On suboptimal policies in multi-class tandem models

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We study two single-server multi-class systems in tandem. As simple optimal policies exist only in special cases, our objective is to study simple policies which perform close to optimality. For one such policy, which is based on the μc rule, we prove that it is optimal in an asymptotic sense. Numerical work comparing this policy with the optimal one is also supplied.

1. INTRODUCTION

Stochastic scheduling is concerned with assigning service to customers in some cost-efficient way. For numerous simple models the policy minimizing some specified cost function has been obtained. See Righter [5] for an overview.

However, for less simple models often there is no hope to derive the optimal policy explicitly, due to its complexity. For these types of models there is a need for simple policies which can be shown to perform close to optimality. Good examples of these types of results can be found in Weiss [6], in which multi-server systems are studied. It is well known that for single-server systems and appropriate cost functions the μc rule is optimal. For multi-server systems this holds only for some special cases, as there are extra conditions on the service times needed, as in [3]. In his paper Weiss shows that the μc rule also performs well for multi-server systems, and bounds on its performance are given.

Here we do not study multi-server systems, but two single-server systems in tandem. Here too there are results for special cases, as shown in Hordijk & Koole [3]. There it is derived that, for exponential service times and instantaneous costs linear in the numbers of customers, the μc rule is optimal in the last center of a tandem system if the service time parameters are ordered. More specifically, assume that μ_j is the service rate of class j customers in the second center, and let c_j

be their holding cost rate, then the μc rule serves the customers in decreasing order of $\mu_j c_j$. With reindexing we assume that $\mu_1 c_1 \geq \cdots \geq \mu_m c_m$ (with m the number of customer classes). Then the μc rule minimizes the expected holding costs if in addition $\mu_1 \leq \cdots \leq \mu_m$. There are no results known on the policy to be employed in the first center.

In this paper we will study a policy, based on the μc rule, which prescribes both the server assignment in the first and in the second center. Using dynamic programming, we will show that this *tandem μc rule* is optimal in an asymptotic sense. It uses the μc rule in the second center, and serves the customers in the first center in decreasing order of $\tilde{\mu}_j(\tilde{c}_j - c_j)$ (variables with (without) a tilde refer to the first (second) center). We show that as long as the second center does not empty (due to the availability of many customers), this policy is optimal. This is used to give a bound on the difference in expected costs at T between the tandem μc rule and the optimal policy. This bound goes to 0 if the number of customers in the second center increases. These results can be found in section 2.

Actually, we consider two models. In the first one the order in which the customers are served by the tandem μc rule may differ between the centers. In this case the number of customers in the highest priority queue in the second center must be big enough. The other model has 2 queues in each center, and equal service orders in both centers. In this case the total number of customers in the second center has to be large.

The same method can also be used to study discounted costs instead of costs incurred over a finite horizon. This is done in Koole [4], where a single server single center model with switching costs is studied. We cannot apply the method yet to average costs. This is an interesting area of further research.

In section 3 we present some numerical results. For computational reasons only discounted and average costs for the infinite time horizon are considered. With discounting the tandem μc rule is practically optimal for the cases considered. In the average cost case however the relative differences between between the tandem μc rule and the optimal policy were significantly larger.

2. RESULTS

Let us start by formulating our model precisely. We have a system of two centers in series, with customers joining the first center according to a marked Markov arrival process (MMAP), introduced below. We choose an MMAP because it is Markovian in nature, and every arrival process can be approximated with it arbitrarily close (Asmussen & Koole [1]). Both centers consist of m queues, and each customer in queue j at center 1 (2) has a service time which is exponentially distributed with parameter $\tilde{\mu}_j$ (μ_j). At each center there is a server, which can be assigned to the customers available at the center in a preemptive manner. Customers who finish service at the first center join the same queue at the second center, customers departing from the second center leave the system. The scheduling of the servers is done on line, based on full state information. The cost structure will be specified later.

2.1. Definition. (marked Markov arrival process) Let Λ be the finite state space of a Markov process with transition intensities λ_{xy} with $x, y \in \Lambda$. When this process moves from x to y, then with probability q^j_{xy} an arrival in class j, $1 \le j \le m$, occurs. The arrival events are mutually exclusive, and $\sum_{j=1}^m q^j_{xy} \le 1$.

First we formulate the dynamic programming (d.p.) equation for our model. For this equation we will then derive certain properties, which are the basis of our main results.

Assume (without restricting generality) that $\sum_y \lambda_{xy} = \gamma$ for all x and that $\gamma + \tilde\mu + \mu \le 1$, where $\mu = \max_j \mu_j$ and $\tilde\mu = \max_j \tilde\mu_j$. This can always be done by scaling of the process. Thus the transition rates can now be interpreted as transition probabilities of an embedded discrete time model. Let $(x, \tilde\imath, i)$ denote the state of our system, with x the state of the MMAP, and $\tilde\imath$ (i) the queue length vector at center 1 (2). Define $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, with the 1 in jth position.

The following transitions are possible: from $(x, \tilde\imath, i)$ to $(x, \tilde\imath - e_j, i + e_j)$, corresponding to a class j customer moving from center 1 to center 2 (which event has transition rate $\tilde\mu_j$, given that the server at center 1 serves class j, and that $\tilde\imath_j > 0$); from $(x, \tilde\imath, i)$ to $(x, \tilde\imath, i - e_j)$, corresponding to a class j customer leaving center 2 (with rate μ_j, given that the server at center 2 serves class j, and that $i_j > 0$); from $(x, \tilde\imath, i)$ to $(y, \tilde\imath + e_j, i)$, corresponding to a transition of the MMAP and an arrival of a class j customer (with rate $\lambda_{xy} q^j_{xy}$); from $(x, \tilde\imath, i)$ to $(y, \tilde\imath, i)$, corresponding to a transition of the MMAP without an arrival (with rate $\lambda_{xy}(1 - \sum_j q^j_{xy})$).

Now consider the following dynamic programming equation:

$$\begin{aligned}
v^{n+1}_{(x,\tilde\imath,i)} =& \min_{\tilde l, l} \Big\{ \sum_y \lambda_{xy} \Big(\sum_{j=1}^m q^j_{xy} v^n_{(y,\tilde\imath+e_j,i)} + (1 - \sum_{j=1}^m q^j_{xy}) v^n_{(y,\tilde\imath,i)} \Big) + \\
&\qquad \tilde\mu_{\tilde l} v^n_{(x,\tilde\imath-e_{\tilde l},i+e_{\tilde l})} + (\tilde\mu - \tilde\mu_{\tilde l}) v^n_{(x,\tilde\imath,i)} + \\
&\qquad \mu_l v^n_{(x,\tilde\imath,i-e_l)} + (\mu - \mu_l) v^n_{(x,\tilde\imath,i)} + (1 - \gamma - \tilde\mu - \mu) v^n_{(x,\tilde\imath,i)} \Big\} = \\
& \sum_y \lambda_{xy} \Big(\sum_{j=1}^m q^j_{xy} v^n_{(y,\tilde\imath+e_j,i)} + (1 - \sum_{j=1}^m q^j_{xy}) v^n_{(y,\tilde\imath,i)} \Big) + \\
& \min_j \Big\{ \tilde\mu_j v^n_{(x,\tilde\imath-e_j,i+e_j)} + (\tilde\mu - \tilde\mu_j) v^n_{(x,\tilde\imath,i)} \Big\} + \\
& \min_j \Big\{ \mu_j v^n_{(x,\tilde\imath,i-e_j)} + (\mu - \mu_j) v^n_{(x,\tilde\imath,i)} \Big\} + (1 - \gamma - \tilde\mu - \mu) v^n_{(x,\tilde\imath,i)}, \\
v^0_{(x,\tilde\imath,i)} =& c_{(x,\tilde\imath,i)}.
\end{aligned} \tag{2.1}$$

The minimization ranges over all non-empty queues, and idleness corresponds to action 0 with $\tilde\mu_0 = \mu_0 = 0$. This equation differs from standard d.p. equations in the sense that there are no direct costs, only salvage costs c. Thus $v^n_{(x,\tilde\imath,i)}$ can be interpreted as the minimal expected costs incurred after n steps of the embedded discrete time model, when starting in $(x, \tilde{\imath}, i)$.

Now we prove certain properties of the value function v^n, assuming that the same properties hold for the costs c. We start by proving monotonicity in both centers, i.e., we formulate conditions on c under which serving an arbitrary customer (if available) is better than idling.

2.2. Lemma. *If*

$$w_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} \leq w_{(x,\tilde{\imath},i)} \text{ for } \tilde{\imath}_{j_1} > 0 \tag{2.2}$$

and

$$w_{(x,\tilde{\imath},i-e_{j_1})} \leq w_{(x,\tilde{\imath},i)} \text{ for } i_{j_1} > 0 \tag{2.3}$$

hold for the cost function $w = c$, then they hold for all v^n, $n \geq 0$.

Proof. By induction. The lemma trivially holds for v^0. Now assume that the lemma holds up to n. We start with (2.2). We will write v^{n+1} in terms of v^n, by inserting the d.p. equation, and then we will prove the inequality by term-by-term comparison. When writing $v^{n+1}_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})}$ and $v^{n+1}_{(x,\tilde{\imath},i)}$ out it is easily seen that the inequalities for terms corresponding to arrivals follow directly by the induction hypothesis.

Consider the departures from the first center. Let j^* be the optimal action in the first center in state $(x, \tilde{\imath}, i)$. If $j^* \neq j_1$, j^* is allowable in state $(x, \tilde{\imath} - e_{j_1}, i + e_{j_1})$ and the term follows by induction. If $j^* = j_1$, and $\tilde{\imath} = e_{j_1}$, then idling is the only action in state in state $(x, \tilde{\imath} - e_{j_1}, i + e_{j_1})$ and the term follows by induction. If there is at least one more customer available, say in queue j_2, and $j^* = j_1$, then

$$\begin{gathered}
\min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_{j_1}-e_j,i+e_{j_1}+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} \right\} \leq \\
\tilde{\mu}_{j_2} v^n_{(x,\tilde{\imath}-e_{j_1}-e_{j_2},i+e_{j_1}+e_{j_2})} + (\tilde{\mu} - \tilde{\mu}_{j_2}) v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} \overset{(2.2)}{\leq} \\
\tilde{\mu}_{j_2} v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} + (\tilde{\mu} - \tilde{\mu}_{j_2}) v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} \overset{(2.2)}{\leq} \\
\tilde{\mu}_{j_1} v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} + (\tilde{\mu} - \tilde{\mu}_{j_1}) v^n_{(x,\tilde{\imath},i)} = \\
\min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_j,i+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath},i)} \right\}.
\end{gathered}$$

Consider the departures from the second center. The optimal action in $(x, \tilde{\imath}, i)$ is allowable in $(x, \tilde{\imath} - e_{j_1}, i + e_{j_1})$. Therefore the term follows easily by induction.

Consider (2.3). The terms regarding arrivals and the departures from the first center follow by induction, because the optimal action in the first center in $(x, \tilde{\imath}, i)$ is also allowable in $(x, \tilde{\imath}, i - e_{j_1})$. Regarding the departures from the second center, the optimal action in $(x, \tilde{\imath}, i)$ is also allowable in $(x, \tilde{\imath}, i - e_{j_1})$, unless it is j_1. But then we can use action 0, which means idleness of the server in the second center, in $(x, \tilde{\imath}, i - e_{j_1})$. □

Now we have the following theorem which follows from lemma 2.2 and uniformization techniques (see e.g. Van Dijk [2]):

2.3. Theorem. *The optimal policy at T is non-idling in both centers for all cost functions satisfying (2.2) and (2.3).*

Without any difficulty we could have added direct costs in the d.p. equation, leading to the same result for the total expected costs incurred from time 0 to time T.

Let us see what the inequalities mean for linear costs, i.e. $c_{(x,\tilde{\imath},i)} = \sum_j \tilde{c}_j \tilde{\imath}_j + \sum_j c_j i_j$. Equation (2.3) requires that $c_j \geq 0$ for all j. It is easily seen that (2.2) requires $\tilde{c}_j - c_j \geq 0$ for all j. This is not surprising, as this number is the cost reduction when a class j customer moves from center 1 to 2. This gives us a conjecture on how an asymptotic optimal policy might be: serve in center 1 the queue with highest $\tilde{\mu}_j(\tilde{c}_j - c_j)$ and follow in center 2 the μc-rule, i.e. serve the queue with highest $\mu_j c_j$. This indeed gives us the tandem μc-rule.

Note that there are, besides linear costs, many interesting cost functions satisfying (2.2) and (2.3).

The tandem μc rule is not optimal in general. Indeed, the tandem μc rule is only concerned with reducing the costs myopically, while an optimal policy would also have to consider objectives like keeping enough work available for the second server. However, there seem to be no other effects than those related to emptiness of the second center withholding the tandem μc rule from being optimal. Therefore, if we assume that the second center never empties, we have the following.

2.4. Lemma. *Assume idleness is not allowed. If*

$$\tilde{\mu}_{j_1} w_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} + (\tilde{\mu} - \tilde{\mu}_{j_1}) w_{(x,\tilde{\imath},i)} \leq \tilde{\mu}_{j_2} w_{(x,\tilde{\imath}-e_{j_2},i+e_{j_2})} + (\tilde{\mu} - \tilde{\mu}_{j_2}) w_{(x,\tilde{\imath},i)}$$

$$\text{for } j_1 < j_2 \text{ and } \tilde{\imath}_{j_1}, \tilde{\imath}_{j_2} > 0 \qquad (2.4)$$

and, for some j^*,

$$\mu_{j^*} w_{(x,\tilde{\imath},i-e_{j^*})} + (\mu - \mu_{j^*}) w_{(x,\tilde{\imath},i)} \leq \mu_{j_1} w_{(x,\tilde{\imath},i-e_{j_1})} + (\mu - \mu_{j_1}) w_{(x,\tilde{\imath},i)}$$

$$\text{for } j_1 \text{ with } i_{j_1} > 0 \qquad (2.5)$$

hold for the cost function $w = c$, then they hold for v^n, $n \geq 0$, whenever $n \leq i_{j^*}$.

Note that it follows from (2.5) and the d.p. equation (2.1) that it is optimal to serve queue j^* in center 2, hence queue j^* has highest priority, and will be served because we assume that there are customers in that queue. The inequality (2.4) together with the d.p. equation then gives that the policy that assigns the server to the non-empty queue with smallest index (the SIP) is optimal for time horizon n in the first center, if the number of customers in the highest priority queue of center 2 is at least n.

If the cost function is linear, i.e., $c_{(x,\tilde{\imath},i)} = \sum_j \tilde{c}_j \tilde{\imath}_j + \sum_j c_j \tilde{\imath}_j$, and satisfies (2.4) then $\tilde{\mu}_1(\tilde{c}_1 - c_1) \geq \cdots \geq \tilde{\mu}_m(\tilde{c}_m - c_m)$. The inequalities (2.5) for this linear cost function imply that $\mu_{j^*} c_{j^*} \geq \mu_j c_j$ for all j with $i_j > 0$. Hence the optimal rule which follows from the inequalities is the tandem μc rule.

Proof of lemma 2.4. By induction. Assume the lemma holds up to n. We start with (2.4). Assume $n + 1 \leq i_{j^*}$. The terms concerning arrivals follow immediately, using induction, because $n < i_{j^*}$. Consider the terms corresponding to departures from center 1. Let j^* be the optimal action in state $\tilde{\imath}$. Because $\tilde{\imath}_{j_1} > 0$, j^* is also optimal in $\tilde{\imath} - e_{j_2}$. If $j^* \neq j_1$, then the terms follow easily by induction. If $j^* = j_1$, then

$$\tilde{\mu}_{j_1} \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_{j_1}-e_j,i+e_{j_1}+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} \right\} +$$

$$(\tilde{\mu} - \tilde{\mu}_{j_1}) \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_j,i+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath},i)} \right\} \leq$$

$$\tilde{\mu}_{j_1} \tilde{\mu}_{j_2} v^n_{(x,\tilde{\imath}-e_{j_1}-e_{j_2},i+e_{j_1}+e_{j_2})} + \tilde{\mu}_{j_1}(\tilde{\mu} - \tilde{\mu}_{j_2}) v^n_{(x,\tilde{\imath}-e_{j_1},i+e_{j_1})} +$$

$$(\tilde{\mu} - \tilde{\mu}_{j_1}) \tilde{\mu}_{j_2} v^n_{(x,\tilde{\imath}-e_{j_2},i+e_{j_2})} + (\tilde{\mu} - \tilde{\mu}_{j_1})(\tilde{\mu} - \tilde{\mu}_{j_2}) v^n_{(x,\tilde{\imath},i)} =$$

$$\tilde{\mu}_{j_2} \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_{j_2}-e_j,i+e_{j_2}+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath}-e_{j_2},i+e_{j_2})} \right\} +$$

$$(\tilde{\mu} - \tilde{\mu}_{j_2}) \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_j,i+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath},i)} \right\}.$$

Consider the second center. By (2.5), serving queue j^* is always optimal. The terms follow by induction. Note that we used (2.5) at step n with at least $n + 1$ customers in queue j^*. Also the dummy term follows easily.

Consider (2.5). Again the terms concerning arrivals and the dummy transition follow easily. The optimal action in the first center of $(x, \tilde{\imath}, i)$ depends only on $\tilde{\imath}$. Because the number of customers in queue j^* in state $(x, \tilde{\imath}, i - e_{j_1})$, $(x, \tilde{\imath}, i - e_{j^*})$ and $(x, \tilde{\imath}, i)$ is $i_{j^*} - 1$ or more, there are at least n customers available, meaning that, by (2.4), the same action is optimal in each state. Therefore also the terms concerning departures from the first center follow easily. Concerning the second center, serving queue j^* is optimal in each state. Also these terms follow easily by induction. □

Lemma 2.4 is the basis of our heavy traffic theorem.

2.5. Theorem. *For all T, cost functions satisfying (2.4) and (2.5), and $\varepsilon > 0$ there is a number N such that the tandem μc-rule in both centers is ε-optimal at T, if there are more than N customers in queue j^* at time 0.*

Proof. Let N_1 denote the fixed number of customers in the first center, at time 0. We compare the costs of two policies: the tandem μc-rule and the optimal policy R^*. Let the r.v. $\Phi_x^T(\mu c)$ and $\Phi_x^T(R^*)$ denote their expected costs, where x is the starting state of the whole system.

If we condition on the times of the jumps, R^* results in an discrete-time policy on these jump times. Note that the tandem μc rule is a stationary policy. Thus, if we assume that the number of jumps is smaller than N, then, by lemma 2.4, we find that the tandem μc performs better than R^*. This leads to $\mathbb{E}(\Phi_x^T(\mu c)|A_N^c) - \mathbb{E}(\Phi_x^T(R^*)|A_N^c) \leq 0$, with A_N the event that there are N or more jumps in $[0,T]$.

This results in

$$\mathbb{E}\Phi_x^T(\mu c) - \mathbb{E}\Phi_x^T(R^*) =$$
$$\Big(\mathbb{E}(\Phi_x^T(\mu c)|A_N) - \mathbb{E}(\Phi_x^T(R^*)|A_N)\Big)\mathbb{P}(A_N) +$$
$$\Big(\mathbb{E}(\Phi_x^T(\mu c)|A_N^c) - \mathbb{E}(\Phi_x^T(R^*)|A_N^c)\Big)\mathbb{P}(A_N^c) \leq$$
$$\Big(\mathbb{E}(\Phi_x^T(\mu c)|A_N) - \mathbb{E}(\Phi_x^T(R^*)|A_N)\Big)\mathbb{P}(A_N).$$

The expected number of arrivals, conditioned on A_N, is smaller than $N + T/\gamma$. Thus the expected number of customers available at T, conditioned on A_N, for both the tandem μc-rule and R^*, is smaller than $N_1 + 2N + T/\gamma$. The expected costs are bounded by $(N_1 + 2N + T/\gamma)c$ for some c. It remains to show that there is a N such that $\mathbb{P}(A_N)(N_1 + 2N + T/\gamma)c \leq \frac{\varepsilon}{2}$. This follows easily as $\mathbb{P}(A_N)$ and $N\mathbb{P}(A_N) \downarrow 0$ as $N \to \infty$. □

If idleness is allowed, we can combine lemma 2.4 and 2.2:

2.6. Theorem. *For all T, cost functions satisfying (2.4), (2.5), (2.2) and (2.3) and $\varepsilon > 0$ there is a number N such that the tandem μc-rule in both centers is ε-optimal at T, if there are more than N customers in queue j^* at time 0.*

Now we restrict ourselves to $m = 2$ centers, and we assume that the tandem μc-rule has the same priority in both centers, i.e. serving queue 1 is optimal in both centers if there are sufficiently many customers in queue 2 of center 2. In lemma 2.7 we relax this requirement to the condition that for fixed time horizon n there are at least n customers in the second center.

2.7. Lemma. *Assume idleness is not allowed. If*

$$\tilde\mu_1 w_{(x,\tilde\imath-e_1,i+e_1)} + (\tilde\mu - \tilde\mu_1)w_{(x,\tilde\imath,i)} \leq \tilde\mu_2 w_{(x,\tilde\imath-e_2,i+e_2)} + (\tilde\mu - \tilde\mu_2)w_{(x,\tilde\imath,i)}$$

$$\text{for } \tilde\imath_1, \tilde\imath_2 > 0 \qquad (2.6)$$

and

$$\mu_1 w_{(x,\tilde\imath,i-e_1)} + (\mu - \mu_1)w_{(x,\tilde\imath,i)} \leq \mu_2 w_{(x,\tilde\imath,i-e_2)} + (\mu - \mu_2)w_{(x,\tilde\imath,i)}$$

for $i_1, i_2 > 0$ (2.7)

hold for the cost function $w = c$*, then they hold for* v^n*,* $n \geq 0$*, whenever* $n \leq i_1 + i_2$*.*

Proof. By induction. Assume the lemma holds up to n. We start with (2.6). Assume $n+1 \leq i_1+i_2$. The terms concerning arrivals follow immediately, using induction, because $n < i_1 + i_2$. Consider the terms corresponding to departures from center 1. In $\tilde{\imath}$ and $\tilde{\imath} - e_2$ it is optimal to serve queue 1. Thus

$$\tilde{\mu}_1 \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_1-e_j,i+e_1+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath}-e_1,i+e_1)} \right\} +$$
$$(\tilde{\mu} - \tilde{\mu}_1) \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_j,i+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath},i)} \right\} \leq$$
$$\tilde{\mu}_1 \tilde{\mu}_2 v^n_{(x,\tilde{\imath}-e_1-e_2,i+e_1+e_2)} + \tilde{\mu}_1 (\tilde{\mu} - \tilde{\mu}_2) v^n_{(x,\tilde{\imath}-e_1,i+e_1)} +$$
$$(\tilde{\mu} - \tilde{\mu}_1) \tilde{\mu}_2 v^n_{(x,\tilde{\imath}-e_2,i+e_2)} + (\tilde{\mu} - \tilde{\mu}_1)(\tilde{\mu} - \tilde{\mu}_2) v^n_{(x,\tilde{\imath},i)} =$$
$$\tilde{\mu}_2 \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_2-e_j,i+e_2+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath}-e_2,i+e_2)} \right\} +$$
$$(\tilde{\mu} - \tilde{\mu}_2) \min_j \left\{ \tilde{\mu}_j v^n_{(x,\tilde{\imath}-e_j,i+e_j)} + (\tilde{\mu} - \tilde{\mu}_j) v^n_{(x,\tilde{\imath},i)} \right\}.$$

Consider the second center. If $i_1 > 0$, serving queue 1 is optimal in $i + e_1$, i and $i + e_2$, using that (2.7) holds for $i_1 + i_2 \geq n + 1$ at stage n. Then

$$\tilde{\mu}_1 \min_j \left\{ \mu_j v^n_{(x,\tilde{\imath}-e_1,i+e_1-e_j)} + (\mu - \mu_j) v^n_{(x,\tilde{\imath}-e_1,i+e_1)} \right\} +$$
$$(\tilde{\mu} - \tilde{\mu}_1) \min_j \left\{ \mu_j v^n_{(x,\tilde{\imath},i-e_j)} + (\mu - \mu_j) v^n_{(x,\tilde{\imath},i)} \right\} =$$
$$\tilde{\mu}_1 \mu_1 v^n_{(x,\tilde{\imath}-e_1,i+e_1-e_1)} + \tilde{\mu}_1 (\mu - \mu_1) v^n_{(x,\tilde{\imath}-e_1,i+e_1)} +$$
$$(\tilde{\mu} - \tilde{\mu}_1) \mu_1 v^n_{(x,\tilde{\imath},i-e_1)} + (\tilde{\mu} - \tilde{\mu}_1)(\mu - \mu_1) v^n_{(x,\tilde{\imath},i)} \stackrel{(2.6)}{\leq}$$
$$\tilde{\mu}_2 \mu_1 v^n_{(x,\tilde{\imath}-e_2,i+e_2-e_1)} + \tilde{\mu}_2 (\mu - \mu_1) v^n_{(x,\tilde{\imath}-e_2,i+e_2)} +$$
$$(\tilde{\mu} - \tilde{\mu}_2) \mu_1 v^n_{(x,\tilde{\imath},i-e_1)} + (\tilde{\mu} - \tilde{\mu}_2)(\mu - \mu_1) v^n_{(x,\tilde{\imath},i)} =$$
$$\tilde{\mu}_2 \min_j \left\{ \mu_j v^n_{(x,\tilde{\imath}-e_2,i+e_2-e_j)} + (\mu - \mu_j) v^n_{(x,\tilde{\imath}-e_2,i+e_2)} \right\} +$$
$$(\tilde{\mu} - \tilde{\mu}_2) \min_j \left\{ \mu_j v^n_{(x,\tilde{\imath},i-e_j)} + (\mu - \mu_j) v^n_{(x,\tilde{\imath},i)} \right\}.$$

We wanted to prove the inequality for all i with $n + 1 \leq i_1 + i_2$. We used at stage n (2.6) with $i_1 + i_2 + 1 > n$ customers in the second center.

If $i_1 = 0$, then $i_2 > 0$. Thus serving queue 2 is optimal in i and $i + e_2$. Then

$$\tilde{\mu}_1 \min_j \left\{ \mu_j v^n_{(x,\tilde{\imath}-e_1,i+e_1-e_j)} + (\mu - \mu_j) v^n_{(x,\tilde{\imath}-e_1,i+e_1)} \right\} +$$

$$
\begin{aligned}
&(\tilde\mu-\tilde\mu_1)\min_j\Big\{\mu_j v^n_{(x,\tilde\imath,i-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath,i)}\Big\}\le\\
&\tilde\mu_1\mu_2 v^n_{(x,\tilde\imath-e_1,i+e_1-e_2)}+\tilde\mu_1(\mu-\mu_2)v^n_{(x,\tilde\imath-e_1,i+e_1)}+\\
&(\tilde\mu-\tilde\mu_1)\mu_2 v^n_{(x,\tilde\imath,i-e_2)}+(\tilde\mu-\tilde\mu_1)(\mu-\mu_2)v^n_{(x,\tilde\imath,i)}\overset{(2.6)}{\le}\\
&\tilde\mu_2\mu_2 v^n_{(x,\tilde\imath-e_2,i+e_2-e_2)}+\tilde\mu_2(\mu-\mu_2)v^n_{(x,\tilde\imath-e_2,i+e_2)}+\\
&(\tilde\mu-\tilde\mu_2)\mu_2 v^n_{(x,\tilde\imath,i-e_2)}+(\tilde\mu-\tilde\mu_2)(\mu-\mu_2)v^n_{(x,\tilde\imath,i)}=\\
&\tilde\mu_2\min_j\Big\{\mu_j v^n_{(x,\tilde\imath-e_2,i+e_2-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath-e_2,i+e_2)}\Big\}+\\
&(\tilde\mu-\tilde\mu_2)\min_j\Big\{\mu_j v^n_{(x,\tilde\imath,i-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath,i)}\Big\}.
\end{aligned}
$$

Also the dummy term follows easily.

Consider (2.7). Again the terms concerning arrivals and the dummy transition follow easily. The optimal action in the first center of $(x,\tilde\imath,i)$ depends only on $\tilde\imath$. Because the number of customers in center 2 in state $(x,\tilde\imath,i-e_1)$, $(x,\tilde\imath,i-e_2)$ and $(x,\tilde\imath,i)$ is i_1+i_2-1 or more, there are at least n customers available, meaning that, by (2.6), the same action is optimal in each state. Therefore also the terms concerning departures from the first center follow easily. Concerning the second center, we have

$$
\begin{aligned}
&\mu_1\min_j\Big\{\mu_j v^n_{(x,\tilde\imath,i-e_1-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath,i-e_1)}\Big\}+\\
&(\mu-\mu_1)\min_j\Big\{\mu_j v^n_{(x,\tilde\imath,i-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath,i)}\Big\}\le\\
&\mu_1\mu_2 v^n_{(x,\tilde\imath,i-e_1-e_2)}+\mu_1(\mu-\mu_2)v^n_{(x,\tilde\imath,i-e_1)}+\\
&(\mu-\mu_1)\mu_2 v^n_{(x,\tilde\imath,i-e_2)}+(\mu-\mu_1)(\mu-\mu_2)v^n_{(x,\tilde\imath,i)}=\\
&\mu_2\min_j\Big\{\mu_j v^n_{(x,\tilde\imath,i-e_2-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath,i-e_2)}\Big\}+\\
&(\mu-\mu_2)\min_j\Big\{\mu_j v^n_{(x,\tilde\imath,i-e_j)}+(\mu-\mu_j)v^n_{(x,\tilde\imath,i)}\Big\}.
\end{aligned}
$$ □

Analogous to theorem 2.5, this results in the following:

2.8. Theorem. *For all T, cost functions satisfying (2.6) and (2.7) (and (2.2) and (2.3) if idleness is allowed) and $\varepsilon>0$ there is a number N such that the tandem μc rule in both centers is ε-optimal at T, if there are more than N customers in the second center at time 0.*

3. COMPUTATIONS

It is interesting to compare the tandem μc rule and the optimal policy numerically for the infinite time horizon. We did this again using dynamic programming for a single set of service and holding cost rates, and for varying arrival rates (we took the arrivals Poisson). Both discounted and average costs were considered. The model we did our computations for can be found in figure 1. One reason for choosing this particular set of parameters is that we cannot expect the μc rule to be optimal in the second center (according to [3] since $\mu_1 c_1 \geq \mu_2 c_2$ and $\mu_1 \geq \mu_2$).

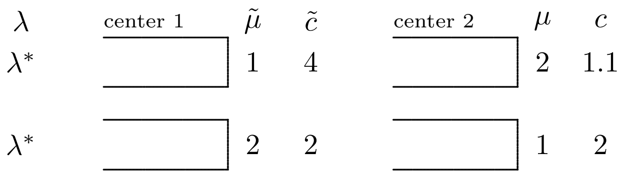

Figure 1

In table 1 the results, using value iteration, for the discounted cost case are summarized. For all combinations we computed the relative difference between the costs under the optimal policy and under the μc-rule, for the starting states with each queue empty (upper entries) and 5 customers in each queue of both centers (lower entries). Of course we had to make the state space finite. We did this by giving an upper bound on the total number of customers in the system. By doing it this way, buffer influences are minimized. Note that the average load is equal to $\frac{3}{2}\lambda^*$, and thus $\lambda^* = 0.6$ gives an average load of 0.9.

For most instances the difference between both policies was below machine precision. Only for higher values of the discount factor β there was some difference. It is remarkable to see that the tandem μc rule works better for the empty initial state. This can be explained by the fact that in this case there are few customers available, giving less choice for the servers.

	$\beta = 0.01$	0.1	0.25	0.5	0.75
$\lambda^* = 0.1$	0 0	0 0	0 0	0 0	$2.6 \cdot 10^{-8}$ $4.3 \cdot 10^{-6}$
0.2	0 0	0 0	0 0	0 0	$4.0 \cdot 10^{-7}$ $4.2 \cdot 10^{-6}$
0.3	0 0	0 0	0 0	0 0	$4.5 \cdot 10^{-7}$ $2.1 \cdot 10^{-6}$
0.4	0 0	0 0	0 0	0 0	0 $<10^{-15}$
0.5	0 0	0 0	0 0	0 0	0 $<10^{-15}$
0.6	0 0	0 0	0 0	0 $<10^{-14}$	$<10^{-14}$ $<10^{-13}$

Table 1. Discounted costs

The results for the average cost case in table 2 indicate that theorem 2.5 does not hold for average costs. The results for high traffic intensities are less accurate (indicated with $\approx$) due to the finite state space, although we had a model with a maximum of 60 customers, giving more than $6 \cdot 10^5$ states.

λ^*	R^*	μc	rel. diff.
0.1	0.886	0.889	0.0034
0.2	2.134	2.171	0.017
0.3	4.024	4.202	0.044
0.4	7.248	≈ 7.939	0.095
0.5	≈ 14.092	≈ 16.862	0.20
0.6	≈ 36.6	≈ 48.5	0.32

Table 2. Average costs

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