Functional Analysis

Functional analysis is a toolkit for solving equations in which the unknowns are functions rather than numbers. For instance, we may want to find a function $f = f(x)$ such that, for every $x \in [0, 1]$,

$$f(x) - \int_0^x \sin(x - t) f(t) dt = \cos(x).$$

This is an example of a linear integral equation. The left hand side defines a function of a function, which we refer to as a (linear) functional acting on the variable function f. In the nonlinear integral equation

$$f(x) - \int_0^x \sin(x - t) f(t)^2 dt = \cos(x),$$

the left hand side defines a nonlinear functional.

Most of the equations we solved in analysis and linear algebra required finding a solution as a number or a finite set of numbers, which, substituted in some given function, would make it zero, or would maximize or minimize it. Consequently we learned in linear algebra and analysis all sorts of things about linear and nonlinear functions defined on subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$, finite dimensional vector spaces over the real or complex numbers, equipped with a natural (inner product) norm. Our lifes were made easy by the fact that bounded closed sets in $\mathbb{R}^n$ and $\mathbb{C}^n$ are compact so that bounded sequences have convergent subsequences. Another fact taking completely for granted was the continuity of linear functions.

In the infinite-dimensional setting needed to solve problems such as the integral equations above, we first need good normed vector spaces in which our solutions are to be found. We will see many different possibilities to assign a norm to a function, leading to different spaces. There are many candidates for $\mathbb{R}^\infty$ so to speak. This will make the theory of even only linear functionals a subtle issue in which linear algebra and analysis (epsilons and delta's) merge.