The Heat and the Wave Equation

by

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PHYSICAL BACKGROUND

1. The wave equation in one dimension

In this section we derive the equations of motion for a vibrating string and a vibrating membrane.

Consider a string which we assume to be described as the graph of a function of x (space) and t (time):

$$y = u(x,t).$$

Vertical external forces acting on a piece of the string between $x = a$ and $x = b$, (a,b) for short, may be described as

$$\int_a^b f(x,t)dx \qquad \text{(in positive } y\text{−direction).}$$

Here $f(x,t)$ is the force per unit of lenght, and $u_x = \partial u/\partial x$ is assumed to be small, so that the arc lenght

$$\sqrt{1+(\frac{\partial u}{\partial x})^2}\ dx \approx dx.$$

Now what are the internal forces acting on (a,b)?

In $x = a$ we have a tangential force proportional to the strain,

$$\vec{F}_a = -\sigma(a)\ \frac{1}{\sqrt{1+u_x(a)^2}}\ \begin{pmatrix} 1 \\ u_x(a) \end{pmatrix}.$$

Similarly, at $x = b$,

$$\vec{F}_b = \sigma(b)\ \frac{1}{\sqrt{1+u_x^2(b)}}\ \begin{pmatrix} 1 \\ u_x(b) \end{pmatrix}.$$

Assuming again that u_x is small, the total internal force acting on (a,b) is given by

$$\vec{F} = \sigma(b)\begin{pmatrix} 1 \\ u_x(b) \end{pmatrix} - \sigma(a)\begin{pmatrix} 1 \\ u_x(a) \end{pmatrix}.$$

Newton's law says that the combined forces determine the change of impuls moment. Ignoring motion in the x-direction, we conclude that $\sigma(a) = \sigma(b)$, and since a, b where arbitrary,

$$\sigma(x) \equiv \sigma \qquad \text{is constant.}$$

Thus the impuls moment of (a,b) has only a y-component, given by

$$\int_a^b \rho(x)\frac{\partial u}{\partial t}(x,t)dx,$$

where $\rho(x)$ is the mass density of the string per unit lenght, so that

$$\frac{d}{dt}\int_a^b \rho(x)\frac{\partial u}{\partial t}(x,t)dx = \sigma(b)\frac{\partial u}{\partial x}(b,t) - \sigma(a)\frac{\partial u}{\partial x}(a,t) + \int_a^b f(x,t)dx,$$

or, assuming also $\rho(x) \equiv \rho$ is a constant,

$$\int_a^b \rho\frac{\partial^2 u}{\partial t^2}(x,t) = \int_a^b \frac{\partial}{\partial x}\sigma\frac{\partial u}{\partial x}(x,t)dx + \int_a^b f(x,t)dx.$$

Again, since a and b are arbitrary, we conclude that

$$\rho\frac{\partial^2 u}{\partial t^2} = \sigma\frac{\partial^2 u}{\partial x^2} + f, \tag{1.1}$$

which is the *one-dimensional inhomogeneous wave equation*.

2. The wave equation in more dimensions

Next we consider a vibrating membrane. We examine where the derivation above has to be adjusted. Instead of $y = u(x,t)$ we have

$$z = u(x,y,t),$$

and instead of (a,b) we take a small open disk D in the (x,y)-plane. The horizontal internal force acting on the piece corresponding to D is given by, again assuming that u_x and u_y are small,

$$\oint_{\partial D} \sigma(x,y)\nu(x,y)dS,$$

Here ν is the outward normal, dS is the arc lenght, ∂D is the boundary of D, and σ is the strain. By the vector valued integral version of the divergence theorem, this equals

$$\int_D \nabla\sigma(x,y)d(x,y),$$

which has to be zero again, because we neglect motion in the horizontal directions. But D is arbitrary so $\nabla\sigma \equiv 0$, i.e. $\sigma(x,y) = \sigma$ is constant. The vertical internal force acting on D is then

$$\sigma\oint_{\partial D} \nabla u(x,y,t)\cdot\nu(x,y)dS =$$

(by the divergence theorem)

$$\sigma\int_D \Delta u(x,y,t)\ d(x,y).$$

Here ∇ and Δ act only on x and y, but not on t. The *inhomogeneous wave equation in two (and in fact any n) dimensions* thus reads

$$\rho \frac{\partial^2 u}{\partial t^2} = \sigma \Delta u + f. \tag{2.1}$$

3. Conservation laws and diffusion

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, i.e. a bounded open connected set. We assume Ω is filled with some sort of *diffusive* material, with concentration given by

$$c = c(x,t) = c(x_1, x_2, x_3, t),$$

where x is space, t is time. Motion is then usually described by the *mass flux*

$$\vec{\Phi} = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \vec{\Phi}(x,t).$$

The direction of $\vec{\Phi}$ coincides with the direction of the motion, and its magnitude says how much mass flows through a plane perpendicular to $\vec{\Phi}$, per unit of surface area.

If we consider any ball B contained in Ω and compute what comes out of B per unit of time, we find

$$\oint_{\partial B} \Phi(x,t)\nu(x)dS(x) = \int_B \mathrm{div}\Phi(x,t)dx =$$

$$\int_B \{ \frac{\partial \Phi_1(x,t)}{\partial x_1} + \frac{\partial \Phi_2(x,t)}{\partial x_2} + \frac{\partial \Phi_3(x,t)}{\partial x_3} \} d(x_1, x_2, x_3).$$

Assuming that new material is being produced in Ω, and that per unit of time the production rate in any disk B is given by

$$\int_B q(x,t)dx,$$

we have by the *conservation of mass principle*

$$\frac{d}{dt}\int_B c(x,t)dx = -\int_B \mathrm{div}\Phi(x,t)dx + \int_B q(x,t)dt.$$

Since B was arbitrary, we find

$$\frac{\partial c}{\partial t} = -\mathrm{div}\Phi(x,t) + q(x,t), \tag{3.1}$$

which is commonly called a *conservation law*.

This conservation law has to be combined with some sort of second relation between the concentration c and the flux Φ in order to arrive at a single equation for c. An example of such a relation is the principle of *diffusion* which says that mass flows from higher to lower concentrations, i.e. the flux Φ and the gradient of the concentration, point in opposite directions:

$$\vec{\Phi} = -D\nabla C. \tag{3.2}$$

Here $D > 0$ is the diffusion coefficient, which may depend on space, time, etc. In the simplest case D is a constant. Substituting this second relation in the conservation law we obtain, if D is a constant,

$$\frac{\partial c}{\partial t} = \mathrm{div}\ D\nabla c + q \ = \ D\Delta c + q. \tag{3.3}$$

Because a similar derivation can be given for the flow of heat in a physical body, this equation is often called the *inhomogeneous heat equation*.

PART 2: THE WAVE EQUATION

4. The Cauchy problem in one space dimension

For $u = u(x,t)$ we consider the equation

$$u_{tt} - c^2 u_{xx} = 0, \tag{4.1}$$

where $c \in \mathbb{R}^+$ is fixed and x and t are real variables. We change variables by setting

$$\xi = x + ct, \quad \eta = x - ct. \tag{4.2}$$

Then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta},$$

so that

$$\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\ \frac{\partial^2}{\partial \xi \partial \eta} + c^2\frac{\partial^2}{\partial \eta^2}$$

$$-c^2\frac{\partial^2}{\partial \xi^2} - 2c^2\frac{\partial^2}{\partial \xi \partial \eta} - c^2\frac{\partial^2}{\partial \eta^2} = -(2c)^2\frac{\partial^2}{\partial \xi \partial \eta},$$

and (4.1) reduces to

$$u_{\xi\eta} = 0. \tag{4.3}$$

Formally then every function of the form

$$u(x,t) = f(\xi) + g(\eta) = f(x+ct) + g(x-ct), \tag{4.4}$$

is a solution. The lines $\xi = \mathit{constant}$ and $\eta = \mathit{constant}$ are called *characteristics* .

Next consider the initial value problem

$$(CP) \begin{cases} u_{tt} - c^2 u_{xx} = 0 & x,t \in \mathbb{R}; \\ u(x,0) = \alpha(x) & x \in \mathbb{R}; \\ u_t(x,0) = \beta(x) & x \in \mathbb{R}. \end{cases}$$

This is usually called the *Cauchy problem* for the wave equation in one space dimension. To solve (CP) for given functions α and β we use (4.4). Thus we have to find f and g such that

$$\alpha(x) = u(x,0) = f(x) + g(x) \quad \text{and} \quad \beta(x) = u_t(x,0) = cf'(x) - cg'(x).$$

It is no restriction to assume that $f(0) - g(0) = 0$. Hence

$$f(x) - g(x) = \frac{1}{c}\int_0^x \beta(s)ds \quad \text{and} \quad f(x) + g(x) = \alpha(x).$$

Solving for f and g we obtain

$$f(x) = \frac{1}{2}\alpha(x) + \frac{1}{2c}\int_0^x \beta(s)ds \quad \text{and} \quad g(x) = \frac{1}{2}\alpha(x) - \frac{1}{2c}\int_0^x \beta(s)ds.$$

Here the only restriction on the functions α and β is that the latter one has to be locally integrable. Using (4.2) and (4.4) we conclude that

$$u(x,t) = \frac{1}{2}\big\{\alpha(x+ct) + \alpha(x-ct)\big\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)ds. \tag{4.5}$$

Clearly, u defined as such, satisfies $u(x,0) = \alpha(x)$, and if α is differentiable, and β continuous, then

$$u_t(x,t) = \frac{1}{2}\big\{c\alpha'(x+ct) - c\alpha'(x-ct)\big\} + \frac{1}{2c}\big\{c\beta(x+ct) + c\beta(x-ct)\big\},$$

so that $u_t(x,0) = \beta(x)$.

For the (1.1) to be satisfied in a classical way, i.e. for u_{tt} and u_{xx} to be continuous, we need α to be twice and β to be once continuously differentiable. We summarize these results in the following theorem.

4.1 Theorem Let $\alpha \in C^2(\mathbb{R})$ and $\beta \in C^1(\mathbb{R})$. Then problem (CP) has a unique solution $u \in C^2(\mathbb{R} \times \mathbb{R})$, given by

$$u(x,t) = \frac{1}{2}\{\alpha(x+ct) + \alpha(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)ds.$$

The right hand side of this expression is defined for all $\alpha : \mathbb{R} \to \mathbb{R}$ and all locally integrable $\beta : \mathbb{R} \to \mathbb{R}$.

Proof The derivation of the formula is correct if u is a twice continuously differentiable solution and it is easy to check that under the hypotheses u as defined in the theorem is indeed such a solution. ∎

4.2 Corollary Suppose $\mathrm{supp}\,\alpha \cup \mathrm{supp}\,\beta \subset [A, B]$. Then $\mathrm{supp}\,u \subset [A-ct, B+ct]$, for $t > 0$,

5. The inhomogeneous wave equation in dimension one

Next we consider the Cauchy problem for the inhomogeneous wave equation,

$$(CP_i) \begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x,t) & x, t \in \mathbb{R}; \\ u(x,0) = \alpha(x) & x \in \mathbb{R}; \\ u_t(x,0) = \beta(x) & x \in \mathbb{R}, \end{cases}$$

for given functions α, β, φ. We assume φ is integrable.

We shall derive a representation formula for the solution of (CP_i). To do so, fix x_0 and $t_0 > 0$, and consider the triangle G in $\mathbb{R} \times \mathbb{R}$ bounded by the segments $C_1 = \{x - x_0 = c(t - t_0), 0 < t < t_0\}$, $C_2 = \{x - x_0 = -c(t - t_0), 0 < t < t_0\}$, and $I = \{t = 0, x_0 - ct_0 < x < x_0 + t_0\}$. Assume u is smooth and satisfies

$$u_{tt} - c^2 u_{xx} = \mathrm{div} \begin{pmatrix} -c^2 u_x \\ u_t \end{pmatrix} = \varphi(x,t). \tag{5.1}$$

Here x is the first, and t the second coordinate. Applying the divergence theorem we have

$$\int\int_G \varphi(x,t)dxdt = \oint_{\partial G} \begin{pmatrix} -c^2 u_x \\ u_t \end{pmatrix} \cdot \nu \; dS =$$

(where ν is the outward normal on ∂G)

$$\oint_{\partial G} (-c^2 u_x \; dt - u_t \; dx) = \int_{C_1} + \int_{C_2} + \int_I (-c^2 u_x \; dt - u_t \; dx) =$$

(using $dx = cdt$ along C_1 and $dx = -cdt$ along C_2)

$$\int_{C_1} (-cu_x\ dx - cu_t\ dt) + \int_{C_2} (cu_x dx + cu_t\ dt) + \int_I -u_t\ dx =$$

$$-c\alpha(x_0 - ct_0) + 2cu(x_0, t_0) - c\alpha(x_0 + ct_0) - \int_{x_0-ct_0}^{x_0+ct_0} \beta(s)ds.$$

Thus problem (CP_i) should have as a solution

$$u(x,t) = \frac{1}{2}\{\alpha(x-ct)+\alpha(x+ct)\}+\frac{1}{2c}\int_{x-ct}^{x+ct} \beta(s)ds+\frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)d\xi d\tau.$$

We have already investigated for which α and β this makes sense, so consider the new term, which we denote by

$$u_p(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)d\xi d\tau. \tag{5.2}$$

For all locally integrable φ the function u_p is well defined as a function of $x \in \mathbb{R}$ and $t \in \mathbb{R}$, and since φ is integrated over a domain in $\mathbb{R} \times \mathbb{R}$ with continuously varying boundary, it is clear that $u_p \in C(\mathbb{R} \times \mathbb{R})$, and that $u_p(x, 0) = 0$ for all $x \in \mathbb{R}$. Also, the measure of G equals ct^2, so that for locally bounded φ,

$$u_p(x,t) = O(t^2) \quad \text{as} \quad t \to 0,$$

uniformly on bounded x-intervals. In particular,

$$\frac{\partial u_p}{\partial t}(x, 0) = 0,$$

for all $x \in \mathbb{R}$.

Next we give conditions on φ for u_p to be a classical solution of the inhomogeneous wave equation. We assume that $\varphi \in C(\mathbb{R} \times \mathbb{R})$. Then

$$u_p(x,t) = \frac{1}{2c}\int_0^t g(x,t,\tau)d\tau; \quad g(x,t,\tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)d\xi,$$

so that

$$\frac{\partial g}{\partial x}\ (x,t,\tau) = \varphi(x + c(t-\tau),\tau) - \varphi(x - c(t-\tau),\tau),$$

and

$$\frac{\partial g}{\partial t}(x,t,\tau) = c\varphi(x + c(t-\tau),\tau) + c\varphi(x - c(t-\tau),\tau).$$

Thus g is differentiable with respect to x and t, with partial derivatives continuous in x, t and τ. Hence

$$\frac{\partial u_p}{\partial t}(x,t) = \frac{1}{2c} g(x,t,t) + \frac{1}{2c}\int_0^t \frac{\partial g}{\partial t}(x,t,\tau)d\tau$$

$$= \frac{1}{2}\int_0^t \{\varphi(x+c(t-\tau),\tau) + \varphi(x-c(t-\tau),\tau\} d\tau,$$

which is continuous because φ is. Similarly we find that

$$\frac{\partial u_p}{\partial x}(x,t) = \frac{1}{2c}\int_0^t \frac{\partial g}{\partial x}(x,t,\tau)d\tau =$$

$$\frac{1}{2c}\int_0^t \{\varphi(x+c(t-\tau),\tau) - \varphi(x-c(t-\tau),\tau)\} d\tau$$

is continuous. We conclude that $u_p \in C^1(\mathbb{R}\times\mathbb{R}))$.

If we want u_p to be in $C^2(\mathbb{R}\times\mathbb{R})$, we need more regularity on φ because we have to differentiate once more under the integral sign. This is allowed if φ_x is continuous. Then

$$\frac{\partial^2 u_p}{\partial t^2}(x,t) = \varphi(x,t) + \frac{1}{2}\int_0^t \{c\varphi_x(x+c(t-\tau),\tau) - c\varphi_x(x-c(t-\tau),\tau)\} d\tau,$$

while

$$\frac{\partial^2 u_p}{\partial x^2}(x,t) = \frac{1}{2c}\int_0^t \{\varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\} d\tau,$$

so that indeed u_p is a solution of the inhomogeneous wave equation.

5.1 Theorem Suppose $\alpha \in C^2(\mathbb{R})$, $\beta \in C^1(\mathbb{R})$, $\varphi \in C(\mathbb{R}\times\mathbb{R})$, and $\varphi_x \in C(\mathbb{R}\times\mathbb{R})$. Then problem (CP_i) has a unique solution $u \in C^2(\mathbb{R}\times\mathbb{R})$, which for $t > 0$ is given by

$$u(x,t) = \frac{1}{2}\{\alpha(x-ct) + \alpha(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \beta(\xi)d\xi + \frac{1}{2c}\int\int_{G(x,t)} \varphi(\xi,\tau)d\xi\ d\tau,$$

where

$$G(x,t) = \{(\xi,\tau), 0 \le \tau \le t,\ |\xi - x| \le c(t-\tau)\}.$$

Proof The derivation above is correct if u is a twice continuously differentiable solution and we have seen that under the hypotheses u as defined in the theorem is indeed such a solution. ▪

6. Initial boundary value problems

We now consider the inhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} + \varphi \tag{6.1}$$

on the strip $\{(x,t) : a < x < b\}$. Initial conditions are again of the form

$$(IC) \quad \begin{cases} u(x,0) = \alpha(x) & x \in (a,b); \\ u_t(x,0) = \beta(x) & x \in (a,b). \end{cases}$$

For (lateral) boundary conditions one can take any of the following four combinations

$$(DD) \quad \begin{cases} u(a,t) = A(t) \\ u(b,t) = B(t) \end{cases} \qquad (DN) \quad \begin{cases} u(a,t) = A(t) \\ u_x(b,t) = B(t) \end{cases}$$

$$(ND) \quad \begin{cases} u_x(a,t) = A(t) \\ u(b,t) = B(t) \end{cases} \qquad (NN) \quad \begin{cases} u_x(a,t) = A(t) \\ u_x(b,t) = B(t) \end{cases}$$

6.1 Theorem For any $T > 0$ there is atmost one solution $u \in C^2([a,b] \times [0,T])$ of (6.1) satisfying the initial conditions (IC) as well as the lateral boundary conditions (DD), (ND), (DN) or (NN).

Proof Assuming the existence of two different solutions we obtain, by subtraction, the existence of a nontrivial solution u with boundary conditions given by $A(t) \equiv B(t) \equiv 0$, and $\alpha(x) \equiv \beta(x) \equiv 0$. Define the "energy" integral

$$E(t) = \frac{1}{2} \int_a^b \{c^2 u_x^2 + u_t^2\} dx.$$

Then for all $t \geq 0$,

$$\frac{dE}{dt}(t) = \int_a^b \{c^2 u_x u_{xt} + u_t u_{tt}\} dx = \int_a^b \{c^2 u_x u_{xt} + u_t c^2 u_{xx}\} dx$$

$$= c^2 \int_a^b \frac{\partial}{\partial x} (u_x u_t) \, dx = c^2 \, [u_x u_t]_{x=a}^{x=b} = 0.$$

Thus $E(t) \equiv E(0) = 0$, so that $u \equiv 0$.
Contradiction, because we assumed u to be nontrivial. ∎

For the construction of solutions we use the following lemma.

6.2 Lemma Let $u \in C^2(\vartheta)$ for some open subset ϑ of $\mathbb{R} \times \mathbb{R}$. Then u is a solution of $u_{tt} = u_{xx}$ in ϑ, if and only if u satisfies the difference equation

$$u(x-k, t-h) + u(x+k, t+h) = \quad u(x-h, t-k) + u(x+h, t+k)$$

for all x, t, k, h such that the rectangle R with vertices $A = (x-k, t-h)$, $B = (x+h, t+k)$, $C = (x+k, t+h)$, and $D = (x-h, t-k)$ is contained in ϑ. (R is called a characteristic rectangle, because its boundary consists of characteristics.)

Proof Suppose u solves $u_{tt} = u_{xx}$. Then u is of the form $u(x,t) = f(x+t) + g(x-t)$. Since

$$f(A) + f(C) = f(x+t-h-k) + f(x+t+h+k) = f(B) + f(D),$$

and

$$g(A) + g(C) = g(x-k-t+h) + g(x+k-t-h) = g(B) + g(D),$$

it follows that $u(A) + u(C) = u(B) + u(D)$.

Conversely, suppose u satisfies the difference equation for all characteristics rectangles $R \subset \vartheta$. Put $h = 0$, then

$$\frac{u\ (x-k,t) - 2u(x,t) + u(x+k,t)}{k^2} = \frac{u\ (x,t-k) - 2u(x,t) + u(x,t+k)}{k^2}.$$

Using Taylor's theorem with respect to the variable k in the numerators, we obtain, as $k \to 0$, that $u_{tt} = u_{xx}$. This completes the proof of the lemma. ∎

With this lemma we can obtain a solution of the inhomogeneous wave equation satisfying initial conditions (IC) and lateral boundary conditions (DD).

6.3 Theorem Let $\alpha \in C^2([a,b])$, $\beta \in C^1([a,b])$, $A, B \in C^2([0,\infty])$, $\varphi, \varphi_x \in C([a,b]) \times [0,\infty])$, and suppose that the following six compatibility conditions are satisfied:

$$A''(0) = c^2\alpha''(a) + \varphi(a,0)\ ;\ \alpha(a) = A(0)\ ;\ A'(0) = \beta(a)\ ;$$

$$B''(0) = c^2\alpha''(b) + \varphi(b,0)\ ;\ \alpha(b) = B(0)\ ;\ B'(0) = \beta(b).$$

Then the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = \varphi & a < x < b,\ t > 0; \\ u(a,t) = A(t)\ ;\ u(b,t) = B(t) & t > 0; \\ u(x,0) = \alpha(x)\ ;\ u_t(x,0) = \beta(x) & a \le x \le b, \end{cases}$$

has a unique solution $u \in C^2([a,b] \times [0,\infty))$.

Proof It suffices to prove existence. First we reduce the problem to the case $\varphi \equiv 0$. To do so we observe that we may assume that φ and φ_x belong to $C(\mathbb{R} \times [0,\infty))$ by setting

$$\varphi(x,t) = \varphi(b,t) + \varphi_x(b,t)(x-b) \text{ for } x \ge b,$$

and

$$\varphi(x,t) = \varphi(a,t) + \varphi_x(a,t)(x-a) \text{ for } x \le a.$$

We also assume without loss of generality that $c = 1$. Taking the difference between the unknown function $u(x,t)$ and

$$\frac{1}{2}\int\int_{G(x,t)} \varphi(\xi,\tau)d\xi\, d\tau,$$

and renaming this difference u again, we obtain a new problem, with new functions A, B, α and β, and with $\varphi = 0$, satisfying the same regularity and compatibility conditions.

We construct a solution for $0 < t \le b-a$. The square $[a,b]\times[0,b-a]$ if subdivided by its diagonals into four triangles, which we number counterclockwise starting at the bottom as I, II, III and IV. To compute u in I, we use the formula

$$u(x,t) = \frac{1}{2}\big\{\alpha(x+t) + \alpha(x-t)\big\} + \frac{1}{2}\int_{x-t}^{x+t} \beta(s)ds.$$

We then define u for every (x,t) in II and IV using the difference equation in Lemma 6.2 for characteristic rectangles with two vertices contained in I, one on the lateral boundary, and the last one at (x,t). Then with u being determined for every point in II and IV, we extend u to III using the difference equation again, now applied to characteristic rectangles with one vertex in each triangle. This defines a function u on $[a,b]\times[0,b-a]$.

Repeating the construction on $[a,b]\times[b-a,2(b-a)]$, etc., we obtain the value of $u(x,t)$ for every (x,t) in $(a,b)\times(0,\infty)$. We claim that $u \in C^2([a,b]\times[0,\infty])$, and that $u_{tt} = u_{xx}$. Clearly, because of the previous results it suffices to establish $u \in C^2([a,b]\times[0,\infty])$ This is left as an exercise. ∎

7. The fundamental solution in one space dimension

We have seen that under appropriate conditions on α, β and φ, the solution of

$$(CP_i)\begin{cases} u_{tt} - u_{xx} = \varphi(x,t) & x,t \in \mathbb{R};\\ u(x,0) = \alpha(x) & x \in \mathbb{R};\\ u_t(x,0) = \beta(x) & x \in \mathbb{R},\end{cases}$$

is given by

$$u(x,t) = u_\alpha(x,t) + u_\beta(x,t) + u_p(x,t), \tag{7.1}$$

where

$$u_\alpha(x,t) = \frac{1}{2}\alpha(x+t) + \frac{1}{2}\alpha(x-t);\quad u_\beta(x,t) = \frac{1}{2}\int_{x-t}^{x+t}\beta(s)ds;$$

$$u_p(x,t) = \frac{1}{2} \int\int_{G(x,t)} \varphi(\xi,\tau) d\xi \; d\tau; \quad G(x,t) = \{(\xi,\tau), 0 \leq \tau \leq t, \; |\xi - x| \leq t - \tau\}.$$

Note that u_α is the solution of $u_{tt} = u_{xx}$ with $u(x,0) = \alpha(x)$ and $u_t(x,0) \equiv 0$, u_β of $u_{tt} = u_{xx}$ with $u(x,0) \equiv 0$ and $u_t(x,0) = \beta(x)$, and u_p of $u_{tt} = u_{xx} + \varphi$ with $u(x,0) \equiv u_t(x,0) \equiv 0$. In fact these three different functions are constructed by means of one (*fundamental*) solution. To see this we have to make a small detour into the theory of distributions.

As an example we consider first the so-called *Heaviside function*:

$$H(s) = \begin{cases} 0 & s < 0; \\ 1 & s > 0. \end{cases}$$

If we look at H as an element of $L^1_{loc}(\mathbb{R})$, $H(0)$ need not be defined. If we look at H as a "*maximal monotone graph*" , we must set $H(0) = [0,1]$. We cannot differentiate H in the class of functions, but we can in the class of distributions. The "*testfunctionspace*" is defined by

$$D(\mathbb{R}) = \{\psi \in C^\infty(\mathbb{R}); \psi \text{ has compact support}\}.$$

We say that for $\psi_n, n = 1, 2...$, and ψ in $D(\mathbb{R})$,

$$\psi_n \to \psi \quad \text{as} \quad n \to \infty \quad \text{in} \quad D(\mathbb{R}),$$

if the supports of $\psi_n^{(k)}$ are uniformly bounded, and if $\psi_n^{(k)} \to \psi^{(k)}$ uniformly on $\mathbb{R}$ for all $k = 0, 1, 2, ...$

7.1 Definition A linear functional $T : D(\mathbb{R}) \to \mathbb{R}$ is called a distribution if $\psi_n \to \psi$ in $D(\mathbb{R})$ implies that $T\psi_n \to T\psi$.

Every $\varphi \in L^1_{loc}(\mathbb{R})$ defines a distribution

$$T_\varphi(\psi) =< \varphi, \psi >= \int_{-\infty}^{\infty} \varphi\psi. \tag{7.2}$$

Now suppose we take for φ a smooth function. Then

$$T_{\varphi'}(\psi) =< \varphi', \psi >= \int_{-\infty}^{\infty} \varphi'\psi = -\int_{-\infty}^{\infty} \varphi\psi' = - < \varphi, \psi' >= -T_\varphi(\psi'). \tag{7.3}$$

In view of this property, the following definition is natural.

7.2 Definition Let $T : D(\mathbb{R}) \to \mathbb{R}$ be a distribution. Define $T' : D(\mathbb{R}) \to \mathbb{R}$ by $T'(\psi) = -T(\psi')$. Then T' is called the *distributional derivative* of T. Note that T' is again a distribution.

7.3 Example Let H be the Heaviside function. Then

$$T_H(\psi) =< H, \psi >= \int_{-\infty}^{\infty} H(s)\psi(s)ds = \int_{0}^{\infty} \psi(s)ds,$$

and

$$(T_H)'(\psi) =< H', \psi >= -\int_{-\infty}^{\infty} H(s)\psi'(s)ds = -\int_{0}^{\infty} \psi'(s)ds = \psi(0).$$

We introduce the *Dirac delta distribution* $\delta = \delta(x)$ by

$$< \delta, \psi >= \int_{-\infty}^{\infty} \delta(x)\psi(x)dx = \psi(0). \tag{7.4}$$

Clearly δ is the distributional derivative of H. Intuitively, δ is a function with

$$\delta(x) = 0 \text{ for } x \neq 0; \ \delta(0) = +\infty; \ \int_{-\infty}^{\infty} \delta(x)dx = 1,$$

but one should always remember that mathematically speaking, δ is not a function. A better and correct way is to say that δ is a measure which assigns the value one to any set containing zero.

Returning to u_β we have that, for $t \geq 0$

$$u_\beta(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \beta(s)ds = \int_{-\infty}^{\infty} \frac{1}{2}H(x+t-s)H(s-x+t)\beta(s)ds =$$

$$\int_{-\infty}^{\infty} E^+(x-s,t)\beta(s)ds,$$

where

$$E^+(x,t) = \frac{1}{2}H(t+x)H(t-x), \ x \in \mathbb{R}, \quad t \geq 0.$$

We extend E^+ to the whole of $\mathbb{R}^2$ by setting $E^+(x,t) = 0$ for $t \leq 0$. Note that we can also write

$$E^+(x,t) = \frac{1}{2}H(t)\{H(x+t) - H(x-t)\}, \tag{7.5}$$

and that supp $E^+ \subset \mathbb{R} \times \bar{\mathbb{R}}^+$. Extending the definitions of distributions and their derivatives in the obvious way from $\mathbb{R}$ to $\mathbb{R}^2$, and in particular defining the Dirac distribution in $\mathbb{R} \times \mathbb{R}$ by

$$< \delta, \psi >= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,t)\psi(x,t)dx = \psi(0,0), \tag{7.6}$$

we claim that

$$E^+_{tt} - E^+_{xx} = \delta(x,t) = \delta(x)\delta(t) \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{7.7}$$

To see this, let ψ be any smooth function with compact support in $\mathbb{R}\times\mathbb{R}$, i.e. $\psi \in D(\mathbb{R}\times\mathbb{R})$, and let γ be the boundary of the triangle $\{(x,t): \ -t < x < t, \ 0 < t < T\}$, where T is so large that the support of ψ is contained in $\{t < T\}$. Then

$$< E^+_{tt} - E^+_{xx}, \psi >= \int\int E^+(x,t)(\psi_{tt} - \psi_{xx})dxdt =$$

$$\frac{1}{2}\int\int_{-t\le x\le t}(\psi_{tt}-\psi_{xx})dxdt = \frac{1}{2}\int\int_{-t\le x\le t}\frac{\partial}{\partial x}(-\psi_x) + \frac{\partial}{\partial t}(\psi_t)dxdt$$

$$= \frac{1}{2}\int\int_{-t\le x\le t} \operatorname{div}\begin{pmatrix}-\psi_x\\ \psi_t\end{pmatrix} dxdt = \frac{1}{2}\oint_\gamma \begin{pmatrix}-\psi_x\\ \psi_t\end{pmatrix}\cdot \nu ds =$$

$$-\frac{1}{2}\oint_\gamma \psi_x \ dt + \psi_t dx = \psi(0,0) =< \delta, \psi > .$$

Next we compute, as distributions on $\mathbb{R}$, for $t > 0$,

$$< E^+(\cdot,t), \psi >= \int_{-\infty}^{\infty} E^+(x,t)\psi(x)\ dx = \int_{-t}^{t}\psi(x)dx,$$

for all $\psi \in D(\mathbb{R})$. Clearly, $< E^+(\cdot,t), \psi > \to 0$ as $t \downarrow 0$. In view of the following definition we say that $E^+(\cdot,t) \to 0$ as $t \downarrow 0$ in the class of distributions on $\mathbb{R}$.

7.4 Definition Let $T_n, n = 1,2,...,$ and T be distributions on an open set $\Omega \subset \mathbb{R}^n$. We say that $T_n \to T$ if $T_n\psi \to T\psi$ for all $\psi \in D(\Omega)$.

Finally we look at E^+_t. Again let $\psi \in D(\mathbb{R}\times\mathbb{R})$. Then

$$< E^+_t, \psi >= - < E^+, \psi_t >= -\frac{1}{2}\int\int_{-t\le x\le t}\psi_t(x,t)dxdt =$$

$$\frac{1}{2}\int_0^\infty \psi(x,x)\ dx + \frac{1}{2}\int_{-\infty}^0 \psi(x,-x)\ dx = \frac{1}{2}\int_0^\infty(\psi(t,t)+\psi(-t,t))\ dt$$

$$= \int_0^\infty < \frac{1}{2}(\delta(x-t)+\delta(x+t)), \psi(x,t) > \ dt.$$

Here we have used the notation

$$< \delta(\cdot \pm t), \psi >= \int \delta(x\pm t)\psi(x)\ dx = \psi(\mp t).$$

Symbolically we write for $t > 0$,

$$E_t^+(x,t) = \frac{1}{2}\delta(x+t) + \frac{1}{2}\delta(x-t). \tag{7.8}$$

Consequently, for $\psi \in D(\mathbb{R})$,

$$< E_t^+(\cdot,t), \psi >= \frac{1}{2}\psi(-t) + \frac{1}{2}\psi(t) \to \psi(0) =< \delta(x), \psi(x) >$$

as $t \downarrow 0$, i.e. $E_t^+(\cdot,t) \to \delta \quad \text{as} \quad t \downarrow 0$.

7.5 Definition E^+ is called the *fundamental solution* of $u_{tt} = u_{xx}$. Its support, the set $\{|x| \le t\}$ is called the *forward light cone*.

7.6 Remark The derivation of the formula above for E_t^+ is formal, but can be made mathematically rigourous, if one considers δ as a measure.

7.7 Definition For $f, g : \mathbb{R} \to \mathbb{R}$ the convolution of f and g is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-s)g(s)ds,$$

whenever this integral exists.

Now recall that for $t > 0$

$$u_\beta(x,t) = \int_{-\infty}^{\infty} E^+(x-s,t)\beta(s)ds, \tag{7.9}$$

i.e. $u_\beta(\cdot,t)$ is the convolution of $E^+(\cdot,t)$ and β.

Next we consider u_α. For $t > 0$ we have

$$u_\alpha(x,t) = \frac{1}{2}\alpha(x+t) + \frac{1}{2}\alpha(x-t) = \int_{-\infty}^{\infty} \frac{1}{2}(\delta(x-s+t) + \delta(x-s-t))\alpha(s)ds$$

$$= \int_{-\infty}^{\infty} E_t^+(x-s,t)\alpha(s)ds,$$

so that formally u_α is the convolution of $E_t^+(\cdot,t)$ and α.

Finally we look at u_p. We have for $t > 0$

$$u_p(x,t) = \frac{1}{2}\int\int_{G(x,t)} \psi(\xi,\tau)d\xi\ d\tau = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} \psi(\xi,\tau)d\xi\ d\tau$$

$$= \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} H\ (\xi - x + t - \tau)\ H\ (x + t - \tau - \xi)\ \psi\ (\xi, \tau)\ d\xi\ d\tau$$

$$= \int_0^t \int_{-\infty}^{\infty} E^+\ (x - \xi, t - \tau)\ \psi\ (\xi, \tau)\ d\xi\ d\tau.$$

Now this is the convolution of E^+ and ψ with respect to both variables in $\mathbb{R} \times \mathbb{R}^+$. Summarizing we have for $t > 0$

$$u_\alpha = E_t^+(\cdot, t) * \alpha \quad \text{and} \quad u_\beta = E^+(\cdot, t) * \beta \quad \text{(convolution in } x\text{);}$$

$$u_p = E^+ * \varphi \quad \text{(convolution in } x \text{ and } t\text{).}$$

8. The fundamental solution in three and two space dimensions

For the wave equation in one dimension, we have constructed the fundamental solution

$$E^+(x, t) = \frac{1}{2}\ H(t)\{H(x + t) - H(x - t)\},$$

which was a distributional solution on $\mathbb{R} \times \mathbb{R}$ of $u_{tt} - u_{xx} = \delta(x, t) = \delta(x)\delta(t)$, with support contained in $\mathbb{R} \times [0, \infty)$.

Next we turn to the 3-dimensional case and try to find the analog of E^+. Thus we try to find a distribution in $\mathbb{R}^3 \times \mathbb{R}$ with support contained in $\{t \geq 0\}$, satisfying

$$u_{tt} - \Delta u = \delta(x_1, x_2, x_3, t) = \delta(x_1)\delta(x_2)\delta(x_3)\delta(t). \tag{8.1}$$

We shall first obtain a solution by formal methods, and then give a rigorous proof.

Because of the radial symmetry in this problem, we look for a solution of the form $u = u(r, t)$. For $t > 0$ this implies

$$u_{tt} = u_{rr} + \frac{2}{r} u_r,$$

or (this trick only works for $N = 3$)

$$(ru)_{tt} = (ru)_{rr}.$$

As in the one dimensional case we conclude that

$$ru(r, t) = v(t - r) + w(t + r).$$

Because the second term reflects signals coming inwards, we neglect it. Thus we consider

$$u(r,t) = \frac{v(t-r)}{r}.$$

Tracing "characteristics" of the form $t - r = c$ backwards in time, we conclude that $v(c) = 0$ if $c \neq 0$. These considerations suggest that $v(t-r) = \delta(t-r)$ (up to a constant).

8.1 Theorem The fundamental solution of the wave equation in $\mathbb{R}^3 \times \mathbb{R}$, i.e. the solution of (8.1) with support in $\mathbb{R}^3 \times [0,\infty]$, is given by

$$E^+(x_1,x_2,x_3,t) = \frac{\delta(t-r)}{4\pi r},$$

which we define as a distribution below.

In order to define E^+ as a distribution, we first compute formally what $< E^+, \psi >$ would be for $\psi \in D(\mathbb{R}^3 \times \mathbb{R})$, using the "rule"

$$\int \varphi(s)\ \delta(t-s)ds = \varphi(t).$$

Thus we evaluate $< E^+, \psi >$ using polar coordinates

$$x_1 = r\sin\theta\cos\varphi; \quad x_2 = r\sin\theta\sin\varphi; \quad x_3 = r\cos\theta.$$

Then

$$< E^+, \psi >=$$

$$\int_{-\infty}^{\infty}\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\delta(t-r)}{4\pi r}\psi(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta, t)\ r^2\sin\theta\ drd\varphi d\theta dt$$

$$= \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{1}{4\pi t}\ \psi(t\sin\theta\cos\varphi, t\sin\theta\sin\varphi, t\cos\theta, t)t^2\sin\theta d\varphi d\theta dt =$$

$$\int_0^{\infty} \frac{1}{4\pi t}\oint_{x_1^2+x_2^2+x_3^2=t^2} \psi(x_1,x_2,x_3,t)\ dS\ dt,$$

and we use this final expression as a definition of E^+.

8.2 Definition We define the distribution E^+ on $\mathbb{R}^3 \times \mathbb{R}$ by

$$< E^+, \psi >= \int_0^{\infty} \frac{1}{4\pi t}\oint_{x_1^2+x_2^2+x_3^2=t^2} \psi(x_1,x_2,x_3,t)\ dS(x_1,x_2,x_3)\ dt$$

for all $\psi \in D(\mathbb{R}^3 \times \mathbb{R})$. We also define $E^+(\cdot,\cdot,\cdot,t)$ as a distribution on $\mathbb{R}^3$ by

$$< E^+(t), \psi >= \frac{1}{4\pi t}\oint_{x_1^2+x_2^2+x_3^2=t^2} \psi(x_1,x_2,x_3)\ dS(x_1,x_2,x_3).$$

Next we prove that E^+ is a fundamental solution.

8.3 Lemma E^+ satisfies $E^+_{tt} - \Delta E^+ = \delta(x_1, x_2, x_3, t)$ in $\mathbb{R}^3 \times \mathbb{R}$.

Proof Let $\psi \in D(\mathbb{R}^3 \times \mathbb{R})$. Since $< \delta(x_1, x_2, x_3, t),\ \psi(x_1, x_3, x_3, t) >= \psi(0,0,0,0)$, and $< E^+_{tt} - \Delta E^+, \psi >=< E^+, \psi_{tt} - \Delta\psi >$, we have to show that $< E^+, \psi_{tt} - \Delta\psi >= \psi(0,0,0,0)$. Again we use polar coordinates. We have

$$\Delta\psi = \frac{1}{r^2}(r^2\psi_r)_r + \frac{1}{r^2\sin\theta}(\sin\theta\psi_\theta)_\theta + \frac{1}{r^2\sin^2\theta}\quad \psi_{\varphi\varphi},$$

so that

$$< E^+, \psi_{tt} - \Delta\psi >=$$

$$\int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{1}{4\pi t}\big[\psi_{tt} - \frac{1}{r^2}\{(r^2\psi_r)_r - \frac{1}{\sin\theta}(\sin\theta\psi_\theta)_\theta - \frac{1}{\sin^2\theta}\psi_{\varphi\varphi}\}\big]_{r=t} t^2\sin\theta d\varphi d\theta dt$$

$$= \int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{1}{4\pi}\ \big[(r\psi)_{tt} - (r\psi)_{rr}\big]_{r=t}\sin\theta d\varphi d\theta dt$$

$$-\int_0^\infty\int_0^\pi\int_0^{2\pi}(\sin\theta\ \psi_\theta)_\theta d\varphi d\theta dt - \int_0^\infty\int_0^\pi\int_0^{2\pi}\ \frac{\psi_{\varphi\varphi}}{\sin\theta}\ d\varphi d\theta dt.$$

Obviously, the last two integrals are zero, so if γ is the curve $\{r = t > 0\}$ in the (r,t)- plane (along which we have $dr = dt$), then

$$< E^+_{tt} - \Delta E^+, \psi >= \int_0^\infty\int_0^\pi\int_0^{2\pi} \frac{1}{4\pi}\big[(r\psi)_{tt} - (r\psi)_{rr}\big]_{r=t}\sin\theta d\varphi d\theta dt =$$

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\int_\gamma\{(r\psi)_{tt} - (r\psi)_{rr}\}\ dt\ d\varphi\ d\theta =$$

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\int_\gamma\{(r\psi)_{tt}dt + (r\psi)_{tr}dr - (r\psi)_{rr}dr - (r\psi)_{rt}dt\}d\varphi\ d\theta =$$

(since ψ has compact support)

$$\frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\sin\theta\big[(r\psi)_r - (r\psi)_t\big]_{r=t=0}d\varphi\ d\theta = \psi(0,0,0,0).$$

This completes the proof. ▪

Formally now, the solution of the equation $u_{tt} - \Delta u = \varphi(x_1, x_2, x_3, t)$ in $\mathbb{R}\times[0,\infty]$ with $u = u_t \equiv 0$ for $t < 0$, should be obtained by taking the convolution of E^+ and φ with respect to all variables, just like in the one-dimensional case. However, here E^+ is no longer a function, so the definition of this convolution is not entirely obvious. We shall restrict ourselves here to the formal computation. Then, with $(x,y,z) = (x_1,x_2,x_3)$, we have for $t > 0$,

$$u(x,y,x,t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x-\xi, y-\eta, z-\zeta, t-\tau)\varphi(\xi,\eta,\zeta,\tau)d\xi d\eta d\zeta d\tau =$$

(writing $P = (x,y,z), Q = (\xi,\eta,\zeta)$, and $r_{PQ} = \sqrt{(x-\xi)^2 + (y-\eta^2) + (z-\zeta)^2)}$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^t \frac{\delta(t-r-r_{PQ})}{4\pi r_{PQ}} \varphi(\xi,\eta,\zeta,\tau)d\tau \; d\xi d\eta d\zeta =$$

(using the "rule" $\int \varphi(\tau) \; \delta(s-t)ds = \varphi(t)$)

$$\frac{1}{4\pi} \int\int\int_{r_{PQ}\leq t} \frac{\varphi(\xi,\eta,\zeta,t-r_{PQ})}{r_{PQ}} d\xi d\eta d\zeta =$$

$$\int\int\int_{G(x,y,z,t)} \frac{\varphi(\xi,\eta,\zeta,t-\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta^2)}}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} d\xi d\eta d\zeta,$$

where

$$G(x,y,z,t) = \{(\xi,\eta,\zeta,\tau) : (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \leq t^2\}.$$

Next we treat (only formally) some special cases.

8.4 Example Consider

$$\varphi(x,y,z,t) = \delta(x)\delta(y)\delta(z)f(t).$$

We find that $u(x,y,z,t) =$

$$\int\int\int_{G(x,y,z,t)} \frac{\delta(\xi)\delta(\eta)\delta(\zeta)f(t-\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta^2)})}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} d\xi d\eta d\zeta$$

$$= \frac{f(t-r)}{4\pi r}.$$

8.5 Example Consider $\varphi(x,y,z,t) = \delta(x)\delta(y)f(t)$. Then $u(x,y,z,t) =$

$$\int\int\int_{G(x,y,z,t)} \frac{\delta(\xi)\delta(\eta)f(t-\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta^2)})}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} d\xi d\eta d\zeta$$

$$= \frac{1}{2\pi} \int_{x^2+y^2+(z-\zeta)^2\leq t^2, \; z-\zeta\geq 0} \frac{f(t-\sqrt{x^2+y^2+(z-\zeta^2)}}{\sqrt{x^2+y^2+(z-\zeta)^2}} \; d\zeta$$

$$= \frac{1}{2\pi}\int_0^{t-r} \frac{f(\tau)d\tau}{\sqrt{(t-\tau)^2-r^2}}.$$

(here $r = \sqrt{x^2+y^2}$, $\tau = t - \sqrt{x^2+y^2+(z-\zeta)^2}$, $d\tau = \frac{-\zeta+z}{\sqrt{x^2+y^2+(z-\zeta)^2}}\ d\zeta$, $(z-\zeta)^2 = (t-\tau)^2 - r^2$)

8.6 Example Consider $\varphi(x,y,z,t) = \delta(x)\delta(y)\delta(t)$ (or $f(t) = \delta(t)$ in the last example), then

$$u(x,y,z,t) = \frac{1}{2\pi}\int_0^{t-r} \frac{\delta(\tau)}{\sqrt{(t-\tau)^2-r^2}}\ \ dt = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}$$

Note however that this last expression is independent of t, so we have found the fundamental solution for the wave equation in two dimensions.

8.7 Proposition Let $E^+(x,y,t)$be given by

$$E^+(x,y,t) = \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}$$

Then E^+ is the fundamental solution of the wave equation in two dimensions, i.e. E^+ has support in $\{t \geq 0\}$ and satisfies $E^+_{tt} - E^+_{xx} - E^+_{yy} = \delta(x,y,t) = \delta(x,y,t)$ on $\mathbb{R}^2 \times \mathbb{R}$ in the sense of distributions.

Proof First note that E^+ is now a function. We have to show that $< E^+_{tt} - E^+_{xx} - E^+_{yy}, \psi > = < E^+, \psi_{tt} - \psi_{xx} - \psi_{yy} > = \psi(0,0,0)$ for all $\psi \in D(\mathbb{R}^2 \times \mathbb{R})$. To do so we introduce polar coordinates on $\mathbb{R}^2$, $x = r\cos\varphi$; $y = r\sin\varphi$. Then

$$\Delta\psi = \psi_{xx} + \psi_{yy} = \frac{1}{r}(r\psi_r)_r + \frac{1}{r^2}\psi_{\varphi\varphi}.$$

Thus

$$< E^+, \psi_{tt} - \Delta\psi > = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\frac{H(t-r)}{\sqrt{t^2-r^2}}\ \ (\psi_{tt} - \Delta\psi)\ \ dx\ dy\ dt$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{t} \frac{\psi_{tt} - r^{-1}(r\psi r)_r - r^{-2}\psi_{\varphi\varphi}}{2\pi\sqrt{t^2-r^2}}\ r\ dr\ d\varphi\ dt$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{t} \frac{r\psi_{tt} - (t\psi_r)_r}{2\pi\sqrt{t^2-r^2}}\ dr\ d\varphi\ dt = \frac{1}{2\pi}\int_0^{2\pi} J(\varphi)\ d\varphi,$$

where

$$J(\varphi) = \int_0^{\infty}\int_0^{t} \frac{r\psi_{tt} - (r\psi_r)_r}{\sqrt{t^2-r^2}}\ dr\ dt =$$

(if supp$\psi \subset \{t \leq T\}$)

$$\int_0^T \int_0^t \frac{r\psi_{rr} - (r\psi_r)_r}{\sqrt{t^2 - r^2}} \; dr \; dt = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^T \int_\varepsilon^t \frac{r\psi_{rr} - (r\psi_r)_r}{\sqrt{t^2 - r^2}} \; dr \; dt =$$

(using the transformation $x = r$, $y = t/r$)

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^T \int_1^{T/x} \{ \frac{-(\psi_y \sqrt{y^2-1})_y}{x} - \frac{(x\psi_x)_x}{\sqrt{y^2-1}} + \frac{2y\psi_{xy}}{\sqrt{y^2-1}} \} \; dy \; dx$$

(here we have used

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x} - \frac{y}{x}\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{1}{x}\,\frac{\partial}{\partial y},$$

to transform the derivatives, and $drdt = xdxdy$)

$$= \lim_{\varepsilon \downarrow 0} \{ \int_\varepsilon^T [-\frac{\psi_y \sqrt{y^2-1}}{x}]_{y=1}^{y=T/x} dx + \int_1^{T/\varepsilon} [\frac{2y\psi_y - x\psi_x}{\sqrt{y^2-1}}]_{x=\varepsilon}^{x=T/y} dy \}$$

$$= \lim_{\varepsilon \downarrow 0} \int_1^{T/\varepsilon} \frac{x\psi_x - 2y\psi_y}{\sqrt{y^2-1}} \; \Big|_{x=\varepsilon} \; dy =$$

(transforming the x- and y-derivatives back to r- and t-derivatives)

$$\lim_{\varepsilon \downarrow 0} \int_1^{T/\varepsilon} \frac{\varepsilon\psi_r - \varepsilon y \psi_t}{\sqrt{y^2-1}} \Big|_{x=\varepsilon} dy =$$

(writing $\psi(r, \varphi, t)$)

$$\lim_{\varepsilon \downarrow 0} \int_1^{T/\varepsilon} \frac{\varepsilon\psi_r(\varepsilon, \varphi, \varepsilon y) - \varepsilon y \psi_t(\varepsilon, \varphi, t)}{\sqrt{y^2-1}} \; dy =$$

(substituting $t = \varepsilon y$)

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^T \frac{\varepsilon\psi_r(\varepsilon, \varphi, t) - t\psi_t(\varepsilon, \varphi, t)}{\sqrt{t^2 - \varepsilon^2}} \; dt =$$

$$\lim_{\varepsilon \downarrow 0} \int_0^{T-\varepsilon} \{ \frac{\varepsilon\psi_r(\varepsilon, \varphi, t+\varepsilon)}{\sqrt{(t+2\varepsilon)t}} - \frac{t+\varepsilon}{\sqrt{(t+2\varepsilon)t}} \; \psi_t(\varepsilon, \varphi, t+\varepsilon) \} \; dt$$

$$= -\int_0^T \psi_t(0, \varphi, t) dt = \psi(0, \varphi, 0),$$

which completes the proof. ▪

PART 3: THE HEAT EQUATION

9. The fundamental solution of the heat equation in dimension one

As a first example we consider the problem

$$(P)\begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = H(x) & x \in \mathbb{R}, \end{cases}$$

where H is the Heaviside function. Now observe that if $u(x,t)$ is a solution of (P), then $u_a(x,t) = u(ax, a^2t)$ is also a solution of (P). Since we expect the solution to be unique, we should have

$$u(ax, a^2t) = u(x,t), \tag{9.1}$$

for all $a > 0$, $x \in \mathbb{R}$, $t > 0$. Thus if we put $a = 1/\sqrt{t}$, we obtain

$$u(x,t) = u(\frac{x}{\sqrt{t}}, 1) = U(\eta); \quad \eta = \frac{x}{\sqrt{t}}. \tag{9.2}$$

Here η is called the *similarity variable*. From (9.2) it follows that

$$u_t = U'(\eta)\frac{\partial \eta}{\partial t} = U'(\eta)\frac{x}{2t\sqrt{t}} = -\frac{\eta U'(\eta)}{2t}; \quad u_{xx} = \frac{U''}{t},$$

so that $u(x,t) = U(\eta)$ is a solution of the heat equation if

$$U''(\eta) + \eta U'(\eta)/2 = 0, \tag{9.3}$$

or

$$(e^{\eta^2/4}U'(\eta))' = 0.$$

Thus

$$e^{\eta^2/4}U'(\eta) = \text{ constant } = A,$$

and

$$U(\eta) = B + A\int_{-\infty}^{\eta} e^{-s^2/4}\, ds = B + 2A\int_{-\infty}^{\eta/2} e^{-y^2} dy.$$

Since for $x < 0$,

$$0 = u(x,0) = \lim_{t\downarrow 0} U(\frac{x}{\sqrt{t}}) = U(-\infty) = B,$$

and for $x > 0$,

$$1 = u(x,0) = \lim_{t\downarrow 0} U(\frac{x}{\sqrt{t}}) = U(+\infty) = B + 2A\int_{-\infty}^{\infty} e^{-\eta^2} d\eta = 2A\sqrt{\pi},$$

we find that

$$U(\eta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\eta/2} e^{-s^2} ds; u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/2\sqrt{t}} e^{-s^2} ds. \tag{9.4}$$

We make the following observations.

(i) $u(x,t)$ is *smooth* for $t > 0$, but not at $t = 0$,

(ii) $\lim_{t \downarrow 0} u(x,t) = \begin{cases} 0 & \text{x} < 0 \\ 1/2 & \text{x} = 0 \\ 1 & \text{x} > 0 \end{cases}$,

(iii) $0 = \min_{x\in\mathbb{R}} \; u(x,0) < u(x,t) < \max_{x\in\mathbb{R}} \; u(x,0) = 1$ (i.e a *strong comparison principle* seems to hold),

(iv) The positivity of u on $\mathbb{R}^+$ for $t = 0$ causes u to become positive immediately for $t > 0$ on the whole of $\mathbb{R}$ (*infinite speed of propagation*, in sharp constrast with the finite speed of propagation for the wave equation),

(v) $u(x,t) = U(x/\sqrt{t})$ is a self similar solution (or *similarity solution*).

Next we compute the solution of the heat equation with the initial value

$$u(x,0) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}.$$

Naturally we obtain

$$u_a(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{(x-a)/2\sqrt{t}} e^{-s^2} ds,$$

so that the solution with initial conditions

$$w(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < a \\ 0 & x > a \end{cases},$$

is given by

$$w(x,t) = u(x,t) - u_a(x,t) = \frac{1}{\sqrt{\pi}} \int_{(x-a)/2\sqrt{t}}^{x/2\sqrt{t}} e^{-s^2} ds,$$

which obviously satisfies

$$|w(x,t)| = |u(x,t) - u_a(x,t)| < \frac{a}{2\sqrt{\pi t}}.$$

Thus $w(x,t) \to 0$ as $t \to \infty$, the *decay order* being $1/\sqrt{t}$. Note that $w(x,0)$ is a bounded integrable function.

Going back to the solution with $u(x,0) = H(x)$, which is given by (9.4), we differentiate it with respect to t, to obtain a new solution

$$E^+(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}. \tag{9.5}$$

Obviously E^+ satisfies $E^+_t = E^+_{xx}$ for $t > 0$, and it is in fact the *fundamental solution for the heat equation*, that is, extending E^+ by $E^+(x,t) = 0$ for $t < 0$, we have

9.1 Proposition The function E^+ satisfies the fundamental equation $E^+_t - E^+_{xx} = \delta(x,t) = \delta(x)\delta(t)$ in $\mathbb{R}^2$.

Proof To check that E^+ is indeed a fundamental solution, we let $\psi \in D(\mathbb{R} \times \mathbb{R})$ and compute

$$< E^+_t - E^+_{xx}, \psi >= - < E^+, \psi_t + \psi_{xx} >= - \int\int_{\mathbb{R}\times\mathbb{R}^+} E^+(\psi_t + \psi_{xx}) d(x,t) =$$

$$- \lim_{\varepsilon \downarrow 0} \int\int_{\mathbb{R}\times(\varepsilon,\infty)} E^+(\psi_t + \psi_{xx}) d(x,t) =$$

$$- \lim_{\varepsilon \downarrow 0} \Big\{ \int_{-\infty}^{\infty} \int_{\varepsilon}^{\infty} E^+ \psi_t dt dx + \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} E^+ \psi_{xx} dx dt \Big\} =$$

$$- \lim_{\varepsilon \downarrow 0} \Big\{ \int_{-\infty}^{\infty} [E^+\psi]_{t=\varepsilon}^{t=\infty} dx - \int_{-\infty}^{\infty} \int_{\varepsilon}^{\infty} E^+_t \psi dt dx + \int_{\varepsilon}^{\infty} \int_{-\infty}^{\infty} E^+_{xx} \psi dx dt \Big\} =$$

$$\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} E^+(x,\varepsilon)\psi(x,\varepsilon) dx = \lim_{t \downarrow 0} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \psi(x,t) dx.$$

To complete the proof we have to show that this limit equals $\psi(0,0)$.

For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x| < \delta$ and $t < \delta$ then $|\psi(x,t) - \psi(0,0)| < \varepsilon$. Thus

$$\Big| \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \psi(x,t) dx - \psi(0,0) \Big| = \Big| \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} (\psi(x,t) - \psi(0,0)) dx \Big|$$

$$\leq \varepsilon \int_{-\delta}^{\delta} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} + 2 \sup |\psi| \int_{|x|\geq\delta} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} dx \leq$$

$$\varepsilon + \frac{\sup|\psi|}{\sqrt{\pi}} \int_{|s|\geq\delta/\sqrt{t}} e^{-s^2/4} ds \to \varepsilon \quad \text{as} \quad t \downarrow 0.$$

Since $\varepsilon > 0$ was arbitrary this completes the proof. ■

10. The Cauchy problem in one dimension

For a given function $u_0 : \mathbb{R} \to \mathbb{R}$ we consider the problem

$$(CP) \begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

Our experience with the wave equation suggests to consider the convolution

$$u(x,t) = (E^+(t) * u_0)(x) = \int_{-\infty}^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-(x-\xi)^2/4t} u_0(\xi)d\xi. \tag{10.1}$$

10.1 Notation Let $Q = \mathbb{R} \times \mathbb{R}^+$. Then

$$C^{2,1}(Q) = \{u : Q \to \mathbb{R};\ u, u_t, u_x, u_{xx} \in C(Q)\}.$$

10.2 Theorem Suppose $u_0 \in C(\mathbb{R})$ is bounded. Then (CP) has a unique bounded classical solution $u \in C^{2,1}(Q) \cap C(\overline{Q})$, given by the convolution (10.1).

Proof of existence Clearly $E^+(\cdot,t) * u_0$ is well defined and bounded for all $(x,t) \in \mathbf{Q}$, because u_0 is bounded and $E^+(x,t)$ decays exponentially fast to zero as $|x| \to \infty$. Since the same holds for all partial derivatives of $E^+(x,t)$, we can differentiate under the integral with respect to x and t. Thus for any $n, l \in \mathbb{N} \cup \{0\}$,

$$(\frac{\partial}{\partial t})^n (\frac{\partial}{\partial x})^l u(x,t) = (\frac{\partial}{\partial t})^n (\frac{\partial}{\partial x})^l \int_{\infty}^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi$$

$$= \int_{-\infty}^{\infty} \frac{\partial^{n+l} E^+(x-\xi,t)}{\partial t^n \partial x^l} u_0(\xi)d\xi = \big(\frac{\partial^{n+l} E^+}{\partial t^n \partial x^l}(\cdot,t) * u_0\big)(x),$$

and in particular

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = (\frac{\partial E^+}{\partial t} - \frac{\partial^2 E^+}{\partial x^2}) * u_0 = 0.$$

Hence $u \in C^\infty(Q)$ satisfies $u_t = u_{xx}$ in Q.

It remains to show that for every $x_0 \in \mathbb{R}$

$$\lim_{\substack{x \to x_0 \\ t \downarrow 0}} u(x,t) = u_0(x).$$

The argument is similar to the proof that E^+ satisfies the fundamental equation.

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|u_0(x) - u_0(x_0)| < \varepsilon$ for $|x - x_0| < \delta$. For $|x - x_0| < \frac{1}{2}\delta$ we have

$$|u(x,t) - u_0(x_0)| = \Big| \int_{-\infty}^{\infty} E^+(x-\xi,t)(u_0(\xi) - u_0(x_0))d\xi \Big| \leq$$

$$\int_{|x-\xi|<\frac{1}{2}\delta} E^+(x-\xi,t)\big|u_0(\xi) - u_0(x_0)\big|d\xi + \int_{|x-\xi|>\frac{1}{2}\delta} E^+(x-\xi,t)\big|u_0(\xi) - u_0(x_0)\big|d\xi$$

(since $|x - \xi| < \frac{1}{2}\delta$ together with $|x - x_0| < \frac{1}{2}\delta$ implies $|\xi - x_0| < \delta$)

$$\leq \varepsilon \int_{|x-\xi|<\frac{1}{2}\delta} E^+(x-\xi,t)d\xi + 2\sup|u_0| \int_{|x-\xi|>\frac{1}{2}\delta} E^+(x-\xi,t)d\xi$$

$$\leq \varepsilon + 2\sup|u_0| \int_{|\xi|>\frac{1}{2}\delta} E^+(\xi,t)d\xi \to \varepsilon \quad \text{as} \quad t \downarrow 0$$

as before. Since $\varepsilon > 0$ was arbitrary, this completes the proof of the existence of a solution. Note that for the continuity of $u(x,t)$ at $(x,t) = (x_0, 0)$ we have only used the continuity of $u_0(x)$ at $x = x_0$. ∎

Proof of uniqueness Suppose there exist two different solutions of (CP) in $C^{2,1}(Q) \cap C(\overline{Q})$. Then the difference is a nontrivial classical solution u of

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

This is impossible in view of a *maximum principle* which we state and prove below.

10.3 Lemma Suppose $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, where $Q_T = \mathbb{R} \times (0,T]$ $(T > 0)$, satisfies

$$u_t \leq u_{xx} \text{ in } Q_T.$$

If (i) $u(x,0) \leq 0$ for all $x \in \mathbb{R}$;
(ii) $u(x,t) \leq Ae^{Bx^2}$ for all $(x,t) \in Q_T$,
where $A > 0$ and B are fixed constants, then

$$u \leq 0 \text{ in } \overline{Q}_T.$$

10.4 Lemma For $-\infty < a < b < \infty$ and $T > 0$ let $Q_T^{a,b} = (a,b) \times (0,T]$, and $\Gamma_T^{a,b} = \overline{Q}_T^{a,b} \backslash Q_T^{a,b}$. $\Gamma_T^{a,b}$ is called the *parabolic boundary* of $Q_T^{a,b}$. Suppose $u \in C^{2,1}(Q_T^{a,b}) \cap C(\overline{Q}_T^{a,b})$ satisfies

$$u_t \leq u_{xx} \text{ in } Q_T^{a,b}.$$

Then

$$\sup_{Q_T^{a,b}} u = \max_{\Gamma_T^{a,b}} u.$$

Proof of Lemma 10.4 First observe that if $u_t < u_{xx}$ in $Q_T^{a,b}$, then u cannot have a (local or global) maximum in $Q_T^{a,b}$. Indeed, if this maximum would be situated at (x_0, t_0) with $a < x_0 < b$ and $0 < t_0 < T$, then at $(x,t) = (x_0, t_0)$ one has $u_{xx} > u_t = u_x = 0$, contradiction. Also a maximum at (x_0, T) is impossible because then $u_{xx} > u_t \geq 0$, again a contradiction.

Next we reduce the case $u_t \leq u_{xx}$ to $u_t < u_{xx}$. Let

$$u_n(x,t) = u(x,t) + \frac{x^2}{2n}.$$

Then obviously

$$u_{nt} = u_t \leq u_{xx} < u_{xx} + \frac{1}{n} = u_{nxx},$$

so that

$$\sup_{Q_T^{a,b}} u_n = \max_{\Gamma_T^{a,b}} u_n.$$

Taking the limit $n \to \infty$ the lemma follows. ∎

Proof of Lemma 10.3 It is sufficient to prove the statement for one fixed $T > 0$. For $\alpha, \beta, \gamma > 0$ let

$$h(x,t) = \exp\left(\frac{\alpha x^2}{1 - \beta t} + \gamma t\right) \qquad x \in \mathbb{R},\ 0 \leq t < \frac{1}{\beta}.$$

Define $u(x,t)$ by $u = hv$. Then

$$0 \geq u_t - u_{xx} = (hv)_t - (hv)_{xx} = hv_t + h_t v - hv_{xx} - 2h_x v_x - h_{xx} v =$$

$$h(v_t - v_{xx} - v_x \frac{2h_x}{h} + v\frac{h_t - h_{xx}}{h} =$$

$$h\big(v_t - v_{xx} - v_x \frac{4\alpha x}{1 - \beta t} + v(\frac{\alpha\beta x^2}{(1-\beta t)^2} + \gamma - (\frac{2\alpha x}{1-\beta t})^2 - \frac{2\alpha}{1-\beta t})\big) =$$

$$h\big(v_t - v_{xx} - \frac{4\alpha x}{1-\beta t}v_x + v(\gamma - \frac{(4\alpha-\beta)\alpha x^2}{(1-\beta t)^2} - \frac{2\alpha}{1-\beta t})\big). \tag{9.6}$$

Choosing $\beta > 4\alpha$ and $\gamma > 4\alpha$ the coefficient of v is positive for $x \in \mathbb{R}$ and $0 \le t \le 1/2\beta$. We then also have

$$v(x,t) = u(x,t)\exp(-\frac{\alpha x^2}{1-\beta t} - \gamma t) \le Ae^{(B-\alpha)x^2},$$

so that, choosing $\alpha > B$,

$$\limsup_{|x|\to\infty} v(x,t) \le 0 \quad \text{uniformly on} \quad [0, \frac{1}{2\beta}]. \tag{9.7}$$

Now suppose the lemma is false for $T = 1/2\beta$. Then u and v achieve positive values on $Q_{1/2\beta}$. In view (9.7) this implies that v must have a positive maximum in $Q_{1/2\beta}$. By the inequality for $u_t - u_{xx}$ and the choice of α, β, γ this implies $v_t < v_{xx}$ at this maximum. But in the proof of Lemma 10.4 we have seen that this is impossible, contradiction. ∎

10.5 Exercise Finish the uniqueness proof. ∎

10.6 Exercise For $u_0 \in C(\mathbb{R})$ satisfying

$$|u_0(x)| \le Ae^{Bx^2}$$

for all $x \in \mathbb{R}$, prove that (CP) has a classical solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ for all $T < 1/4B$, and give a growth condition which determines the solution uniquely.

Next we consider the equation

$$u_t = u_{xx} + \varphi \qquad x \in \mathbb{R}, \quad 0 < t \le T,$$

where $\varphi : \mathbb{R} \times (0,T) \to \mathbb{R}$. If φ is measurable and bounded, we can try as a particular solution

$$u_p(x,t) = \int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau. \tag{9.8}$$

Clearly, u_p is well defined, because the integral is dominated by

$$\int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau) \sup_{Q_T} |\varphi| \; d\xi d\tau \le t \sup_{Q_T} |\varphi|,$$

so that in particular $u_p(x,t) \to 0$ uniformly in x as $t \downarrow 0$.

One would like to have $u_p \in C^{2,1}(Q_T)$, which is however rather technical to establish and unfortunately requires more than just the continuity of φ. Here we just restrict ourselves to

10.7 Proposition Let $\varphi \in L^\infty(Q_T)$. Then

$$u_p(x,t) = \int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau$$

defines a bounded function which is a solution of $u_t = u_{xx} + \varphi$ in the sense of distributions on $\mathbb{R} \times (0,T)$, and tends to zero uniformly on $\mathbb{R}$ as $t \downarrow 0$.

Proof If we set $E^+(x,t) \equiv \varphi(x,t) \equiv 0$ for all $t < 0$, then

$$u_p(x,t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau.$$

Let $\psi \in D(\mathbb{R} \times (0,T))$, and extend ψ to $\mathbb{R} \times \mathbb{R}$ by $\psi(x,t) \equiv 0$ for $t \leq 0$ and $t \geq T$. Then

$$< \frac{\partial u_p}{\partial t} - \frac{\partial^2 u_p}{\partial x^2}, \psi >= - < u_p, \psi_t + \psi_{xx} >=$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)(\psi_t(x,t) + \psi_{xx}(x,t))d\xi d\tau dx dt =$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)(\psi_t(x,t) + \psi_{xx}(x,t))dx dt\}\varphi(\xi,\tau)d\xi d\tau =$$

(as in the proof that E^+ is a fundamental solution)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(\xi,\tau)\psi(\xi,\tau)d\xi d\tau =< \varphi, \psi >,$$

so that u_p is a solution of the inhomogeneous heat equation in the sense of distributions. ▪

We can now write down the solution of

$$(CP_i) \begin{cases} u_t = u_{xx} + \varphi & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

as

$$u(x,t) = \int_{-\infty}^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi + \int_0^t \int_{-\infty}^{\infty} E^+(x-\xi, t-\tau)\varphi(\xi,\tau)d\xi d\tau,$$

but we do not give the precise hypothesis here on u_0 and φ that guarantee that this formula defines a *classical* solution, i.e.

$$u \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}} \times \overline{\mathbb{R}}^+).$$

10.8 Exercise Show that (CP_i) has at most one bounded classical solution.

11. Initial boundary value problems

First we indicate how one can generalize results for

$$(CP)\begin{cases} u_t = u_{xx} & x \in \mathbb{R}\ ,\ t > 0; \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

to

$$(CD)\begin{cases} u_t = u_{xx} & x > 0\ ,\ t > 0; \\ u(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0, \end{cases}$$

and

$$(CN)\begin{cases} u_t = u_{xx} & x > 0\ ,\ t > 0; \\ u_x(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0. \end{cases}$$

For (CD) and (CN) we consider (CP) with odd and even initial data respectively.

We begin with (CD). Extending u_0 to the whole of $\mathbb{R}$ by $u_0(-x) = -u_0(x)$, the integral representation of solutions gives

$$u(x,t) = -\int_{-\infty}^{0} E^+(x-\xi,t)u_0(-\xi)d\xi + \int_0^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi =$$

$$\int_0^{\infty} \{E^+(x-\xi,t) - E^+(x+\xi,t)\}u_0(\xi)d\xi = \int_0^{\infty} G_1(x,\xi,t)u_0(\xi)d\xi, \qquad (11.1)$$

where

$$G_1(x,\xi,t) = E^+(x-\xi,t) - E^+(x+\xi,t) \qquad (11.2)$$

is called the *Green's function of the first kind.*

For (CN) we extend u_0 by $u_0(-x) = u_0(x)$, and thus

$$u(x,t) = \int_{-\infty}^{0} E^+(x-\xi,t)u_0(-\xi)d\xi + \int_0^{\infty} E^+(x-\xi,t)u_0(\xi)d\xi =$$

$$\int_0^{\infty} \{E^+(x-\xi,t) + E^+(x+\xi,t)\}u_0(\xi)d\xi = \int_0^{\infty} G_2(x,\xi,t)u_0(\xi)d\xi, \qquad (11.3)$$

where

$$G_2(x,\xi,t) = E^+(x-\xi,t) + E^+(x+\xi,t) \qquad (11.3)$$

is called the *Green's function of the second kind.*

11.1 Exercise Let $u_0 \in C(\overline{\mathbb{R}}^+)$ be bounded, and let $u_0(0) = 0$. Prove that (CD) has a unique bounded solution $u \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+)$.

11.2 Exercise Let $u_0 \in C(\overline{\mathbb{R}}^+)$ be bounded. Prove that (CN) has a unique bounded solution $u \in C^{2,1}(\overline{\mathbb{R}}^+ \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}}^+ \times \overline{\mathbb{R}}^+)$.

11.3 Exercise Derive formal integral representations for the solutions of

$$(CD_i) \begin{cases} u_t = u_{xx} + \varphi & x > 0,\ t > 0; \\ u(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0, \end{cases}$$

and

$$(CN_i) \begin{cases} u_t = u_{xx} + \varphi & x > 0\ ,\ t > 0; \\ u_x(0,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & x \geq 0. \end{cases}$$

Next we consider what is usually called the *Dirichlet problem* for the heat equation on $(0,1)$:

$$(D) \begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u(0,t) = u(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \leq x \leq 1. \end{cases}$$

To find an integral representation for the solution of (D) we extend u_0 to a 2-periodic function $\tilde{u}_0 : \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{u}_0 \equiv u_0 \text{ on } (0,1);\ \tilde{u}_0(x) = -\tilde{u}_0(-x);\ \tilde{u}_0(1+x) = -\tilde{u}_0(1-x).$$

For the Cauchy problem with initial dat $\tilde{u}_0$ we then have

$$u(x,t) = \int_{-\infty}^{\infty} E^+(x-\xi,t)\tilde{u}_0(\xi)d\xi = \sum_{k=-\infty}^{\infty} \int_k^{k+1} E^+(x-\xi,t)\tilde{u}_0(\xi)d\xi =$$

$$\sum_{k=-\infty}^{\infty} \int_0^1 E^+(x-\xi-k,t)\tilde{u}_0(\xi+k)d\xi =$$

$$\sum_{n=-\infty}^{\infty} \{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi+2n)d\xi + \int_0^1 E^+(x-\xi-2n-1,t)\tilde{u}_0(\xi+2n+1)d\xi\} =$$

$$\sum_{n=-\infty}^{\infty} \{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi)d\xi + \int_0^1 E^+(x-\xi-2n-1,t)\tilde{u}_0(\xi+1)d\xi\} =$$

$$\sum_{n=-\infty}^{\infty} \{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi)d\xi - \int_0^1 E(x-\xi-2n-1,t)\tilde{u}_0(1-\xi)d\xi\} =$$

$$\sum_{n=-\infty}^{\infty}\{\int_0^1 E^+(x-\xi-2n,t)\tilde{u}_0(\xi)d\xi - \int_0^1 E^+(x+\xi-1-2n-1,t)\tilde{u}_0(\xi)d\xi\}$$

$$= \int_0^1 \sum_{n=-\infty}^{\infty} \{E^+(x-\xi-2n,t) - \ E^+(x+\xi-2n,t)\}\tilde{u}_0(\xi)$$

$$= \int_0^1 G_D(x,\xi,t)\tilde{u}_0(\xi)d\xi, \tag{11.4}$$

where

$$G_D(x,\xi,t) = \sum_{n=-\infty}^{\infty} \{E^+(x-\xi-2n,t) - E^+(x+\xi-2n,t)\}. \tag{11.5}$$

(Note that this sum is absolutely convergent for $t > 0$, uniformly in x.)

11.4 Theorem Let $u_0 \in C([0,1]), u_0(0) = u_0(1) = 0$, and let $Q_T = (0,1) \times (0,T]$. Then for every $T > 0$ there exists a unique bounded solution $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ of (D), given by

$$u(x,t) = \int_0^1 G_D(x,\xi,t)u_0(\xi)d\xi.$$

Proof Exercise, for the uniqueness part, the maximum principle has to be used again. ∎

G_D is called the *Green's function for the Dirichletproblem*.

For the *Neumannproblem*, that is

$$(N) \begin{cases} u_t = u_{xx} & 0 < x < 1, \ t > 0; \\ u_x(0,t) = u_x(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \le x \le 1, \end{cases}$$

we extend u_0 to a 2-periodic function $\tilde{u}_0 : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{u}_0 \equiv u_0 \text{ on } [0,1]; \ \tilde{u}_0(x) = \tilde{u}_0(-x); \ \tilde{u}_0(1+x) = \tilde{u}_0(1-x).$$

We now obtain

$$u(x,t) = \int_0^1 G_N(x,\xi,t)u_0(\xi)d\xi, \tag{11.6}$$

where

$$G_N(x,\xi,t) = \sum_{n=-\infty}^{\infty} \{E^+(x-\xi-2n,t) + E^+(x+\xi-2n,t)\} \tag{11.7}$$

is *Green's function for the Neumannproblem.*

11.5 Theorem Let $u_0 \in C([0,1])$. Then for every $T > 0$ there exists a unique bounded classical solution of (N), given by

$$u(x,t) = \int_0^1 G_N(x,\xi,t)u_0(\xi)d\xi.$$

11.6 Exercise Give a suitable definition of a classical solution of (N) and prove this theorem.

11.7 Exercise Derive a representation formula for solutions of the mixed problem

$$(DN)\begin{cases} u_t = u_{xx} & 0 < x < 1,\ t > 0; \\ u(0,t) = u_x(1,t) = 0 & t > 0; \\ u(x,0) = u_0(x) & 0 \le x \le 1, \end{cases}$$

and formulate and prove a uniqueness/existence theorem.

11.8 Exercise Give formal derivations for integral representations of solutions to the problems above with $u_t = u_{xx}$ replaced by the inhomogeneous equation $u_t = u_{xx} + \varphi$.

REFERENCES

[J] John, F., Partial Differential Equations, 4th edition, Springer 1986.

K] Kevorkian, J., Partial Differential Equations, Analytical Solution Techniques, Brooks/Cole 1989.

[Sm] Smoller, J., Shock-Waves and Reaction-Diffusion Equations, Springer 1983.

[T] Treves, F., Basic Linear Partial Differential Equations, Academic Press 1975.