On the Integrability of Scalar Evolution Equations

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1 Introduction

This paper was motivated by the observation that after quickly finding a number of hierarchies (mKdV, Sawada-Kotera, Kaup-Kuperschmidt) soon after finding that KdV was integrable, nothing more was found for polynomial scalar evolution equations linear in the highest order derivative. In this paper we prove that under some mild conditions on the equations one can put a uniform bound on the order of the recursion operator of any such hierarchy.

We do this using the symbolic method, introduced by Gel'fand-Dikii [GD75]. This method was used in [TQ81] and [QT82] to produce the following results. give results

The basic idea is very old, probably dating from the time when the position of index and power were not as fixed as they are today. In fact, the symbolic calculus of classical invariant theory relies on it. The idea is simply to replace u_i, where i is an index, in our case counting the number of derivatives, by ξ^i, where ξ is now a symbol. We see that the basic operation of differentation, i.e. replacing u_i by u_{i+1}, is now replaced by multiplation with ξ, as is the case in Fourier transformation theory. If one has multiple u's, as in $u_i u_j$, one replaces this by $\frac{1}{2}\left(\xi_1^i \xi_2^j + \xi_1^j \xi_2^i\right)$. We have averaged over the permutation group Σ_2 to retain complete equality among the symbols, reflecting the fact that $u_i u_j = u_j u_i$. Differentiation now becomes multiplication with $\xi_1 + \xi_2$.

With this method one can readily translate solvability questions into divisibility questions and we can use generating functions to handle infinitely many orders at once.

2 Symbolic Notation

Notation 1 *1. Throughout this paper all polynomials have coefficients in the complex field.*

2. *Let $\mathcal{A}_n^k$ be the set of polynomials f of degree k in $n+1$ variables and $\tilde{\mathcal{A}}_n^k$ be the set of its symmetrized elements $\tilde{f} \stackrel{\text{def}}{=} < f > [n+1]$. Here*

$$< f > [n](x_1, \cdots, x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f(\sigma(x_1), \cdots, \sigma(x_n)),$$

where Σ_n is the permutation group on n elements.

3. *For brevity, $[u]$ is used to denote the set of arguments $u, u_1, u_2, \cdots$. We denote by $\mathcal{U}_n^k (n \geq 0, k \geq 0)$ the set of polynomials in $[u]$ of degree $k+1$ and index n, that is*

$$\mathcal{U}_n^k = \{f | f = \sum_{\|\alpha\|=n, |\alpha|=k+1} C_{\alpha_0 \cdots \alpha_m} u^{\alpha_0} u_1^{\alpha_1} \cdots u_m^{\alpha_m}\},$$

where $|\alpha| \stackrel{\text{def}}{=} \sum_{i=0}^m \alpha_i$ and $\|\alpha\| \stackrel{\text{def}}{=} \sum_{i=0}^m i\alpha_i$. The space of all polynomials of $[u]$ is denoted by $\mathcal{U}$ and $\mathcal{U} = \sum_{n \geq 0, k \geq 0} \mathcal{U}_n^k$.

4. *$\tilde{\mathcal{A}}_n$, $\tilde{\mathcal{A}}^k$, $\mathcal{U}_n$ and $\mathcal{U}^k$ make sense. e.g. $\mathcal{U}^k$ is the set of $[u]$'s polynomials of degree $k+1$.*

Remark 1 *Notice that we consider $k \geq 0$ which excludes the constant case, i.e. $1 \notin \mathcal{U}$. This case will be treated separately.*

With each polynomial in $\mathcal{U}_n^k$ we associate a form in $\mathcal{A}_k^n$ by the following rule:

$$u^{\alpha_0} u_1^{\alpha_1} \cdots u_m^{\alpha_m} \longrightarrow \xi_1^0 \cdots \xi_{\alpha_0}^0 \xi_{\alpha_0+1}^1 \cdots \xi_{\alpha_0+\alpha_1}^1 \cdots \xi_{k-\alpha_m+2}^m \cdots \xi_{k+1}^m.$$

Definition 1 *The Gel'fand-Dikii transformation [GD75] maps $f \in \mathcal{U}_n^k$ to $\tilde{f} \in \tilde{\mathcal{A}}_k^n$. For a monomial it is defined as*

$$u^{\alpha_0} u_1^{\alpha_1} \cdots u_m^{\alpha_m} \longmapsto < \xi_1^0 \cdots \xi_{\alpha_0}^0 \xi_{\alpha_0+1}^1 \cdots \xi_{\alpha_0+\alpha_1}^1 \cdots \xi_{k-\alpha_m+2}^m \cdots \xi_{k+1}^m > [k+1].$$

For any $f \in \mathcal{U}^k$, two important properties of Gel'fand-Dikii transformation are

$$\begin{array}{l} < D_x f > [k+1](\xi_1, \cdots, \xi_{k+1}) = \tilde{f}(\xi_1, \cdots, \xi_{k+1}) \sum_{i=1}^{k+1} \xi_i, \\ < \frac{\partial f}{\partial u_i} > [k+1](\xi_1, \cdots, \xi_{k+1}) = \frac{k+1}{i!} \frac{\partial^i \tilde{f}}{\partial \xi_{k+1}^i}(\xi_1, \cdots, \xi_k, 0). \end{array} \qquad (1)$$

Proposition 1 *Let $f \in \mathcal{U}_r^m$ and $g \in \mathcal{U}_s^n$, then $D_f(g) \in \mathcal{U}_{r+s}^{m+n}$ and*

$$\begin{aligned} & < D_f(g) > [m+n+1] \\ = \quad & (m+1) < \tilde{f}(\xi_1, \cdots, \xi_m, \sum_{i=1}^{n+1} \xi_{m+i}) \tilde{g}(\xi_{m+1}, \cdots, \xi_{m+n+1}) > [m+n+1]. \end{aligned}$$

Proof Using (1), we compute

$$
\begin{aligned}
& < D_f(g) >=< \sum_j \frac{\partial f}{\partial u_j} D_x^i g > \\
= \quad & \sum_j < \frac{m+1}{j!} \frac{\partial^j \tilde{f}}{\partial \xi_{m+1}^j}(\xi_1, \cdots, \xi_m, 0)(\zeta_1 + \cdots + \zeta_{n+1})^j \tilde{g}(\zeta_1, \cdots, \zeta_{n+1}) > \\
= \quad & (m+1) < \tilde{f}(\xi_1, \cdots, \xi_m, \zeta_1 + \cdots + \zeta_{n+1}) \tilde{g}(\zeta_1, \cdots, \zeta_{n+1}) > .
\end{aligned}
$$

Proposition 2 *Let $f \in \mathcal{U}_r^m$ and $g \in \mathcal{U}_s^n$, and define ξ_0 by $\xi_0 + \cdots + \xi_{n+m+1} = 0$. Then $D_f^*(g) \in \mathcal{U}_{r+s}^{m+n}$ and*

$$
< D_f^*(g) >= (m+1) < \tilde{f}(\xi_1, \cdots, \xi_m, \xi_0) \tilde{g}(\xi_{m+1}, \cdots, \xi_{m+n+1}) > .
$$

Proof Using (1), we compute

$$
\begin{aligned}
& < D_f^*(g) >=< \sum_j (-1)^j D_x^j (\frac{\partial f}{\partial u_j} g) > \\
= \quad & \sum_j (-\sum_{i=1}^m \xi_i - \sum_{s=1}^{n+1} \zeta_s)^j < \frac{m+1}{j!} \frac{\partial^j \tilde{f}}{\partial \xi_{m+1}^j}(\xi_1, \cdots, \xi_m, 0) \tilde{g}(\zeta_1, \cdots, \zeta_{n+1}) > \\
= \quad & (m+1) < \tilde{f}(\xi_1, \cdots, \xi_m, -(\sum_{i=1}^m \xi_i + \sum_{s=1}^{n+1} \zeta_s)) \tilde{g}(\zeta_1, \cdots, \zeta_{n+1}) > \\
= \quad & (m+1) < \tilde{f}(\xi_1, \cdots, \xi_m, \xi_0) \tilde{g}(\zeta_1, \cdots, \zeta_{n+1}) > .
\end{aligned}
$$

Definition 2 *For any $f, g \in \mathcal{U}$, we define*

$$
[\tilde{f}, \tilde{g}] =< [f, g] >=< D_f(g) > - < D_g(f) > .
$$

Proposition 3 *Let $f \in \mathcal{U}_r^m$ and $g \in \mathcal{U}_s^n$, then*

$$
\begin{aligned}
& [\tilde{f}, \tilde{g}](\xi_1, \cdots, \xi_{m+n}, 0) \\
= \quad & \frac{m+1}{m+n+1} [\tilde{f}(\xi_1, \cdots, \xi_m, 0), \tilde{g}] + \frac{n+1}{m+n+1} [\tilde{f}, \tilde{g}(\xi_1, \cdots, \xi_n, 0)]
\end{aligned}
$$

Proof This can be proved using $\frac{\partial}{\partial u}[f, g] = [\frac{\partial f}{\partial u}, g] + [f, \frac{\partial g}{\partial u}]$ and formula (1).

Proposition 4 *Let $Q, K \in \mathcal{U}$ and $Q = \sum Q_r^i$, $K = \sum K_s^j$, where $Q_r^i, K_r^i \in \mathcal{U}_r^i$. Then Q is a (co-)symmetry of the equation $u_t = K$ iff*

$$
\sum_{i+j=p, r+s=q} L_{\tilde{K_s^j}} \tilde{Q_r^i} = 0, \ (p \geq 0; q \geq 0).
$$

Proof We know that $Q[u]$ is a (co-)symmetry of the equation $u_t = K[u]$ iff $D_Q K - D_K Q = 0$ ($D_Q K + D_K^* Q = 0$). By Proposition 1 and 2, this can be proved directly.

Notation 2 $(\mathcal{C})\mathcal{S}_f = \{g[u] | g \in \mathcal{U}$ *is a (co-)symmetry of the equation* $u_t = f[u]\}$.

We give the following result as an application of this proposition.

Proposition 5 *Consider linear evolution equation* $u_t = f = \sum_{j=1}^{p} \lambda_j u_j$, *where the* λ_j *are constants and* $\lambda_p \neq 0$.

- $\mathcal{S}_f = \mathcal{U}$ *iff* $p = 1$;
- $\mathcal{S}_f = \mathcal{U}^0$ *iff* $p > 1$.

Proof Notice $\sum_{j=1}^{p} \lambda_j u_j \in \mathcal{U}^0$ and $< \sum_{j=1}^{p} \lambda_j u_j >= \sum_{j=1}^{p} \lambda_j \xi_1^j$. Let $Q \in \mathcal{U}$ and $Q = \sum Q^i$, where $Q^i \in \mathcal{U}^i$. By Proposition 4, Q is a symmetry of this equation iff $[\tilde{Q}^i, \sum_{j=1}^{p} \lambda_j \xi_1^j] = 0$, for any $i \geq 0$. So

$$(i+1) < \tilde{Q}^i(\xi_1, \cdots, \xi_{i+1}) \sum_{j=1}^{p} \lambda_j \xi_{i+1}^j >= \sum_{j=1}^{p} \lambda_j (\xi_1 + \cdots + \xi_{i+1})^j \tilde{Q}^i(\xi_1, \cdots, \xi_{i+1}).$$

This implies

$$\sum_{j=1}^{p} \lambda_j (\xi_1^j + \cdots + \xi_{i+1}^j) = \sum_{j=1}^{p} \lambda_j (\xi_1 + \cdots + \xi_{i+1})^j.$$

Under the assumption, it holds iff either $p = 1$ or $p \neq 1$ and $i = 0$.

Proposition 6 *Consider linear evolution equation* $u_t = f = \sum_{j=1}^{p} \lambda_j u_j$, *where* λ_j *are constants and* $\lambda_p \neq 0$. *Let* ξ_0 *be defined by* $\sum_{j=0}^{i+1} \xi_j = 0$.

- $\mathcal{CS}_f = \mathcal{U}$ *iff* $p = 1$;
- $\mathcal{CS}_f = \mathcal{U}^0$ *iff* $p > 1$ *and all* j *are odd..*
- $\mathcal{CS}_f = c$: *constant iff* $p > 1$ *and at least one of* j *is even..*

Proof Notice $\sum_{j=1}^{p} \lambda_j u_j \in \mathcal{U}^0$ and $< \sum_{j=1}^{p} \lambda_j u_j >= \sum_{j=1}^{p} \lambda_j \xi_1^j$. Let $Q \in \mathcal{U}$ and $Q = \sum Q^i$, where $Q^i \in \mathcal{U}^i$. By Proposition 4, Q is a co-symmetry of this equation iff $D_{\tilde{Q}^i}(\sum_{j=1}^{p} \lambda_j \xi_1^j) + D_{\sum_{j=1}^{p} \lambda_j \xi_1^j} \tilde{Q}^i = 0$, for any $i \geq 0$. So

$$(i+1) < \tilde{Q}^i(\xi_1, \cdots, \xi_{i+1}) \sum_{j=1}^{p} \lambda_j \xi_{i+1}^j > + \sum_{j=1}^{p} \lambda_j \xi_0^j \tilde{Q}^i(\xi_1, \cdots, \xi_{i+1}) = 0.$$

This implies

$$\sum_{j=1}^{p} \lambda_j \sum_{k=0}^{i+1} \xi_k^j = 0.$$

Under the assumption, it holds iff either $p = 1$ or $p \neq 1$ and $i = 0$ when j are odd.

3 Symmetries of equations

Notation 3 $G_k^l = (\sum_{i=1}^l \xi_i)^k - \sum_{i=1}^l \xi_i^k$, *where* $k > 0$ *and* $l > 0$ *are integers. By defining* ξ_0 *through the relation* $\sum_{k=0}^l \xi_k = 0$, *we can express* G_k^l *as* $-\sum_{i=0}^l \xi_i^k$. *We see that* G_k^l *is invariant under the natural action of the permutation group* Σ_{l+1} *on the coordinates* $\xi_0, \cdots, \xi_l$. *For fixed* l, *we denote the expressions* G_k^l, $k = 2, \cdots, l+1$ *after elimination of* ξ_0 *by* c_k *(Chern classes).*

$$u_t = \sum_{\kappa i + \lambda j = n} K_j^i \tag{2}$$

where $K_n^0 = u_n$.

4 Reducibility of the G_k^m

In this section, we study the the polynomials $G_k^m = (\sum_{i=1}^m \xi_i)^k - \sum_{i=1}^m \xi_i^m$, where $k > 1$.

Proposition 7 $G_k^m = t_k^m q_k^m$, *where* q_k^m *is* **irreducible** *in* $\mathbf{Q}[\xi_1, \cdots, \xi_m]$ *and* t_k^m *is one of the following cases.*

- $m = 2$:
  - $k = 0 \pmod 2$: $\xi_1 \xi_2$
  - $k = 3 \pmod 6$: $\xi_1 \xi_2 (\xi_1 + \xi_2) = c_3^{(2)}$
  - $k = 5 \pmod 6$: $\xi_1 \xi_2 (\xi_1 + \xi_2)(\xi_1^2 + \xi_1 \xi_2 + \xi_2^2) = c_3^{(2)} c_2^{(2)}$
  - $k = 1 \pmod 6$: $\xi_1 \xi_2 (\xi_1 + \xi_2)(\xi_1^2 + \xi_1 \xi_2 + \xi_2^2)^2 = c_3^{(2)} (c_2^{(2)})^2$
- $m = 3$:
  - $k = 0 \pmod 2$: 1
  - $k = 1 \pmod 2$: $(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3) = c_3^{(3)}$
- $m > 3$: 1

Proof We denote by $\mathcal{F}_k^m$ the number of irreducible factors of G_k^m. Notice that $G_k^m = G_k^{m+1}|_{\xi_{m+1}=0}$. Therefore, $\mathcal{F}_k^{m+1} \leq \mathcal{F}_k^m$.

For $m = 2$, this is proved by F. Beukers [Beu97] using diophantine approximation theory. Despite the innocent look of the polynomials involved, we have not been able to find a simpler way of proving this case.

Next we consider $m = 3$ and $k = 1 \pmod 2$. The ring of Σ_3, Σ_4-invariants is freely generated by $c_l^{(3)}, l = 2, 3, 4$. Since G_k^3 has odd degree, it must contain at least one factor of the only odd degree generator $c_3^{(3)}$, which restricts to a factor $c_3^{(2)}$ when putting $\xi_3 = 0$. This proves the case $k = 3 \pmod 6$. The only other possible divisors now must be of degree 2 or 4, since they restrict to $c_2^{(2)}$ or $(c_2^{(2)})^2$. That $c_2^{(3)} \not| G_k^3$ is then proved in appendix A.3. If $c_4^{(3)} - \alpha {c_2^{(3)}}^2 | G_k^3$, then this would imply ${c_2^{(2)}}^2 | G_k^2$, since $c_4^{(3)}|_{\xi_3=0} = 0$. Therefore $k = 1 \pmod 6$. We refer to appendix A.2 for the computational details of the proof that this is impossible. Any homogeneous factor t_k^3 can be written as $t_k^3 = p_k^3(c_2^{(3)}, c_3^{(3)}) + c_4^{(3)} r_k^3(c_2^{(3)}, c_3^{(3)}, c_4^{(3)})$, which reduces to $p_k^3|_{\xi_3=0}$, and so should be of degree ≤ 4 by the results for $m = 2$. This shows there can be no other divisors (but $c_3^{(3)}$) of order ≤ 4.

We now turn to the case $m = 3$ and $k = 0 \pmod 2$. This is treated in appendix A.1.

Finally we consider the case $m > 3$. We show that G_k^4 is irreducible in appendix A.4, where we only need to consider $k = 1 \pmod 2$, since the other case is automatically true since then G_k^3 is irreducible. The case $m > 4$ follows immediately for the same reason.

5 Analyzing the symmetry equation

We use the following notation convention. Fix $\lambda, \mu \in \mathbb{N}$. Then we denote by $\sum_{\{i=\alpha\}}^{\{m\}} Q^i$ the sum over all positive j and positive $i \geq \alpha$ of Q_j^i such that $\lambda i + \mu j = \mu m$. We consider the equation

$$u_t = \sum_{\{i=0\}}^{\{m\}} K^i$$

Let $\tilde{Q}^0 = \xi_1^r$ and let $\tilde{Q} = \sum_{\{p=0\}}^{\{r\}} \tilde{Q}^p$. Writing out the expression for the Lie derivative of Q with respect to the equation $u_t = K$, we obtain

Definitie G, L derivative

$$L_{\tilde{K}} \tilde{Q} = -\sum_{\{p=1\}}^{\{r\}} G_m^{p+1} \tilde{Q}^p + \sum_{\{i=1\}}^{\{m\}} \sum_{\{p=0\}}^{\{r\}} L_{\tilde{K}^i} \tilde{Q}^p$$

Suppose the equation $L_{\tilde{K}} \tilde{Q} = 0$ is solved for all $\tilde{Q}^p$ with $p < l$. Then it reduces at level l to

$$0 = L_{\tilde{K}} \tilde{Q} = -G_m^{l+1} \tilde{Q}^l + \sum_{\{i=1\}}^{\{m\}} L_{\tilde{K}^i} \tilde{Q}^{l-i}$$

We now write $\tilde{Q}^p = \mathcal{M}^p \tilde{Q}^0$. Then define operators $\mathcal{M}^p$ by $\mathcal{M}^0 \tilde{Q}^0 = \tilde{Q}^0$, $\mathcal{M}^l \tilde{Q}^0 = 0$ if $l < 0$, and for $l > 0$

$$\mathcal{M}^l \tilde{Q}^0 = \frac{1}{G_m^{l+1}} \sum_{\{i=1\}}^{\{m\}} L_{\tilde{K}^i} \mathcal{M}^{l-i} \tilde{Q}^0.$$

Proposition 8 *This equation implicitly depends on r. Let $\mathcal{R}_l$ be the set of r for which $\mathcal{M}^p$ exists (i.e. $\mathcal{M}^p \tilde{Q}^0 \in \mathcal{A}_{p+1}$) for $p \leq l$. Then $\mathcal{R}_{l+1} = \mathcal{R}_l$ for $l \geq 1$.*

Proposition 9 *Suppose $\mathcal{M}^{l+1}\tilde{S}^0$ exists. Then $\mathcal{M}^{l+1}\tilde{Q}^0$ exists, Moreover,*

$$\sum_{i=1}^{l}[\mathcal{M}^i\tilde{Q}^0, \mathcal{M}^{l+1-i}\tilde{S}^0] = [\tilde{S}^0, \mathcal{M}^{l+1}\tilde{Q}^0] + [\mathcal{M}^{l+1}\tilde{S}^0, \tilde{Q}^0]$$

Proof

$$\begin{aligned}
-G_m^{l+2}\sum_{i=1}^{l}[\mathcal{M}^i\tilde{Q}^0, \mathcal{M}^{l+1-i}\tilde{S}^0] &= \sum_{i=1}^{l}[[\mathcal{M}^i\tilde{Q}^0, \mathcal{M}^{l+1-i}\tilde{S}^0], \tilde{K}^0] \\
&= -\sum_{i=1}^{l}[[\mathcal{M}^{l+1-i}\tilde{S}^0, \tilde{K}^0], \mathcal{M}^i\tilde{Q}^0] - \sum_{i=1}^{l}[[\tilde{K}^0, \mathcal{M}^i\tilde{Q}^0], \mathcal{M}^{l+1-i}\tilde{S}^0] \\
&= \sum_{i=1}^{l}[\sum_{\{j=1\}}^{\{m\}} L_{\tilde{K}^j}\mathcal{M}^{l+1-i-j}\tilde{S}^0, \mathcal{M}^i\tilde{Q}^0] - \sum_{i=1}^{l}[\sum_{\{j=1\}}^{\{m\}} L_{\tilde{K}^j}\mathcal{M}^{i-j}\tilde{Q}^0, \mathcal{M}^{l+1-i}\tilde{S}^0] \\
&= \sum_{\{j=1\}}^{\{m\}}\sum_{i=1}^{l+1-j}[L_{\tilde{K}^j}\mathcal{M}^{l+1-i-j}\tilde{S}^0, \mathcal{M}^i\tilde{Q}^0] - \sum_{\{j=1\}}^{\{m\}}\sum_{i=j}^{l}[L_{\tilde{K}^j}\mathcal{M}^{i-j}\tilde{Q}^0, \mathcal{M}^{l+1-i}\tilde{S}^0] \\
&= \sum_{\{j=1\}}^{\{m\}}\sum_{i=1}^{l+1-j}[L_{\tilde{K}^j}\mathcal{M}^{l+1-i-j}\tilde{S}^0, \mathcal{M}^i\tilde{Q}^0] + \sum_{\{j=1\}}^{\{m\}}\sum_{i=0}^{l-j}[\mathcal{M}^{l+1-i-j}\tilde{S}^0, L_{\tilde{K}^j}\mathcal{M}^i\tilde{Q}^0] \\
&= \sum_{\{j=1\}}^{\{m\}}\left([L_{\tilde{K}^j}\tilde{S}^0, \mathcal{M}^{l+1-j}\tilde{Q}^0] + [\mathcal{M}^{l+1-j}\tilde{S}^0, L_{\tilde{K}^j}\tilde{Q}^0] + L_{\tilde{K}^j}\sum_{i=1}^{l-j}[\mathcal{M}^{l+1-i-j}\tilde{S}^0, \mathcal{M}^i\tilde{Q}^0]\right) \\
&= \sum_{\{j=1\}}^{\{m\}}\Big([L_{\tilde{K}^j}\tilde{S}^0, \mathcal{M}^{l+1-j}\tilde{Q}^0] + [\mathcal{M}^{l+1-j}\tilde{S}^0, L_{\tilde{K}^j}\tilde{Q}^0] - L_{\tilde{K}^j}([\tilde{S}^0, \mathcal{M}^{l+1-j}\tilde{Q}^0] + [\mathcal{M}^{l+1-j}\tilde{S}^0, \tilde{Q}^0])\Big) \\
&= \sum_{\{j=1\}}^{\{m\}} -[\tilde{S}^0, L_{\tilde{K}^j}\mathcal{M}^{l+1-j}\tilde{Q}^0] - [L_{\tilde{K}^j}\mathcal{M}^{l+1-j}\tilde{S}^0, \tilde{Q}^0] \\
&= -G_s^{l+2}\sum_{\{j=1\}}^{\{m\}} L_{\tilde{K}^j}\mathcal{M}^{l+1-j}\tilde{Q}^0 + G_r^{l+2}L_{\tilde{K}^j}\mathcal{M}^{l+1-j}\tilde{S}^0
\end{aligned}$$

6 Symmetries of KdV-like equations

We consider $2n+1$-th order KdV-like equations in the form of

$$u_t = \sum_{i=0}^{n} K^i_{2n+1-2i}, \quad (K^i_{2n+1-2i} \in \mathcal{U}^i_{2n+1-2i}), \tag{3}$$

where $n \geq 2$, $K^0_{2n+1} = u_{2n+1}$, $K^1_{2n-1} = \sum_{i=0}^{n-1}\alpha_i u_i u_{2n-1-i}$ and $K^n_1 = \beta_0 u^n u_1$. Here α_i, β_0 are constant and $\alpha_0\beta_0 \neq 0$. If $Q \in \mathcal{U}$ is an order m symmetry of (3) $\quad \alpha_0 \neq 0$??

by Proposition 4, the following formula holds

$$\sum_{i=0}^{n}[\tilde{Q}_{j-2n+2i}^{l-i},\tilde{K}_{2n+1-2i}^{i}]=0,\quad (j\geq 0,\ l\geq 0). \tag{4}$$

where $Q_j = 0$ for $j < 0$ or $j > m$ and $Q^l = 0$ for $l < 0$. Taking $j = 0$ in (4), we obtain $[\tilde{Q}_0,\ \tilde{K}_1^n] = 0$, $(Q_0 \in \mathcal{U}_0)$. This implies that Q_0 is a symmetry of the equation

$$u_t = \beta_0 u^n u_1,\quad (n\geq 2). \tag{5}$$

We know there is no zero-order symmetry for (5). In other words, $\tilde{Q}_0 = 0$. Substituting this into (4) and taking $j = 2$, we have

$$[\tilde{Q}_2,\ \tilde{K}_1^n]=0,\quad (Q_2\in\mathcal{U}_2).$$

We know that $\mathcal{S}_{\beta_0 u^n u_1} = \mathcal{U}_1$ from Appendix B. It follows that $\tilde{Q}_2 = 0$. By induction, we conclude that $\tilde{Q}_j = 0$ when j is even. Therefore $Q = \sum_{r=0}^{p} Q_{2r+1}$, where $Q_{2r+1} \in \mathcal{U}_{2r+1}$. In the meantime we may substitute $2j+1$ into (4) for j, so that it becomes

$$\sum_{i=0}^{n}[\tilde{Q}_{2j-2n+2i+1}^{l-i},\tilde{K}_{2n+1-2i}^{i}]=0,\quad (j\geq 0,\ l\geq 0). \tag{6}$$

Taking $j = p + n$ in (6), since $Q_r = 0$ when $r > 2p + 1$, we get

$$[\tilde{Q}_{2p+1}^{l},\tilde{K}_{2n+1}^{0}]=0.$$

Therefore if $l = 0$ we have a symmetry of order $m = 2p+1$ and $\tilde{Q}_{2p+1}^0 = \lambda\xi_1^{2p+1}$ by Proposition 5. We take $\lambda = 1$ without loss of generality. When $l = 1$ and $j = p + n - 1$, (6) leads to

$$[\tilde{Q}_{2p-1}^{1},\tilde{K}_{2n+1}^{0}]+[\tilde{Q}_{2p+1}^{0},\tilde{K}_{2n-1}^{1}]=0. \tag{7}$$

By Proposition 1, we obtain, adding ξ_0 by the relation $\xi_0 + \xi_1 + \xi_2 = 0$,

$$\tilde{Q}_{2p-1}^{1} = \tilde{K}_{2n-1}^{1}\frac{\xi_0^{2p+1}+\xi_1^{2p+1}+\xi_2^{2p+1}}{\xi_0^{2n+1}+\xi_1^{2n+1}+\xi_2^{2n+1}} = \frac{\tilde{K}_{2n-1}^{1}G_{2p+1}^{2}}{G_{2n+1}^{2}}. \tag{8}$$

We know $Q_{2p-1}^1 \in \mathcal{U}_{2p-1}^1$. So the necessary condition for equation (3) to possess a $2p+1$ order symmetry is that the right hand side of (8) is polynomial. This leads to the following proposition.

Proposition 10 *If there exists an order m symmetry of equation (3), then $m = 2p+1$ and $G_{2n+1}^2 | \tilde{K}_{2n-1}^1 G_{2p+1}^2$ for $p \geq 0$.*

Remark 2 *For the polynomials $p_k(\xi_1,\xi_2) = (\xi_1+\xi_2)^k - \xi_1^k - \xi_2^k$, with $k > 1$, F. Beukers [Beu97] proved that $p_k = t_k q_k$, where q_k is* **irreducible** *in $\mathbf{Q}[\xi_1,\xi_2]$ and t_k is one of the following cases.*

- $k = 0 \pmod 2$: $\xi_1\xi_2$
- $k = 3 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)$
- $k = 5 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)$
- $k = 1 \pmod 6$: $\xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)^2$

Applying Remark 2 to the polynomial G_k^2, for $m \geq 0$ we have,

$$\begin{aligned} G^2_{6m+2s+1} &= \xi_1\xi_2(\xi_1+\xi_2)(\xi_1^2+\xi_1\xi_2+\xi_2^2)^s\hat{G}^2_{6m+2s+1} \\ &= c_2^s c_3 \bar{G}^2_{6m+2s+1}(c_2, c_3), \end{aligned} \tag{9}$$

where $s = 0, 1, 2$.

Therefore, if equation (3) possesses any nontrivial symmetries, i.e. apart from u_1, u_t, then $\hat{G}^2_{6m+2s+1}|\tilde{K}^1_{2n-1}$, where $n = 3m+s$ for some m and $s \in 0, 1, 2$. For different p, c_3 may also divide $\tilde{K}^1_{2n-1}$. For example, we need $c_2^s|\tilde{K}^1_{2n-1}$ when $2p+1 = 3 \pmod 6$. This puts conditions on K^1_{2n-1}, i.e. on the differential equation. check this

Corollary 1 *If there exists an order $6m+3, m \geq 0$ nontrivial symmetry for equation (3) or a nontrivial symmetry for order $2n+1 = 3 \pmod 6$ equation, then $\alpha_0 \neq 0$ and $\tilde{K}^1_{2n-1} = \frac{\alpha_0 G^2_{2n+1}}{2(2n+1)\xi_1\xi_2}$.*

Proof This can be proved using Remark 9 and comparing the degree of ξ_1 and the coefficient of the highest degree term between numerator and denominator of $\tilde{Q}^1_{2p-1}$. We continue the procedure and take $l = 2$ and $j = p+n-2$ in (6) for $n \geq 2$. This leads to

$$[\tilde{Q}^2_{2p-3}, \tilde{K}^0_{2n+1}] + [\tilde{Q}^1_{2p-1}, \tilde{K}^1_{2n-1}] + [\tilde{Q}^0_{2p+1}, \tilde{K}^2_{2n-3}] = 0. \tag{10}$$

By Proposition 1, we obtain

$$\tilde{Q}^2_{2p-3} = \frac{\tilde{K}^2_{2n-3}G^3_{2p+1} + [\tilde{Q}^1_{2p-1}, \tilde{K}^1_{2n-1}]}{G^3_{2n+1}} \tag{11}$$

Notice that this procedure may go on and that by putting conditions on the equation we can find all the equations which possess a $2p+1$ symmetry. By taking different l and j, step by step, and solving equations with respect to $\tilde{Q}^i_{2p+1-2i}$ for $i = 0, \cdots, p$, we can find either the symmetry $Q = \sum_{i=0}^p Q^i_{2p+1-2i}$ or the obstruction to its existence.

Conjecture 1 *The period 6 of G^2_{2p+1} determines the order of recursion operator for KdV-like equations (3) if it exists. Moreover, the hereditary operator starting with D_x^6 determines the complete KdV hierarchy.*

7 Co-symmetries of KdV-like equations

If $Q \in \mathcal{U}$ is an order m co-symmetry of (3) by Proposition 4, the following formula holds

$$\sum_{i=0}^{n} L_{\tilde{K}^i_{2n+1-2i}} \tilde{Q}^{l-i}_{j-2n+2i} = 0, \quad (j \geq 0,\ l \geq 0). \tag{12}$$

where $Q_j = 0$ for $j < 0$ or $j > m$ and $Q^l = 0$ for $l < 0$. Taking $j = 1$ in (12), we obtain $L_{\tilde{K}^n_1} \tilde{Q}_1 = 0, \quad (Q_1 \in \mathcal{U}_1)$. This implies Q_1 is a co-symmetry of the equation

$$u_t = \beta_0 u^n u_1, \quad (n \geq 2). \tag{13}$$

We know there is no first-order co-symmetry for (13). This is equivalent to $\tilde{Q}_1 = 0$. Substituting this into (12) and taking $j = 3$, we have

$$L_{\tilde{K}^n_1} \tilde{Q}_3 = 0, \quad (Q_3 \in \mathcal{U}_3).$$

We know that $\mathcal{CS}_{\beta_0 u^n u_1} = \mathcal{U}_0$ from Appendix B. It follows that $\tilde{Q}_3 = 0$. By induction, we conclude $\tilde{Q}_j = 0$ when j is odd. Therefore $Q = \sum_{r=0}^{p} Q_{2r}$, where $Q_{2r} \in \mathcal{U}_{2r}$. In the meantime we may substitute $2j$ into (12) instead of j, it becomes

$$\sum_{i=0}^{n} L_{\tilde{K}^i_{2n+1-2i}} \tilde{Q}^{l-i}_{2j-2n+2i} = 0, \quad (j \geq 0,\ l \geq 0). \tag{14}$$

Taking $j = p + n$ in (14), since $Q_r = 0$ when $r > 2p$, we get

$$L_{\tilde{K}^0_{2n+1}} \tilde{Q}^l_{2p} = 0.$$

Therefore $l = 0$ i.e. co-symmetry order $m = 2p$ and $\tilde{Q}^0_{2p} = \lambda \xi_1^{2p}$ by Proposition 6. We take $\lambda = 1$ without loss of generality. When $l = 1$ and $j = p + n - 1$, (14) leads to

$$L_{\tilde{K}^0_{2n+1}} \tilde{Q}^1_{2p-1} + L_{\tilde{K}^1_{2n-1}} \tilde{Q}^0_{2p+1} = 0. \tag{15}$$

By Proposition 2, we obtain, adding ξ_0 by the relation $\xi_0 + \xi_1 + \xi_2 = 0$,

$$\tilde{Q}^1_{2p-2} = -\frac{\tilde{K}^1_{2n-1}(\xi_1, \xi_2)\xi_0^{2p} + \tilde{K}^1_{2n-1}(\xi_1, \xi_0)\xi_2^{2p} + \tilde{K}^1_{2n-1}(\xi_2, \xi_0)\xi_1^{2p}}{\xi_0^{2n+1} + \xi_1^{2n+1} + \xi_2^{2n+1}}. \tag{16}$$

We know that $Q^1_{2p-2} \in \mathcal{U}^1_{2p-2}$. So the necessary condition for equation (3) to possess a $2p$ order co-symmetry is that the right hand side of (16) is a polynomial.

A Proof of irreducibility

In the following sections we denote c_i^3 by c_i, $i = 1, \cdots, 4$.

A.1 $k = 0 \pmod 2$

Proof that c_2 does not divide G_k^3 $(k = 2j)$.

```
> with(grobner):
> X:=[x,y,z,c_1,c_2,c_3]:
```

F will be the Gröbner basis for Σ_3 invariants.

```
> F:=[c_1-(x+y+z),c_2-(x*y+x*z+y*z),c_3-(x*y*z)]:
> F:=gbasis(F,X,plex);
```

$$F := [-c_1 + x + y + z,\ c_2 + y\,z - c_1\,y - c_1\,z + y^2 + z^2,\ -c_3 + z\,c_2 - c_1\,z^2 + z^3]$$

GH is the generating function of G_k^3 $(k = 2j)$, i.e. $GH = \sum_{j=0}^{\infty} G_{2j}^3 t^{2j}$.

```
> f:=p->(x+y+z)^(2*p)-x^(2*p)-y^(2*p)-z^(2*p);
```

$$f := p \to (x + y + z)^{(2\,p)} - x^{(2\,p)} - y^{(2\,p)} - z^{(2\,p)}$$

```
> GH:=normal((x*t+y*t+z*t)^2/(1-(x*t+y*t+z*t)^2 )-(x*t)^2/(1-(x*t)^2)-(y*t)^2/(1-(y*
> Gn:=numer(GH):
```

Assume it does divide G_k^3, so the kth coefficient of the generating function vanishes if we put $c_2 = 0$.

```
> Hn:=normalf(Gn,F,X);
```

$$Hn := 2\,t^4\,{c_2}^2 + 2\,t^6\,c_3\,{c_1}^3 - 4\,t^4\,c_3\,c_1 + 2\,t^8\,{c_3}^2\,{c_1}^2 - 3\,t^6\,{c_3}^2 + 2\,t^2\,c_2 - t^6\,{c_2}^2\,{c_1}^2$$

```
> Gd:=expand(denom(GH)):
> Hd:=normalf(Gd,F,X);
```

$$Hd := 1 - 2\,t^2\,{c_1}^2 + t^4\,{c_1}^4 - 2\,t^4\,c_3\,c_1 + 2\,t^6\,c_3\,{c_1}^3 + t^4\,{c_2}^2 + t^8\,{c_3}^2\,{c_1}^2 - 2\,t^4\,c_2\,{c_1}^2 - t^6\,{c_2}^2\,{c_1}^2 - t^6\,{c_3}^2 + 2\,t^2\,c_2$$

```
> Q:=unapply(Hn/Hd,c_2);
```

$$Q := c_2 \to (2\,t^4\,{c_2}^2 + 2\,t^6\,c_3\,{c_1}^3 - 4\,t^4\,c_3\,c_1 + 2\,t^8\,{c_3}^2\,{c_1}^2 - 3\,t^6\,{c_3}^2 + 2\,t^2\,c_2 - t^6\,{c_2}^2\,{c_1}^2)/(1 - 2\,t^2\,{c_1}^2 + t^4\,{c_1}^4 - 2\,t^4\,c_3\,c_1 + 2\,t^6\,c_3\,{c_1}^3 + t^4\,{c_2}^2 + t^8\,{c_3}^2\,{c_1}^2 - 2\,t^4\,c_2\,{c_1}^2 - t^6\,{c_2}^2\,{c_1}^2 - t^6\,{c_3}^2 + 2\,t^2\,c_2)$$

```
> P:=factor(subs(c_3=0,simplify(Q(0)/c_3)));
```

$$P := 2\,\frac{t^4\,c_1\,(t^2\,{c_1}^2 - 2)}{(t\,c_1 - 1)^2\,(t\,c_1 + 1)^2}$$

```
> q1:=simplify(sum((-2*j)*c_1^(2*j-3)*t^(2*j),j= 2..infinity));
```

$$q1 := 2\,\frac{t^4\,c_1\,(t^2\,{c_1}^2 - 2)}{(-1 + t^2\,{c_1}^2)^2}$$

Since $P = q1$ and $q1 = \sum_{j=2}^{\infty}(-2j)c_1^{2j-3}t^{2j}$, we have reached a contradiction. It follows that c_2 does not divide G_{2j}^3 for any $j > 1$.

A.2 $k = 1 \pmod 6$

We give an edited Maple session here which does all the necessary computations to prove that $c_4 - ac_2^2$ does not divide G_k^3 ($k = 6j+1$). We assume that it does and we reach a contradiction as follows.

```
> with(grobner):
> X:=[c_4,c_3,c_2,x,y,z]:
```

F will be the Gröbner basis for Σ_3, Σ_4 invariants.

```
> F:=[c_4-x*y*z*(x+y+z),c_3-(x+y)*(x+z)*(y+z),c_2- (x^2+y^2+z^2+x*y+x*z+y*z)]:
> F:=gbasis(F,X);
```

$$F := [-c_2 + x^2 + y^2 + z^2 + x\,y + x\,z + y\,z,\ -c_4 + z\,c_3 - c_2\,z^2 + z^4,\\ c_3 + y^2\,z + y\,z^2 - c_2\,y - c_2\,z + y^3 + z^3]$$

Generating function of G_k^3 ($k = 6j+1$), i.e. $GH = \sum_{j=0}^{\infty} G_{6j+1}t^{6j+1}$.

```
> GH:=normal((x*t+y*t+z*t)/(1-(x*t+y*t+z*t)^6)- x*t/(1-(x*t)^6)-y*t/(1-(y*t)^6)-z*t/
> Gn:=numer(GH):
```

Assume it does divide G_k^3, so the kth coefficient of the generating function vanishes if we put $c_4 = ac_2^2$.

Note that $c_3c_2^2$ is a common factor in the numerator:

```
> Hn:=normal(subs(c_4=a*c_2^2,normalf(Gn,F,X))/( c_3*c_2^2));
```

$$Hn := t^7(-8\,t^6\,a\,{c_3}^2 + 5\,t^6\,{c_3}^2 - 5\,t^{12}\,a^2\,{c_2}^3\,{c_3}^2 - 4\,t^6\,a\,{c_2}^3 + t^{12}\,a\,{c_3}^4 - t^6\,{c_2}^3 + 5\,t^{12}\,a^3\,{c_2}^6\\ + 7\,a - 3\,t^6\,a^2\,{c_2}^3 + 7 + 5\,t^{12}\,a^4\,{c_2}^6)$$

```
> Gd:=expand(denom(GH)):
> Hd:=subs(c_4=a*c_2^2,normalf(Gd,F,X));
```

$$Hd := -6\,t^6\,a\,{c_2}^3 + t^{24}\,a^6\,{c_2}^{12} + 6\,t^{18}\,a\,{c_2}^3\,{c_3}^4 + 6\,t^{18}\,a^4\,{c_2}^9 - 6\,t^{18}\,a^3\,{c_2}^6\,{c_3}^2 - 3\,t^6\,{c_3}^2\\ - 2\,t^6\,{c_2}^3 + 1 + 3\,t^{12}\,{c_3}^4 + 6\,t^{12}\,a\,{c_2}^6 - 9\,t^{18}\,a^2\,{c_2}^6\,{c_3}^2 + 2\,t^{18}\,a^3\,{c_2}^9 - t^{18}\,{c_3}^6 - 2\,t^{12}\,a^3\,{c_2}^6\\ + t^{12}\,{c_2}^6 - 6\,t^{12}\,{c_3}^2\,{c_2}^3 + 9\,t^{12}\,a^2\,{c_2}^6$$

We now have the numerator and denumerator of the generating function of the G^3_{6j+1}, divided by $c_3c_2^2$ and expressed in invariants, modulo the relation $c_4 = ac_2^2$, where a is supposed to be j dependent.

```
> Q:=unapply(Hn/Hd,c_2);
```

$$Q := c_2 \to t^7(-8\,t^6\,a\,c_3{}^2 + 5\,t^6\,c_3{}^2 - 5\,t^{12}\,a^2\,c_2{}^3\,c_3{}^2 - 4\,t^6\,a\,c_2{}^3 + t^{12}\,a\,c_3{}^4 - t^6\,c_2{}^3 + 5\,t^{12}\,a^3\,c_2{}^6 + 7\,a - 3\,t^6\,a^2\,c_2{}^3 + 7 + 5\,t^{12}\,a^4\,c_2{}^6)/(-6\,t^6\,a\,c_2{}^3 + t^{24}\,a^6\,c_2{}^{12} + 6\,t^{18}\,a\,c_2{}^3\,c_3{}^4 + 6\,t^{18}\,a^4\,c_2{}^9 - 6\,t^{18}\,a^3\,c_2{}^6\,c_3{}^2 - 3\,t^6\,c_3{}^2 - 2\,t^6\,c_2{}^3 + 1 + 3\,t^{12}\,c_3{}^4 + 6\,t^{12}\,a\,c_2{}^6 - 9\,t^{18}\,a^2\,c_2{}^6\,c_3{}^2 + 2\,t^{18}\,a^3\,c_2{}^9 - t^{18}\,c_3{}^6 - 2\,t^{12}\,a^3\,c_2{}^6 + t^{12}\,c_2{}^6 - 6\,t^{12}\,c_3{}^2\,c_2{}^3 + 9\,t^{12}\,a^2\,c_2{}^6)$$

```
> P:=simplify(Q(0));
```

$$P := -\frac{t^7\,(-8\,t^6\,a\,c_3{}^2 + 5\,t^6\,c_3{}^2 + 7 + t^{12}\,a\,c_3{}^4 + 7\,a)}{-1 + 3\,t^6\,c_3{}^2 - 3\,t^{12}\,c_3{}^4 + t^{18}\,c_3{}^6}$$

We now write P in the form of a series expansion. Since P is supposed to be zero, this allows us to equate the coefficients in the series to zero.

```
> q1:=simplify(sum((6*j+1)*(a)*c_3^(2*j-2)*t^(6* j+1),j=1..infinity));
```

$$q1 := -\frac{a\,t^7\,(t^6\,c_3{}^2 - 7)}{(t^6\,c_3{}^2 - 1)^2}$$

```
> q2:=simplify(sum((6*j+1)*(j)*c_3^(2*j-2)*t^(6* j+1),j=1..infinity));
```

$$q2 := -\frac{t^7\,(5\,t^6\,c_3{}^2 + 7)}{(t^6\,c_3{}^2 - 1)^3}$$

```
> simplify(q1+q2-P);
```

$$0$$

This shows that $Q(0) = \sum_{j=1}^{\infty}(6*j+1)(a+j)c_3^{2j-2}t^{6j+1}$. It follows from the divisibility assumption that $P = 0$, so $a + j = 0$. We now consider $PP = Q'''(0)/6$:

```
> PP:=simplify(D[1,1,1](Q)(0)/6);
> h1:=l->sum((2*(k))^2,k=1..l):
> h2:=l->sum(4*k+3,k=1..l-1)+3:
> h:=k->sum(ll^2,ll=1..k)*(2*k+3)/5:
> hh:=l->simplify(h2(l)*(-l-1)^2+h1(l)*(-l-1)+h (l));
```

$$hh := l \to \mathrm{simplify}(\mathrm{h2}(l)\,(-l-1)^2 + \mathrm{h1}(l)\,(-l-1) + \mathrm{h}(l))$$

```
> p_1:=simplify(sum(
(6*(j+1)+1)*h2(j)*c_3^(2*j-2)*a^2*t^(6*j+7),
j=1..infinity));
```

$$p_1 := -\frac{a^2\,t^{13}\,(-39 - 34\,t^6\,c_3{}^2 + t^{12}\,c_3{}^4)}{(t^6\,c_3{}^2 - 1)^4}$$

```
> p_2:=simplify(sum(
(6*j+7)*h1(j)*c_3^(2*j-2)*a*t^(6*j+7),
j=1..infinity));
```

$$p_2 := -4\,\frac{a\,t^{13}\,(5\,t^{12}\,c_3{}^4 + 30\,t^6\,c_3{}^2 + 13)}{(t^6\,c_3{}^2 - 1)^5}$$

```
> p_3:=simplify(sum(
(6*j+7)*h(j)*c_3^(2*j-2)*t^(6*j+7),
j=1..infinity));
```

$$p_3 := \frac{t^{13}\,(13 + 55\,t^6\,c_3{}^2 + 27\,t^{12}\,c_3{}^4 + t^{18}\,c_3{}^6)}{(t^6\,c_3{}^2 - 1)^6}$$

We see that $PP = p_1 + p_2 + p_3$ and the jth coefficient in PP equals

$$\frac{1}{30}\,c_3^{2\,j-2}\,j\,(12\,j+13)\,(2\,j+1)\,(6\,j+7)\,(j+1)$$

This shows that $Q'''(0) = \frac{1}{5}\sum_{j=1}^{\infty} c_3^{2j-2} j(12j+13)(2j+1)(6j+7)(j+1) \neq 0$. Therefore the divisibility assumption must be wrong.

A.3 $m = 3, k = 1 \pmod 2$

```
> with(grobner):
> X:=[c_3,c_4,x,y,z]:
```

F will be the Gröbner basis for Σ_3, Σ_4 invariants.

```
> F:=gbasis([x^2+y^2+z^2+x*y+x*z+y*z,(x+y)*(x+ z)*(y+z)-c_3,x*y*z*(x+y+z)-c_4],X);
```

$$F := [-c_4 + z\,c_3 + z^4,\ y^2\,z + y\,z^2 + c_3 + y^3 + z^3,\ x^2 + y^2 + z^2 + x\,y + x\,z + y\,z]$$

```
> for m by 2 to 5 do
> GH:=normal((x*t+y*t+z*t)^m/(1-(x*t+y*t+z*t)^6 )
- (x*t)^m/(1-x^6*t^6)-(y*t)^m/(1-y^6*t^6)-(z*t)^m/(1-z^6*t^6)):
> Hd:=collect(normalf(denom(GH),F,X),t);
> Hn:=collect(normalf(numer(GH),F,X),t);
> Q:=unapply(Hn/Hd,c_3,c_4);
> if m=1 then
> Q1:=simplify(D[2](Q)(c_3,0));
> P1:=simplify(sum((6*p+1)*c_3^(2*p-1)*t^(6*p+1), p=1..infinity));
> fi:
> if m=3 then
> Q3:=simplify(Q(c_3,0));
> P3:=simplify(sum(3*c_3^(2*p+1)*t^(6*p+3),p=0..i nfinity));
```

```
> fi:
> if m=5 then
> Q5:=simplify(D[2,2](Q)(c_3,0));
> P5:=simplify(sum((6*p+5)*2*p*c_3^(2*p-1)*t^(6*p +5),p=1..infinity));
> fi:
> printf(`Q%a=%a\n`,m,Q.m):
> printf(`Test for m=%a =%a\n`,m,simplify(Q.m-P.m)):
> od:
```

```
Q1=-(t^6*c_3^2-7)*t^7*c_3/(c_3^4*t^12-2*t^6*c_3^2+1)
Test for m=1=0
Q3=-3*t^3*c_3/(-1+t^6*c_3^2)
Test for m=3=0
Q5=-2*t^11*c_3*(t^6*c_3^2+11)/(-1+t^4}18*c_3^6-3*c_3^4*t^12+3*t^6*c_3^2)
Test for m=5=0
```

A.4 $m = 4, k = 1 \pmod 2$

In this section we denote c_i^4 by c_i, $i = 2, 3, 4$.

```
> with(grobner):
> X:=[c_4,c_3,c_2,x,y,z,s]:
```

F will be the Gröbner basis for Σ_4, Σ_5 invariants.

```
> F:=gbasis([x^2+y^2+z^2+s^2+x*y+x*z+y*z+x*s+y*s+z*s-c_2,
2*x*y*z+2*z*y*s+2*s*y*x+2*z*s*x+x^2*z+x*z^2+y^2*z+y*z^2
+s^2*z+s*z^2+x*s^2+x^2*s+s^2*y+s*y^2+x^2*y+x*y^2,
3*x*y*z*s+x^2*y*z+x*y^2*z+x*y*z^2+z*s^2*x+z*s*x^2+z^2*s*x
+z*y*s^2+z*y^2*s+z^2*y*s+s^2*y*x+s*y*x^2+s*y^2*x-c_3,
x*y*z*s*(x+y+z+s)-c_4],X);
```

$$F := [x^2 + y^2 + z^2 + s^2 + x\,y + x\,z + y\,z + x\,s + y\,s + z\,s - c_2,\\ -c_2\,z + s^2\,y + s\,y^2 + z\,y\,s + s^3 - c_2\,s + s\,z^2 + s^2\,z + y^3 + y\,z^2 + y^2\,z + z^3 - c_2\,y,\\ -s\,c_3 + c_4 + s^5 - s^3\,c_2,\ -c_3 - c_2\,z^2 - c_2\,s^2 + s^4 + z^4 + z^2\,s^2 + z^3\,s + s^3\,z - c_2\,z\,s]$$

```
> GH:=simplify((x*t+y*t+z*t+s*t)/(1-(x*t+y*t+z*t+s*t)^2)
-x*t/(1-x^2*t^2)-y*t/(1-y^2*t^2)
-z*t/(1-z^2*t^2))-s*t/(1-s^2*t^2):
> Hd:=collect(normalf(denom(GH),F,X),t);
```

$$Hd := t^{10}\,c_4^2 - t^8\,c_3^2 - 2\,t^6\,c_3\,c_2 + (-c_2^2 + 2\,c_3)\,t^4 + 2\,t^2\,c_2 - 1$$

```
> Hn:=collect(normalf(numer(GH),F,X),t);
```

$$Hn := t^9\,c_4\,c_3 + 3\,t^7\,c_4\,c_2 - 5\,t^5\,c_4$$

```
> Q1:=simplify(D[3](unapply(Hn/Hd,c_2,c_3,c_4))(c_2,0,0 ));
```

$$Q1 := -\frac{t^5\,(3\,t^2\,c_2-5)}{1+t^4\,c_2^2-2\,t^2\,c_2}$$

```
> P1:=simplify(sum((2*p+1)*c_2^(p-2)*t^(2*p+1),p= 2..infinity));
```

$$P1 := -\frac{t^5\,(3\,t^2\,c_2-5)}{(t^2\,c_2-1)^2}$$

```
> printf(‘Test=%a‘,simplify(Q1-P1));
Test=0
```

B Polynomial symmetries of $u_t = u^n u_1 (n \neq 0)$

Proposition 11 $\mathcal{S}_{u^n u_1} = \mathcal{U}_1$, *where* $n \neq 0$ *[QT82] [TQ81].*

Proof Suppose Q is an order $q (q \geq 2)$ symmetry, so $[u^n u_1, Q] = 0$. Write it out

$$\begin{aligned} 0 &= u^n D_x Q + n u^{n-1} u_1 Q - D_Q(u^n u_1) \\ &= u^n \sum_{i=0}^{q} \frac{\partial Q}{\partial u_i} u_{i+1} + n u^{n-1} u_1 Q - \sum_{i=0}^{q} \frac{\partial Q}{\partial u_i} D_x^i (u^n u_1) \end{aligned} \tag{17}$$

Differentiating (17) with respect to u_q, when $q \geq 2$, we get

$$0 = u^n D_x(\frac{\partial Q}{\partial u_q}) - D_{\frac{\partial Q}{\partial u_q}}(u^n u_1) - q n u^{n-1} u_1 \frac{\partial Q}{\partial u_q}$$

Therefore

$$0 = \frac{\partial [u^n u_1, Q]}{\partial u_q} = [u^n u_1, \frac{\partial Q}{\partial u_q}] - (q+1) n u^{n-1} u_1 \frac{\partial Q}{\partial u_q} \tag{18}$$

Assume $P = \frac{\partial Q}{\partial u_q}$ and $|P| = r \geq 2$. By formula (18), it follows

$$[u^n u_1, \frac{\partial P}{\partial u_r}] = \frac{\partial [u^n u_1, P]}{\partial u_r} + (r+1) n u^{n-1} u_1 \frac{\partial P}{\partial u_r} = (q+r+2) n u^{n-1} u_1 \frac{\partial P}{\partial u_r}$$

Since Q is polynomial, we can continue the preceding procedure till $r \leq 1$. By induction, we have

$$[u^n u_1, P] = M n u^{n-1} u_1 P \tag{19}$$

where $|P| = 1$ and $M \geq 1 + q \geq 3$ is constant.

Computing (19), it leads the equation $\frac{\partial P}{\partial u_1} u_1 = (1-M)P$. So

$$P = H(u) u_1^{1-M},$$

where H is a function of u. Notice P is polynomial only when $M \leq 1$. This contradicts (since $M \geq 3$) our basic assumption that $q \geq 2$. Therefore this implies $|Q| \leq 1$ i.e. Q only depends on u and u_1. It is easy to know that the solution of $[u^n u_1, Q(u, u_1)] = 0$ is $Q = H(u)u_1$, where H is any polynomial of u. By the same method, we can prove the similar result for its co-symmetries.

Proposition 12 $\mathcal{CS}_{u^n u_1} = \mathcal{U}_0$, *where* $n \neq 0$. *Especially,* 1 *is also a co-symmetry.*

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