A FULLY NONLINEAR EQUATION FOR THE FLAME FRONT IN A QUASI-STEADY COMBUSTION MODEL

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This paper is dedicated to Roger Temam on the occasion of his 70th birthday

Abstract. We revisit the Near Equidiffusional Flames (NEF) model introduced by Matkowsky and Sivashinsky in 1979 and consider a simplified, quasi-steady version of it. This simplification allows, near the planar front, an explicit derivation of the front equation. The latter is a pseudodifferential fully nonlinear parabolic equation of the fourth-order. First, we study the (orbital) stability of the null solution. Second, introducing a parameter ε, we rescale both the dependent and independent variables and prove rigorously the convergence to the solution of the Kuramoto-Sivashinsky equation as $\varepsilon \to 0$.

1. **Introduction.** Paradigms of two-dimensional Free Boundary evolution problems are often formulated in a strip, say $\Omega = \mathbb{R} \times [-\ell/2, \ell/2]$ with coordinates denoted respectively by x and y. The other independent variable t stands for the time. A moving front represented by $x = \xi(t, y)$ separates two regions or phases. The dependent variables are, on the one hand, the function ξ and on the other hand a vector-valued function, say $\mathbf{u}(t, x, y)$, whose components correspond to physical quantities such as temperature, mass fraction, enthalpy, etc. Usually, those quantities are continuous at the front, whereas there is a jump in their gradient since the interface corresponds to a thin reaction zone. On both sides of the front, the restriction of $\mathbf{u}$ satisfies a system of reaction-diffusion equations with convenient boundary

conditions at $y = \pm \ell/2$ (in the sequel periodicity will be assumed). Finally, there is a set of Free Boundary conditions at the front $x = \xi(t, y)$ and convenient initial conditions.

The case of a planar front, namely $\xi(t, y) = -ct$, $c > 0$, is of special interest. It corresponds to a family of one-dimensional Traveling Waves which propagate with velocity $-c$. Its (orbital) stability with respect to small initial perturbations is an important question. It leads to the study of corrugated fronts $\xi(t, y)$ in some neighborhood of the planar front, namely small perturbations of the form $\varphi(t, y) = \xi(t, y) + ct$. In the case of instabilities, the dynamics of φ is a crucial issue.

A very challenging problem is the derivation of a single equation for φ, which may capture most of the dynamics and, as a consequence, yields a reduction of the effective dimensionality of the system. In this spirit, one of the authors in [23] derived asymptotically from the NEF system in combustion theory the Kuramoto-Sivashinsky equation (K–S)

$$\Phi_\tau + 4\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0, \tag{1.1}$$

in a set of conveniently rescaled dependent and independent variables. Since then, this equation has received considerable attention from the mathematical community, especially for its ability to generate a cellular structure and chaotic behavior in appropriate range of parameters [13]. We refer to the book [26] and its extensive bibliography.

Other dissipative systems with similar dynamics have been a subject of discussion in recent years. Let us mention the Burgers-Sivashinsky (B–S) equation, a model pertinent to the flame front dynamics subject to the buoyancy effect [1], and the Q–S equation derived in [3] as a quasi-steady version of the $\kappa - \theta$ model in flame theory.

Several generalizations of K–S have been considered in the literature. For instance in [21], it was observed that D^4 can be replaced by an elliptic pseudo-differential operator of order $2m$ and D^2 by the opposite of an elliptic pseudo-differential operator of order less than $2m$. Recently in [11], the K–S equation has been generalized as the following one:

$$\varphi_t + \mathscr{G}(\varphi_y) = \mathscr{L}\varphi, \tag{1.2}$$

where $\mathscr{L}$ is a linear pseudo-differential operator of order $2m$ and $\mathscr{G}$ is a sufficiently smooth real valued function which satisfies $c_{\min}\zeta^2 \le \mathscr{G}(\zeta^2) \le C_{\max}\zeta^2$ for $|\zeta|$ large.

In [4], we have considered a two-dimensional Stefan problem, a simplified version of a solid-liquid interface model. We have derived an equation for the front of the form:

$$\frac{\partial}{\partial t}\mathscr{B}\varphi + \mathscr{F}((\varphi_y)^2) = \mathscr{S}\varphi, \tag{1.3}$$

where $\mathscr{B}$ and $\mathscr{F}$ are pseudo-differential operators of the second-order and $\mathscr{S}$ is a pseudo-differential operator of the fourth-order. Actually, $\mathscr{B}$ can be inverted. Hence, Equation (1.3) can be reformulated as a second order parabolic equation similar to (1.2). However, in contrast to [11], $\mathscr{G}$ is not a quadratic function of φ_y but a bounded linear operator acting on $(\varphi_y)^2$. Indeed the equation for the front reads:

$$\varphi_t + \mathscr{G}((\varphi_y)^2) = \mathscr{L}\varphi. \tag{1.4}$$

In this paper, we derive a new front equation which sticks as close as possible to the original NEF combustion model. We get to an equation which belongs to

the class (1.3), (1.4). However, whereas $\mathscr{S}$ (resp. $\mathscr{L}$) is still a pseudo-differential operator of the fourth-order (resp. second-order), it turns out that $\mathscr{F}$ (resp. $\mathscr{G}$) is a pseudo-differential operator of the third-order (resp. first-order) acting on $(\varphi_y)^2$. Therefore, the nonlinear part is of the same order as the linear part: this is, by definition, a *fully nonlinear problem* (see [18]), in contrast to (1.1) and (1.2) which are semilinear.

The paper is organized as follows. First, in Section 2, we consider as a paradigm Free Boundary Problem a simplified quasi-steady version of the NEF model, present our results and introduce some mathematical setting. Next, in Section 3 we derive formally the front equations (1.3), (1.4). We refer to [4, 5] for a rigorous derivation of the front equation as a solvability condition in a Stefan problem. Section 4 is devoted to prove the first main result of this paper (Theorem 2.1), i.e., the study of the stability of the front. For this purpose, we first study the symbols of the operators $\mathscr{B}$, $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{L}$. In particular, we characterize the spectrum of the realization of the operator $\mathscr{L}$ in $L^2(-\ell/2, \ell/2)$.

Section 5 is devoted to prove the second main result of this paper (Theorem 2.2), i.e., we provide a rigorous derivation of the K–S equation (1.1). For this purpose, we set $\alpha = 1 + \varepsilon$ and introduce the rescaled variables

$$t = \tau/\varepsilon^2, \quad y = \eta/\sqrt{\varepsilon}, \quad \varphi = \varepsilon\psi.$$

This leads us to a fully nonlinear equation of the form

$$\frac{\partial}{\partial \tau}\mathscr{B}_\varepsilon(\psi) = \mathscr{S}(\psi) + \mathscr{F}_\varepsilon((D_\eta\psi)^2). \tag{1.5}$$

The core of this section consists of the *a priori* estimates in Subsection 5.2 for solutions to (1.5) which, together with Lemma 5.6, are the main tools needed to prove Theorem 2.2.

Finally, for the reader's convenience, in the appendices we give a brief introduction to the Near Equidiffusional Flames model and provide a rather detailed proof of the existence, uniqueness and regularity properties of the solution to the K–S equation (1.1).

2. Setting up the problem, main results and some mathematical setting.
In this section we set up our problem, state our main results and introduce some notation which are kept throughout the paper.

2.1. Setting up the problem.
Our starting point is the following NEF system (see the Appendix) for the temperature θ, the enthalpy S and the moving flame front, defined by $x = \xi(t, y)$, which reads

$$\frac{\partial \theta}{\partial t} = \Delta\theta, \quad x < \xi(t, y), \tag{2.1}$$

$$\theta = 1, \quad x \geq \xi(t, y), \tag{2.2}$$

$$\frac{\partial S}{\partial t} = \Delta S - \alpha\Delta\theta, \quad x \neq \xi(t, y). \tag{2.3}$$

For some mathematical results about this problem, see e.g., [6, 14, 15, 16, 17, 10]. Here, we consider only the case where α is positive, i.e., the case of high mobility of the deficient reactant. It will be convenient to assume periodicity in y with period

ℓ, and restrict attention to $y \in [-\ell/2, \ell/2]$. At the front, θ and S are continuous and the following jump conditions occur for the normal derivatives:

$$\left[\frac{\partial \theta}{\partial n}\right] = -\exp(S), \tag{2.4}$$

$$\left[\frac{\partial S}{\partial n}\right] = \alpha \left[\frac{\partial \theta}{\partial n}\right]. \tag{2.5}$$

System (2.1)-(2.5) admits a planar travelling wave (TW) solution, with velocity -1,

$$\overline{\theta}(x) = \begin{cases} \exp x, & x \leq 0, \\ 1, & x \geq 0 \end{cases} \qquad \overline{S}(x) = \begin{cases} \alpha x \exp x, & x \leq 0, \\ 0, & x \geq 0. \end{cases}$$

As usual one fixes the free boundary. We set $\xi(t, y) = -t + \varphi(t, y)$, $x' = x - \xi(t, y)$. In this new framework:

$$\theta_t + (1 - \varphi_t)\theta_{x'} = \Delta_\varphi \theta, \qquad x' < 0, \tag{2.6}$$

$$\theta(x') = 1, \qquad x' > 0, \tag{2.7}$$

$$S_t + (1 - \varphi_t)S_{x'} = \Delta_\varphi S - \alpha \Delta_\varphi \theta, \qquad x' \neq 0, \tag{2.8}$$

where

$$\Delta_\varphi = (1 + (\varphi_y)^2)D_{x'x'} + D_{yy} - \varphi_{yy}D_{x'} - 2\varphi_y D_{x'y}.$$

The front is now fixed at $x' = 0$. The first condition (2.4) reads:

$$\sqrt{1 + (\varphi_y)^2} \left[\frac{\partial \theta}{\partial x'}\right] = -\exp(S),$$

the second one (2.5) becomes

$$\left[\frac{\partial S}{\partial x'}\right] = \alpha \left[\frac{\partial \theta}{\partial x'}\right].$$

We will consider a quasi-steady version of the NEF model. As a matter of fact, it has been observed in similar problems (see [3]) that not far from the instability threshold the time derivatives in the temperature and enthalpy equations have a relatively small effect on the solution. The dynamics appears to be essentially driven by the front. Based on this observation one can define a *quasi-steady* NEF model replacing (2.6)-(2.8) by

$$(1 - \varphi_t)\theta_{x'} = \Delta_\varphi \theta, \qquad x' < 0,$$

$$\theta = 1, \qquad x' > 0,$$

$$(1 - \varphi_t)S_{x'} = \Delta_\varphi S - \alpha \Delta_\varphi \theta, \qquad x' \neq 0.$$

Next we consider the perturbations of temperature u and enthalpy v:

$$\theta = \overline{\theta} + u, \quad S = \overline{S} + v.$$

Writing for simplicity x instead of x', the problem for the triplet (u, v, φ) reads:

$$(1 - \varphi_t)u_x - \Delta_\varphi u - \varphi_t \overline{\theta}_x = (\Delta_\varphi - \Delta)\overline{\theta}, \qquad x < 0,$$

$$u = 0, \qquad x > 0,$$

$$(1 - \varphi_t)v_x - \Delta_\varphi(v - \alpha u) - \varphi_t \overline{S}_x = (\Delta_\varphi - \Delta)(\overline{S} - \alpha \overline{\theta}), \qquad x \neq 0,$$

where

$$(\Delta_\varphi - \Delta)(\overline{\theta}) = ((\varphi_y)^2 - \varphi_{yy})\overline{\theta}_x,$$

$$(\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}) = \alpha((\varphi_y)^2\overline{S}_x - \varphi_{yy}\overline{S}).$$

As in [4, 5], we introduce further simplifications: we keep only linear and second-order terms for the perturbation of the front φ, and first-order terms for the perturbations of temperature u and enthalpy v. The skipped terms contribute to higher order perturbations only. This leads to the equations:

$$u_x - \Delta u - \varphi_t\overline{\theta}_x = (\Delta_\varphi - \Delta)\overline{\theta}, \qquad x < 0,$$

$$v_x - \Delta(v - \alpha u) - \varphi_t\overline{S}_x = (\Delta_\varphi - \Delta)(\overline{S} - \alpha\overline{\theta}), \quad x \neq 0.$$

At $x = 0$ there are several conditions. First

$$[u] = [v] = 0,$$

however, since $u(x) = 0$ for $x > 0$, this is equivalent to

$$u(0^-) = [v] = 0.$$

Second,

$$\sqrt{1 + (\varphi_y)^2}\,[\overline{\theta}_x + u_x] = -\exp(\overline{S} + v),$$

hence up to the second-order:

$$-1 + [u_x] = -(1 + (\varphi_y)^2)^{-\frac{1}{2}}\,e^v \sim -\left(1 - \frac{1}{2}(\varphi_y)^2\right)\left(1 + v(0) + \frac{1}{2}(v(0))^2\right)$$

and keeping only the first-order for v yields:

$$-u_x(0) + v(0) = \frac{1}{2}(\varphi_y)^2.$$

Moreover, the condition $[S_x] = \alpha[\theta_x]$ yields

$$[v_x] = -\alpha u_x(0).$$

Therefore, the final system reads:

$$\begin{cases} u_x - \Delta u - \varphi_t\overline{\theta}_x = ((\varphi_y)^2 - \varphi_{yy})\overline{\theta}_x, & x < 0, \\ v_x - \Delta(v - \alpha u) - \varphi_t\overline{S}_x = (\varphi_y)^2\overline{S}_x - \varphi_{yy}\overline{S}, & x \neq 0, \\ u(0) = [v] = 0, \\ v(0) - u_x(0) = \frac{1}{2}(\varphi_y)^2, \\ [v_x] = -\alpha u_x(0). \end{cases} \qquad (2.9)$$

We remark that the equation for u associated with the boundary condition $u(0) = 0$ entirely determines u when φ is given. Therefore, it can be viewed as a kind of *pseudo-differential Stefan condition*. We will take advantage of this remark in Section 3.

2.2. **Our results.** The goal of this paper is to show that this simplified NEF model still contains the dynamics of the system. It is simple enough to be integrated explicitly via a discrete Fourier transform in the variable y and therefore it allows a separation of the dependent variables. We get to a self-consistent pseudo-differential equation for the front φ which reads:

$$(X_k^2 + \alpha X_k - \alpha)\widehat{\varphi}_t(t, k) = (-4\lambda_k^2 + (\alpha - 1)\lambda_k)\widehat{\varphi}(t, k)$$
$$+ \frac{1}{4}(X_k^3 - 3X_k^2 - 4\alpha X_k + 4\alpha)\widehat{(\varphi_y)^2}(t, k), \quad k = 0, 1, \ldots, \qquad (2.10)$$

where the $-\lambda_k$'s are the non-positive eigenvalues of the operator D_{yy} with periodic boundary conditions at $y = \pm \ell/2$ (that we denote below by A) and

$$X_k = \sqrt{1 + 4\lambda_k}, \qquad k = 0, 1, \ldots,$$

is the symbol of operator $\sqrt{1 - 4D_{yy}}$.

Equation (2.10) can be written in the more abstract form:

$$\frac{\partial}{\partial t}\varphi = \mathscr{L}(\varphi) + \mathscr{G}((\varphi_y)^2), \qquad (2.11)$$

where $\mathscr{L}$ is a pseudodifferential operator whose leading part is D_{yy} and $\mathscr{G}$ is a nonlinear operator whose leading term is $\frac{1}{4}\sqrt{1 - 4D_{yy}}$. This makes (2.11) a strongly nonlinear equation, more precisely it is a *fully nonlinear parabolic equation*: in the L^2-setting the nonlinear part is exactly of the same order as the linear operator. This is one of the main issues of this paper. Note that the realization of the operator $\sqrt{1 - 4D_{yy}}$ in the space of continuous and ℓ-periodic functions (say $C_\sharp$) is defined only in a *proper* subspace of $C_\sharp^1$ (the space of all the ℓ-periodic C^1-functions). Hence, in the $C_\sharp$-setting, the nonlinear term $\mathscr{G}((\varphi_y)^2)$ represents the leading part of the right-hand side of (2.11). This would make the study of (2.11) more difficult than in the L^2-setting, where we confine our analysis.

In the case where φ is smoother, we can rewrite Equation (2.11) as a fourth-order equation as follows:

$$\frac{\partial}{\partial t}\mathscr{B}\varphi = \mathscr{S}(\varphi) + \mathscr{F}((\varphi_y)^2), \qquad (2.12)$$

where $\mathscr{S}$ is nothing but the usual fourth-order differential operator

$$\mathscr{S}(\varphi) = -\varphi_{yyyy} - (\alpha - 1)\varphi_{yy}.$$

Operators $\mathscr{B}$ and $\mathscr{F}$ are pseudo-differential ones with symbols, respectively,

$$b_k = X_k^2 + \alpha X_k - \alpha, \quad f_k = \frac{1}{4}(X_k^3 - 3X_k^2 - 4\alpha X_k + 4\alpha).$$

Therefore,

$$\mathscr{B} = I - 4D_{yy} + \alpha\left(\sqrt{I - 4D_{yy}} - I\right),$$
$$\mathscr{F} = \frac{1}{4}(I - 4D_{yy})^{\frac{3}{2}} - \frac{3}{4}(I - 4D_{yy}) - \alpha\left(\sqrt{1 - 4D_{yy}} - I\right).$$

The main feature of Equation (2.12) is that the nonlinear part is rather unusual. Actually, it has a fourth-order leading term, as $\mathscr{S}$ has. Therefore (2.12) is also a fully nonlinear problem.

The first main result of the paper is the following one.

Theorem 2.1. *Let*

$$\alpha_c = 1 + \frac{16\pi^2}{\ell^2}. \tag{2.13}$$

Then, the following properties are satisfied.

(a) *If $\alpha < \alpha_c$, then, the null solution to Equation (2.12) is (orbitally) stable, with asymptotic phase, with respect to sufficiently smooth and small perturbations.*
(b) *If $\alpha > \alpha_c$, then the null solution to Equation (2.12) is unstable.*

An important question, that we address in Section 5, is the link between (2.12) and K–S. Following [23], we introduce a small parameter $\varepsilon > 0$, setting

$$\alpha = 1 + \varepsilon,$$

and define the rescaled dependent and independent variables accordingly:

$$t = \tau/\varepsilon^2, \quad y = \eta/\sqrt{\varepsilon}, \quad \varphi = \varepsilon\psi.$$

We see that ψ solves the equation

$$\frac{\partial}{\partial\tau}\left\{I - 4\varepsilon D_{\eta\eta} + (1+\varepsilon)\left(\sqrt{I - 4\varepsilon D_{\eta\eta}} - 1\right)\right\}\psi$$

$$= -4D_{\eta\eta\eta\eta}\psi - D_{\eta\eta}\psi$$

$$+ \frac{1}{4}\left\{(I - 4\varepsilon D_{\eta\eta})^{\frac{3}{2}} - 3(I - 4\varepsilon D_{\eta\eta}) - 4(1+\varepsilon)\left(\sqrt{1 - 4\varepsilon D_{\eta\eta}} - I\right)\right\}(D_\eta\psi)^2.$$

Then, we anticipate, in the limit $\varepsilon \to 0$, that $\psi \sim \Phi$, where Φ solves (1.1). More precisely, we take for ℓ:

$$\ell_\varepsilon = \ell_0/\sqrt{\varepsilon},$$

which blows up as $\varepsilon \to 0$; hence $\alpha_c = 1 + \frac{16\pi^2}{\ell_0^2}\varepsilon$. Thus, ℓ_0 becomes the new bifurcation parameter. We shall assume that $\ell_0 > 4\pi$ in order to have $\alpha_c \in (1, 1+\varepsilon)$, i.e., $\alpha > \alpha_c$, otherwise the trivial solution is stable and the dynamics is trivial.

The second main result of the paper is the following.

Theorem 2.2. *Let $\Phi_0 \in H^m$ be a periodic function of period ℓ_0. Further, let Φ be the periodic solution of (1.1) (with period ℓ_0) on a fixed time interval $[0, T]$, satisfying the initial condition $\Phi(0, \cdot) = \Phi_0$. Then, if m is large enough, there exists $\varepsilon_0 = \varepsilon_0(T) \in (0, 1)$ such that, for $0 < \varepsilon \le \varepsilon_0$, Problem (2.12) admits a unique classical solution φ on $[0, T/\varepsilon^2]$, which is periodic with period $\ell_0/\sqrt{\varepsilon}$ with respect to y, and satisfies*

$$\varphi(0, y) = \varepsilon\Phi_0(y\sqrt{\varepsilon}), \qquad |y| \le \frac{\ell_0}{2\sqrt{\varepsilon}}.$$

Moreover, there exists a positive constant C, independent of $\varepsilon \in (0, \varepsilon_0]$, such that

$$|\varphi(t, y) - \varepsilon\Phi(t\varepsilon^2, y\sqrt{\varepsilon})| \le C\varepsilon^2, \qquad 0 \le t \le \frac{T}{\varepsilon^2}, \ |y| \le \frac{\ell_0}{2\sqrt{\varepsilon}},$$

for any $\varepsilon \in (0, \varepsilon_0]$.

In other words, starting from the same configuration, the solution of (2.12) remains on a fixed time interval close to the solution of K–S up to some renormalization, uniformly in ε sufficiently small. Note that the initial condition for φ is of special type, compatible with Φ_0 and (1.1) at $\tau = 0$. Initial conditions of this type have been already considered in [2, 3, 5].

Although energy methods are known to be usually inefficient in fully nonlinear problems, here we may take advantage of the special structure of $\mathscr{F}$. It allows us to

establish sharp a priori estimates on the remainder (more precisely on its derivative) when ε is small enough. A key point is an extension of a lemma that we already successfully used in [2, 5].

2.3. **Some mathematical setting.** In this subsection we briefly introduce some notation, the functional spaces and operators we will use below. We will mainly use the discrete Fourier transform with respect to the variable y. For this purpose, given a function $f : (-\ell/2, \ell/2) \to \mathbb{C}$, we denote by $\widehat{f}(k)$ its k-th Fourier coefficient, that is, we write

$$f(y) = \sum_{k=0}^{+\infty} \widehat{f}(k) w_k(y), \qquad y \in (-\ell/2, \ell/2),$$

where $\{w_k\}$ is a complete set of (complex valued) eigenfunctions of the operator

$$A = D_{yy} : D(A) = H^2(-\ell/2, \ell/2) \to L^2(-\ell/2, \ell/2),$$

with ℓ-periodic boundary conditions, corresponding to the non-positive eigenvalues

$$0, -\frac{4\pi^2}{\ell^2}, -\frac{4\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{16\pi^2}{\ell^2}, -\frac{36\pi^2}{\ell^2}, \dots$$

We shall find it convenient to label this sequence as

$$0 = -\lambda_0(\ell) > -\lambda_1(\ell) = -\lambda_2(\ell) > -\lambda_3(\ell) = -\lambda_4(\ell) > \dots$$

Quite often we simply write λ_k instead of $\lambda_k(\ell)$.

When f depends also on t and/or x, we denote by $\widehat{f}(\cdot, k)$ the k-th Fourier coefficient of f with respect to y. For instance, for fixed t and x, $\widehat{f}(t, x, k)$ will denote the k-th Fourier coefficient of the function $f(t, x, \cdot)$.

For integer or arbitrary real s, we denote by $H^s_\sharp$ the usual Sobolev space of order s consisting of ℓ-periodic (generalized) functions, which we will conveniently represent as

$$H^s_\sharp = \left\{ w = \sum_{k=0}^{+\infty} a_k w_k : \sum_{k=0}^{+\infty} \lambda_k^s a_k^2 < +\infty \right\},$$

with norm

$$\|w\|_s^2 = \sum_{k=0}^{+\infty} \lambda_k^s a_k^2.$$

For $k = 0$, we simply write L^2 instead of $H^0_\sharp$ and $|\cdot|_2$ instead of $\|\cdot\|_0$.

We recall that for any $\beta > 0$ and $\gamma \in (0, 1)$ the operator $(I - \beta A)^\gamma$ has $H^{2\gamma}_\sharp$ as a domain and it is defined by its symbol $((1 + \beta\lambda_k)^\gamma)$ (see e.g., [19, Thm. 4.33]).

Next, for any $n = 0, 1, \dots$ and any $\beta \in [0, 1)$, we set

$$C^{n+\beta}_\sharp = \{ f \in C^{n+\beta}([-\ell/2, \ell/2]) : f^{(j)}(-\ell/2) = f^{(j)}(\ell/2), \ j \leq n \}.$$

$C^{n+\beta}_\sharp$ is endowed with the Euclidean norm of $C^{n+\beta}([-\ell/2, \ell/2])$. Finally, we denote by $\|\cdot\|_\infty$ the sup-norm.

3. **The derivation of a self-consistent equation for the front.** The aim of this section is the derivation of a self-consistent equation (in the Fourier variables) for the front φ. For this purpose, we rewrite Problem (2.9), making $\overline{\theta}$ and $\overline{S}$ explicit. We get

$$\begin{cases} u_x - \Delta u = (\varphi_t + (\varphi_y)^2 - \varphi_{yy})e^x, & x < 0, \\ v_x - \Delta(v - \alpha u) = \alpha(\varphi_t + (\varphi_y)^2)(x+1)e^x - \alpha\varphi_{yy}xe^x, & x < 0, \\ v_x - \Delta v = 0, & x > 0, \\ u(0) = [v] = 0, \\ v(0) - u_x(0) = \frac{1}{2}(\varphi_y)^2, \\ [v_x] = -\alpha u_x(0). \end{cases} \tag{3.1}$$

In what follows, we assume that (u, v, φ) is a sufficiently smooth solution to Problem (3.1) such that the function $x \mapsto e^{-x/2}u(t, x, y)$ is bounded in $(-\infty, 0]$ and the function $x \mapsto e^{-x/2}v(t, x, y)$ is bounded in $\mathbb{R}$. As it has been stressed in Section 2, we use the first equation in (3.1) and the boundary condition $u(\cdot, 0, \cdot) = 0$ as a pseudo-differential Stefan condition. We solve the problem for u via discrete Fourier transform. This leads us to a system of infinitely many equations

$$\widehat{u}_x(t, x, k) - \widehat{u}_{xx}(t, x, k) + \lambda_k \widehat{u}(t, x, k) = \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) + \lambda_k \widehat{\varphi}(t, k) \right) e^x, \tag{3.2}$$

for $k = 0, 1, 2, \ldots$, where we recall that $-\lambda_k = -\lambda_k(\ell)$ is the k-th eigenvalue of the realization of the operator D_{yy} in L^2. For notational convenience we set $\nu_k = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\lambda_k}$ for any $k = 0, 1, \ldots$ A straightforward computation reveals that the solution to (3.2) which vanishes at $x = 0$ and tends to 0 as $x \to -\infty$ not slower than $e^{-x/2}$ is given by

$$\widehat{u}(t, x, 0) = -\left(\widehat{\varphi}_t(t, 0) + \widehat{(\varphi_y)^2}(t, 0) \right) x e^x, \qquad x \le 0,$$

$$\widehat{u}(t, x, k) = \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) + \lambda_k \widehat{\varphi}(t, k) \right) (e^x - e^{\nu_k x}), \qquad x \le 0, \ k = 1, 2, \ldots$$

Let us now consider the problem for v, where we disregard (for the moment) the condition $v(\cdot, 0, \cdot) - u_x(\cdot, 0, \cdot) = \frac{1}{2}(\varphi_y)^2$. Taking the Fourier transform (with respect to the variable y), we get the Cauchy problems

$$\begin{cases} \widehat{v}_x(t, x, 0) - \widehat{v}_{xx}(t, x, 0) = \alpha \left(\widehat{\varphi}_t(t, 0) + \widehat{(\varphi_y)^2}(t, 0) \right) (2x+3)e^x, & x < 0, \\ \widehat{v}_x(t, x, 0) - \widehat{v}_{xx}(t, x, 0) = 0, & x > 0, \\ [\widehat{v}(t, \cdot, 0)] = 0, \\ [\widehat{v}_x(t, \cdot, 0)] = -\alpha\widehat{u}_x(t, 0, 0) = \alpha \left(\widehat{\varphi}_t(t, 0) + \widehat{(\varphi_y)^2}(t, 0) \right), \end{cases}$$

for $k = 0$, and

$$
\begin{cases}
\widehat{v}_x(t, x, k) - \widehat{v}_{xx}(t, x, k) + \lambda_k \widehat{v}(t, x, k) \\
\quad = \alpha \left(x + 2 - \dfrac{1}{\lambda_k} \right) \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) \right) e^x + \alpha \lambda_k \left(x + 1 - \dfrac{1}{\lambda_k} \right) \widehat{\varphi}(t, k) e^x \\
\qquad + \dfrac{\alpha \nu_k}{\lambda_k} \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) + \lambda_k \widehat{\varphi}(t, k) \right) e^{\nu_k x}, & x < 0, \\[2mm]
\widehat{v}_x(t, x, k) - \widehat{v}_{xx}(t, x, k) + \lambda_k \widehat{v}(t, x, k) = 0, & x > 0, \\
[\widehat{v}(t, \cdot, k)] = 0, \\
[\widehat{v}_x(t, \cdot, k)] = -\alpha \widehat{u}_x(t, 0, k) = \alpha \nu_k^{-1} \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) + \lambda_k \widehat{\varphi}(t, k) \right),
\end{cases}
$$

for $k \geq 1$.

It is easy to show that

$$
\widehat{v}(t, x, 0) = -\alpha \left(\widehat{\varphi}_t(t, 0) + \widehat{(\varphi_y)^2}(t, 0) \right) x(x + 1) e^x, \qquad x < 0,
$$

$$
\widehat{v}(t, x, 0) = 0, \qquad x > 0.
$$

and

$$
\widehat{v}(t, x, k) = c_{1,k} e^{\nu_k x} + \frac{\alpha}{\lambda_k} \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) \right) (x + 2) e^x + \alpha \widehat{\varphi}(t, k)(x + 1) e^x
$$

$$
\qquad + \frac{\alpha}{\lambda_k} \frac{\nu_k}{1 - 2\nu_k} \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) + \lambda_k \widehat{\varphi}(t, k) \right) x e^{\nu_k x}, \qquad x < 0,
$$

$$
\widehat{v}(t, x, k) = c_{2,k} e^{(1 - \nu_k) x}, \qquad x \geq 0,
$$

where

$$
c_{1,k} = \frac{\alpha}{1 - 2\nu_k} \left(1 + \nu_k + \frac{\nu_k}{1 - 2\nu_k} + \frac{\lambda_k}{\nu_k} \right) \widehat{\varphi}(t, k)
$$

$$
\qquad + \frac{\alpha}{1 - 2\nu_k} \left(\frac{1}{\lambda_k} + \frac{2\nu_k}{\lambda_k} + \frac{1}{\nu_k} + \frac{1}{\lambda_k} \frac{\nu_k}{1 - 2\nu_k} \right) \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) \right),
$$

$$
c_{2,k} = \frac{\alpha}{1 - 2\nu_k} \left(2 + \frac{\nu_k}{1 - 2\nu_k} + \frac{\lambda_k}{\nu_k} - \nu_k \right) \widehat{\varphi}(t, k)
$$

$$
\qquad - \frac{\alpha}{1 - 2\nu_k} \left(\frac{2\nu_k}{\lambda_k} - \frac{3}{\lambda_k} - \frac{1}{\lambda_k} \frac{\nu_k}{1 - 2\nu_k} - \frac{1}{\nu_k} \right) \left(\widehat{\varphi}_t(t, k) + \widehat{(\varphi_y)^2}(t, k) \right).
$$

Now, we are in a position to determine the equation for the front. Indeed, rewriting the boundary condition

$$
v(0) - u_x(0) = \frac{1}{2} (\varphi_y)^2,
$$

in Fourier variables, and using the above results, we get to the following equations for the front (in the Fourier coordinates):

$$
\widehat{\varphi}_t(t, 0) + \frac{1}{2} \widehat{(\varphi_y)^2}(t, 0) = 0,
$$

$$
\left\{ \frac{\alpha}{1 - 2\nu_k} \left(2 + \frac{\nu_k}{1 - 2\nu_k} + \frac{\lambda_k}{\nu_k} - \nu_k \right) + \frac{\lambda_k}{\nu_k} \right\} \widehat{\varphi}(t, k)
$$

$$+\left\{-\frac{\alpha}{1-2\nu_k}\left(\frac{2\nu_k}{\lambda_k}-\frac{3}{\lambda_k}-\frac{1}{\lambda_k}\frac{\nu_k}{1-2\nu_k}-\frac{1}{\nu_k}\right)+\frac{1}{\nu_k}\right\}\widehat{\varphi}_t(t,k)$$

$$+\left\{-\frac{\alpha}{1-2\nu_k}\left(\frac{2\nu_k}{\lambda_k}-\frac{3}{\lambda_k}-\frac{1}{\lambda_k}\frac{\nu_k}{1-2\nu_k}-\frac{1}{\nu_k}\right)+\frac{1}{\nu_k}-\frac{1}{2}\right\}\widehat{(\varphi_y)^2}(t,k)=0.$$

Let us set $X_k=\sqrt{1+4\lambda_k}$. Then, the equation for φ reads (in terms of X_k) as follows:

$$\frac{(X_k-1)(X_k^2-\alpha)}{2X_k^2}\widehat{\varphi}(t,k)+\frac{2(X_k^2+\alpha X_k-\alpha)}{X_k^2(X_k+1)}\widehat{\varphi}_t(t,k)$$

$$-\frac{X_k^3-3X_k^2-4\alpha X_k+4\alpha}{2X_k^2(X_k+1)}\widehat{(\varphi_y)^2}(t,k)=0,$$

for any $k=0,1,2,\ldots$, or, equivalently,

$$4\widehat{\varphi}_t(t,k)=\frac{(1-X_k^2)(X_k^2-\alpha)}{X_k^2+\alpha X_k-\alpha}\widehat{\varphi}(t,k)+\frac{X_k^3-3X_k^2-4\alpha X_k+4\alpha}{X_k^2+\alpha X_k-\alpha}\widehat{(\varphi_y)^2}(t,k),\quad(3.3)$$

or

$$(X_k^2+\alpha X_k-\alpha)\widehat{\varphi}_t(t,k)=\frac{1}{4}(1-X_k^2)(X_k^2-\alpha)\widehat{\varphi}(t,k)$$

$$+\frac{1}{4}(X_k^3-3X_k^2-4\alpha X_k+4\alpha)\widehat{(\varphi_y)^2}(t,k)$$

$$=(-4\lambda_k^2+(\alpha-1)\lambda_k)\widehat{\varphi}(t,k)$$

$$+\frac{1}{4}(X_k^3-3X_k^2-4\alpha X_k+4\alpha)\widehat{(\varphi_y)^2}(t,k),$$

for any $k=0,1,\ldots$ Hence, we can conclude that φ solves the equations

$$\frac{d}{dt}\mathscr{B}(\varphi)=\mathscr{S}(\varphi)+\mathscr{F}((\varphi_y)^2)\tag{3.4}$$

and

$$\varphi_t=\mathscr{B}^{-1}\mathscr{S}(\varphi)+\mathscr{B}^{-1}\mathscr{F}((\varphi_y)^2):=\mathscr{L}(\varphi)+\mathscr{G}((\varphi_y)^2),\tag{3.5}$$

where the operators $\mathscr{B}$, $\mathscr{S}$ and $\mathscr{F}$ are defined through their symbols

$$b_k=X_k^2+\alpha X_k-\alpha,\tag{3.6}$$

$$s_k=-4\lambda_k^2+(\alpha-1)\lambda_k,\tag{3.7}$$

$$f_k=\frac{1}{4}(X_k^3-3X_k^2-4\alpha X_k+4\alpha),\tag{3.8}$$

for any $k=0,1,\ldots$

4. **Stability of the front.** In this section we are interested in the stability and instability properties of the null solution to the Equations (3.4) and (3.5). In this respect we need to study the symbols appearing in (3.3).

4.1. **Study of the symbols.** In this subsection, we study the main properties of the operators $\mathscr{B}$, $\mathscr{G}$, $\mathscr{S}$, and $\mathscr{L}$, $\mathscr{F}$, whose symbols are respectively defined by (3.6)-(3.8) and by

$$l_k=\frac{(1-X_k^2)(X_k^2-\alpha)}{4(X_k^2+\alpha X_k-\alpha)},$$

$$g_k=\frac{X_k^3-3X_k^2-4\alpha X_k+4\alpha}{4(X_k^2+\alpha X_k-\alpha)},$$

for any $k = 0, 1, \ldots$ Even if all these operators depend on α, we prefer not to stress explicitly the dependence on α to avoid cumbersome notations.

Proposition 4.1. *The following properties are satisfied.*

(i) *The operator $\mathscr{L}$ admits a realization L in L^2 which is a sectorial operator. Moreover, its spectrum consists of the sequence (l_k). In particular, 0 is a simple eigenvalue of L. The spectral projection associated with this eigenvalue is the operator Π defined by*

$$\Pi(\psi) = \frac{1}{\ell} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \psi(y) dy, \qquad \psi \in L^2.$$

Finally, $\sigma(L) \setminus \{0\} \subset (-\infty, 0)$ if and only if $\alpha < \alpha_c$ (see (2.13)).

(ii) *The operator $\mathscr{B}$ admits a bounded realization B mapping $H_\sharp^2$ into L^2. Moreover, B is invertible.*

(iii) *The operator $\mathscr{F}$ admits a bounded realization F mapping $H_\sharp^3$ into L^2.*

(iv) *The operator $\mathscr{G}$ admits a bounded realization G mapping $H_\sharp^1$ into L^2.*

(v) *The realization of the operator $\mathscr{S}$ in L^2 is the operator*

$$S = -4D_{yyyy} + (\alpha - 1)D_{yy},$$

with $H_\sharp^4$ as domain.

Proof. (i). To begin with, we observe that

$$l_k = -\frac{\lambda_k(4\lambda_k + 1 - \alpha)}{\alpha\sqrt{4\lambda_k + 1} + 4\lambda_k + 1 - \alpha}.$$

Hence, we can split

$$l_k = -\lambda_k + \frac{\alpha\lambda_k\sqrt{1 + 4\lambda_k}}{\alpha\sqrt{1 + 4\lambda_k} + 4\lambda_k + 1 - \alpha} := -\lambda_k + l_{1,k},$$

for any $k = 0, 1, \ldots$ Note that $l_{1,k} \sim \frac{\alpha}{2}\sqrt{\lambda_k}$ as $k \to +\infty$. Hence, from the above splitting of the symbol (l_k) it follows at once that the operator $\mathscr{L}$ admits a realization L in L^2 with domain $D(L) = H_\sharp^2$ which can be split as $L = A + L_1$, where L_1 is a bounded operator from $H_\sharp^1$ into L^2, and A is the realization of D_{yy} in L^2 with domain $H_\sharp^2$. Since $H_\sharp^1$ is an intermediate space of class $J_{1/2}$ between L^2 and $D(A)$, [18, Prop. 2.4.1(i)] applies and shows that L is sectorial.

Let us now compute the spectrum of the operator L. For this purpose, we observe that, since $D(L)$ is compactly embedded into L^2, $\sigma(L)$ consists of eigenvalues only. Further, if λ is an eigenvalue of L, then there exists a not identically vanishing function ψ such that $L\psi = \lambda\psi$. In the Fourier variables, the previous equation leads to the system of infinitely many equations:

$$\lambda\widehat{\psi}(k) - l_k\widehat{\psi}(k) = 0, \qquad k = 0, 1, 2, \ldots$$

If $\lambda \neq l_k$, then $\widehat{\psi}(k) = 0$. Hence, if λ is not an element of the sequence (l_k), λ is in the resolvent set of L. On the other hand, it is clear that the sequence (l_k) consists of eigenvalues of L. So $\sigma(L) = \{l_k : k = 0, 1, \ldots\}$.

Since $l_k \to -\infty$ as $k \to +\infty$, 0 is an isolated point of the spectrum of L and the corresponding eigenspace is one-dimensional. Let us prove that Π is the spectral projection associated with such an eigenvalue. For this purpose, we prove that 0

is a simple pole of the function $\lambda \mapsto R(\lambda, L)$ and compute the residual at 0. Note that for any $\lambda \notin \sigma(L)$ and any $\psi \in H^2_\sharp$ it holds that

$$R(\lambda, L)\psi = \sum_{k=0}^{+\infty} \frac{1}{\lambda - l_k} \widehat{\psi}(k) w_k.$$

Hence,

$$\lambda R(\lambda, L)\psi = \widehat{\psi}(0) w_0 + \sum_{k=1}^{+\infty} \frac{\lambda}{\lambda - l_k} \widehat{\psi}(k) w_k = \Pi\psi + \sum_{k=1}^{+\infty} \frac{\lambda}{\lambda - l_k} \widehat{\psi}(k) w_k.$$

Hence, for $|\lambda| \leq \frac{1}{2} \min\limits_{k=1,2,\dots} |l_k|$, we can estimate

$$|\lambda R(\lambda, L)\psi - \Pi\psi|_2^2 \leq \sum_{k=1}^{+\infty} \left| \frac{\lambda}{\lambda - l_k} \right|^2 |\widehat{\psi}(k)|^2 \leq \frac{2|\lambda|^2}{l_{\min}} \sum_{k=1}^{+\infty} |\widehat{\psi}(k)|^2 \leq \frac{2|\lambda|^2}{l_{\min}} |\psi|_2^2,$$

where $l_{\min} = \min_{n=1,2,\dots} |l_n| > 0$. This shows that $R(\lambda I - L)$ has a simple pole at $\lambda = 0$ and its residual is the operator Π, which turns out to be spectral projection associated with the eigenvalue 0, which is simple. For more details, we refer the reader to e.g., [18, Prop. A.1.2 & A.2.1].

To conclude the proof of point (i), we observe that $l_k < 0$, for $k \geq 1$, if and only if $1 + 4\lambda_k - \alpha > 0$. Since (λ_k) is a nondecreasing sequence, $l_k < 0$ for any $k = 1, 2, \dots$, if and only if $4\lambda_1 + 1 - \alpha > 0$, i.e., if and only if $\alpha < \alpha_c$.

(ii), (iii) & (iv). It is enough to observe that $b_k \sim 4\lambda_k$, $f_k \sim 2\lambda_k^{3/2}$, $g_k \sim \frac{1}{2}\sqrt{\lambda_k}$ as $k \to +\infty$ and $b_k \neq 0$ for any $k = 0, 1, \dots$

(v). It is immediate and, hence, omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Proof of Theorem** 2.1. The proof is rather classical and is based on the results in Propositions 4.1. Nevertheless, for the reader's convenience we go into details. We split the proof in two steps: in the first one we deal with Equation (3.5) and in the second one we consider Equation (3.4).

Step 1. Using classical arguments based on a fixed point argument, one can show that for any $\alpha \in \mathbb{R}$ and any $T > 0$, there exists $r_0 > 0$ such that, if $\|\varphi_0\|_2 \leq r_0$, the Cauchy problem

$$\begin{cases} \varphi_t(t,y) = (L\varphi(t,\cdot))(y) + (G((\varphi_y(t,\cdot))^2))(y), & t > 0, \quad |y| \leq \frac{\ell}{2}, \\ \varphi(t, -\ell/2) = \varphi(t, \ell/2), & t > 0, \\ \varphi_y(t, -\ell/2) = \varphi_y(t, \ell/2), & t > 0, \\ \varphi(0, y) = \varphi_0(y), & |y| \leq \frac{\ell}{2}, \end{cases} \qquad (4.1)$$

admits a unique solution $\psi \in \bigcup_{\theta \in (0,1)} \mathscr{X}_\theta(T)$, where

$$\mathscr{X}_\theta(T) = \left\{ \psi \in C([0,T]; H^2_\sharp) \cap C^1([0,T]; L^2) : \sup_{0 < \varepsilon < T} \varepsilon^\theta [\psi]_{C^\theta([\varepsilon,T];H^2_\sharp)} \right\}.$$

This can be proved slightly adapting the proof of [18, Thm. 8.1.1]. The crucial point is the estimate

$$s^\theta |G((\psi_y(t,\cdot))^2) - G((\psi_y(s,\cdot))^2)|_2 \leq C_1 s^\theta |\psi_y(s,\cdot)|_2 [\psi]_{C^\theta([s,T];L^2)} |t - s|^\theta, \quad (4.2)$$

for any $0 < s < t \leq T$, some positive constant C_1 and any $\psi \in \mathscr{X}_\theta(T)$ $(\theta \in (0,1))$. To prove this estimate it suffices to observe that, by Proposition 4.1(iv)

$$|G((\psi_y(t,\cdot))^2) - G((\psi_y(s,\cdot))^2)|_2^2$$

$$\leq C|(\psi_y(t,\cdot))^2 - (\psi_y(s,\cdot))^2|_2 + C|D_y(\psi_y(t,\cdot))^2 - D_y(\psi_y(s,\cdot))^2|_2^2$$
$$\leq C|\psi_y(t,\cdot) - \psi_y(s,\cdot)|_2 \|\psi_y(t,\cdot) + \psi_y(s,\cdot)\|_\infty$$
$$\quad + C|\psi_{yy}(t,\cdot)|_2 \|\psi_y(t,\cdot) - \psi_y(s,\cdot)\|_\infty$$
$$\quad + C\|\psi_y(s,\cdot)\|_\infty |\psi_{yy}(t,\cdot) - \psi_{yy}(s,\cdot)|_2$$
$$\leq C|\psi_y(t,\cdot) - \psi_y(s,\cdot)|_2 |\psi_{yy}(t,\cdot) + \psi_{yy}(s,\cdot)|_2$$
$$\quad + C\left(|\psi_{yy}(t,\cdot)|_2 + C|\psi_y(s,\cdot)|_2\right)|\psi_{yy}(t,\cdot) - \psi_{yy}(s,\cdot)|_2,$$

for any $0 < s < T$, where the last side of the previous chain of inequalities follows from the Poincaré-Wirtinger inequality, and C denotes a positive constant, independent of s, t and ψ, which may vary from line to line. Estimate (4.2) now follows at once.

Let us now prove properties (a) and (b). It is convenient to split the solution φ to Equation (3.5) along $\Pi(L^2)$ and $(I-\Pi)(L^2)$. We get $\varphi(t,y) = p(t)w_0 + \psi(t,y)$ for any $t > 0$ and any $y \in [-\ell/2, \ell/2]$. Since Π commutes with both the time and the spatial derivatives, $\Pi(D_t\varphi) = D_t\Pi(\varphi) = p'$ and $(I - \Pi)(D_y\varphi) = D_y(I - \Pi)(\varphi) = D_y w$. Moreover, for any $\psi \in H^1_\sharp$, $G(\psi) = \sum_{k=0}^{+\infty} g_k \widehat{\psi}(k) w_k$, so that

$$\Pi G(\psi) = g_0 \widehat{\psi}(0) = -\frac{1}{2}\Pi\psi.$$

Hence, projecting the Cauchy problem (4.1) along $\Pi(L^2)$ and $(I - \Pi)(L^2)$, we get the two self-consistent equations for p and ψ:

$$\begin{cases} p'(t) = -\frac{1}{2}\Pi((\psi_y(t,\cdot))^2), & t > 0, \\ p(0) = \Pi(\varphi_0), \end{cases} \tag{4.3}$$

and

$$\begin{cases} \psi_t(t,y) = (L\psi(t,\cdot))(y) + (I-\Pi)(G((\psi_y(t,\cdot))^2))(y), & t > 0, \quad |y| \leq \frac{\ell}{2}, \\ \psi(t,-\ell/2) = \psi(t,\ell/2), & t > 0, \\ \psi_y(t,-\ell/2) = \psi_y(t,\ell/2), & t > 0, \\ \psi(0,y) = ((I - \Pi)(\varphi_0))(y), & |y| \leq \frac{\ell}{2}. \end{cases} \tag{4.4}$$

Clearly, the stability of the null solution to Equation (3.5) depends only on the stability of the null solution to the equation $\psi_t = L\psi + (I - \Pi)(G((\psi_y)^2))$, set in $(I - \Pi)(L^2)$.

Note that the part of the operator L in $(I - \Pi)(L^2)$ is still a sectorial operator, and its spectrum is $\sigma(L) \setminus \{0\} = \{l_k : k = 1, 2, \ldots\}$. In particular, all the elements of $\sigma(L) \setminus \{0\}$ lie in $(-\infty, 0)$. Hence, the linearized stability principle applies to this situation. More specifically, in the case where $\alpha < \alpha_c$ all the eigenvalues of the part of L in $(I - \Pi)(L^2)$ are contained in the plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$. Hence, up to replacing r_0 with a smaller value (if needed), for any $\varphi \in B(0, r) \subset L^2$, the solution ψ to Problem (4.4) exists for all positive times. Moreover, for any $\omega > \max\{l_k : k = 1, 2, \ldots\}$, there exists a positive constant C_ω such that

$$|\psi_t(t,\cdot)|_2 + \|\psi(t,\cdot)\|_2 \leq C_\omega e^{\omega t}\|\varphi_0\|_2, \qquad t > 0.$$

As a byproduct, we can infer that the solution to Problem (4.3) exists for all positive times and

$$\lim_{t \to +\infty} p(t) = p_\infty := \Pi\varphi_0 - \frac{1}{2}\int_0^{+\infty} \Pi((\psi_y(t,\cdot))^2)dt.$$

Coming back to Problem (4.1), the above results show that, if $\alpha < \alpha_c$, this problem admits a unique solution, defined for all positive times. Moreover,

$$|\varphi_t(t,\cdot)|_2 + \|\varphi(t,\cdot) - p_\infty\|_\infty + \|\varphi_y(t,\cdot)\|_\infty + |\varphi_{yy}(t,\cdot)|_2 \leq P_\omega e^{\omega t}\|\varphi_0\|_2,$$

for any $t > 0$, any ω as above and some positive constant P_ω independent of s, φ_0 and φ, i.e., the null solution to Equation (3.5) is (orbitally) stable with asymptotic phase.

In the case where $\alpha > \alpha_c$ the spectrum of $L_{|(I-\Pi)(L^2)}$ contains (a finite number of) eigenvalues with positive real part. Hence, the equation $\psi_t = L\psi + (I-\Pi)(G((\psi_y)^2))$ admits a backward solution, exponentially decreasing to 0 at $-\infty$ and this implies that the null solution to Problem (4.4) and, consequently, the null solution to Problem (4.1) are unstable. For further details, we refer the reader to e.g., [12] and [18, Thm. 9.1.2 & 9.1.3].

Step 2. We focus on the case where $\alpha < \alpha_c$, the other case being simpler. Of course, we just need to deal with the function $\psi = (I - \Pi)\varphi$. We assume that $\varphi_0 \in H^4_\sharp$. We are going to show that for any $\omega \in (0, \max_{k=1,2,\dots} l_k)$, it holds that

$$\sup_{t>0} e^{-\omega t}\|\varphi(t,\cdot)\|_4 + \sup_{t>0} e^{-\omega t}\|\varphi_t(t,\cdot)\|_2 < +\infty.$$

For this purpose, let us consider the differentiated problem

$$\begin{cases} \rho_t(t,y) = (L\rho(t,\cdot))(y) + (D_{yy}(I-\Pi)(G((\mathscr{P}(\rho(t,\cdot)))^2)))(y), & t > 0, \quad |y| \leq \frac{\ell}{2}, \\ \rho(t,-\ell/2) = \rho(t,\ell/2), & t > 0, \\ \rho_y(t,-\ell/2) = \rho_y(t,\ell/2), & t > 0, \\ \rho(0,y) = D_{yy}\varphi_0(y), & |y| \leq \frac{\ell}{2}, \end{cases} \tag{4.5}$$

for the unknown $\rho = \psi_{yy}$. Here,

$$\mathscr{P}(\zeta) = (I - \Pi)\left(y \mapsto \int_{-\frac{\ell}{2}}^{y} \zeta(\eta)d\eta\right), \qquad \zeta \in L^2. \tag{4.6}$$

This problem has the same structure as Problem (4.4), and, by assumptions, $D_{yy}\varphi_0 \in H^2_\sharp$. Therefore, up to taking a smaller r_0 (if necessary), if $\|D_{yy}\varphi_0\|_2 \leq r_0$, Problem (4.5) has a solution ρ which belongs to $C^1([0,T];L^2) \cap C([0,T];H^2_\sharp)$ for any $T > 0$. Moreover,

$$\sup_{t>0} e^{-\omega t}\|\rho(t,\cdot)\|_2 < +\infty,$$

and $\rho(t,\cdot) = (I - \Pi)\rho(t,\cdot)$ for any $t > 0$.

Let us show that $\psi(t,\cdot) = \mathscr{P}^2(\rho(t,\cdot))$ for any $t > 0$. Clearly, the function $\Psi = \mathscr{P}^2(\rho)$ belongs to $C([0,+\infty);H^4_\sharp) \cap C^1([0,+\infty);L^2)$. Moreover, it belongs to $\mathscr{X}_{1/2}(T)$ for any $T > 0$. Indeed, $H^2_\sharp$ belongs to the class $J_{1/2}$ between L^2 and $H^4_\sharp$. This means that

$$\|\Psi(t,\cdot) - \Psi(s,\cdot)\|_2 \leq C\|\Psi(t,\cdot) - \Psi(s,\cdot)\|_2^{\frac{1}{2}}\|\Psi(t,\cdot) - \Psi(s,\cdot)\|_4^{\frac{1}{2}}$$

$$\leq \sqrt{2}C\|\Psi_t\|_{C([0,T];L^2)}^{\frac{1}{2}}\|\Psi\|_{C([0,T];H^4_\sharp)}^{\frac{1}{2}}|t - s|^{\frac{1}{2}},$$

for any $0 \leq s \leq t \leq T$ and some positive constant C, independent of s, t and Ψ. From this estimate, it is clear that $\Psi \in C^{1/2}([0,T];H^2_\sharp) \subset \mathscr{X}_{1/2}(T)$ for any $T > 0$.

Further, $D_{yy}\Psi = \rho$ and $D_t\Psi = \mathscr{P}^2(D_t\rho)$, so that $D_{yy}D_t\Psi = D_tD_{yy}\Psi = \rho_t$. It turns out that

$$(i)\ D_{yy}(D_t\Psi - L\Psi - (I - \Pi)G((\Psi_y)^2)) \equiv 0, \qquad (ii)\ \Psi(0,\cdot) \equiv (I - \Pi)\varphi_0.$$

Hence, $D_t\Psi - L\Psi - (I-\Pi)G((\Psi_y)^2) = a(t) + b(t)y$ for some functions $a, b : [0, +\infty) \to \mathbb{R}$. Since Ψ_t, $L\Psi$ and $G((\Psi_y)^2)$ are continuous functions in $[0, +\infty) \times [-\ell_0/2, \ell_0/2]$ and are periodic with respect to y, it follows that $D_t\Psi - L\Psi - (I - \Pi)G((\Psi_y)^2)$ is periodic with respect to y as well. Moreover, this latter function belongs to $(I - \Pi)(L^2)$ since Ψ does. Hence, $a = b \equiv 0$, implying that Ψ and ψ actually coincide. We have thus proved that $\psi \in C([0, +\infty); H_\sharp^4)$ and $D_t\psi \in C([0, +\infty); H_\sharp^2)$. Moreover,

$$\sup_{t>0} e^{-\omega t}|\psi_{tyy}(t,\cdot)|_2 + \sup_{t>0} e^{-\omega t}\|\psi(t,\cdot)\|_{H_\sharp^4} < +\infty.$$

To complete the proof it suffices to show that φ solves Equation (3.4), but this follows immediately observing that φ is in the domain of both the operators B (see Proposition 4.1(ii)) and S, and $(\varphi_y)^2$ is in the domain of the operator F (see Proposition 4.1(iii)). Further, $L = B^{-1}S$ and $G = B^{-1}F$ in $H_\sharp^4$. Since φ solves the differential equation $\varphi_t - L\varphi - G((\varphi_y)^2) = 0$, applying B to both sides of the equation, it now follows immediately that φ solves Equation (3.4).

5. **Rigorous derivation of the Kuramoto-Sivashinsky equation.** In this section we are interested in proving Theorem 2.2.

5.1. **Rescaling and equation for the remainder.** Let φ be a solution to (2.12). We set $\alpha = 1 + \varepsilon$ and define the rescaled dependent and independent variables:

$$t = \tau/\varepsilon^2, \quad y = \eta/\sqrt{\varepsilon}, \quad \varphi = \varepsilon\psi.$$

The spatial period is now $\ell_\varepsilon = \ell_0/\sqrt{\varepsilon}$, for some $\ell_0 > 4\pi$ fixed, see the Introduction. A straightforward computation reveals that the function ψ satisfies the equation

$$\frac{\partial}{\partial\tau}\mathscr{B}_\varepsilon(\psi) = \mathscr{S}(\psi) + \mathscr{F}_\varepsilon((D_\eta\psi)^2), \tag{5.1}$$

where

$$\mathscr{B}_\varepsilon = I - 4\varepsilon D_{\eta\eta} + (1 + \varepsilon)\left(\sqrt{I - 4\varepsilon D_{\eta\eta}} - 1\right),$$

$$\mathscr{S} = -4D_{\eta\eta\eta\eta} - D_{\eta\eta},$$

$$\mathscr{F}_\varepsilon = \frac{1}{4}\left\{(I - 4\varepsilon D_{\eta\eta})^{\frac{3}{2}} - 3(I - 4\varepsilon D_{\eta\eta}) - 4(1 + \varepsilon)\left(\sqrt{1 - 4\varepsilon D_{\eta\eta}} - I\right)\right\}(D_\eta\cdot)^2.$$

Note that, if we denote by $(\lambda_k(\ell))$ the sequence of the eigenvalues of the second-order derivative with periodic boundary conditions in $[-\ell/2, \ell/2]$, it turns out that $\lambda_k(\ell_\varepsilon) = \varepsilon\lambda_k(\ell_0) := \varepsilon\lambda_k$, for any $k = 0, 1, \ldots$. Hence, the symbols of the operators $\mathscr{B}_\varepsilon$, $\mathscr{S}$ and $\mathscr{F}_\varepsilon$ are

$$b_{\varepsilon,k} = X_{\varepsilon,k}^2 + (1 + \varepsilon)X_{\varepsilon,k} - 1 - \varepsilon,$$

$$s_k = -\lambda_k(4\lambda_k - 1),$$

$$f_{\varepsilon,k} = \frac{1}{4}(X_{\varepsilon,k}^3 - 3X_{\varepsilon,k}^2 - 4(1 + \varepsilon)X_{\varepsilon,k} + 4 + 4\varepsilon), \tag{5.2}$$

for any $k = 0, 1, \ldots$, where

$$X_{\varepsilon,k} = \sqrt{1 + 4\varepsilon\lambda_k}, \qquad k = 0, 1, \ldots$$

Hence, the equation for the function ψ (in Fourier coordinates) reads

$$b_{\varepsilon,k}\widehat{\psi}_\tau(\tau,k) = -\lambda_k(4\lambda_k - 1)\widehat{\psi}(\tau,k) + f_{\varepsilon,k}\widehat{(\psi_\eta)^2}(\tau,k),$$

for any $k = 0, 1, \ldots$ Note that the leading terms (at order 0 in ε) of $b_{\varepsilon,k}$ and $f_{\varepsilon,k}$ are 1 and $-1/2$, respectively. Hence, at the zero-order, we recover the K–S equation

$$\Phi_\tau + 4\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0.$$

As we recall in the Introduction, this equation has been thoroughly studied by many authors. For our purposes, we need the following classical result. For the reader's convenience we provide a rather detailed proof in Appendix B.

Theorem 5.1. *Let $\Phi_0 \in H_\sharp^m$ for some $m \geq 4$ and fix $T > 0$. Then, the Cauchy problem*

$$
\begin{cases}
\Phi_\tau(\tau,\eta) = -4\Phi_{\eta\eta\eta\eta}(\tau,\eta) - \Phi_{\eta\eta}(\tau,\eta) - \frac{1}{2}(\Phi_\eta(\tau,\eta))^2, & \tau \geq 0, \quad |\eta| \leq \frac{\ell_0}{2}, \\
D_\eta^k\Phi(\tau,-\ell_0/2) = D_\eta^k\Phi(\tau,\ell_0/2), & \tau \geq 0, \quad k = 0,1,2,3, \\
\Phi(0,\eta) = \Phi_0(\eta), & |\eta| \leq \frac{\ell_0}{2},
\end{cases}
\tag{5.3}
$$

admits a unique solution $\Phi \in C([0,T]; H_\sharp^m)$ such that $\Phi_\tau \in C([0,T]; H_\sharp^{m-4})$.

The above (heuristical) arguments suggest to split ψ as follows:

$$\psi = \Phi + \varepsilon\rho_\varepsilon.$$

To avoid cumbersome notation, we usely write ρ for ρ_ε. By assumptions (see Theorem 2.2), the initial condition for ρ is

$$\rho(0,\cdot) = 0.$$

Replacing ψ in (5.1) we get, after simplifying by ε,

$$\frac{\partial}{\partial\tau}\mathscr{B}_\varepsilon(\rho) + \mathscr{H}_\varepsilon(\Phi_\tau) = \mathscr{S}(\rho) + \mathscr{M}_\varepsilon((\Phi_\eta)^2) + \varepsilon\mathscr{F}_\varepsilon((\rho_\eta)^2) + 2\mathscr{F}_\varepsilon(\Phi_\eta\rho_\eta), \tag{5.4}$$

for any $k = 0, 1, \ldots$, where the symbols of the operators $\mathscr{H}_\varepsilon$ and $\mathscr{M}_\varepsilon$ are

$$h_{\varepsilon,k} = \frac{1}{\varepsilon}(X_{\varepsilon,k}^2 + (1+\varepsilon)X_{\varepsilon,k} - 2 - \varepsilon), \tag{5.5}$$

$$m_{\varepsilon,k} = \frac{1}{4\varepsilon}(X_{\varepsilon,k}^3 - 3X_{\varepsilon,k}^2 - 4(1+\varepsilon)X_{\varepsilon,k} + 6 + 4\varepsilon), \tag{5.6}$$

for any $k = 0, 1, \ldots$

Proposition 5.2. *Fix $\varepsilon \in (0,1]$. Then there exists a positive constant C_* such that the following properties are satisfied:*

(a) *for any $s = 2, 3, \ldots$, the operators $\mathscr{B}_\varepsilon$ and $\mathscr{H}_\varepsilon$ admit bounded realizations B_ε and H_ε, respectively, mapping $H_\sharp^s$ into $H_\sharp^{s-2}$. Moreover*

$$\|B_\varepsilon\|_{L(H_\sharp^s, H_\sharp^{s-2})} + \|H_\varepsilon\|_{L(H_\sharp^s, H_\sharp^{s-2})} \leq C_*,$$

for any $\varepsilon \in (0,1]$ and any s as above. Finally, the operator B_ε is invertible both from $H_\sharp^s$ to $H_\sharp^{s-2}$ for any $s = 2, 3, \ldots$;

(b) *for any $s \geq 3$, the operators $\mathscr{F}_\varepsilon$ and $\mathscr{M}_\varepsilon$ admit bounded realizations F_ε and M_ε, respectively, mapping H^s into H^{s-3}. Moreover,*

$$\|F_\varepsilon\|_{L(H_\sharp^s, H_\sharp^{s-3})} + \|M_\varepsilon\|_{L(H_\sharp^s, H_\sharp^{s-3})} \leq C_*,$$

for any $\varepsilon \in (0,1]$ and any $s = 3, 4, \ldots$

Proof. (a). A straightforward computation shows that

$$|h_{\varepsilon,k}| = 4\lambda_k + \frac{4(\varepsilon+1)\lambda_k}{\sqrt{1+4\varepsilon\lambda_k}+1} \le 4\lambda_k + 2(\varepsilon+1)\lambda_k = (6+2\varepsilon)\lambda_k,$$

for any $k = 0, 1, \ldots$ and any $\varepsilon \in (0, 1]$. This shows that $\mathscr{H}_\varepsilon$ admits a bounded realization mapping $H_\sharp^s$ into $H_\sharp^{s-2}$ for any $s \ge 2$ and its norm can be bounded by a constant, independent of $\varepsilon \in (0, 1]$.

Since $b_{\varepsilon,k} = \varepsilon h_{\varepsilon,k} + 1$ for any $k = 0, 1, \ldots$, the boundedness of the operator B_ε from $H_\sharp^s$ to $H_\sharp^{s-2}$ follows at once.

Showing that the operator B_ε is invertible from $H_\sharp^s$ into $H_\sharp^{s-2}$ is an easy task. It suffices to observe that $b_{\varepsilon,k} \ge 4\varepsilon\lambda_k + 1$ for any $k = 0, 1, \ldots$.

(b). Since $f_k = \varepsilon m_{\varepsilon,k} - 1/2$ for any $k = 0, 1, \ldots$, we can limit ourselves to considering the operator $\mathscr{M}_\varepsilon$. A simple computation shows that

$$\begin{aligned}
|m_{\varepsilon,k}| &\le \frac{(1+4\varepsilon\lambda_k)^{\frac{3}{2}}-1}{4\varepsilon} + 3\lambda_k + 4\frac{1+\varepsilon}{\varepsilon}\left(\sqrt{1+4\varepsilon\lambda_k}-1\right) \\
&\le \frac{16\varepsilon^2\lambda_k^3 + 12\varepsilon\lambda_k^2 + 3\lambda_k}{(1+4\varepsilon\lambda_k)^{\frac{3}{2}}+1} + 19\lambda_k \\
&\le \frac{16\varepsilon^2\lambda_k^3}{(1+4\varepsilon\lambda_k)^{\frac{3}{2}}} + \frac{12\varepsilon\lambda_k^2}{1+4\varepsilon\lambda_k} + 22\lambda_k \\
&\le 2\sqrt{\varepsilon}\lambda_k^{\frac{3}{2}} + 25\lambda_k,
\end{aligned}$$

for any $k = 0, 1, 2, \ldots$ Hence, M_ε is well defined (and bounded) in $H_\sharp^s$ with values in $H_\sharp^{s-3}$ for any $s \ge 3$. Since its symbol can be estimated from above uniformly with respect to $\varepsilon \in (0, 1]$, the assertion follows immediately. □

Since all the operators appearing in (5.4) commute with D_η, the *differentiated* problem for $\zeta := \rho_\eta$ reads as follows:

$$\frac{\partial}{\partial\tau}\mathscr{B}_\varepsilon(\zeta) + \mathscr{H}_\varepsilon(\Psi_\tau) = \mathscr{S}(\zeta) + \mathscr{M}_\varepsilon((\Psi^2)_\eta) + \varepsilon\mathscr{F}_\varepsilon((\zeta^2)_\eta) + 2\mathscr{F}_\varepsilon((\Psi\zeta)_\eta), \quad (5.7)$$

where we have set $\Psi = \Phi_\eta$. Obviously (5.7) is to be solved with zero initial condition at time $\tau = 0$. For simplicity, we denote D_η by D. For an integer $n \ge 1$, D^n is the differentiation operator of order n. We also set $D^0 = \mathrm{Id}$.

5.2. **Formal a priori estimates.** For any $n = 0, 1, 2, \ldots$ and any $T > 0$, we set

$$\mathscr{Y}_n(T) = \left\{ \zeta \in C([0,T]; H_\sharp^{4\vee 2n}) \cap C^1([0,T]; L^2) : \zeta_\tau \in C([0,T]; H_\sharp^{2\vee(n+1)}) \right\},$$

where $a \vee b := \max\{a, b\}$. The main result of this subsection is contained in the following theorem.

Theorem 5.3. *Fix an integer $n \ge 0$ and $T > 0$. Further, fix m in Theorem 5.1 large enough such that $\Psi \in C([0,T]; H_\sharp^{n+4}) \cap C^1([0,T]; L^2)$ and $\Psi_\tau \in C([0,T]; H_\sharp^{n+2})$. Then, there exist $\varepsilon_1 = \varepsilon_1(n, T) \in (0, 1)$ and $K_n = K_n(n, T) > 0$ such that, if $\zeta \in \mathscr{Y}_n(T_1)$ is a solution on the time interval $[0, T_1]$ of Equation (5.7) for some $T_1 \le T$, then*

$$\sup_{\tau\in[0,T_1]}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}|D^n\zeta(\tau,\cdot)|^2 d\eta + \varepsilon \sup_{\tau\in[0,T_1]}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}|D^{n+1}\zeta(\tau,\cdot)|^2 d\eta \le K_n, \quad (5.8)$$

whenever $0 < \varepsilon \le \varepsilon_1$.

To prove (5.8), we multiply both sides of the equation (5.7) by $(-1)^n D^{2n}\zeta$ and integrate by parts over $(-\ell_0/2, \ell_0/2)$. We thus get

$$(-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\zeta_\tau(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta - (-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} S(\zeta(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta$$

$$= -(-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \left(H_\varepsilon(\Psi_\tau(\tau,\cdot)) - M_\varepsilon((\Psi^2)_\eta(\tau,\cdot))\right) D^{2n}\zeta(\tau,\cdot)d\eta$$

$$+ (-1)^n \varepsilon \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\zeta^2)_\eta(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta$$

$$+ 2(-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\zeta)_\eta(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta, \tag{5.9}$$

where $S = -4D_{\eta\eta\eta\eta} - D_{\eta\eta}$.

In the following lemmata we estimate all the terms appearing in the previous equation. We first deal with the left-hand side of (5.9) which consists of the "benign" terms.

Lemma 5.4. *Fix $n = 0, 1, \ldots$, $\varepsilon > 0$, $T_1 \le T$ and $\zeta \in \mathscr{Y}_n(T_1)$. Then,*

$$(-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\zeta_\tau(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta - (-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} S(\zeta(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta$$

$$= \frac{1}{2}\frac{d}{d\tau} \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^n\zeta(\tau,\cdot)|^2 d\eta + 2\varepsilon \frac{d}{d\tau} \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+1}\zeta(\tau,\cdot)|^2 d\eta$$

$$+ \frac{(1+\varepsilon)}{2}\frac{d}{d\tau} \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \langle R_\varepsilon D^n\zeta(\tau,\cdot), D^n\zeta(\tau,\cdot)\rangle d\eta$$

$$+ 4\int_{\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+2}\zeta(\tau,\cdot)|^2 d\eta - \int_{\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+1}\zeta(\tau,\cdot)|^2 d\eta, \tag{5.10}$$

where $R_\varepsilon : H_\sharp^1 \to L^2$ is the positive operator whose symbol is $(X_{\varepsilon,k} - 1)$.

Proof. For any $\zeta \in \mathscr{Y}_n(T_1)$, we can estimate

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\zeta_\tau(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta = (-1)^n \sum_{k=1}^{+\infty} b_{\varepsilon,k}\widehat{\zeta_\tau}(\tau,k)\lambda_k^n \widehat{\zeta}(\tau,k)$$

$$= (-1)^n \sum_{k=1}^{+\infty} \lambda_k^n \widehat{\zeta_\tau}(\tau,k)\widehat{\zeta}(\tau,k) + 4(-1)^n\varepsilon \sum_{k=1}^{+\infty} \lambda_k^{n+1}\widehat{\zeta_\tau}(\tau,k)\widehat{\zeta}(\tau,k)$$

$$+ (-1)^n(1+\varepsilon) \sum_{k=1}^{+\infty} \lambda_k^n (X_{\varepsilon,k} - 1)\widehat{\zeta_\tau}(\tau,k)\widehat{\zeta}(\tau,k)$$

$$= \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \zeta_\tau(\tau,\cdot)D^{2n}\zeta(\tau,\cdot)d\eta - 4\varepsilon \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \zeta_{\tau\eta\eta}(\tau,\cdot)D^{2n}\zeta(\tau,\cdot)d\eta$$

$$+ (1+\varepsilon) \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \zeta_\tau(\tau,\cdot)(\sqrt{I - 4\varepsilon D_{\eta\eta}} - I)D^{2n}\zeta(\tau,\cdot)d\eta$$

$$
\begin{aligned}
=&(-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} (D^n\zeta)_\tau(\tau,\cdot) D^n\zeta(\tau,\cdot) d\eta + 4(-1)^n\varepsilon \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} D^{n+1}(\zeta_\tau(\tau,\cdot)) D^{n+1}\zeta(\tau,\cdot) d\eta \\
&+ (-1)^n(1+\varepsilon) \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \langle R_\varepsilon D^n\zeta(\tau,\cdot), (D^n\zeta)_\tau(\tau,\cdot)\rangle d\eta \\
&- (-1)^n(1+\varepsilon) \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} (D^n\zeta)_\tau(\tau,\cdot) D^n\zeta(\tau,\cdot) d\eta \\
=&(-1)^n\left\{ \frac{1}{2}\frac{d}{d\tau}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^n\zeta(\tau,\cdot)|^2 d\eta + 2\varepsilon\frac{d}{d\tau}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+1}\zeta(\tau,\cdot)|^2 d\eta \right. \\
&\left. + \frac{(1+\varepsilon)}{2}\frac{d}{d\tau}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \langle R_\varepsilon D^n\zeta(\tau,\cdot), D^n\zeta(\tau,\cdot)\rangle d\eta \right\},
\end{aligned}
$$

$$(5.11)$$

for any $\tau \in [0, T_1]$. On the other hand, a straightforward computation shows that

$$
\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} S(\zeta(\tau,\cdot)) D^{2n}\zeta(\tau,\cdot) d\eta = -4(-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+2}\zeta(\tau,\cdot)|^2 d\eta \tag{5.12}
$$

$$
+ (-1)^n \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+1}\zeta(\tau,\cdot)|^2 d\eta,
$$

for any $\tau \in [0, T_1]$. Combining (5.11) and (5.12), Estimate (5.10) follows at once.
□

We now deal with the other terms in (5.9).

Lemma 5.5. *Fix $n = 0, 1, \ldots$, $T_1 \leq T$ and assume that $\Psi \in C([0, T_1]; H_\sharp^{n+4})$ and $\Psi_\tau \in C([0, T_1]; H_\sharp^{n+2})$. Then, there exist a positive constant C_n, independent of $\varepsilon \in (0, 1]$ and T_1, and constants $K_1'(n, \Psi)$ and $K_2'(n, \Psi)$ such that the following estimates hold*

$$
\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \left(H_\varepsilon(\Psi_\tau(\tau,\eta)) - M_\varepsilon((\Psi^2)_\eta(\tau,\cdot)) \right) D^{2n}\zeta(\tau,\cdot) d\eta \right| \leq K_1'(n,\Psi) + |D^n\zeta(\tau,\cdot)|_2^2;
$$

$$(5.13)$$

$$
\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\zeta^2)_\eta) D^{2n}\zeta(\tau,\cdot) d\eta \right| \leq C_n\varepsilon^{\frac{3}{2}}|D^n\zeta(\tau,\cdot)|_2 |D^{n+2}\zeta(\tau,\cdot)|_2^2 + C_n|D^n\zeta(\tau,\cdot)|_2^4
$$

$$(5.14)$$

$$
+ C_n\varepsilon^2|D^{n+1}\zeta(\tau,\cdot)|_2^4 + C_n|D^{n+2}\zeta(\tau,\cdot)|_2^2;
$$

$$
\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\zeta)_\eta(\tau,\cdot)) D^{2n}\zeta(\tau,\cdot) d\eta \right| \tag{5.15}
$$

$$
\leq K_2'(n,\Psi) \left(\varepsilon|D^{n+2}\zeta(\tau,\cdot)|_2^2 + \varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2^2 + |D^n\zeta(\tau,\cdot)|_2^2 \right) + \frac{1}{4}|D^{n+2}\zeta(\tau,\cdot)|_2^2,
$$

for any $\tau \in [0, T_1]$ and any $\zeta \in \mathscr{Y}_n(T_1)$.

Proof. Fix $n = 0, 1, \ldots$ Throughout the proof C denotes a positive constant depending on n, but being independent of τ, Ψ and ζ, which may vary from line to line.

Estimate (5.13) follows immediately from Proposition 5.2, the Poincaré-Wirtinger and Cauchy-Schwarz inequalities, which allow us to estimate

$$
\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \left(H_\varepsilon(\Psi_\tau(\tau, \cdot)) - M_\varepsilon((\Psi^2)_\eta(\tau, \cdot)) \right) D^{2n}\zeta(\tau, \cdot) d\eta \right|
$$

$$
= \left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} D^n \left(H_\varepsilon(\Psi_\tau(\tau, \cdot)) - M_\varepsilon((\Psi^2)_\eta(\tau, \cdot)) \right) D^n\zeta(\tau, \cdot) d\eta \right|
$$

$$
\leq K_1'(n, \Psi) + |D^n\zeta(\tau, \cdot)|_2^2,
$$

for any $\tau \in [0, T_1]$.

Let us now prove Estimate (5.14). For this purpose, we observe that

$$
|f_{\varepsilon,k}| \leq 2\sqrt{2}\varepsilon^{\frac{3}{2}}\lambda_k^{\frac{3}{2}} + 3\varepsilon\lambda_k + 2(\varepsilon+1)\sqrt{\varepsilon}\lambda_k^{\frac{1}{2}} + \frac{3+\sqrt{2}}{4}, \qquad k = 0, 1, \dots
$$

For the convenience of the reader, we note the splittings we use below:

$$
\frac{3}{2} + n = \frac{2+n}{2} + \frac{1+n}{2}, \qquad 1 + n = \frac{1+n}{2} + \frac{1+n}{2},
$$

$$
\frac{1}{2} + n = \frac{n-1}{2} + \frac{2+n}{2}, \qquad n = \frac{n-1}{2} + \frac{n+1}{2},
$$

for $n \geq 1$. (The case $n = 0$ can be handled likewise with very few slight and straightforward changes.) Hence, for any $\chi \in C([0, T_1]; H_\sharp^{4 \vee 2n})$ we can estimate

$$
\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon(\chi_\eta) D^{2n}\zeta d\eta \right| = \sum_{k=0}^{+\infty} \lambda_k^n |n_{\varepsilon,k}| |\widehat{\chi_\eta}(\tau, k)| |\widehat{\zeta}(\tau, k)| \tag{5.16}
$$

$$
\leq 2\sqrt{2}\varepsilon^{\frac{3}{2}} \sum_{k=0}^{+\infty} \lambda_k^{\frac{3}{2}+n} |\widehat{\chi_\eta}(\tau, k)| |\widehat{\zeta}(\tau, k)| + 3\varepsilon \sum_{k=0}^{+\infty} \lambda_k^{1+n} |\widehat{\chi_\eta}(\tau, k)| |\widehat{\zeta}(\tau, k)|
$$

$$
+ 2\sqrt{\varepsilon}(1+\varepsilon) \sum_{k=0}^{+\infty} \lambda_k^{\frac{1}{2}+n} |\widehat{\chi_\eta}(\tau, k)| |\widehat{\zeta}(\tau, k)|
$$

$$
+ \frac{3+\sqrt{2}}{4} \sum_{k=0}^{+\infty} \lambda_k^n |\widehat{\chi_\eta}(\tau, k)| |\widehat{\zeta}(\tau, k)|
$$

$$
\leq 2\sqrt{2}\varepsilon^{\frac{3}{2}} \left(\sum_{k=0}^{+\infty} \lambda_k^{1+n} |\widehat{\chi_\eta}(\tau, k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{+\infty} \lambda_k^{2+n} |\widehat{\zeta}(\tau, k)|^2 \right)^{\frac{1}{2}}
$$

$$
+ 3\varepsilon \left(\sum_{k=0}^{+\infty} \lambda_k^{1+n} |\widehat{\chi_\eta}(\tau, k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{+\infty} \lambda_k^{1+n} |\widehat{\zeta}(\tau, k)|^2 \right)^{\frac{1}{2}}
$$

$$
+ 2\sqrt{\varepsilon}(1+\varepsilon) \left(\sum_{k=0}^{+\infty} |\lambda_k^{n-1} \widehat{\chi_\eta}(\tau, k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{+\infty} \lambda_k^{2+n} |\widehat{\zeta}(\tau, k)|^2 \right)^{\frac{1}{2}}
$$

$$
+ \frac{3+\sqrt{2}}{4} \left(\sum_{k=0}^{+\infty} |\lambda_k^{n-1} \widehat{\chi_\eta}(\tau, k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k=0}^{+\infty} \lambda_k^{n+1} |\widehat{\zeta}(\tau, k)|^2 \right)^{\frac{1}{2}}
$$

$$
= 2\sqrt{2}\varepsilon^{\frac{3}{2}} |D^{n+2}\chi(\tau, \cdot)|_2 |D^{n+2}\zeta(\tau, \cdot)|_2 + 3\varepsilon |D^{n+2}\chi(\tau, \cdot)|_2 |D^{n+1}\zeta(\tau, \cdot)|_2
$$

$$
+ 2\sqrt{\varepsilon}(1+\varepsilon) |D^n\chi(\tau, \cdot)|_2 |D^{n+2}\zeta(\tau, \cdot)|_2
$$

$$+ \frac{3+\sqrt{2}}{4}|D^n\chi(\tau,\cdot)|_2|D^{n+1}\zeta(\tau,\cdot)|_2,$$

for any $\tau \in [0, T_1]$.

Now, we are in a position to prove Estimate (5.14). For this purpose, we observe that, using the Leibniz formula and the Poincaré-Wirtinger inequality, we get:

$$|D^{n+2}(\zeta(\tau,\cdot))^2|_2 \le C(|D^{n+2}\zeta(\tau,\cdot)|_2|D^n\zeta(\tau,\cdot)|_2 + |D^{n+1}\zeta(\tau,\cdot)|_2^2) \qquad (5.17)$$

$$|D^n(\zeta(\tau,\cdot))^2|_2 \le C|D^n\zeta(\tau,\cdot)|_2^2, \qquad (5.18)$$

for any $\tau \in [0, T_1]$. Replacing Estimates (5.17), (5.18) in (5.16) (with $\chi = \zeta^2$), and using the Cauchy-Schwarz inequality and, again, the Poincaré-Wirtinger inequality, we get

$$\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\zeta^2)_\eta(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta \right|$$

$$\le C\varepsilon^{\frac{3}{2}}|D^n\zeta(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C\varepsilon^{\frac{3}{2}}|D^{n+1}\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2$$

$$+ C\varepsilon|D^n\zeta_\eta(\tau,\cdot)|_2|D^{n+1}\zeta(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2 + C\varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2^3$$

$$+ C\sqrt{\varepsilon}(1+\varepsilon)|D^n\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2 + C|D^n\zeta(\tau,\cdot)|_2^2|D^{n+1}\zeta(\tau,\cdot)|_2$$

$$\le C\varepsilon^{\frac{3}{2}}|D^n\zeta(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C\varepsilon^{\frac{3}{2}}|D^{n+1}\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2$$

$$+ C\varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2 + C\varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2$$

$$+ C\sqrt{\varepsilon}(1+\varepsilon)|D^n\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2 + C|D^n\zeta(\tau,\cdot)|_2^2|D^{n+2}\zeta(\tau,\cdot)|_2$$

$$\le C\varepsilon^{\frac{3}{2}}|D^n\zeta(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C\left(\varepsilon^2|D^{n+1}\zeta(\tau,\cdot)|_2^4 + \varepsilon|D^{n+2}\zeta(\tau,\cdot)|_2^2\right)$$

$$+ C|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C\varepsilon^2|D^{n+1}\zeta(\tau,\cdot)|_2^2 + C\varepsilon^2|D^{n+1}\zeta(\tau,\cdot)|_2^4 + C|D^{n+2}\zeta(\tau,\cdot)|_2^2$$

$$+ C|D^n\zeta(\tau,\cdot)|_2^4 + C\varepsilon|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C|D^n\zeta(\tau,\cdot)|_2^4 + C|D^{n+2}\zeta(\tau,\cdot)|_2^2,$$

for any $\tau \in [0, T_1]$ and any $\varepsilon \in (0, 1]$. Now, Estimate (5.14) follows immediately.

To complete the proof, let us prove Estimate (5.15). From (5.16) and the estimates

$$|D^n(\Psi\zeta)(\tau,\cdot)|_2 \le C|D^n\zeta(\tau,\cdot)|_2|D^n\Psi(\tau,\cdot)|_2,$$

$$|D^{n+2}(\Psi\zeta)(\tau,\cdot)|_2 \le C|D^{n+2}\zeta(\tau,\cdot)|_2|D^{n+2}\Psi(\tau,\cdot)|_2,$$

(which can be proved using the same argument as in the proof of (5.17) and (5.18)) we get

$$\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\zeta)_\eta(\tau,\cdot))D^{2n}\zeta(\tau,\cdot)d\eta \right|$$

$$\le \sqrt{2}\varepsilon^{\frac{3}{2}}|D^{n+2}\zeta(\tau,\cdot)|_2|D^{n+2}(\Psi\zeta)(\tau,\cdot)|_2 + 3\varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2|D^{n+2}(\Psi\zeta)(\tau,\cdot)|_2$$

$$+ 2\sqrt{\varepsilon}(1+\varepsilon)|D^{n+2}\zeta(\tau,\cdot)|_2|D^n(\Psi\zeta)(\tau,\cdot)|_2$$

$$+ \frac{3+\sqrt{2}}{4}|D^{n+1}\zeta(\tau,\cdot)|_2|D^n(\Psi\zeta)(\tau,\cdot)|_2$$

$$\le C\varepsilon^{\frac{3}{2}}|D^{n+2}\Psi(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C\varepsilon|D^{n+2}\Psi(\tau,\cdot)|_2|D^{n+1}\zeta(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2$$

$$+ C\sqrt{\varepsilon}|D^n\Psi(\tau,\cdot)|_2|D^n\zeta(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2$$

$$+ C|D^n\Psi(\tau,\cdot)|_2|D^n\zeta(\tau,\cdot)|_2|D^{n+1}\zeta(\tau,\cdot)|_2$$

$$\le C\varepsilon^{\frac{3}{2}}|D^{n+2}\Psi(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2 + C\varepsilon|D^{n+2}\Psi(\tau,\cdot)|_2|D^{n+1}\zeta(\tau,\cdot)|_2^2$$

$$+ C\varepsilon|D^{n+2}\Psi(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2$$
$$+ C|D^n\Psi(\tau,\cdot)|_2|D^n\zeta(\tau,\cdot)|_2^2 + C\varepsilon|D^n\Psi(\tau,\cdot)|_2|D^{n+2}\zeta(\tau,\cdot)|_2^2$$
$$+ C\delta^{-1}|D^n\Psi(\tau,\cdot)|_2^2|D^n\zeta(\tau,\cdot)|_2^2 + C\delta|D^{n+2}\zeta(\tau,\cdot)|_2^2,$$

for any $\tau \in [0, T_1]$, any $\varepsilon \in (0, 1]$ and any $\delta > 0$. Estimate (5.15) follows taking $C\delta = 1/4$. This completes the proof. $\qquad\square$

We are almost ready to write the crucial *a priori* estimate satisfied by $\zeta(\tau, \cdot)$. For this purpose, we recall that

$$|D^{n+1}\psi|_2 \le |D^n\psi|_2^{\frac{1}{2}}|D^{n+2}\psi|_2^{\frac{1}{2}}, \qquad \psi \in H_{\sharp}^{n+2}.$$

Applying this estimate to $D^{n+1}\zeta(\tau, \cdot)$, together with Young's inequality, yields

$$|D^{n+1}\zeta(\tau,\cdot)|_2^2 \le |D^n\zeta(\tau,\cdot)|_2^2 + \frac{1}{4}|D^{n+2}\zeta(\tau,\cdot)|_2^2, \tag{5.19}$$

for any $\tau \in [0, T_1]$. Combining Lemmata 5.4, 5.5 and Estimate (5.19) allows us to estimate

$$\frac{1}{2}\frac{d}{d\tau}\left(|D^n\zeta(\tau,\cdot)|_2^2 + 4\varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2^2 + (1+\varepsilon)|\sqrt{R_\varepsilon}D^n\zeta(\tau,\cdot)|_2^2\right)$$
$$+ \left(\frac{15}{4} - C_n\varepsilon - \varepsilon K_2'(n,\Psi) - C_n\varepsilon^{\frac{5}{2}}|D^n\zeta(\tau,\cdot)|_2\right)|D^{n+2}\zeta(\tau,\cdot)|_2^2$$
$$\le K_1'(n,\Psi) + (2 + K_2'(n,\Psi))|D^n\zeta(\tau,\cdot)|_2^2 + \varepsilon K_2'(n,\Psi)|D^{n+1}\zeta(\tau,\cdot)|_2^2$$
$$+ C_n\varepsilon|D^n\zeta(\tau,\cdot)|_2^4 + C_n\varepsilon^3|D^{n+1}\zeta(\tau,\cdot)|_2^4, \tag{5.20}$$

for any $\tau \in [0, T_1]$. If we set

$$A_\varepsilon(\tau) = |D^n\zeta(\tau,\cdot)|_2^2 + 4\varepsilon|D^{n+1}\zeta(\tau,\cdot)|_2^2 + (1+\varepsilon)|\sqrt{R_\varepsilon}D^n\zeta(\tau,\cdot)|_2^2, \qquad \tau \in [0, T_1],$$

$$c_1 = 2K_1'(n, \Psi),$$

$$c_2 = 4 + 2K_2'(n, \Psi),$$

$$c_3 = 2C_n,$$

and assume ε small enough such that

$$C_n\varepsilon + \varepsilon K_2'(n, \Psi) < \frac{3}{4},$$

we can rewrite Inequality (5.20) in the more compact form

$$A_\varepsilon'(\tau) + \left(6 - 2C\varepsilon^{\frac{5}{2}}A_\varepsilon(\tau)\right)|D^{n+2}\zeta(\tau,\cdot)|_2^2 \le c_1 + c_2 A_\varepsilon(\tau) + c_3\varepsilon(A_\varepsilon(\tau))^2,$$

for any $\tau \in [0, T_1]$.

The following lemma allows us to estimate the function A_ε.

Lemma 5.6. *Let A_0, c_0, c_1, c_2, c_3, ε, T_0, T_1 be positive constants with $T_1 < T_0$. Further, let $f_\varepsilon : [0, T_1] \to \mathbb{R}$ and A_ε be positive functions of class $C([0, T_1])$ and $C^1([0, T_1])$, respectively, that satisfy the inequalities*

$$\begin{cases} A_\varepsilon'(\tau) + (c_0 - \varepsilon A_\varepsilon(\tau))f_\varepsilon(\tau) \le c_1 + c_2 A_\varepsilon(\tau) + c_3\varepsilon(A_\varepsilon(\tau))^2, & \tau \in [0, T_1], \\ A_\varepsilon(0) = 0. \end{cases}$$

Then, there exist $\varepsilon_1 = \varepsilon_1(T_0) \in (0, 1)$ and a constant $K = K(T_0)$ such that $A_\varepsilon(\tau) \le K$ for any $\tau \in [0, T_1]$ and any $\varepsilon \in (0, \varepsilon_1]$.

Proof. The proof follows basically from [2, Lemma 3.1], which deals with the case where $f_\varepsilon \equiv 0$. Repeating the arguments in that proof, we can easily show that $A_\varepsilon(\tau) \leq 4c_1 e^{c_2 T_0}/(3c_2)$ for any $\tau \in [0, T_1]$ and any $\varepsilon \in (0, \varepsilon_2(T_0)]$, where $\varepsilon_2(T_0) = 3c_2^2/(16c_1 c_3(e^{c_2 T_0} - 1))$.

Let us now consider the general case where f_ε does not identically vanish in $[0, T_1]$. We fix $\varepsilon_1(T_0) \leq \varepsilon_2(T_0)$ such that $3c_0 c_2 - 4c_1 e^{c_2 T_0} \varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0(T)]$. Since $A_\varepsilon(0) = 0$, there exists a maximal interval $[0, T_\varepsilon)$ where $c_0 - \varepsilon A_\varepsilon > 0$. We are going to prove that $T_\varepsilon = T_1$. For this purpose, we observe that in $[0, T_\varepsilon)$ the function A_ε satisfies the inequality $A'_\varepsilon \leq c_1 + c_2 A_\varepsilon + c_3 \varepsilon A_\varepsilon^2$. Hence, from the above result it follows that $A_\varepsilon(\tau) \leq (4c_1 e^{c_2 T_0})/(3c_2)$ for any $\tau \in [0, T_\varepsilon]$, so that $c_0 - \varepsilon A_\varepsilon(T_\varepsilon) > 0$. This clearly implies that $T_\varepsilon = T_1$. $\qquad\square$

We are now in position to prove Theorem 5.3. Applying Lemma 5.6 it follows immediately that

$$\sup_{\tau \in [0,T]} \left(|D^n \zeta(\tau, \cdot)|_2^2 + 4\varepsilon |D^{n+1} \zeta(\tau, \cdot)|_2^2 + (1+\varepsilon)|\sqrt{R_\varepsilon} D^n \zeta(\tau, \cdot)|_2^2 \right) \leq K_{1,n},$$

for any $n = 0, 1, \ldots$, from which (5.10) follows at once.

5.3. Existence and uniqueness of a solution to Equation (5.4) vanishing at $\tau = 0$.
In this subsection we are concerned with the proof of the following theorem.

Theorem 5.7. *For any $T > 0$, there exists $\varepsilon_0(T) > 0$ such that, for any $0 < \varepsilon \leq \varepsilon_0(T)$, Equation (5.7) has a unique classical solution ζ on $[0, T]$, which vanishes at $\tau = 0$.*

Existence part. We prove the existence of a solution ζ to Equation (5.4), vanishing at 0, by a standard Faedo-Galerkin method. Let us fix $\xi \in (I - \Pi)(H_\sharp^s)$ and expand it as a Fourier series (see Section 2.3) as follows:

$$\xi = \sum_{k=1}^{+\infty} \widehat{\xi}(k) w_k.$$

For $N = 1, 2, \ldots$, we denote by $\Xi_N = P_N((I - \Pi)(H_\sharp^s))$ the projection of $(I - \Pi)(H_\sharp^s)$ along the vector space spanned by the functions $w_1, \ldots, w_N$.

Let us look for a solution $\zeta_N \in \Xi_N$ to the variational problem

$$\frac{\partial}{\partial \tau} \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\zeta_N) \xi d\eta = \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} S(\zeta_N) \xi d\eta + \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} \left\{ M_\varepsilon((\Psi^2)_\eta) - H_\varepsilon(\Psi_\tau) \right\} \xi d\eta$$

$$+ \varepsilon \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\zeta_N^2)_\eta) \xi d\eta + 2 \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi \zeta_N)_\eta) \xi d\eta. \quad (5.21)$$

for all $\xi \in \Xi_N$. The problem is subject to the initial condition $\zeta_N(0, \cdot) \equiv 0$. In terms of Fourier series, the variational formulation (5.9) reads as follows:

$$\frac{d}{d\tau} \sum_{k=1}^{+\infty} b_{k,\varepsilon} \widehat{\zeta}_N(\cdot, k) \widehat{\xi}(k) = \sum_{k=1}^{+\infty} s_k \widehat{\zeta}_N(\cdot, k) \widehat{\xi}(k)$$

$$+ \sum_{k=1}^{+\infty} \left\{ m_{\varepsilon,k} \widehat{(\Psi^2)_\eta}(\cdot, k) - h_{\varepsilon,k} \widehat{\Psi_\tau}(\cdot, k) \right\} \widehat{\xi}(k)$$

$$+ \varepsilon \sum_{k=1}^{+\infty} f_{k,\varepsilon} \widehat{(\zeta_N^2)_\eta}(\cdot, k) \widehat{\xi}(k) + 2 \sum_{k=1}^{+\infty} f_{\varepsilon,k} \widehat{(\Psi \zeta_N)_\eta}(\cdot, k) \widehat{\xi}(k).$$

$$(5.22)$$

Taking $\xi = w_j$ $(j = 1, \ldots, N)$ in (5.22), we see that the $\widehat{\zeta}_N(\cdot, k)$'s verify a system of N ordinary differential equations with zero initial data. Hence, there exists a unique solution to System (5.22), defined on some maximal time interval $[0, T_N)$, where T_N may also depend on ε.

Next, we take $\xi = \zeta_N$ in (5.21). The estimates of Section 5.2 remain valid also for the function ζ_N. Writing such estimates, taking as T_1 any number less than T_N and then letting $T' \to T_N$, we thus get

$$\sup_{\tau \in [0, T_N)} \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^n \zeta_N(\tau, \cdot)|^2 d\eta + \varepsilon \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} |D^{n+1} \zeta_N(\tau, \cdot)|^2 d\eta \leq K_n, \qquad (5.23)$$

for any $n \geq 1$. From this estimate we infer that, whenever $0 < \varepsilon \leq \varepsilon_1(T)$, the solution of the ODE system can be extended up to T.

We now let $N \to +\infty$. For this purpose, we use Estimate (5.23) with $n = 4$. It leads to the following facts:

(i) the sequence $(\zeta_N)_{N \in \mathbb{N}}$ is bounded in $C([0, T]; H_\sharp^5)$, with a bound possibly depending on $\varepsilon \in (0, \varepsilon_1(T)]$;

(ii) the sequence $((\zeta_N)_\tau)_{N \in \mathbb{N}}$ is bounded in $C([0, T]; H_\sharp^3)$ with a bound possibly depending on ε.

Property (i) follows immediately from (5.23). (Note that, if $n \geq 5$, then the bound is uniform in $0 < \varepsilon \leq \varepsilon_1(T)$.) To prove property (ii), we observe that (5.21) may be rewritten as:

$$\frac{\partial}{\partial \tau} B_\varepsilon(\zeta_N) = P_N \left\{ S(\zeta_N) + M_\varepsilon((\Psi^2)_\eta) - H_\varepsilon(\Psi_\tau) + \varepsilon F_\varepsilon((\zeta_N^2)_\eta) + 2 F_\varepsilon((\Psi \zeta_N)_\eta) \right\},$$

and we use (i) and Proposition 5.2.

Now, we can make the compactness argument work for any arbitrarily fixed $\varepsilon \in (0, \varepsilon_1(T)]$. By the Sobolev embedding theorem, the sequences $(\zeta_N)_{N \in \mathbb{N}}$ and $((\zeta_N)_\tau)_{N \in \mathbb{N}}$ are bounded in $C([0, T]; C_\sharp^{9/2})$ and in $C([0, T]; C_\sharp^{5/2})$, respectively. In particular, by interpolation we easily see that $(D_\eta^l(\zeta_N)_\tau)_{N \in \mathbb{N}}$ $(l = 0, \ldots, 4)$ are bounded in $C^{1/9}([0, T] \times [-\ell_0/2, \ell_0/2])$. Indeed, $C_\sharp^4$ belongs to the class $J_{8/9}$ between $C_\sharp$ and $C_\sharp^{9/2}$. Hence, we can estimate

$$\|\zeta_N(\tau_2, \cdot) - \zeta_N(\tau_1, \cdot)\|_{C_\sharp^4} \leq \|\zeta_N(\tau_2, \cdot) - \zeta_N(\tau_1, \cdot)\|_\infty^{\frac{1}{9}} \|\zeta_N(\tau_2, \cdot) - \zeta_N(\tau_1, \cdot)\|_{C_\sharp^{9/2}}^{\frac{8}{9}}$$

$$\leq \|D_\tau \zeta_N\|_\infty^{\frac{1}{9}} \sup_{\tau \in [0, T]} \|\zeta_N(\tau, \cdot)\|_{C_\sharp^{9/2}}^{\frac{8}{9}} |\tau_2 - \tau_1|^{\frac{1}{9}},$$

for any $\tau_1, \tau_2 \in [0, T]$. Since the sequence (ζ_N) is bounded both in $C^{1/9}([0, T]; C_\sharp^4)$ and in $C([0, T]; C_\sharp^{9/2})$, it is bounded in $C^{1/9}([0, T] \times [-\ell_0/2, \ell_0/2])$, as well. The Arzelà-Ascoli theorem then implies that, up to a subsequence, ζ_N converges in $C([0, T]; C_\sharp^4)$ to a function $\zeta \in C([0, T]; C_\sharp^{9/2})$. Similarly, $D_\eta^l((\zeta_\tau)_N)_{N \in \mathbb{N}}$ $(l = 0, 1, 2)$ converges, up to a subsequence, to $D_\eta^l(\zeta_\tau)$ $(l = 0, 1, 2)$. Clearly, the function ζ solves the equation (5.4) and vanishes at $\tau = 0$.

Uniqueness part. Assume that ζ_1 and ζ_2 are two classical solutions to Equation (5.7) which vanish at $\tau = 0$. Then, the function $\chi := \zeta_1 - \zeta_2$ turns out to solve the equation

$$\frac{\partial}{\partial \tau} B_\varepsilon(\chi) = S(\chi) + \varepsilon F_\varepsilon((\chi(\zeta_1 + \zeta_2))_\eta) + 2F_\varepsilon((\Psi\chi)_\eta). \tag{5.24}$$

We multiply (5.24) by χ and integrate over $[-\ell_0/2, \ell_0/2]$. We get

$$\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} B_\varepsilon(\chi(\tau, \cdot))\chi(\tau, \cdot)d\eta = \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} S(\chi(\tau, \cdot))\chi(\tau, \cdot)d\eta$$

$$+ \varepsilon \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\chi(\tau, \cdot)(\zeta_1(\tau, \cdot) + \zeta_2(\tau, \cdot)))_\eta)\chi(\tau, \cdot)d\eta$$

$$+ 2 \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\Psi\chi(\tau, \cdot))_\eta)\chi(\tau, \cdot),$$

for any $\tau \in [0, T]$. All the terms but $\int_{-\ell_0/2}^{\ell_0/2} F_\varepsilon((\chi(\tau, \cdot)(\zeta_1(\tau, \cdot) + \zeta_2(\tau, \cdot)))_\eta)\chi(\tau, \cdot)d\eta$ have been already estimated in Lemmata 5.4 and 5.5. Hence, we just need to estimate the latter term. For this purpose, we observe that (5.16) implies that

$$\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\chi(\zeta_1 + \zeta_2))_\eta(\tau, \cdot))\chi(\tau, \cdot)d\eta \right| \tag{5.25}$$

$$\leq 2\sqrt{2}\varepsilon^{\frac{3}{2}}|D^2(\chi(\zeta_1 + \zeta_2))(\tau, \cdot)|_2|D^2\chi(\tau, \cdot)|_2 + 3\varepsilon|D^2(\chi(\zeta_1 + \zeta_2))(\tau, \cdot)|_2|D\chi(\tau, \cdot)|_2$$

$$+ 4\sqrt{\varepsilon}|\chi(\tau, \cdot)(\zeta_1 + \zeta_2)(\tau, \cdot)|_2|D^2\chi(\tau, \cdot)|_2$$

$$+ \frac{3 + \sqrt{2}}{4}|\chi(\tau, \cdot)(\zeta_1 + \zeta_2)(\tau, \cdot)|_2|D\chi(\tau, \cdot)|_2,$$

for any $\tau \in [0, T]$. By the a priori estimates (5.8) with $n = 1$, we infer that

$$|D(\zeta_1 + \zeta_2)(\tau, \cdot)|_2^2 + \varepsilon|D^2(\zeta_1 + \zeta_2)(\tau, \cdot)|_2^2 \leq 2K_1, \qquad \tau \in [0, T].$$

Therefore,

$$|\chi(\tau, \cdot)(\zeta_1 + \zeta_2)(\tau, \cdot)|_2 \leq C_1|\chi(\tau, \cdot)|_2|D(\zeta_1 + \zeta_2)(\tau, \cdot)|_2 \leq 2C_1K|\chi(\tau, \cdot)|_2,$$

$$|D^2(\chi(\zeta_1 + \zeta_2))(\tau, \cdot)|_2 \leq C_1|D^2(\zeta_1 + \zeta_2)(\tau, \cdot)|_2|D\chi(\tau, \cdot)|_2$$

$$+ C_1|D(\zeta_1 + \zeta_2)(\tau, \cdot)|_2|D^2\chi(\tau, \cdot)|_2$$

$$\leq 2C_1K\varepsilon^{-1}|D\chi(\tau, \cdot)|_2 + 2C_1K|D^2\zeta(\tau, \cdot)|_2,$$

for any $\tau \in [0, T]$ and some positive constant C_1, depending on ℓ_0 only. We can thus continue (5.25) getting

$$\left| \int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}} F_\varepsilon((\chi(\zeta_1 + \zeta_2))_\eta(\tau, \cdot))\chi(\tau, \cdot)d\eta \right|$$

$$\leq C_K\sqrt{\varepsilon}|D\chi(\tau, \cdot)|_2|D^2\chi(\tau, \cdot)|_2 + C_K\varepsilon^{\frac{3}{2}}|D^2\chi(\tau, \cdot)|_2^2 + C_K\sqrt{\varepsilon}|\chi(\tau, \cdot)|_2|D\chi(\tau, \cdot)|_2$$

$$+ C_K|D\chi(\tau, \cdot)|_2^2 + C_K|\chi(\tau, \cdot)|_2|D\chi(\tau, \cdot)|_2$$

$$\leq C_K\varepsilon|D^2\chi(\tau, \cdot)|_2^2 + C_K|D\chi(\tau, \cdot)|_2^2 + C_K|D^2\chi(\tau, \cdot)|_2^2,$$

for any $\tau \in [0, T]$ and some positive constant C_K, depending on K only. Hence, combining this estimate with (5.10) and (5.15) yields

$$\frac{1}{2}\frac{d}{d\tau}|\chi(\tau, \cdot)|_2^2 + 2\varepsilon\frac{d}{d\tau}|D\chi(\tau, \cdot)|_2^2 + \frac{1+\varepsilon}{2}\frac{d}{d\tau}|\sqrt{R_\varepsilon}\chi(\tau, \cdot)|_2^2 + M_{\varepsilon,K}|D^2\chi(\tau, \cdot)|_2^2$$
$$\leq (K_2(0, \Psi) + C_K\varepsilon + 1)\,|\chi(\tau, \cdot)|_2^2 + \varepsilon\,(C_K + K_2(0, \Psi))\,|D\chi(\tau, \cdot)|_2^2,$$

for any $\tau \in [0, T]$, where $M_{\varepsilon,K} = \frac{15}{4} - C_K\varepsilon^2 - K_2(0, \psi)\varepsilon^{\frac{3}{2}}$. Up to replacing $\varepsilon_1(T)$ with a smaller value $\varepsilon_0(T)$, if needed, we can assume that $M_{\varepsilon,K} \leq 0$ for any $\varepsilon \in (0, \varepsilon_0(T)]$. Now, Gronwall's Lemma applies and yields $\zeta \equiv 0$ since $\zeta(0, \cdot) = 0$.

5.4. **Proof of Theorem** 2.2. We now return to ρ and to Problem (5.4). This can be done as in the proof of Theorem 2.1. The idea is simple: we look for ρ as $\rho(\tau, \eta) = \chi(\tau, \eta) + p(\tau)w_0$, where χ has zero average. More precisely, we set $\chi = \mathscr{P}(\zeta)$, where the operator $\mathscr{P}$ is defined by (4.6). A simple computation shows that

$$B_\varepsilon(\chi) + H_\varepsilon(\Phi_\eta) - S(\chi) - M_\varepsilon((\Phi_\eta)^2) - \varepsilon F_\varepsilon((\chi_\eta)^2) - 2F_\varepsilon(\Phi_\eta\chi_\eta)$$

is independent of η. Since $\chi \in (I - \Pi)(L^2)$, this means that

$$B_\varepsilon(\chi) + H_\varepsilon(\Phi_\eta) = S(\chi) + (I - \Pi)(M_\varepsilon((\Phi_\eta)^2)) + \varepsilon(I - \Pi)(F_\varepsilon((\chi_\eta)^2))$$
$$+ 2(I - \Pi)(F_\varepsilon(\Phi_\eta\chi_\eta)).$$

Let us now denote by $p : [0, T] \to \mathbb{R}$ the solution to the Cauchy problem

$$\begin{cases} \dfrac{dp}{d\tau} = -\Pi(H_\varepsilon(\Phi_\eta)) + \Pi(M_\varepsilon((\Phi_\eta)^2)) + \varepsilon\Pi(F_\varepsilon((\chi_\eta)^2)) + 2\Pi(F_\varepsilon(\Phi_\eta\chi_\eta)), \\ p(0) = 0. \end{cases}$$
$$\tag{5.26}$$

If we set $\rho = p + \chi$, we immediately see that $\rho(0, \cdot) = 0$ and ρ solves equation (5.4).

Clearly, this function is the unique solution to Equation (5.4) which vanishes at $\tau = 0$. Indeed, if ρ_1 and ρ_2 are two such solutions, then the functions $\zeta_1 := D_\eta\rho_1$ and $\zeta_2 := D_\eta\rho_2$ solve Equation (5.7) and vanish at $\tau = 0$. By the above results, ζ_1 and ζ_2 agree. This means that $(I - \Pi)(\rho_1) \equiv (I - \Pi)(\rho_2)$. But then also $\Pi(\rho_1)$ and $\Pi(\rho_2)$ agree, since, as Problem (5.26) shows, $\Pi(\rho_1)$ and $\Pi(\rho_2)$ are uniquely determined by $(I - \Pi)(\rho_1)$.

To complete the proof of Theorem 2.2, let us check that there exists $M > 0$ such that

$$\sup_{\substack{\tau \in [0, T] \\ \eta \in [-\ell_0/2, \ell_0/2]}} |\rho(\tau, \eta)| \leq M, \tag{5.27}$$

uniformly in $0 < \varepsilon \leq \varepsilon_0(T)$. Applying the a priori estimates in Theorem 5.3 (here $n = 0$ is enough) and using (4.6), one can easily show that

$$\|(I - \Pi)(\rho)\|_\infty = \|\mathscr{P}(\zeta)\|_\infty \leq (1 + \ell_0)\sqrt{\ell_0 K_0}.$$

As far as the component of ρ along $\Pi(L^2)$ is concerned (which we still denote by p), we observe that (see (5.2), (5.5) and (5.6))

$$\Pi(H_\varepsilon(\Phi_\eta)) = \Pi(M_\varepsilon((\Phi_\eta)^2)) = 0,$$
$$\Pi(F_\varepsilon((\chi_\eta)^2)) = -\frac{1}{2}\Pi((\chi_\eta)^2),$$
$$\Pi(F_\varepsilon(\Phi_\eta\chi_\eta)) = -\frac{1}{2}\Pi(\Phi_\eta\chi_\eta),$$

and we can estimate

$$|\Pi(F_\varepsilon((\chi_\eta(\tau,\cdot))^2))| \leq \frac{1}{2}|\chi_\eta(\tau)|_2 \leq \frac{1}{2}\sup_{\tau\in[0,T]}|\zeta(\tau,\cdot)|_2^2 \leq \frac{1}{2}K_0,$$

$$\begin{aligned}|\Pi(F_\varepsilon(\Phi_\eta(\tau,\cdot)\chi_\eta(\tau,\cdot)))| \leq &\frac{1}{2}\int_{-\frac{\ell_0}{2}}^{\frac{\ell_0}{2}}|\Phi_\eta(\tau,\cdot)\chi_\eta(\tau,\cdot)|d\eta\\ \leq &\frac{1}{2}|\Phi_\eta(\tau,\cdot)|_2|\zeta(\tau,\cdot)|_2\\ \leq &\frac{1}{2}\sqrt{K_0}\sup_{\tau\in[0,T]}|\Phi_\eta(\tau,\cdot)|_2,\end{aligned}$$

for any $\tau \in [0,T]$. It thus follows from (5.26) that

$$\begin{aligned}|p(\tau)| \leq \varepsilon\int_0^\tau &|\Pi(F_\varepsilon((\chi_\eta(\tau,\cdot))^2)) + 2\Pi(F_\varepsilon(\Phi_\eta(\tau,\cdot)\chi_\eta(\tau,\cdot)))|d\tau\\ \leq &\frac{1}{2}K_0T + \sqrt{K_0}T\sup_{\tau\in[0,T]}|\Phi_\eta(\tau,\cdot)|_2,\end{aligned}$$

for any $\tau \in [0,T]$. Estimate (5.27) now follows immediately.

Finally, coming back to Problem (2.12) and setting $\ell_\varepsilon = \ell_0/\sqrt{\varepsilon}$ and $T_\varepsilon = T/\varepsilon^2$, one can easily conclude that, for any $\varepsilon \in (0,\varepsilon_0]$, such a problem admits a unique classical solution φ. Moreover,

$$\|\varphi(t,\cdot) - \varepsilon\Phi(t\varepsilon^2,\sqrt{\varepsilon}\,\cdot)\|_{C([-\ell_\varepsilon/2,\ell_\varepsilon/2])} \leq \varepsilon^2M, \qquad t \in [0,T_\varepsilon].$$

This accomplishes the proof of Theorem 2.2.

Appendix A. The NEF model. Flames constitute a complex physical system involving fluid dynamics, multistep chemical kinetics, as well as molecular and radiative transfer. The laminar flames of low-Lewis-number premixtures are known to display diffusive-thermal instability responsible for the formation of a non-steady cellular structure (see [24]). However, the cellular instability is quite robust with respect to these aero-thermo-chemical complexities and may be successfully captured by a model involving only two equations: the heat equation for the system's temperature and the diffusion equation for the deficient reactant's concentration. In suitably chosen units, the so-called thermal-diffusional model reads, see e.g., [7]:

$$\Theta_t = \Theta_{xx} + \Theta_{yy} + \Omega(Y,\Theta), \tag{A.1}$$

$$Y_t = Le^{-1}(Y_{xx} + Y_{yy}) - \Omega(Y,\Theta), \tag{A.2}$$

$$\Omega = \frac{1}{2}Le^{-1}\beta^2 Y\exp[\beta(\Theta-1)/(\sigma+(1-\sigma)\Theta)]. \tag{A.3}$$

Here, $\Theta = (T - T_u)/(T_{ad} - T_u)$ is the scaled temperature, where T_u and T_{ad} correspond to, respectively, the temperature of the unburned gas and the adiabatic temperature of combustion products; $Y = C/C_u$ is the scaled concentration of the deficient reactant with C_u being its value in the unburned gas; x, y, t are the scaled spatiotemporal coordinates referred to D_{th}/U, and D_{th}/U^2, respectively, where D_{th} is the thermal diffusivity of the mixture and U is the velocity of the undisturbed planar flame; Le is the Lewis number (the ratio of thermal and molecular diffusivities); $\sigma = T_u/T_{ad}$; $\beta = T_a(1-\sigma)/T_{ad}$ is the Zeldovich number, assumed to be large, where T_a is the activation temperature; Ω is the scaled reaction rate, where

the normalizing factor $\frac{1}{2}Le^{-1}\beta^2$ ensures that at $\beta \gg 1$ the planar flame propagates at velocity close to unity.

Due to the distributed nature of the reaction rate Ω, Equations (A.1) and (A.2) are still difficult for a theoretical exploration. One therefore turns to the conventional high activation energy limit $(\beta \gg 1)$ which converts the reaction rate term into a localized source distributed over a certain interface $x = \xi(t,y)$, the flame front. Intensity of the source varies along the front as $\exp\left(\frac{1}{2}(\Theta_f - 1)\right)$ (see [22]). Here, Θ_f is the scaled temperature at the curved front, which may differ from unity $(T = T_{ad})$ by a quantity of the order of β^{-1}. Due to the strong temperature dependence of the reaction rate $(\beta \gg 1)$, even slight changes of Θ_f may considerably affect its intensity, and thereby also the local flame speed. The study of flame propagation is thus reduced to a free-interface problem. To ensure that the emerging free-interface model does not involve large parameters one should combine the limit of large activation energy $(\beta \gg 1)$ with the requirement that the product $\alpha = \frac{1}{2}\beta(1 - Le)$ remains finite, i.e., the ratio of thermal and molecular diffusivities (Le) should be closed to unity. This is the Near Equidiffusive Flames model, in short NEF, introduced in [20]. As a result, instead of the reaction diffusion problem for Θ and Y, one ends up with the free-interface problem

$$\frac{\partial \theta}{\partial t} = \Delta\theta, \quad x < \xi(t,y),$$

$$\theta = 1, \quad x \geq \xi(t,y),$$

$$\frac{\partial S}{\partial t} = \Delta S - \alpha\Delta\theta, \quad x \neq \xi(t,y),$$

$$\left[\frac{\partial \theta}{\partial n}\right] = -\exp(S), \qquad \left[\frac{\partial S}{\partial n}\right] = \alpha\left[\frac{\partial \theta}{\partial n}\right],$$

for the new scaled temperature $\theta = \lim_{\beta\to+\infty}\Theta$ and the reduced enthalpy $S = \lim_{\beta\to+\infty}\beta^{-1}(\Theta + Y - 1)$. For some rigorous mathematical justification, see [9, 8].

Appendix B. Proof of Theorem 5.1. Showing that Problem (5.3) admits a unique solution $\Phi \in C^1([0,T_0]; L^2) \cap C([0,T_0]; H_\sharp^4)$ for some $T_0 > 0$ is an easy task. Indeed, the operator $A : H_\sharp^2 \to L^2$ is sectorial in L^2 as has been already remarked. By [18, Prop. 2.4.1 & 2.4.4] the operator $S = -4A^2 - A$ is sectorial in L^2 with domain $H_\sharp^4$. Classical results for semilinear equations associated with sectorial operators show that the Cauchy problem (5.3) admits a unique solution Φ with the above regularity properties. (See e.g., [18, Prop. 7.1.10].) Φ turns out to be a fixed point of the operator Γ, formally defined by

$$(\Gamma(\Phi))(\tau,\cdot) = e^{\tau S}\Phi_0 + \int_0^\tau e^{(\tau-s)S}(\Phi_\eta(s,\cdot))^2 ds, \qquad \tau > 0,$$

where $\{e^{\tau S}\}$ denotes the semigroup generated by S.

Using a classical continuation argument, we can extend Φ to a maximal domain $[0, T_{\max})$ with a function (still denoted by Φ) which belongs to $C^1([0, T_{\max}); L^2) \cap C([0, T_{\max}); H_\sharp^4)$.

Let us regularize Φ. Suppose that $\Phi_0 \in H_\sharp^5$. Note that S commutes with D_η. Hence,

$$\Phi_\eta(\tau,\cdot) = e^{\tau S}(D_\eta\Phi_0) + \int_0^\tau e^{(\tau-s)S}D_\eta(\Phi_\eta(s,\cdot))^2 ds, \qquad \tau \in [0, T_{\max}).$$

Since $\Phi \in C^1([0, T_{\max}); L^2) \cap C([0, T_{\max}); H^4_\sharp)$ and $H^k_\sharp$ belongs to the class $J_{k/4}$ between L^2 and H^4, we can estimate

$$\|\Phi(\tau_2, \cdot) - \Phi(\tau_1, \cdot)\|_k \leq |\Phi(\tau_2, \cdot) - \Phi(\tau_1, \cdot)|_2^{1-\frac{k}{4}} \|\Phi(\tau_2, \cdot) - \Phi(\tau_1, \cdot)\|_4^{\frac{k}{4}}$$

$$\leq 2\|\Phi_\tau\|_{C([0,T_1];L^2)} \|\Phi\|_{C([0,T_1];H^4_\sharp)}^{\frac{k}{4}} |\tau_2 - \tau_1|^{1-\frac{k}{4}},$$

for any $\tau_1, \tau_2 \in [0, T_1]$ and any $T_1 < T_{\max}$. Therefore, by the Sobolev embedding theorem, we can estimate

$$|D_\eta(\Phi_\eta(\tau_2, \cdot))^2 - D_\eta(\Phi_\eta(\tau_1, \cdot))^2|_2 \leq |\Phi_\eta(\tau_2, \cdot)|_\infty |\Phi_{\eta\eta}(\tau_2, \cdot) - \Phi_{\eta\eta}(\tau_1, \cdot)|_2$$

$$+ |\Phi_{\eta\eta}(\tau_2, \cdot)|_\infty |\Phi_\eta(\tau_2, \cdot) - \Phi_\eta(\tau_1, \cdot)|_2$$

$$\leq C_{T_1} |\tau_2 - \tau_1|^{\frac{1}{2}},$$

for any τ_1 and τ_2 as above. This shows that $D_\eta(\Phi_\eta(\tau_2, \cdot))^2$ belongs to $C^{1/2}([0, T_1]; L^2)$ for any $T_1 < T_{\max}$. Theorem 4.3.1 of [18] implies that $\Phi_\eta \in C^1([0, T_{\max}); L^2) \cap C([0, T_{\max}); H^4_\sharp)$. In particular, Φ_τ belongs to $C([0, T_{\max}); H^1_\sharp)$. It follows that $\Phi_{\tau\eta} \equiv \Phi_{\eta\tau}$. Iterating this argument shows that, if $\Phi_0 \in H^m_\sharp$ for some $m \in \mathbb{N}$ such that $m > 4$, then $\Phi \in C([0, T_{\max}); H^m_\sharp)$ and $\Phi_\tau \in C([0, T_{\max}); H^{m-4}_\sharp)$.

The rest of the proof is devoted to show that $T_{\max} = +\infty$. We adapt the arguments in [25, Thm. 2.4]. The main step is the *a priori estimate*

$$|\Phi_\eta(\tau, \cdot)|_2 \leq e^{\frac{13}{6}\tau} |D_\eta \Phi_0|_2, \qquad \tau \in [0, T_{\max}). \tag{B.1}$$

For this purpose, we introduce the function v, defined by $v(\tau, \eta) = e^{-2\tau} \Phi_\eta(\tau, \eta)$ for any $(\tau, \eta) \in [0, T_{\max}) \times [-\ell_0/2, \ell_0/2]$. The smoothness of Φ implies that $v \in C^{1,4}([0, T_{\max}) \times [-\ell_0/2, \ell_0/2])$, solves the parabolic equation

$$v_\tau = -3v_{\eta\eta\eta\eta} - v_{\eta\eta} - e^{2\tau} v v_\eta - 2v, \tag{B.2}$$

and satisfies the boundary conditions $D_\eta^{(k)} v(\tau, -\ell_0/2) = D_\eta^{(k)} v(\tau, \ell_0/2)$ for any $\tau \in [0, T)$ and $k = 0, 1, 2, 3$. Multiplying both sides of (B.2) by $v(\tau, \cdot)$, integrating on $(-\ell_0/2, \ell_0/2)$ and observing that the integral over $(-\ell_0/2, \ell_0/2)$ of $(v(\tau, \cdot))^2 v_\eta(\tau, \cdot)$ vanishes for any $\tau \in [0, T_{\max})$, we get

$$\frac{d}{d\tau} |v(\tau, \cdot)|_2^2 + 3|v_{\eta\eta}(\tau, \cdot)|_2^2 - |v_\eta(\tau, \cdot)|_2^2 + 2|v(\tau, \cdot)|_2^2 = 0, \qquad \tau \in [0, T_{\max}). \tag{B.3}$$

In view of the estimate

$$|v_\eta(\tau, \cdot)|_2^2 \leq |v(\tau, \cdot)|_2 |v_{\eta\eta}(\tau, \cdot)|_2 \leq 3|v_{\eta\eta}(\tau, \cdot)|_2^2 + \frac{5}{3}|v(\tau, \cdot)|_2^2, \qquad \tau \in [0, T_{\max}),$$

Formula (B.3) leads us to the inequality

$$\frac{d}{d\tau} |v(\tau, \cdot)|_2^2 + \frac{1}{3}|v(\tau, \cdot)|_2^2 \leq 0, \qquad \tau \in [0, T_{\max}),$$

from which Estimate (B.1) follows at once.

We can now complete the proof. For this purpose, let us consider the function Ψ, defined by $\Psi(\tau, \eta) = \Phi(\tau, \eta) - \Pi(\Phi(\tau, \cdot))$ for any $\tau \in [0, T_{\max})$ and any $\eta \in [-\ell_0/2, \ell_0/2]$. Applying the Poincaré inequality, we get

$$|\Phi(\tau, \cdot) - \Pi(\Phi(\tau, \cdot))|_2 \leq \sqrt{\ell_0} e^{\frac{13}{6}\tau} |D_\eta \Phi_0|_2, \qquad \tau \in [0, T_{\max}). \tag{B.4}$$

Let us now show that the function $\tau \mapsto \Pi(\Phi(\tau, \cdot))$ satisfies a similar estimate. For this purpose, we fix $\tau \in [0, T_{\max})$ and apply the operator Π to both sides of

(5.3). Since Φ and its derivatives satisfy periodic boundary conditions,

$$\frac{d}{d\tau}\Pi(\Phi(\tau,\cdot)) = \Pi(\Phi_\tau(\tau,\cdot)) = -\frac{1}{2\ell_0}\Pi((\Phi_\eta(\tau,\cdot))^2),$$

for any $\tau \in [0, T_{\max})$. Taking (B.1) into account, we can then estimate

$$\left|\frac{d}{d\tau}\Pi(\Phi(\tau,\cdot))\right| \leq \frac{1}{2\ell_0}e^{\frac{13}{3}\tau}|D_\eta\Phi_0|_2^2, \qquad \tau \in [0, T_{\max}).$$

Hence,

$$|\Pi(\Phi(\tau))| \leq |\Pi(\Phi_0)| + \int_0^\tau \left|\frac{d}{d\tau}\Pi(\Phi(\tau,\cdot))\right| d\tau \leq |\Pi(\Phi_0)| + \frac{3}{26\ell_0}|D_\eta\Phi_0|_2^2 e^{\frac{13}{3}\tau}, \quad \text{(B.5)}$$

for any $\tau \in [0, T_{\max})$. Estimates (B.4) and (B.5) show that Φ is bounded in $[0, T_{\max})$ with values in L^2. Therefore, we can apply [18, Prop. 7.2.2] with $\gamma = 1/2$, $\alpha = 1/4$, $X_{1/4} = H_\sharp^1$, which implies that $T_{\max} = +\infty$.

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