PDE2006, exercise set 3

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1. This exercise summarises the abstract framework used in the weak solution approach in Chapter 6. Recall that, if H is a real Hilbert space and $f : H \to I\!R$ is a linear function, continuity of f in any point is equivalent to

$$||f|| = \sup\{|f(x)| : ||x||_H \leq 1\} < \infty.$$

Boundedness of this norm $||f||$ is called boundedness of f. This defines a norm on the space H' of continous linear real valued functions on H. The Riesz representation theorem states that H' is isometrically isomorphic to H: the continous linear real valued functions $f : H \to I\!R$ are precisely the functions $f : x \to (x, y)$, where $y \in H$ is fixed. In particular $||f|| = ||y||$.

Now let H and V be Hilbert spaces such that $V \subset H$. The inner product on H is denoted by single brackets, the inner product on V by double brackets. We shall write

$$(u, u) = |u|^2 \quad \text{for} \quad u \in H \qquad \text{and} \qquad ((u, u)) = ||u||^2 \quad \text{for} \quad u \in V$$

We assume that V is dense in H and that the inclusion map is continuous.

(i) Let $f \in H$. Prove that there exists a unique $u \in V$ such that $((u, v)) = (f, v)$ for all $v \in V$. Denote $u = Af$.

(ii) Prove that $A : H \to V$ is injective.

(iii) Prove that $A : H \to H$ is linear, symmetric (meaning $(Au, v) = (u, Av)$) and continuous.

(iv) Prove that also $A : V \to V$ is linear, symmetric and continuous.

(v) Prove that $A : H \to H$ and $A : V \to V$ are positive, i.e. $(Af, f) > 0$ if $f \neq 0$.

We also assume that V is compactly embedded in H, meaning that bounded sequences in V have convergent subsequences in H.

(vi) Prove that $A : H \to H$ is compact (if u_n is a bounded sequence in H then Au_n has a convergent subsequence in H). Prove that $A : V \to V$ is compact.

If $A : H \to H$ is a positive, symmetric, compact linear operator, then H has an orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ of eigenvectors of A corresponding to positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, with $\lambda_n \to 0$ as $n \to \infty$, where

$$\lambda_1 = \max_{f \in H} \frac{(Af, f)}{(f, f)}$$

and, more generally, for $n > 1$,

$$\lambda_n = \max_{f \in H, (f,\phi_1) = \cdots = (f, \phi_{n-1}) = 0} \frac{(Af, f)}{(f, f)}$$

The proof of this statement is based on the existence of a maximizing vector ϕ_1 for

$$\max_{f \in H} \frac{(Af, f)}{(f, f)}$$

combined with the observation that every such maximizing vector is an eigenvector of A. This produces λ_1 and ϕ_1. The proof is completed by induction. Since A maps $H_n = \{u \in H : (u, \phi_1) = \cdots = (u, \phi_n) = 0\}$ to it self, the same argument produces λ_{n+1} and ϕ_{n+1}.

(vii) The above statement applies to $A : H \to H$ defined in (i) but also to $A : V \to V$. Relate the resulting orthonormal bases to oneanother. Evaluate the eigenvalue formula's for $A : V \to V$ in terms of norms only, i.e. without A appearing in the formula's.

2. Let

$$H = L^2(0,1) = \{f : (0,1) \to I\!R \,|\, f \text{ is measurable, } \int_0^1 f^2 < \infty\},$$

equipped with the inner product $(f, g) = \int_0^1 fg$. Let

$$V = \{f \in C^1([0,1]) \,|\, f(0) = f'(0) = f(1) = f'(1) = 0,\ f'' \text{ exists, } f'' \in L^2(0,1)\},$$

with inner product $((f, g)) = \int_0^1 f''g''$. Define A as in Exercise 1. Which boundary value problem does this A solve? Show that this problem has a unique solution $u \in C^4([0,1])$, provided $f \in C([0,1])$. Compute a function $D(\lambda)$ such that the zero's of $D(\lambda)$ are the eigenvalues of A.

3. Let Ω be a bounded domain in $I\!R^m$ with smooth boundary $\partial\Omega$. Denote the outward normal on $\partial\Omega$ by ν. The divergence theorem says that for $v \in C^1(\overline{\Omega}, I\!R^m)$

$$\int_\Omega \nabla \cdot v = \int_{\partial\Omega} v \cdot \nu$$

Consider the problem

$$-\Delta u = f \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{2}$$

(i) Suppose that $u \in C^2(\overline{\Omega})$ is a classical solution of Problem (1,2). Derive an integral condition (IC) that f must satisfy. Show that Problem (1,2) also has a solution which satisfies (IC).

(ii) Suppose that also $\phi \in C^2(\overline{\Omega})$. Evaluate

$$\int_\Omega \nabla u \cdot \nabla \phi \tag{3}$$

Writing $\nabla u = (D_1 u, \ldots, D_m u)$, let

$$H^1(\Omega) = \{u \in L^2(\Omega) : \ D_1 u, \ldots, D_m u \in L^2(\Omega)\}$$

with the (standard Sobolev) inner product norm

$$||u||_1 = \left(\int_\Omega \left(|u|^2 + |\nabla u|^2 \right) \right)^{\frac{1}{2}}$$

This space is compactly embedded in $L^2(\Omega)$, meaning that a sequence which is bounded in $H^1(\Omega)$, has a subsequence convergent in $L^2(\Omega)$.

(iii) Which integral equality for u and arbitrary $\phi \in H^1(\Omega)$ would you suggest as the defining property for a function $u \in H^1(\Omega)$ to be a weak solution of Problem (1,2)?

(iv) Show that (3) defines an inner product on

$$\tilde{H}^1(\Omega) = \{u \in H^1(\Omega) : \ u \text{ satisfies (IC)}\}$$

The inner product norm corresponding to (3) will be equivalent to the norm $|| \cdot ||_1$ on $\tilde{H}^1(\Omega)$, provided there exists a constant C such that for all $u \in \tilde{H}^1(\Omega)$ the following inequality holds:

$$\int_\Omega |u|^2 \leq C \int_\Omega |\nabla u|^2$$

(v) Show, arguing by contradiction and using the compactness of the embedding $\tilde{H}^1(\Omega) \to L^2(\Omega)$, that there is no sequence $u_n \in \tilde{H}^1(\Omega)$ which has $\int_\Omega |u_n|^2 = 1$ and $\int_\Omega |\nabla u_n|^2 \to 0$. Deduce that indeed both norms are equivalent on $\tilde{H}^1(\Omega)$.

(vi) Let $f \in L^2(\Omega)$ satisfy (IC). Show, applying the Riesz representation theorem in $\tilde{H}^1(\Omega)$, that Problem (1,2) has a weak solution $u \in H^1(\Omega)$ which is unique up to an additive constant.

(vii) The construction of u in the proof of (vi) works equally well without the assumption that f satisfies (IC). Which problem does u solve then?

4. With Ω as above consider the problem

$$\Delta\Delta u = f \quad \text{in } \Omega \tag{4}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{5}$$

Let

$$H^2(\Omega) = \{u \in L^2(\Omega) : \, D_i u, D_{ij} u \in L^2(\Omega), \, i, j = 1 \dots m\}$$

with the (standard Sobolev) inner product norm

$$||u||_2 = \left(\int_\Omega \left(|u|^2 + \sum_{i=1}^m |D_i u|^2 + \sum_{i,j=1}^m |D_{ij} u|^2 \right) \right)^{\frac{1}{2}}$$

The space $H_0^2(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^2(\Omega)$. Discuss how you would formulate (and establish the unique) existence of weak solutions of Problem (4,5) in $H_0^2(\Omega)$. Hint: show for functions $u, v \in C_c^\infty(\Omega)$ that $\int_\Omega \sum_{i,j=1}^m D_{ij} u D_{ij} v = \int_\Omega \Delta u \Delta v$, that on $C_c^\infty(\Omega)$ the corresponding norm is equivalent to the $|| \cdot ||_2$-norm, and apply the Riesz theorem to the appropriate weak formulation in $H_0^2(\Omega)$.