In this course we are concerned with mathematical tools for studying Partial Differential Equations (PDE's) of the type

$$u_t + Lu = f(x, u).$$

Here $u = u(x, t)$ is the unknown real valued solution as a function of the spatial variable x and the time variable t. Subscript t denotes partial derivative with respect to time, L is a second order elliptic operator, the standard example being $L = -\Delta$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots \frac{\partial^2}{\partial x_n^2},$$

and $f(x, u)$ is a (given) nonlinear function of x and u.

We will largely restrict the attention to problems set on a bounded spatial domain U in n-dimensional Euclidean space, with smooth boundary ∂U. The time variable will usually be taken in some interval $[0, T]$, where T may be arbitrarily large. Thus we look for solutions $u(x, t)$ defined on $U \times [0, T]$ which satisfy certain initial data at $t = 0$, and certain boundary data for $x \in \partial U$, such as e.g.

$$u(x, 0) = u_0(x) \quad (x \in U); \quad u(x, t) = 0 \quad (x \in \partial U),$$

where u_0 is a given function defined on U. Such problems are called Initial Boundary Value Problems (IBVP's). The PDE above is an example of an evolution equation.

The first issue to be settled is to formulate conditions which guarantee the existence of a unique solution. Here we can distinguish between a theory for classical solutions in which the PDE is satisfied pointwise and a theory for weak solutions in which the IBVP is first rewritten as an integral equation for the solution u and an arbitrary test function, where the crux is that the integral equation has to be satisfied for all test functions in a suitable chosen class. We will work with this weak concept of a solution.

To settle this issue it is first necessary to understand equations of the type

$$Lu = f(x),$$

in which the unknown function $u = u(x)$ is now only a function of x. For this much simpler equation we only impose boundary values, say $u(x) = 0$ for $x \in \partial U$. Such problems are called Boundary Value Problems (BVP's).

For the weak solution approach of BVP's we need the appropriate function spaces in which weak solutions live. These are the Sobolev spaces. Standard spaces we shall work with are $W^{1,p}(U)$ with $1 \leq p \leq \infty$, consisting of functions u for which the integrals

$$\int_U |u|^p, \quad \int_U |\frac{\partial u}{\partial x_i}|^p$$

are all finite. Here the derivatives are no longer defined as limits of difference quotients. Chapter 5 of Evans' book gives a thorough introduction of these spaces. A key feature of

Theorems about Sobolev spaces is that they are proved via approximation of nonsmooth functions with smooth functions. This is done using mollifier techniques discussed in Appendix C.4. We will not have time to give the more technical proofs.

After introducing the Sobolev spaces we solve BVP's in Chapter 6. Here we rely on some ideas from functional analysis, such as the Riesz represenatation Theorem. Appendix D contains the relevant definitions and theorems. The weak solutions we construct can be shown to be as smooth as we want, depending of course on the assumptions, but this will be stated without much proof. Once solutions are smooth we can establish maximum principles, which limit the possibilities for solutions to have extrema other than extrema on the boundary. Spectral theory, i.e. understanding the eigenvalue problem

$$Lu = \lambda u \quad \text{in} \;\; U, \quad u = 0 \quad \text{on} \;\; \partial U,$$

is of interest in itself and necessary for what follows.

In Chapter 7 we restrict our attention to 7.1 and 7.4. In 7.1 weak solutions of the IBVP for

$$u_t + Lu = f(x)$$

are discussed. This is done by approximating the infinite dimensional Sobolev spave in which the weak solution is to be found, by finite dimensional spaces spanned by eigenfunctions, i.e. solutions of the eigenvalue problem above, typically with a different but nice choice for L. An alternative approach is using semigroups in 7.4, relying again on the spectral theory for L.

The first nonlinear problems appear in Chapter 8, where in 8.1, 8.4 and 8.5 we shall study stationary solutions of the IBVP we started out with in our discussion above. Here we shall reformulate problems as finding extremal values of real valued "energy" functions defined on suitable Sobolev spaces. This requires an infinite dimensional calculus which we introduce. Finally, Chapter 9, we discuss the full evolution problem using fixed point arguments in 9.2.