K^{(0,0)}_n=a_{10}u_n\frac{\partial}{\partial u}+a_{20}v_n\frac{\partial}{\partial v}$ and $K^{(-1,1)}_{n-\lambda_2+\lambda_1}=a_{00}v_{n-\lambda_2+\lambda_1}\frac{\partial}{\partial u}$. We remark that the three indices of $K$ are all gradings of the Lie algebra of evolutionary vectorfields.

Assumption 3.2. *We assume that the linear part of the system equals*

$$K^{(0,0)}_n=a_{10}u_n\frac{\partial}{\partial u}+a_{20}v_n\frac{\partial}{\partial v},\quad a_{10}a_{20}\neq 0,\quad a_{10}\neq a_{20},\quad n\geq 2.$$

*This guarantees nonlinear injectiveness (cf. def. A.1), as required in Theorem A.3.*

Under this assumption, we know that any symmetry of the system starts with linear terms ([BSW98]), that is, if the system has an $m^{\mathrm{th}}$-order symmetry $Q$, it has to begin with

$$b_{10}u_m\frac{\partial}{\partial u}+b_{20}v_m\frac{\partial}{\partial v},\quad (b_{10},b_{20})\in\mathbb{CP}^1,\quad n,m\geq 2,\quad m\neq n.$$

Since the linear part is diagonal, it acts semisimply on polynomial vectorfields. This simplifies the analysis considerably. Let us compute the action of the diagonal linear part on vectorfields of $Q^{(i,j)}$ using the symbolic method.

$$[Q^{(i,j)},\widehat{\begin{pmatrix}a_{10}u_n\\ a_{20}v_n\end{pmatrix}}]=\begin{pmatrix}f^{(i,j)}_{u;n}(a_{10},a_{20};x;y) & 0\\ 0 & f^{(i,j)}_{v;n}(a_{10},a_{20};x;y)\end{pmatrix}\hat{Q}^{(i,j)},$$

where $\hat{Q}^{(i,j)}$ is the symbolic expression of $Q^{(i,j)}$ and

$$\begin{aligned} f_{u;n}^{(i,j)}(a_{10},a_{20};x;y) &= a_{10}(\sum_{l=1}^{i+1} x_l + \sum_{k=1}^{j} y_k)^n - a_{10}\sum_{l=1}^{i+1} x_l^n - a_{20}\sum_{k=1}^{j} y_k^n; \\ f_{v;n}^{(i,j)}(a_{10},a_{20};x;y) &= a_{20}(\sum_{l=1}^{i} x_l + \sum_{k=1}^{j+1} y_k)^n - a_{10}\sum_{l=1}^{i} x_l^n - a_{20}\sum_{k=1}^{j+1} y_k^n. \end{aligned} \tag{3}$$

These are two important polynomials for the later analysis and related by

$$f_{u;n}^{(i,j)}(a_{10},a_{20};x;y) = f_{v;n}^{(j,i)}(a_{20},a_{10};y;x). \tag{4}$$

Next we want to find out whether we can verify the technical conditions in the implicit function Theorem A.3. The most important condition is the relatively $l$-prime condition. If we translate this condition into the symbolic language, we try to find minimum $l \in \mathbb{N}$ for a given $n, m \in \mathbb{N}$, such that

$$(f_{u;n}^{i,j}(a_{10},a_{20};x;y), f_{u;m}^{i,j}(b_{10},b_{20};x;y)) = 1, \quad i+j \geq l$$

where $a_{10}, a_{20}$ are the eigenvalues of the system, and $b_{10}, b_{20}$ of the symmetry (This also explains the terminology).

## 4. QUARTIC TERMS

The following theorem is due to Frits Beukers. It can be applied to the $N$-component problem to show that any polynomial (in $u, v, \cdots$ and their $x$-derivatives) system of order $n > 1$ with nonzero diagonal linear part and without quadratic and cubic terms cannot have higher order symmetries.

THEOREM 4.1. *For any positive integer $m$ the polynomial*

$$h_{a,m} = (x+y+z+w)^m - a_1^{m-1}x^m - a_2^{m-1}y^m - a_3^{m-1}z^m - a_4^{m-1}w^m,$$

*where $\Pi_{i=1}^4 a_i \neq 0$, is irreducible over $\mathbb{C}$.*

REMARK 4.2. *The condition $\Pi_{i=1}^4 a_i \neq 0$ is automatically satisfied in our two-component case since the $a_i$ are from the eigenvalues of the linear part of the system and we assume both of them to be nonzero.*

*Proof.* Suppose $h_{a,m} = A \cdot B$ with $A, B$ polynomial of positive degree. Then the projective hypersurface $S$ given by $h_{a,m} = 0$ consists of two

components $S_1, S_2$ given by $A = 0, B = 0$ respectively. These components intersect in an infinite number of points, which should be singularities of $S$. Thus it suffices to show that $S$ has finitely many singular points.
We compute the singular points by setting the partial derivatives of $h_{a,m}$ equals to zero, i.e.,

$$\begin{aligned}(x+y+z+w)^{m-1} - (a_1x)^{m-1} = 0\\ (x+y+z+w)^{m-1} - (a_2y)^{m-1} = 0\\ (x+y+z+w)^{m-1} - (a_3z)^{m-1} = 0\\ (x+y+z+w)^{m-1} - (a_4w)^{m-1} = 0\end{aligned}$$

From these equation follows in particular that

$$x = \xi_1/a_1,\ y = \xi_2/a_2,\ z = \xi_3/a_3,\ w = \xi_4/a_4$$

where $\xi_i^{m-1} = 1$ and $\xi_1/a_1 + \xi_2/a_2 + \xi_3/a_3 + \xi_4/a_4 = 1$. For given $a_i, i = 1 \cdots 4$, we get finitely many singular points. ▮

This theorem implies that when a system (2) has no quadratic and cubic terms, i.e., $K^{(i,j)} = 0$ $(1 \leq i + j \leq 2)$, it is not integrable.

## 5. CUBIC TERMS

From now on we concentrate on the case $n = 2$, that is, the second order system. In this section, we study the divisibility properties of some symbolic expressions, which are essential for the analysis of relatively 1-primeness. We first give the general notation. Let

$$\mathcal{G}_k^m[\underline{c}](x) = c_0(\sum_{i=1}^{k} x_i)^m - \sum_{i=1}^{k} c_i x_i^m, \quad \underline{c} \in \mathbb{CP}^k. \tag{5}$$

Then $\mathcal{G}_3^2[\underline{a}]$ represents the *ad*–action of the linear part of a second order equation on the cubic terms of the candidate symmetry, $\mathcal{G}_3^m[\underline{b}]$ does the same for the action of the candidate symmetry of order $m$ on the equation. Since the $a_i, i = 0, \cdots, k$ are eigenvalues of the linear part of the system, none of them are zero.
We now study the common factors of $\mathcal{G}_3^2[\underline{a}]$ and $\mathcal{G}_3^m[\underline{b}]$. Since the linear part of the equation is assumed to be diagonal, and the elements $u^m v^n \frac{\partial}{\partial u}$ and $u^k v^l \frac{\partial}{\partial v}$ form a basis for the symbolic vectorfields (with polynomials of the symbols as coefficients), we can always assume $\underline{a}$ and $\underline{b}$ to be of the same form, for instance $(a, 1, 1, 1)$ and $(b, 1, 1, 1)$.

Lemma 5.1. *Assume $a_1 = a_0$. A necessary condition for $\mathcal{G}_3^2[\underline{a}]$ and $\mathcal{G}_3^m[\underline{b}]$ $(m > 2)$ to have a common divisor is that one of the following relations* *hold:*

$$\begin{cases} 1. \quad a_0 = a_2, & b_0 = b_2, & a_0 = -a_3 \\ 2. \quad a_0 = -a_2, & b_1 = (-1)^{m-1} b_2, & a_0 = a_3 \\ 3. \quad b_0(a_0+a_2)^{m-2} = & b_1(a_2-a_0)^{m-2}, & a_3 = (m-2)a_2. \end{cases} \tag{6}$$

*Proof.* Let us compute the resultant $A$ between $\mathcal{G}_3^m[\underline{b}]$ and $\mathcal{G}_3^2[\underline{a}]$ with respect to $x$. First $\mathcal{G}_3^2[\underline{a}](x,y,z) = 2a_0(y+z)x + a_0(y+z)^2 - a_2y^2 - a_3z^2 = Rx + Q$. It is easy to see that

$$A = \sum_{j=0}^{m} (-1)^{m-j} P_j(b,y,z) Q^j R^{m-j} = (-1)^m \mathcal{G}_3^m[\underline{b}](-Q, yR, zR), \tag{7}$$

where $P_l$ is the coefficient of $x^l$ in $\mathcal{G}_3^m[\underline{b}]$. We want to compute all $(\underline{b}, m)$ such that $A$ is zero. Let $A = \sum_{i=0}^{\infty} A_i z^i$, and use the similar notation for $Q$ and $R$. Then $Q_0 = (a_0 - a_2)y^2$ and $R_0 = 2a_0y$. Since

$$\mathcal{G}_3^m[\underline{b}](x,y,z) \;=\; b_0(x+y+z)^m - b_1x^m - b_2y^m - b_3z^m,$$

we see that

$$\begin{aligned} A_0 &= (-1)^m \mathcal{G}_3^m[\underline{b}](-Q_0, yR_0, 0) \\ &= (-1)^m \left(b_0(-Q_0 + yR_0)^m - b_1(-Q_0)^m - b_2(R_0y)^m\right) \\ &= (-1)^m \left(b_0((a_2-a_0)y^2 + 2a_0y^2)^m - b_1(a_2-a_0)^m y^{2m} - b_2 y^{2m}(2a_0)^m\right) \\ &= (-1)^m y^{2m} \left(b_0(a_0+a_2)^m - b_1(a_2-a_0)^m - b_2(2a_0)^m\right). \end{aligned}$$

It follows that

$$b_0(a_0+a_2)^m - b_1(a_2-a_0)^m = b_2(2a_0)^m. \tag{8}$$

Next we compute $A_1$, coefficient of $z$ in (7). Recall that $Q_0 = (a_0 - a_2)y^2$ and $R_0 = 2a_0y$. And one has $Q_1 = 2a_0y$ and $R_1 = 2a_0$ . Therefore, ignoring terms of $O(z^2)$ (we denote this with '$\equiv$'),

$$\begin{aligned} (-1)^m A &= \mathcal{G}_3^m[\underline{b}](-Q, yR, zR) \\ &\equiv \mathcal{G}_3^m[\underline{b}](-Q_0 - Q_1z, y(R_0 + R_1z), zR_0) \\ &= b_0((a_2-a_0)y^2 - 2a_0yz + y(2a_0y + 2a_0z) + 2a_0yz)^m \\ &\quad -b_1((a_2-a_0)y^2 - 2a_0yz)^m - b_2y^m(2a_0y + 2a_0z)^m - b_3(2a_0yz)^m \\ &\equiv b_0((a_0+a_2)y^2 + 2a_0yz)^m - b_1((a_2-a_0)y^2 - 2a_0yz)^m \\ &\quad -b_2(2a_0y)^m(y+z)^m \\ &\equiv 2a_0mzy^{2m-1}\left(b_0(a_0+a_2)^{m-1} + b_1(a_2-a_0)^{m-1} - (2a_0)^{m-1}b_2\right). \end{aligned}$$

It follows that $b_0(a_0+a_2)^{m-1}+b_1(a_2-a_0)^{m-1}=(2a_0)^{m-1}b_2$. Combining this with (8) we see that one of the following relations hold:

$$\begin{cases} 1. & a_0=a_2, & b_0=b_2 \\ 2. & a_0=-a_2, & b_1=(-1)^{m-1}b_2 \\ 3. & b_0(a_0+a_2)^{m-2}= & b_1(a_2-a_0)^{m-2}. \end{cases} \tag{9}$$

Finally we compute $A_2$. Recall that $Q=(a_0-a_2)y^2+2a_0yz+(a_0-a_3)z^2$ and $R=2a_0y+2a_0z$. Then

$$\begin{aligned}
(-1)^m A &= \mathcal{G}_3^m[\underline{b}](-Q,yR,zR)\\
&= \mathcal{G}_3^m[\underline{b}]((a_2-a_0)y^2-2a_0yz+(a_3-a_0)z^2,2a_0y(y+z),2a_0z(y+z))\\
&= b_0((a_0+a_2)y^2+2a_0yz+(a_0+a_3)z^2)^m\\
&\quad -b_1((a_2-a_0)y^2-2a_0yz+(a_3-a_0)z^2)^m\\
&\quad -b_2(2a_0y)^m(y+z)^m-b_3(2a_0z)^m(y+z)^m\\
&\equiv mb_0(a_0+a_2)^{m-2}y^{2m-2}\left((a_0+a_2)(a_0+a_3)+2(m-1)(a_0)^2\right)z^2\\
&\quad -mb_1(a_2-a_0)^{m-2}y^{2m-2}\left((a_2-a_0)(a_3-a_0)+2(m-1)(a_0)^2\right)z^2\\
&\quad -mb_2y^{2m-2}\frac{(m-1)(2a_0)^m}{2}z^2\\
&= mb_0(a_0+a_2)^{m-2}y^{2m-2}\left((a_0+a_2)(a_0+a_3)+2(m-1)(a_0)^2\right)z^2\\
&\quad -mb_1(a_2-a_0)^{m-2}y^{2m-2}\left((a_2-a_0)(a_3-a_0)+2(m-1)(a_0)^2\right)z^2\\
&\quad -m(m-1)a_0y^{2m-2}\left(b_0(a_0+a_2)^{m-1}+b_1(a_2-a_0)^{m-1}\right)z^2
\end{aligned}$$

If $a_0=a_2$ then

$$(-1)^m A_2 = mb_0(a_0+a_2)^{m-1}y^{2m-2}(a_0+a_3)z^2$$

and it leads to $a_3=-a_0$.
If $a_0=-a_2$ then

$$(-1)^m A_2 = -mb_1(a_2-a_0)^{m-2}y^{2m-1}(a_3-a_0)z^2$$

and it leads to $a_3=a_0$.
If $b_0(a_0+a_2)^{m-2}=b_1(a_2-a_0)^{m-2}$, then

$$(-1)^m A_2 = 2ma_0b_1(a_2-a_0)^{m-2}y^{2m-2}\left(a_3-(m-2)a_2\right)z^2$$

and we must have $a_3=(m-2)a_2$. ∎

THEOREM 5.2. *Assume $a_1=a_0$. Suppose that there exists more than $m\in\mathbb{N}$ with $\underline{b}$ such that $\mathcal{G}_3^2[\underline{a}]$ divides $\mathcal{G}_3^m[\underline{b}]$. Then we are in one of the following cases,*

$\underline{a} = (1, 1, 1, -1)a_0$. *Then* $m \in \mathbb{N}_{\geq 2}$ *arbitrary and* $\underline{b} = (1, 1, 1, (-1)^{m-1})b_0$.
$\underline{a} = (1, 1, -1, 1)a_0$. *Then* $m \in \mathbb{N}_{\geq 2}$ *arbitrary and* $\underline{b} = (1, 1, (-1)^{m-1}, 1)b_0$.

*Proof.* We only need to determine whether $\mathcal{G}_3^2[\underline{a}]$ divides $\mathcal{G}_3^m[\underline{b}]$, where $\underline{b} = (b_0, b_1, b_0, b_3)$ and $\underline{a} = (a_0, a_0, a_0, -a_0)$ or $\underline{b} = (b_0, b_1, (-1)^{m-1}b_1, b_3)$ and $\underline{a} = (a_0, a_0, -a_0, a_0)$ from Lemma 5.1. This lemma gives us three cases to consider. We work out the first case. The second case can be done in a similar way. Note that

$$\mathcal{G}_3^2[\underline{a}](x, y, z) = a_0\left((x + y + z)^2 - x^2 - y^2 + z^2\right) = 2a_0(y + z)(x + z).$$

Therefore

$$0 \;=\; \mathcal{G}_3^m[\underline{b}]|_{y+z=0} = (b_0 - b_1)x^m - (b_0 - (-1)^{m-1}b_3)y^m.$$

This leads to $b_1 = b_0$ and $b_3 = (-1)^{m-1}b_0$. Under this condition, we have $\mathcal{G}_3^n[\underline{b}]|_{x+z=0} = 0$.
The third case does is not possible under our assumptions: From the relations $a_3 = (m-2)a_2$ we can immediately conclude that if there is more than one higher order on which a symmetry exists, $a_2 = a_3 = 0$, and this is excluded by our basic assumptions on the spectrum of the linear part of the equation. ∎

So far we only studied divisibility between $\mathcal{G}_3^2[\underline{a}]$ with $a_1 = a_0$ and $\mathcal{G}_3^m[\underline{b}]$. When we compute the Lie bracket between the linear and cubic terms, it also leads to the polynomials of the form (cf. (3))

$$\mathcal{G}_3^k[(b, 1, 1, 1)](x, y, z) = b(x + y + z)^k - x^k - y^k - z^k, \quad b \in \mathbb{C},\ k \in \mathbb{N}.$$

Now we study the divisibility between such polynomials and $\mathcal{G}_3^2[(a, 1, 1, 1)]$.

Proposition 5.3. *For all $a \in \mathbb{C}$ and $a \neq \frac{1}{3}$ the polynomial $\mathcal{G}_3^2[(a, 1, 1, 1)]$ is irreducible over $\mathbb{C}$.*

*Proof.* If a polynomial can be written as the product of two lower order polynomials, it must have singular points. This implies

$$\begin{aligned} a(x + y + z) - x &= 0 \\ a(x + y + z) - y &= 0 \\ a(x + y + z) - z &= 0. \end{aligned}$$

These equations have no solution unless $a = \frac{1}{3}$. ∎

It is easy to check that

$$\frac{1}{3}(x+y+z)^2 - x^2 - y^2 - z^2 = -\frac{2}{3}(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z), \quad (10)$$

where $1+\omega+\omega^2=0$, that is, $\omega$ is an elementary cubic root.

PROPOSITION 5.4. *We have* $(\mathcal{G}_3^m[(b,1,1,1)], \mathcal{G}_3^2[(a,1,1,1)]) = 1$ $\forall a,b \in \mathbb{C}$ *and* $m>2$.

*Proof.* We know that $\mathcal{G}_3^2[(a,1,1,1)]$ is irreducible over $\mathbb{C}$ unless $a=1/3$. So when $a \neq \frac{1}{3}$ we only need to show that $\mathcal{G}_3^2[(a,1,1,1)]$ does not divide $\mathcal{G}_3^m[(b,1,1,1)]$. To do so, we use a generating function. First we introduce

$$e_1 = x+y+z, \quad e_2 = xy+yz+xz, \quad e_3 = xyz,$$

the basis of the symmetric polynomials in $x,y,z$. Notice

$$\mathcal{G}_3^2[(a,1,1,1)](x,y,z) = (a-1)e_1^2 + 2e_2.$$

If $x^m+y^m+z^m$ is a polynomial independent of $e_3$, we can find $b$ such that $(\mathcal{G}_3^m[(b,1,1,1)], \mathcal{G}_3^2[(a,1,1,1)]) = \mathcal{G}_3^2[(a,1,1,1)]$. Let $S_m = x^m+y^m+z^m$. We compute

$$\sum_{j=0}^{\infty} S_{3j}t^j|_{e_1=0,e_2=0} = \frac{3}{1-te_3} = 3\sum_{j=0}^{\infty} e_3^j t^j$$
$$\sum_{j=0}^{\infty} S_{3j+1}t^j|_{e_1=0(e_2=0)} = \frac{te_2^2(te_3+2)}{(1-te_3)^3} = \sum_{j=0}^{\infty} e_2^2 \frac{(j+1)(3j+4)}{2} e_3^j t^{j+1}$$
$$\sum_{j=0}^{\infty} S_{3j+2}t^j|_{e_1=0(e_2=0)} = \frac{-e_2(te_3+2)}{(1-te_3)^2} = \sum_{j=0}^{\infty} -e_2(3j+2)e_3^j t^j,$$

where $e_1=0(e_2=0)$ means that we take only $e_1=0$ for the numerator while we take them both zero for the denominator.
This show that $x^m+y^m+z^m$ is dependent on $e_3$ when $m>2$. Therefore there are no $b$ and $m>2$ such that $\mathcal{G}_3^2[(a,1,1,1)]$ divides $\mathcal{G}_3^m[(b,1,1,1)]$.
When $a=\frac{1}{3}$, it follows from formula (10) we need to prove $\mathcal{G}_3^m[(b,1,1,1)]$ has neither $x+\omega y+\omega^2 z$ nor $x+\omega^2 y+\omega z$ as a factor. Assume that $x+\omega y+\omega^2 z$ is a factor of $\mathcal{G}_3^m[(b,1,1,1)]$ (the same method works for the

other factor). Then

$$\begin{aligned} 0 &= b(-\omega y - \omega^2 z + y + z)^m - (-\omega y - \omega^2 z)^m - y^m - z^m \\ &= b(1-\omega)^m (y + (1+\omega) z)^m - (-\omega)^m (y + \omega z)^m - y^m - z^m \\ &= \sum_{j=0}^{m} \binom{m}{j} \left( b(1-\omega)^m (-\omega^2)^j - (-\omega)^m \omega^j \right) y^{m-j} z^j - y^m - z^m. \end{aligned}$$

When $m > 3$, we have $b(1-\omega)^m = (-\omega)^m + 1$ by taking $j = 0$ and $-b(1-\omega)^m = (-\omega)^m$ by taking $j = 3$. This implies $1 + 2(-\omega)^m = 0$, which never happens. When $m = 3$, instead of taking $j = 3$, we take $j = 2$ and we also get a contradiction. Therefore, $x + \omega y + \omega^2 z$ is not a factor of $\mathcal{G}_3^m[(b, 1, 1, 1)]$ for all $b \in \mathbb{C}$ and $m \geq 3$. ▮

From the above analysis, we can draw the following conclusions:

- Under the assumptions in section 3, if the $2^{\text{nd}}$-order system without quadratic terms is integrable, then $a = -1$ and it has arbitrary order symmetries. Moreover, it is of the form

$$\left\{ \begin{array}{l} u_t = -u_2 \\ v_t = \ \ \ v_2 \end{array} \right. + K^{(1,1)} + \sum_{i+j \geq 3} K^{(i,j)}.$$

- Even when the system has quadratic terms, many cases are relatively 1-prime, cf. appendix A, as follows from Theorem 5.2 and Proposition 5.4. The only exception occurs when the result of the Lie brackets between the quadratic terms of the system and its symmetry is of the form $K^{(1,1)}$ and $a = -1$. Such a system is of the form

$$\left\{ \begin{array}{l} u_t = -u_2 \\ v_t = \ \ \ v_2 \end{array} \right. + K^{(-1,2)} + K^{(2,-1)} + K^{(1,1)} + \sum_{i+j \geq 3} K^{(i,j)}. \tag{11}$$

We come back to this in subsection 6.1.

- This analysis is also useful for the $N$-component case. Only in the 2-component case it is complete: we only have two eigenvalues, so either $a_1, a_2$ or $a_3$ is equal to $a_0$, or they are all different from $a_0$ and therefore equal to one another.
- Although in this paper we only consider the integrability problem, Lemma 5.1 equally applies to almost integrable systems (that is, systems with only a finite number of nontrivial generalized symmetries) of depth $\geq 2$, cf. [vdKS99].

## 6. QUADRATIC TERMS

In this section we classify all homogeneous integrable systems under our previous assumptions. In order to do so, we need to compute all the possible weights for $u$ and $v$ first.

THEOREM 6.1. *Any equation with nonzero quadratic or cubic terms and the order of the nonlinear terms less than* 2*, with* $0 < \lambda_1 \leq \lambda_2$ *and* $\lambda_2 - \lambda_1 \notin 1, 2, 3, \cdots$ *, has weight coefficients in the following list of pairs* $(\lambda_1, \lambda_2)$.

| | | | | | |
|---|---|---|---|---|---|
| $(\frac{1}{4},\ \frac{3}{4})$ | | | | | |
| $(\frac{1}{3},\ \frac{2}{3})$ | | | | | |
| $(\frac{1}{2},\ \frac{1}{2})$ | $(\frac{1}{2},\ \frac{3}{4})$ | $(\frac{1}{2},\ \frac{5}{6})$ | $(\frac{1}{2},\ 1)$ | $(\frac{1}{2},\ \frac{5}{4})$ | $(\frac{1}{2},\ 2)$ |
| $(\frac{3}{5},\ \frac{4}{5})$ | | | | | |
| $(\frac{5}{8},\ \frac{7}{8})$ | | | | | |
| $(\frac{2}{3},\ 1)$ | $(\frac{2}{3},\ \frac{4}{3})$ | | | | |
| $(\frac{3}{4},\ \frac{5}{4})$ | | | | | |
| $(\frac{4}{5},\ \frac{7}{5})$ | | | | | |
| $(1,\ 1)$ | $(1,\ \frac{3}{2})$ | | | | |
| $(\frac{6}{5},\ \frac{8}{5})$ | | | | | |
| $(\frac{4}{3},\ \frac{5}{3})$ | $(\frac{4}{3},\ 2)$ | | | | |
| $(\frac{3}{2},\ 2)$ | | | | | |
| $(2,\ 2)$ | | | | | |

*Proof.* This can be done by a simple Maple [CGG+91] computation. ▮

From the implicit function Theorem A.3 we know that under some conditions we only need to find one symmetry of the system to determine whether it is integrable. To do this we first should determine its order and spectrum, which depend on the quadratic terms of the system (or, if they are zero, on the cubic terms). The order is needed so that we at least know where to look for the symmetry, the spectrum is used to carry out the computer algebra calculation successfully.

In this section, we consider concrete families of systems, determined by the weights listed in Theorem 6.1. We work out the details for the most complicated case $\lambda_1 = \lambda_2 = 1$ (Remark that in this case we can still interchange $u$ and $v$ when necessary). The other cases can be done by the same method. Therefore, we only list the integrable systems with their weights in order to keep the length of this paper within reasonable bounds. If for some weights there are no new integrable systems we do not mention them again.

### 6.1. The case $\lambda_1 = \lambda_2 = 1$

Consider $2^{\mathrm{nd}}$-order homogeneous system with $\lambda_1=\lambda_2=1$:

$$\left\{\begin{array}{rcl} u_t &=& au_2+a_{11}uu_1+a_{12}uv_1+a_{13}vu_1+a_{14}vv_1\\ && +a_{15}u^3+a_{16}u^2v+a_{17}uv^2+a_{18}v^3\\ v_t &=& v_2+a_{21}uu_1+a_{22}uv_1+a_{23}vu_1+a_{24}vv_1\\ && +a_{25}u^3+a_{26}u^2v+a_{27}uv^2+a_{28}v^3 \end{array}\right.$$

The symbolic expression of quadratic terms are

$$\begin{array}{rcl} \tilde{K}^{(-1,2)} &=& \frac{a_{14}}{2}(y_1+y_2)v^2\frac{\partial}{\partial u};\\ \tilde{K}^{(1,0)} &=& \frac{a_{11}}{2}(x_1+x_2)u^2\frac{\partial}{\partial u}+(a_{22}y_1+a_{23}x_1)uv\frac{\partial}{\partial v};\\ \tilde{K}^{(2,-1)} &=& \frac{a_{21}}{2}(x_1+x_2)u^2\frac{\partial}{\partial v};\\ \tilde{K}^{(0,1)} &=& (a_{12}y_1+a_{13}x_1)uv\frac{\partial}{\partial u}+\frac{a_{24}}{2}(y_1+y_2)v^2\frac{\partial}{\partial v}. \end{array}$$

Assume that the system has an $m^{\mathrm{th}}$-order symmetry $Q$ $(m>2)$. We know it has to start with

$$b_{10}u_m\frac{\partial}{\partial u}+b_{20}v_m\frac{\partial}{\partial v},\quad (b_{10},b_{20})\in\mathbb{CP}^1.$$

The quadratic terms of the symmetry can be written as

$$\begin{array}{rcl} \tilde{Q}^{(-1,2)} &=& \frac{a_{14}(y_1+y_2)}{2}\ \frac{b_{10}(y_1+y_2)^m-b_{20}y_1^m-b_{20}y_2^m}{a(y_1+y_2)^2-y_1^2-y_2^2}v^2\frac{\partial}{\partial u};\\ \tilde{Q}^{(1,0)} &=& \frac{a_{11}(x_1+x_2)}{2}\ \frac{b_{10}}{a}\frac{(x_1+x_2)^m-x_1^m-x_2^m}{(x_1+x_2)^2-x_1^2-x_2^2}u^2\frac{\partial}{\partial u}\\ && +\ (a_{22}y_1+a_{23}x_1)\frac{b_{20}(x_1+y_1)^m-b_{10}x_1^m-b_{20}y_1^m}{(x_1+y_1)^2-ax_1^2-y_1^2}uv\frac{\partial}{\partial v};\\ \tilde{Q}^{(2,-1)} &=& \frac{a_{21}}{2}(x_1+x_2)\ \frac{b_{20}(x_1+x_2)^m-b_{10}x_1^m-b_{10}x_2^m}{(x_1+x_2)^2-ax_1^2-ax_2^2}u^2\frac{\partial}{\partial v};\\ \tilde{Q}^{(0,1)} &=& (a_{12}y_1+a_{13}x_1)\frac{b_{10}(x_1+y_1)^m-b_{10}x_1^m-b_{20}y_1^m}{a(x_1+y_1)^2-ax_1^2-y_1^2}uv\frac{\partial}{\partial u}\\ && +\ \frac{a_{24}}{2}(y_1+y_2)b_{20}\frac{(y_1+y_2)^m-y_1^m-y_2^m}{(y_1+y_2)^2-y_1^2-y_2^2}v^2\frac{\partial}{\partial v}. \end{array}$$

The necessary condition for the existence of the symmetry $Q$ is that the above expressions are either polynomial or zero. We analyze them one by one. When $b_{20}\neq 0$, we can take $b_{20}=1$ and $b_{10}=b$ without loss of generality (since $\lambda_1=\lambda_2$). Here we list its conditions.

| | | | |
|---|---|---|---|
| $\tilde{Q}^{(-1,2)}$ | $a_{14}=0$ | $b=\frac{\alpha^m+1}{(\alpha+1)^m},\quad a=\frac{\alpha^2+1}{(\alpha+1)^2}$ | |
| $\tilde{Q}^{(1,0)}$ | $a_{22}=a_{23}=0$ | $b=(\frac{a+1}{2})^m-(\frac{a-1}{2})^m$ | $a=1-\frac{2a_{23}}{a_{22}}$, $a_{22}a_{23}\neq 0$ |
| $\tilde{Q}^{(2,-1)}$ | $a_{21}=0$ | $b=\frac{(\beta+1)^m}{\beta^m+1},\quad a=\frac{(\beta+1)^2}{\beta^2+1}$ | |
| $\tilde{Q}^{(0,1)}$ | $a_{12}=a_{13}=0$ | $b=\frac{(2a)^m}{(a+1)^m-(1-a)^m}$ | $a=\frac{a_{13}}{a_{13}-2a_{12}}$, $a_{12}a_{13}\neq 0$ |

Remark 6.2. *We can transfer the case $b_{20}=0$ into the case $b_{20}\neq 0$ under the transformation: $u\mapsto v$ and $v\mapsto u$ since the linear terms of equation transfer into $\frac{1}{a}u_2\frac{\partial}{\partial u}+v_2\frac{\partial}{\partial v}$ and the starting terms of its symmetries into $b_{20}u_m\frac{\partial}{\partial u}+b_{10}v_m\frac{\partial}{\partial v}$. However, in practice, we do need to pay some attention. For an equation, it may have no symmetry starting with $bu_m\frac{\partial}{\partial u}+v_m\frac{\partial}{\partial v}$ since the denominator of $b$ equals zero. But it could have a symmetry starting with $u_m\frac{\partial}{\partial u}$.*

Lemma 6.3. *Assume $a_{14}\neq 0$ and $a\neq 1-2a_{23}/a_{22}$. Suppose that the system is symmetry-integrable. The necessary condition is one of the following:*

1. *$a=-1$ and $b=(-1)^{m+1}$ for $m\equiv 1 (\mathrm{mod}\ 3)$ or $m\equiv 2(\mathrm{mod}\ 3)$.*
2. *$a=-1\pm 2i$ and $b=i^m\mp(1-i)^m$ for $m\equiv 2(\mathrm{mod}\ 4)$.*

*Proof.* Under the assumption, there are infinitely many $m$ such that

$$b=\frac{\alpha^m+1}{(\alpha+1)^m}=(\frac{a+1}{2})^m-(\frac{a-1}{2})^m \tag{12}$$

where $\alpha$ is a root of $\mathcal{G}_2^2[(a,1,1)](x,1)$ (that is $a=(\alpha^2+1)/(\alpha+1)^2$). When $a\neq -1,1$, applying Theorem A.4 (cf. Corollary 3.3 in [BSW98]) the ratios

$$\frac{2\alpha}{(\alpha+1)(a+1)},\ \frac{2}{(\alpha+1)(a-1)};\quad or\quad \frac{2\alpha}{(\alpha+1)(a-1)},\ \frac{2}{(\alpha+1)(a+1)}$$

are roots of unity. We see that under the transformation $\alpha\mapsto\frac{1}{\alpha}$ the first pair is mapped to the second pair, and vice versa. Thus we restrict our attention to the first pair.
The condition $|\frac{2\alpha}{(\alpha+1)(a+1)}|=1$ implies $|\alpha(\alpha+1)|=|\alpha^2+\alpha+1|$, i.e., $\Re(\alpha(\alpha+1))=-\frac{1}{2}$. The condition $|\frac{2}{(\alpha+1)(a-1)}|=1$ implies $|\alpha|=|\alpha+1|$,

i.e., $\Re\alpha = -\frac{1}{2}$. Together these imply $\alpha = -\frac{1}{2} \mp \frac{i}{2}$. Then $a = -1 \pm 2i$. Since $a$ is invariant under $\alpha \mapsto \frac{1}{\alpha}$ the second pair gives the same values for $a$. We define

$$\begin{aligned}\Delta(\alpha, m) &= (\frac{a+1}{2})^m - (\frac{a-1}{2})^m - \frac{\alpha^m + 1}{(\alpha+1)^m} \\ &= \frac{1}{(\alpha+1)^{2m}} \left((\alpha^2+\alpha+1)^m - (-\alpha)^m - (\alpha^2+\alpha)^m - (1+\alpha)^m\right),\end{aligned}$$

where $\alpha = -\frac{1}{2} \mp \frac{i}{2}$. Its value only depends on $a$ since $\Delta(\frac{1}{\alpha}, m) = \Delta(\alpha, m)$. Notice that $\Delta(\alpha, m) = 0$ if and only if

$$\begin{aligned}0 &= (\alpha^2+\alpha+1)^m - (-\alpha)^m - (\alpha^2+\alpha)^m - (1+\alpha)^m \\ &= (\frac{1}{2})^m - (\frac{1}{2} - \frac{i}{2})^m - (-\frac{1}{2})^m - (\frac{1}{2} + \frac{i}{2})^m \\ &= (\frac{1}{2})^m - (-\frac{1}{2})^m - 2(\frac{\sqrt{2}}{2})^m \cos\frac{m\pi}{4}.\end{aligned}$$

Solving it, we obtain $m \equiv 2 (\mathrm{mod}\ 4)$ or $m = 1$. It follows $b = i^m \mp (1-i)^m$ when $a = -1 \pm 2i$ from formula (12).
When $a = -1$, it is easy to see that formula (12) is valid for the values of $m$ listed in the lemma. ▮

Using formula (4), we have the following corollary.

COROLLARY 6.4. *Assume $a_{21} \neq 0$ and $a \neq a_{13}/(a_{13} - 2a_{12})$. Suppose that the system is symmetry-integrable. The necessary condition is one of the following:*

1. $a = -1$ *and* $b = (-1)^{m+1}$ *for* $m \equiv 1 (\mathrm{mod}\ 3)$ *or* $m \equiv 2 (\mathrm{mod}\ 3)$.
2. $a = -\frac{1}{5} \pm \frac{2}{5} i$ *and* $\frac{1}{b} = i^m \pm (1-i)^m$ *for* $m \equiv 2 (\mathrm{mod}\ 4)$.

LEMMA 6.5. *Assume $a_{21}a_{14} \neq 0$. If the system is symmetry-integrable, the necessary condition is:*

$$a = -1 \text{ and } b = (-1)^{m+1} \text{ for } m \equiv 1 (\mathrm{mod}\ 3) \text{ or } m \equiv 2 (\mathrm{mod}\ 3).$$

*Proof.* Under the assumption, there are infinitely many $m$ such that

$$b = \frac{\alpha^m + 1}{(\alpha+1)^m} = \frac{(\beta+1)^m}{\beta^m + 1}, \tag{13}$$

where $\alpha$ is a root of $\mathcal{G}_2^2[(a,1,1)](x,1)$ and $\beta$ is a root of $\mathcal{G}_2^2[(1,a,a)](x,1)$, that is

$$(\alpha\beta)^m + \alpha^m + \beta^m + 1 = (\alpha\beta + \alpha + \beta + 1)^m$$

First we note that $\alpha\beta + \alpha + \beta + 1 \neq 0$ and $\alpha\beta \neq 0$ when $a \neq 1$. Applying Theorem A.4 the ratios

$$\alpha, \quad \beta, \quad \alpha\beta + \alpha + \beta + 1$$

are roots of unity. Together with the relation $a = \frac{\alpha^2+1}{(\alpha+1)^2} = \frac{(\beta+1)^2}{\beta^2+1}$, we obtain $\alpha = \beta = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, which implies that $a = -1$. Moreover, formula (13) is valid only when $m \equiv 1 (\mathrm{mod}\, 3)$ or $m \equiv 2 (\mathrm{mod}\, 3)$ and $b = (-1)^{m+1}$. ▮

Lemma 6.6. *Assume $a \neq a_{13}/(a_{13} - 2a_{12})$ and $a \neq 1 - 2a_{23}/a_{22}$. Suppose that the system is symmetry-integrable. The necessary condition is one of the following:*

*1. $a = -1$ and $b = (-1)^{m+1}$ for $m \equiv 1 (\mathrm{mod}\ 3)$ or $m \equiv 2 (\mathrm{mod}\ 3)$.*

*2. $a = (2 \pm \sqrt{3})i$ or $a = -(2 \pm \sqrt{3})i$ and $b = (\frac{a+1}{2})^m - (\frac{a-1}{2})^m$ for $m \not\equiv 0 (\mathrm{mod}\ 4)$ and $m \not\equiv 0 (\mathrm{mod}\ 3)$.*

*Proof.* Under the assumption, there are infinitely many $m$ such that

$$b = \frac{(2a)^m}{(a+1)^m - (1-a)^m} = (\frac{a+1}{2})^m - (\frac{a-1}{2})^m, \tag{14}$$

that is, $(4a)^m + (1 - (-1)^m)(a^2 - 1)^m = (a+1)^{2m} + (-(a-1)^2)^m$. When $a \notin \{-1, 0, 1\}$, applying Theorem A.4 the ratios $\frac{4a}{(a+1)^2}$, $\frac{4a}{(a-1)^2}$ are roots of unity. The solutions are $a = (-2 \pm \sqrt{3})i$ and $a = (2 \pm \sqrt{3})i$, which are the roots of $a^4 + 14a^2 + 1$. We define

$$\Delta(n) = \frac{((a+1)^n - (a-1)^n)((1+a)^n - (1-a)^n)}{(4a)^n}\Bigg|_{a^4+14a^2+1=0}$$

We claim that this is a function from $\mathbb{Z}/12 \to \mathbb{Z}$. One has

$$\begin{array}{llll} \Delta(0) = 0, & \Delta(1) = 1, & \Delta(2) = 1, & \Delta(3) = -2, \\ \Delta(4) = -3, & \Delta(5) = 1, & \Delta(6) = 4, & \end{array}$$

and the properties $\Delta(n + 12) = \Delta(12)$ and $\Delta(12 - i) = \Delta(i)$. This is proved as follows. Computing modulo $a^4 + 14a^2 + 1$ we see that

$$\begin{aligned} (1+a)^{12} &\equiv 2^{12}(14 + 195a^2) \\ (1-a)^{12} &\equiv 2^{12}(14 + 195a^2) \\ (4a)^6 &\equiv 2^{12}(14 + 195a^2) \end{aligned}$$

It immediately follows that $\Delta(n+12) = \Delta(n)$. That $\Delta(n)$ is independent of the choice of the root of $a^4 + 14a^2 + 1$ can be shown by by the following Maple code, which gives us the value table.

```
f:=(n,x)->normal(simplify((((x+1))^n-((x-1))^n)/((4*x)^n/
                ((x+1)^n-(1-x)^n))));
BB:=[x];
GG:=gbasis([x^4+14*x^2+1],BB);
g:=(n,x)->simplify(normalf(numer(f(n,x)),GG,BB)/
                normalf(denom(f(n,x)),GG,BB));
for i to 12 do print(i,g(i,x)) od;
```

Moreover, we obtain when $m \not\equiv 0 (\text{mod } 4)$ and $m \not\equiv 0 (\text{mod } 3)$, that formula (14) is valid.
When $a = -1$, it is easy to see that formula (14) is valid for the values of $m$ listed in the lemma. ∎

Lemma 6.7. *Assume $a_{21} \neq 0$ and $a \neq 1 - 2a_{23}/a_{22}$. Suppose that the system is symmetry-integrable. The necessary condition is:*

$$a = -1 \text{ and } b = (-1)^{m+1} \text{ for } m \equiv 1 (\text{mod } 3) \text{ or } m \equiv 2 (\text{mod } 3).$$

*Proof.* Under the assumption, there are infinitely many $m$ such that

$$b = \frac{(\beta+1)^m}{\beta^m+1} = (\frac{a+1}{2})^m - (\frac{a-1}{2})^m, \tag{15}$$

where $\beta$ is a root of $\mathcal{G}_2^2[(1,a,a)](x,1)$, that is,

$$(2\beta+2)^m + (\beta a - \beta)^m + (a-1)^m = (\beta a + \beta)^m + (a+1)^m$$

When $a \notin \{-1, 0, 1\}$, applying Theorem A.4 the ratios $\frac{a-1}{a+1}$, $\beta$ are roots of unity. This can never happen, that is, there are no solutions in this case.
When $a = -1$, it is easy to see that formula (15) is valid for the values of $m$ listed in the lemma. ∎

Using formula (4), we have the following corollary.

Corollary 6.8. *Assume $a_{14} \neq 0$ and $a \neq a_{13}/(a_{13} - 2a_{12})$. Suppose that the system is symmetry-integrable. The necessary condition is:*

$$a = -1 \text{ and } b = (-1)^{m+1} \text{ for } m \equiv 1 (\text{mod } 3) \text{ or } m \equiv 2 (\text{mod } 3).$$

As we mentioned in the end of last section, in the most situations the system is relatively 1-prime. This implies that we only need to find one symmetry and the existence of the first step of infinitily many symmetries, i.e., quadratic terms (if they are zero, cubic terms) of the symmetries to determine that the system is integrable.
Before we do this, let us first consider the exceptional case of the form (11)

$$\begin{cases} u_t = -u_2 + a_{14}vv_1 + a_{16}u^2v \\ v_t = v_2 + a_{21}uu_1 + a_{27}uv^2 \end{cases}$$

If $a_{14}a_{21} = 0$, one of the equations has no quadratic terms. The analysis is the same as the system without the quadratic terms.
When $a_{14}a_{21} \neq 0$, by Lemma 6.5, we know its symmetries has orders $m \equiv 1 (\mathrm{mod}\ 3)$ or $m \equiv 2 (\mathrm{mod}\ 3)$ and $b = (-1)^{m+1}$. So

$$\begin{aligned} \tilde{Q}^{(-1,2)} &= \frac{a_{14}(y_1+y_2)}{2}\ \frac{(-1)^{m+1}(y_1+y_2)^m - y_1^m - y_2^m}{-(y_1+y_2)^2 - y_1^2 - y_2^2} v^2 \frac{\partial}{\partial u}; \\ \tilde{Q}^{(2,-1)} &= \frac{a_{21}}{2}(x_1+x_2)\ \frac{(x_1+x_2)^m - (-1)^{m+1}(x_1^m + x_2^m)}{(x_1+x_2)^2 + x_1^2 + x_2^2} u^2 \frac{\partial}{\partial v}; \end{aligned}$$

We know that $[K^{(-1,2)},\ Q^{(-1,2)}] = 0$ and $[K^{(2,-1)},\ Q^{(2,-1)}] = 0$. Using formula (1), we compute

$$\begin{aligned} & [\tilde{K}^{(2,-1)},\ \tilde{Q}^{(-1,2)}] \\ =\ & [u^2 \frac{a_{21}}{2}(x_1+x_2)\frac{\partial}{\partial v},\ v^2 \frac{a_{14}(y_1+y_2)}{2}\ \frac{(-1)^{m+1}(y_1+y_2)^m - y_1^m - y_2^m}{-(y_1+y_2)^2 - y_1^2 - y_2^2} \frac{\partial}{\partial u}] \\ =\ & u^2 v \frac{a_{14}a_{21}}{2}(x_1+x_2)(y_1+x_1+x_2) \frac{(-1)^{m+1}(y_1+x_1+x_2)^m - y_1^m - (x_1+x_2)^m}{-(y_1+x_1+x_2)^2 - y_1^2 - (x_1+x_2)^2} \frac{\partial}{\partial u} \\ -\ & uv^2 \frac{a_{14}a_{21}}{2}(x_1+y_1+y_2)(y_1+y_2) \frac{(-1)^{m+1}(y_1+y_2)^m - y_1^m - y_2^m}{-(y_1+y_2)^2 - y_1^2 - y_2^2} \frac{\partial}{\partial v} \end{aligned}$$

and

$$\begin{aligned} & [\tilde{K}^{(-1,2)},\ \tilde{Q}^{(2,-1)}] \\ =\ & [v^2 \frac{a_{14}(y_1+y_2)}{2} \frac{\partial}{\partial u},\ u^2 \frac{a_{21}}{2}(x_1+x_2)\ \frac{(x_1+x_2)^m - (-1)^{m+1}(x_1^m + x_2^m)}{(x_1+x_2)^2 + x_1^2 + x_2^2} \frac{\partial}{\partial v}] \\ =\ & -u^2 v \frac{a_{14}a_{21}}{2}(y_1+x_1+x_2)(x_1+x_2)\ \frac{(x_1+x_2)^m - (-1)^{m+1}(x_1^m + x_2^m)}{(x_1+x_2)^2 + x_1^2 + x_2^2} \frac{\partial}{\partial u} \\ +\ & uv^2 \frac{a_{14}a_{21}}{2}(y_1+y_2)(x_1+y_1+y_2) \frac{(x_1+y_1+y_2)^m - (-1)^{m+1}(x_1^m + (y_1+y_2)^m)}{(x_1+y_1+y_2)^2 + x_1^2 + (y_1+y_2)^2} \frac{\partial}{\partial v}. \end{aligned}$$

If the $\frac{\partial}{\partial u}$ part of $[\tilde{K}^{(2,-1)},\ \tilde{Q}^{(-1,2)}]+[\tilde{K}^{(-1,2)},\ \tilde{Q}^{(2,-1)}]$ has the factor

$$-(x_1+x_2+y_1)^2+x_1^2+x_2^2-y_1^2=-2(x_1+y_1)(x_2+y_1)$$

and the $\frac{\partial}{\partial v}$ part has the factor

$$(x_1+y_1+y_2)^2+x_1^2-y_1^2-y_2^2=-2(x_1+y_1)(x_1+y_2),$$

according to Theorem 5.2, we can prove the system is also relatively 1-prime. We only write out the proof for the $\frac{\partial}{\partial u}$ part (the other part can be proved in a similar way). Let

$$\begin{aligned} H &= \frac{(-1)^{m+1}(y_1+x_1+x_2)^m-y_1^m-(x_1+x_2)^m}{-(y_1+x_1+x_2)^2-y_1^2-(x_1+x_2)^2} \\ &\quad -\frac{(x_1+x_2)^m-(-1)^{m+1}(x_1^m+x_2^m)}{(x_1+x_2)^2+x_1^2+x_2^2}. \end{aligned}$$

Since $H$ is symmetric with respect to $x_1$ and $x_2$, we only need to check

$$\begin{aligned} H|_{x_1+y_1=0} &= \frac{(-1)^{m+1}x_2^m-y_1^m-(-y_1+x_2)^m}{-x_2^2-y_1^2-(-y_1+x_2)^2} \\ &\quad -\frac{(-y_1+x_2)^m-(-1)^{m+1}((-y_1)^m+x_2^m)}{(-y_1+x_2)^2+y_1^2+x_2^2} \\ &= 0. \end{aligned}$$

We have now handled all cases. The above lemmas and corollaries give us the orders of the symmetries under specific conditions on the quadratic terms of the system. For the other cases, there are no conditions on the orders of the symmetries, i.e., if the system is integrable, it has a symmetry in every order. Moreover, we also proved the system is always relatively 1-prime. This implies we only need to compute one nontrivial symmetry (of lowest possible order), which is done using computer algebra program written in Maple by Peter H. van der Kamp.

We are now in the position to give the complete list of integrable systems with $\lambda_1=\lambda_2=1$. The reader should be aware that some of the parameters can still be scaled to 1 when they are not zero. The following systems have symmetries on every order, with arbitrary parameters $\sigma_i, i=1,2,3\in\mathbb{C}$, and by adding the same systems but with $u$ and $v$ interchanged we can complete this list (the same remark holds true for all lists with $\lambda_1=\lambda_2$; it will not be repeated):

$$\left\{\begin{array}{l} u_t = \sigma_1 u_2 + \sigma_2 u_1 v + \frac{\sigma_2(\sigma_1 - 1)}{2\sigma_1} u v_1 + \frac{\sigma_2(\sigma_2 - \sigma_3)}{4\sigma_1} u v^2 \\ v_t = v_2 + \sigma_3 v_1 v \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{rcl} u_t &=& \sigma_1 u_2 + \sigma_1\sigma_2 u_1 u + \sigma_1\sigma_3 u_1 v + \frac{1}{2}\sigma_3(\sigma_1 - 1) u v_1 \\ && + \frac{1}{4}\sigma_3\sigma_2(\sigma_1 - 1) u^2 v + \frac{1}{4}\sigma_3^2(\sigma_1 - 1) u v^2 \\ v_t &=& v_2 + \sigma_3 v_1 v + \sigma_2 u v_1 - \frac{1}{2}\sigma_2(\sigma_1 - 1) u_1 v - \frac{1}{4}\sigma_2^2(\sigma_1 - 1) u^2 v \\ && - \frac{1}{4}\sigma_3\sigma_2(\sigma_1 - 1) u v^2 \end{array}\right. \tag{17}$$

$$\left\{\begin{array}{l} u_t = -u_2 + (\sigma_1 + \sigma_2) u_1 v + \sigma_2 u v_1 - \frac{1}{4}\sigma_2(\sigma_1 + \sigma_2) u v^2, \\ v_t = v_2 + \sigma_1 v_1 v \end{array}\right. \tag{18}$$

$$\left\{\begin{array}{l} u_t = -u_2 + \sigma_2 u_1 u + \sigma_1 u_1 v + \sigma_1 u v_1 \\ v_t = v_2 + \sigma_1 v_1 v \end{array}\right. \tag{19}$$

$$\left\{\begin{array}{l} u_t = -u_2 + \sigma_1 u_1 u + \sigma_2 u_1 v \\ v_t = v_2 + \sigma_2 v_1 v + \sigma_1 u v_1 \end{array}\right. \tag{20}$$

$$\left\{\begin{array}{l} u_t = -u_2 - \sigma_1 u^2 v \\ v_t = v_2 + \sigma_1 u v^2 \end{array}\right. \tag{21}$$

$$\left\{\begin{array}{l} u_t = -u_2 + \sigma_2 u_1 u + \sigma_1 u_1 v + \sigma_1 u v_1 \\ v_t = v_2 + \sigma_1 v_1 v + \sigma_2 u_1 v + \sigma_2 u v_1 \end{array}\right. \tag{22}$$

$$\left\{\begin{array}{rcl} u_t &=& \sigma_1 u_2 + \frac{1}{2}\sigma_1\sigma_3 u_1 v + \frac{1}{4}\sigma_3(\sigma_1 - 1) u v_1 \\ && - \frac{1}{8}\sigma_2\sigma_3 u^3 + \frac{1}{16}\sigma_3^2(\sigma_1 - 2) u v^2 \\ v_t &=& v_2 + \sigma_2 u_1 u + \sigma_3 v_1 v \end{array}\right. \tag{23}$$

$$\left\{\begin{array}{rcl} u_t &=& \sigma_1 u_2 + \sigma_2 v_1 v + 2\sigma_1\sigma_3 u_1 v + \sigma_3(\sigma_1 - 1) u v_1 + \frac{1}{2}\sigma_2\sigma_3 v^3 \\ && + \frac{1}{2}\sigma_3^2(2\sigma_1 - 1) u v^2 \\ v_t &=& v_2 + \sigma_3 v_1 v \end{array}\right. \tag{24}$$

For equation (23-24) we see that putting $\sigma_3 = 0$ brings us in the Bakirov case [BSW98, BSW01].

The following systems have order $m \equiv 1 (\mathrm{mod}\ 3)$ and $m \equiv 2 (\mathrm{mod}\ 3)$ symmetries:

$$\left\{\begin{array}{l} u_t = -u_2 + \sigma_1 v_1 v \\ v_t = v_2 + \sigma_2 u_1 u \end{array}\right. \tag{25}$$

$$\left\{\begin{array}{l} u_t = -u_2 + \sigma_1 v_1 v + \sigma_2 u v_1 + \frac{\sigma_2^4}{6\sigma_1^2} u^3 + \frac{1}{6}\sigma_1\sigma_2 v^3 + \frac{\sigma_2^3}{2\sigma_1} u^2 v + \frac{1}{2}\sigma_2^2 u v^2 \\ v_t = v_2 + \frac{\sigma_2^3}{\sigma_1^2} u_1 u + \frac{\sigma_2^2}{\sigma_1} u_1 v - \frac{\sigma_2^5}{6\sigma_1^3} u^3 - \frac{1}{6}\sigma_2^2 v^3 - \frac{\sigma_2^4}{2\sigma_1^2} u^2 v - \frac{\sigma_2^3}{2\sigma_1} u v^2 \end{array}\right. \tag{26}$$

$$\left\{\begin{array}{l} u_t = -u_2 + \sigma_1 v_1 v - 2\sigma_2 u_1 v - 3\sigma_2 u v_1 + \frac{1}{3}\sigma_2\sigma_1 v^3 - \frac{3}{2}\sigma_2^2 u v^2 \\ v_t = v_2 + \sigma_2 v_1 v \end{array}\right. \tag{27}$$

### 6.2. The case $\lambda_1 = \lambda_2 = 2$

Consider the $2^{\text{nd}}$-order homogeneous system with $\lambda_1 = \lambda_2 = 2$:

$$\begin{cases} u_t = au_2 + a_{11}u^2 + a_{12}uv + a_{13}v^2 \\ v_t = v_2 + a_{21}u^2 + a_{22}uv + a_{23}v^2 \end{cases}$$

The symbolic expression of quadratic terms are

$$\begin{aligned} \tilde{K}^{(-1,2)} &= a_{13}v^2\frac{\partial}{\partial u};\ \tilde{K}^{(1,0)} = a_{11}u^2\frac{\partial}{\partial u} + a_{22}uv\frac{\partial}{\partial v};\\ \tilde{K}^{(2,-1)} &= a_{21}u^2\frac{\partial}{\partial v};\ \tilde{K}^{(0,1)} = a_{12}uv\frac{\partial}{\partial u} + a_{23}v^2\frac{\partial}{\partial v}. \end{aligned}$$

We do the same analysis as in subsection 6.1 and list the necessary condition for the existence of an $m^{\text{th}}$-order nondegenerate symmetry $Q$ starting with $bu_m\frac{\partial}{\partial u} + v_m\frac{\partial}{\partial v},\ b \neq 0,\ m > 2$ in the following table.

| | | |
|---|---|---|
| $\tilde{Q}^{(-1,2)}$ | $a_{13} = 0$ | $b = \frac{\alpha^m+1}{(\alpha+1)^m}, \quad a = \frac{\alpha^2+1}{(\alpha+1)^2}$ |
| $\tilde{Q}^{(1,0)}$ | $a_{22} = 0$ | $b = (\frac{a+1}{2})^m - (\frac{a-1}{2})^m$ |
| $\tilde{Q}^{(2,-1)}$ | $a_{21} = 0$ | $b = \frac{(\beta+1)^m}{\beta^m+1}, \quad a = \frac{(\beta+1)^2}{\beta^2+1}$ |
| $\tilde{Q}^{(0,1)}$ | $a_{12} = 0$ | $b = \frac{(2a)^m}{(a+1)^m-(1-a)^m}$ |

Notice the expression for $b$ is the same as in subsection 6.1. We can use the same lemmas and corollaries.
The only equation with symmetries in every order is the Bakirov case [BSW01]. The nontrivial case is the following:

$$\begin{cases} u_t = -u_2 + \frac{\sigma_2^2}{\sigma_1}u^2 - 2\sigma_2 uv + \sigma_1 v^2 \\ v_t = v_2 + \frac{\sigma_2^3}{\sigma_1^2}u^2 - \frac{2\sigma_2^2}{\sigma_1}uv + \sigma_2 v^2 \end{cases} \tag{28}$$

with order $m \equiv 1 (\text{mod } 3)$ and $m \equiv 2 (\text{mod } 3)$ symmetries.

### 6.3. The case $\lambda_1 = \lambda_2 = \frac{1}{2}$

Consider $2^{\text{nd}}$-order homogeneous system with $\lambda_1 = \lambda_2 = \frac{1}{2}$:

$$\begin{cases} u_t = au_2 + \left(a_{11}u^2 + a_{12}uv + a_{13}v^2\right)u_1 + \left(a_{14}u^2 + a_{15}uv + a_{16}v^2\right)v_1 \\ \qquad +a_{17}u^5 + a_{18}u^4v + a_{19}u^3v^2 + a_{1,10}u^2v^3 + a_{1,11}uv^4 + a_{1,12}v^5 \\ v_t = v_2 + \left(a_{21}u^2 + a_{22}uv + a_{23}v^2\right)u_1 + \left(a_{24}u^2 + a_{25}uv + a_{26}v^2\right)v_1 \\ \qquad +a_{27}u^5 + a_{28}u^4v + a_{29}u^3v^2 + a_{2,10}u^2v^3 + a_{2,11}uv^4 + a_{2,12}v^5 \end{cases}$$

Since the system has no quadratic terms, we know if it is integrable, then $a = -1$ and it has arbitrary order symmetries. We obtain the following list.

$$\begin{cases} u_t = -u_2 + \sigma_2 uvu_1 + \sigma_1 u^2 v_1 - \frac{\sigma_1}{4}(\sigma_2 - 2\sigma_1)u^3 v^2 \\ v_t = v_2 + \sigma_1 v^2 u_1 + \sigma_2 uvv_1 + \frac{\sigma_1}{4}(\sigma_2 - 2\sigma_1)u^2 v^3 \end{cases} \tag{29}$$

$$\begin{cases} u_t = -u_2 + \sigma_1 uvu_1 + \sigma_1 u^2 v_1 + \frac{1}{4}\sigma_1\sigma_2 u^3 v^2 \\ v_t = v_2 + \sigma_2 v^2 u_1 + \sigma_2 uvv_1 - \frac{1}{4}\sigma_1\sigma_2 u^2 v^3 \end{cases} \tag{30}$$

### 6.4. The case $\lambda_1 = \frac{2}{3}$ and $\lambda_2 = \frac{1}{3}$

Consider 2[nd]-order homogeneous system with $\lambda_1 = \frac{2}{3}$ and $\lambda_2 = \frac{1}{3}$):

$$\begin{cases} u_t = au_2 + a_{11}v_1^2 + a_{12}uu_1 v + a_{13}u^2 v_1 + a_{14}u^4 + a_{15}u_1 v^3 + a_{16}uv^2 v_1 \\ \qquad + a_{17}v^4 v_1 + a_{18}u^3 v^2 + a_{19}u^2 v^4 + a_{1,10}uv^6 + a_{1,11}v^8 \\ v_t = v_2 + a_{21}uu_1 + a_{22}u_1 v^2 + a_{23}uvv_1 + a_{24}v^3 v_1 \\ \qquad + a_{25}u^3 v + a_{26}u^2 v^3 + a_{27}uv^5 + a_{28}v^7 \end{cases}$$

When $a_{11}a_{21} \neq 0$, we directly use Lemma 6.5.
When $a_{11}a_{21} = 0$, following from Theorem 5.2 we have $a = -1$ and $b = (-1)^{m+1}$. If both $a_{11}$ and $a_{21}$ equal to zero, we get the same results as subsection 6.3. Assume $a_{11} = 0$ and $a_{21} \neq 0$, and one of $a_{ij}, j > 1$ nonzero (In case all $a_{ij}, j > 1$ are zero, this brings us in the Bakirov case [BSW98, BSW01]). Then the eigenvalue of its $m^{\text{th}}$-order symmetry $b = \frac{(\beta+1)^m}{\beta^m+1}$, where $\frac{(\beta+1)^2}{\beta^2+1} = a = -1$, i.e., $\beta^2 + \beta + 1 = 0$. Therefore $b = \frac{(-\beta^2)^m}{\beta^m+1} = (-1)^{m+1}$ only when $m \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$.
The following systems have order $m \equiv 1 \pmod 3$ and $m \equiv 2 \pmod 3$ symmetries:

$$\begin{cases} u_t = -u_2 - \frac{1}{2}\sigma_1 uu_1 v - \frac{1}{2}\sigma_1 u^2 v_1 - \frac{1}{12}\sigma_1\sigma_2 u^4 - \frac{1}{8}\sigma_1^2 u^3 v^2 \\ v_t = v_2 + \sigma_2 uu_1 + \sigma_1 u_1 v^2 + \sigma_1 uvv_1 + \frac{1}{12}\sigma_1\sigma_2 u^3 v + \frac{1}{8}\sigma_1^2 u^2 v^3 \end{cases} \tag{31}$$

$$\begin{cases} u_t = -u_2 + \sigma_1 v_1^2 - 2\sigma_2 uu_1 v - 2\sigma_2 u^2 v_1 - \frac{1}{3}\sigma_1\sigma_2 u_1 v^3 + \frac{1}{6}\sigma_1^2\sigma_2 v^4 v_1 \\ \qquad -\frac{1}{2}\sigma_2^2 u^3 v^2 - \frac{1}{4}\sigma_1\sigma_2^2 u^2 v^4 + \frac{1}{72}\sigma_1^2\sigma_2^2 uv^6 + \frac{1}{144}\sigma_1^3\sigma_2^2 v^8 \\ v_t = v_2 + \sigma_2 u_1 v^2 + \sigma_2 uvv_1 + \frac{1}{6}\sigma_1\sigma_2 v^3 v_1 + \frac{1}{2}\sigma_2^2 u^2 v^3 - \frac{1}{72}a_{11}^2\sigma_2^2 v^7 \end{cases} \tag{32}$$

## 7. COMPARISON WITH LISTS IN THE LITERATURE

Several groups have been working on the classification of 2[nd]-order systems with two components. In this section, we compare our results with theirs.

We do not intend to write a review of the subject here; for this we refer to [SM93].
The authors of [MS85, MS86, MSY87] treat the classification of Schrödinger type equations, that is, the $a=-1$ case if the system is homogeneous. In fact, all the systems of the form

$$u_t = A(u)u_2 + F(u, u_1), \quad det(A(u)) \neq 0, \quad u = (u, v)$$

possessing higher conservation laws can be reduced to this type.
The author of [Svi89] studied the Burgers equations, which is $a=1$ case (cf. the excellent review paper [MSS91]). Although we made the assumption $a \neq 1$, some integrable systems in their list can be obtained from our list by taking $a_1 = 1$. If we take $a_1 = 1$, $a_9 = -2$ and $a_{10} = -4$, this leads to the only one nontriangular case

$$\begin{cases} u_t = u_2 - 2vu_1 - u^3 - uv^2 \\ v_t = v_2 - 2uu_1 - 4vv_1 \end{cases}$$

After these two special cases, the authors of [BP89] turned their attention to more general case: the integrable systems with distinct eigenvalues. An interesting example

$$\begin{cases} u_t = \gamma_1 u_2 + 2\gamma_1(u+v)u_1 + (\gamma_1 - \gamma_2)uv_1 + (\gamma_1 - \gamma_2)(u+c)uv \\ v_t = \gamma_2 u_2 + 2\gamma_2(u+v)v_1 + (\gamma_2 - \gamma_1)vu_1 + (\gamma_2 - \gamma_1)(u+c)uv \end{cases}$$

can be obtained from (17) by taking $a_{10} = a_{12}$.
Foursov ([Fou99], see also [Fou00]) considered the coupled Burgers-type equations of the symmetric form in his thesis. He gave the complete list of such type integrable system. After linear transformation, the only case satisfying our assumption is $\begin{cases} u_t = K(u, v) \\ v_t = K(v, u) \end{cases}$, with

$$\begin{aligned} K(u,v) &= (1-\alpha)u_2 + \alpha v_2 + (12 - 8\alpha - \beta)uu_1 + \beta vu_1 \\ &\quad -(4 - 4\alpha - \beta)uv_1 + (8 + 4\alpha - \beta)vv_1 - (8 + 2\alpha - \beta)u^3 \\ &\quad +(16 - 10\alpha - 3\beta)u^2 v - (16 - 10\alpha - 3\beta)uv^2 + (8 + 2\alpha - \beta)v^3, \end{aligned}$$

which can also be obtained from (17).
Recently, the authors of [SW99] noticed there are also systems have infinitely many symmetries, but without higher order conservation laws. They give complete list of integrable systems of the form

$$\begin{cases} u_t = u_2 + A_1(u,v)u_1 + A_2(u,v)v_1 + A_0(u,v) \\ v_t = -v_2 + B_1(u,v)u_1 + B_2(u,v)v_1 + B_0(u,v), \end{cases}$$

without any restrictions on the functions $A_i(u,v)$ and $B_i(u,v)$. Our list covers all their homogeneous integrable systems; system (32) does not appear in their list due to the appearance of the term of $v_1^2$. We remark that the choice of eigenvalues is a happy one as follows from the lemmas and corollaries in section 6.1

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## APPENDIX A

### An implicit function theorem

Let $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \cdots$ be a filtered Lie algebra, where we make the assumption that $\bigcap_{p=0}^{\infty} \mathfrak{g}^{(p)} = 0$. The following theorem states that under certain technical conditions the existence of a symmetry $S$ of an equation $K$, i.e., an $S$ such that $ad(K)S = 0$ for given $K \in \mathfrak{g}^{(0)}$, is enough to show finite determinacy, i.e., from a finite order computation one can conclude the existence of a solution of the equation $ad(K)Q = 0$.

Definition A.1. *We call* $ad(S^0)$, $S^0 \in \mathfrak{g}^{(0)}$ **relatively $l$-prime** *with respect to* $ad(K^0)$, $K^0 \in \mathfrak{g}^{(0)}$ *if* $ad(S^0)v^j \in \operatorname{Im} ad(K^0) \pmod{\mathfrak{g}^{(j+1)}}$, *then it implies that* $v^j \in \operatorname{Im} ad(K^0)|_{\mathfrak{g}^{(j)}} \pmod{\mathfrak{g}^{(j+1)}}$ *for all* $j \geq l$ *and* $v^j \in \mathfrak{g}^{(j)}$.

Definition A.2. *We call* $ad(K^0)$, $K^0 \in \mathfrak{g}^{(0)}$, **nonlinear injective** *if for all* $v^l \in \mathfrak{g}^{(l)}, l > 0$, $ad(K^0)v^l \in \mathfrak{g}^{(l+1)} \Rightarrow v^l \in \mathfrak{g}^{(l+1)}$.

Theorem A.3. *Let* $K^i, S^i \in \mathfrak{g}^{(i)}$, $i = 0, 1$. *Put* $K = K^0 + K^1$ *and* $S = S^0 + S^1$. *Suppose that*

- $ad(K)S = 0$,
- $ad(K^0)$ *is nonlinear injective,*
- $ad(S^0)$ *is relatively* $l+1$*-prime with respect to* $ad(K^0)$ *(this implies* $S \neq K$*),*

*and there exists some* $\hat{Q} \in \mathfrak{g}^{(0)}$ *such that*

- $ad(K)\hat{Q} \in \mathfrak{g}^{(l+1)}$ *and* $ad(S)\hat{Q} \in \mathfrak{g}^{(1)}$.

*Then there exists a unique* $Q = \hat{Q} + Q^{l+1}, Q^{l+1} \in \mathfrak{g}^{(l+1)}$ *such that*

$$ad(K)Q = ad(S)Q = 0.$$

For the proof we refer to [SW98].
Finally, we cite Lech-Macher theorem from number theory [Lec53].

Theorem A.4 (Lech, Mahler). *Let* $A_1, A_2, \ldots, A_n \in \mathbb{C}$ *be non-zero complex numbers and similarly for* $a_1, a_2, \ldots, a_n$. *Suppose that none of the ratios* $\frac{A_i}{A_j}$ *with* $i \neq j$ *is a root of unity. Then the equation*

$$a_1 A_1^k + a_2 A_2^k + \cdots + a_n A_n^k = 0$$

*in the unknown integer k has finitely many solutions.*

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