Stochastic Bounds for Queueing Systems with Multiple On-Off Sources

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Abstract

Consider a queueing system where the input traffic consists of background traffic, modeled by a Markov Arrival Process (MAP), and foreground traffic modeled by $N \geq 1$ homogeneous on-off sources. The queueing system has an increasing and concave service rate, which includes as a particular case multi-server queueing systems. Both the infinite-capacity and the finite-capacity buffer cases are analyzed. We show that the queue length in the infinite-capacity buffer system (respectively the number of losses in finite-capacity buffer system) is larger in the increasing convex order sense (respectively the strong stochastic order sense) than the queue length (respectively the number of losses) of the queueing system with the same background traffic and MN homogeneous on-off sources of the same total intensity as the foreground traffic, where M is an arbitrary integer. As a consequence, the queue length and the loss with a foreground traffic of multiple homogeneous on-off sources is upper bounded by that with a single on-off source and lower bounded by a Poisson source, where the bounds are obtained in the increasing convex order (respectively the strong stochastic order). We also compare $N \geq 1$ homogeneous arbitrary two-state Markov Modulated Poisson Process (MMPP) sources. We prove the monotonicity of the queue length in the transition rates and its convexity in the arrival rates. Standard techniques could not be used due to the different state spaces that we compare. We propose a new approach for the stochastic comparison of queues using dynamic programming which involves initially stationary arrival processes.

Keywords: Markov Modulated Poisson Process, Markov Arrival Process, On-off source, stochastic comparison, dynamic programming.

1 Introduction

The model of on-off sources, and more generally two-state Markov Modulated Poisson Process (MMPP) sources, is one of the most used in the performance analyses of communication networks, due in part to its small number of parameters and to its good characterization of new applications like audio and video traffic. Another interest of such a model comes from observations and the belief that on-off sources are the most resource consuming traffic, see Bean [4] and Boyer et al. [5].

We investigate the comparison between on-off sources in queueing systems with both infinite-capacity and finite-capacity buffers. In order to understand in which sense the on-off sources are the worst-case traffic model, we compare a single on-off source with multiple on-off sources with the same total intensity.

The queueing model under consideration is the following. The input traffic consists of background traffic, modeled by a Markov Arrival Process (MAP), and foreground traffic modeled by $N \geq 1$ homogeneous on-off sources. The queueing system has an increasing and concave service rate, which includes as a particular case multi-server queueing systems.

In the literature many results have been obtained on the comparison of queues. The reader is referred to the books of Ross [14], Stoyan [16], Baccelli and Brémaud [3], and Shaked and Shanthikumar [15]. The results most closely related to our model are those on the comparison of queueing systems with Doubly Stochastic Poisson (DSP) processes, see Ross [13], Rolski [11, 12], and Svoronos and Green [17]. More recently, Chang and Pinedo [6] obtained monotonicity results for the blocking probabilities in a DSP/M/1 queue. Chang et al. [7, 8] obtained comparison results on the queue length in an infinite-capacity multi-server system and in a tandem queueing system with exponential service times. Such results can be applied to MMPP sources where the transition rates from state i to state j are independent of i. They are therefore applicable to the case of a single on-off source (or a single two-state MMPP source), but are not applicable to multiple on-off sources. The reader is referred to these papers for further references on that matter.

We show that the queue length in the infinite-capacity buffer system (respectively the number of losses in finite-capacity buffer system) is larger in the increasing convex order sense (respectively the strong stochastic order sense) than the queue length (respectively number of losses) of the queueing system with the same background traffic and MN homogeneous on-off sources of the same total intensity as the foreground traffic, where M is an arbitrary integer. As a consequence, the queue length and the loss with foreground traffic consisting of multiple homogeneous on-off sources is upper bounded by that with a single on-off source and lower bounded by a Poisson source, where the comparison is in the increasing convex sense (respectively the strong stochastic sense).

We also compare $N \geq 1$ homogeneous arbitrary two-state Markov Modulated Poisson Process (MMPP) sources. We prove the monotonicity of the queue length in the transition rates and its convexity in the arrival rates.

We compare our queueing systems using dynamic programming. This approach has already been successfully applied to other queueing models, see e.g. van Dijk and Lamond [18], van Dijk and van der Wal [19], and Adan et al. [1]. In these papers the comparison results hold essentially for the same state space, and additional relations concerning boundary issues are needed when the state spaces are different but one is a subset of the other. Such an approach is impossible here. By assuming that the initial states (belonging to different state spaces) are stationary, we are able to compare systems with a different number of on-off sources. This approach might be useful to other model as well.

As far as the methodology is concerned, our approach bridges the gap between the stochastic comparison and dynamic programming. Indeed, for Markov decision processes it is usual to concentrate on long-run average and discounted costs. Here we consider queue lengths at and losses upto any time T, which is usual for stochastic scheduling results. We provide a general relation which permits to derive stochastic orders between transitive state variables using dynamic programming technique formalism.

The paper is organized as follows. In Section 2 we present our technique for establishing stochastic comparison results of Markov chains. In Section 3 we present the infinite-capacity queueing model and its convexity properties. We then derive stochastic comparison results of the queue lengths for different input traffic. In Section 4 we analyze the finite-capacity queueing model and derive comparison results. Finally in Section 5 we point out extensions of the results and further research directions.

2 Dynamic Programming and Stochastic Comparison

2.1 Notions of Stochastic Orders

We start with introducing the notation of (integral) stochastic orderings and recalling their basic properties. The reader is referred to [3, 10, 14, 16] for more details. Throughout this paper, increasing, convex, concave and positive are understood to be non-strict, unless otherwise stated.

Let n be an arbitrary strictly positive integer. Let $\mathcal{C}_{\mathcal{L}}$ be a class of functions from $I\!R^n$ to $I\!R$.

Let $\boldsymbol{X}, \boldsymbol{Y} \in I\!R^n$ be two random vectors.

The random vector $\boldsymbol{X}$ is said to be smaller than the random vector $\boldsymbol{Y}$ in the sense of $\leq_{\mathcal{L}}$ (noted $\boldsymbol{X} \leq_{\mathcal{L}} \boldsymbol{Y}$), if for all $f \in \mathcal{C}_{\mathcal{L}},\ E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$, provided that the expectations exist. The binary relation $\leq_{\mathcal{L}}$ is called the integral stochastic ordering, or simply the stochastic ordering, generated by the class of functions $\mathcal{C}_{\mathcal{L}}$.

In this paper we will be particularly interested in the strong stochastic order $\leq_{\text{st}}$ (which will be simply referred to as stochastic order hereafter) and the increasing convex order $\leq_{\text{icx}}$, which are generated by the classes of increasing, increasing and convex functions from $I\!R^n$ into $I\!R$, respectively. It is clear that when $\mathcal{C}_{\mathcal{L}1} \supseteq \mathcal{C}_{\mathcal{L}2}$, $\leq_{\mathcal{L}_1} \Rightarrow \leq_{\mathcal{L}_2}$, where the symbol $\Rightarrow$ denotes the implication between orderings. Thus, $\leq_{\text{st}} \ \Rightarrow \ \leq_{\text{icx}}$.

Other examples include the classes of convex, concave, increasing and concave functions (see e.g. [16]), the class of Schur convex functions (see e.g. [10]), the class of convex symmetric functions, etc.

Of particular interest is the convergence property of these orders.

Lemma 2.1 (Proposition 1.2.3 of Stoyan [16]) *Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables which converge to X and Y in distribution. If for all n, $X_n \leq_{\text{st}} Y_n$, then $X \leq_{\text{st}} Y$.*

Lemma 2.2 (Proposition 1.3.2 of Stoyan [16]) *Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables which converge to X and Y in distribution. Assume that $EX^+ := E[\max(X,0)]$ and EY^+ are finite and that $EX_n^+ \to EX^+$ and $EY_n^+ \to EY^+$ when $n \to \infty$. If for all n, $X_n \leq_{\text{icx}} Y_n$, then $X \leq_{\text{icx}} Y$.*

2.2 Markov Chains and Dynamic Programming Formulation

Consider a continuous-time Markov Chain (MC) (X_t, Y_t, Z_t) on $\mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3 \subseteq I\!R^l \times I\!R^m \times I\!R^n$ with initial state (X_0, Y_0, Z_0) and bounded transition rates, where $l \geq 0$, $m \geq 0$ and $n \geq 1$. Without loss of generality we can assume that the sum of the transition rates is upper bounded by 1 in each state. Thus, we can apply the technique of *uniformization* to obtain a discrete-time MC whose possible transition times are generated by some Poisson process representing the uniformization. See e.g. Çinlar [9], pp. 236–237.

Let (X_k, Y_k, Z_k) be the state of the MC after k jumps. If the MC has stationary initial state

(X_0, Y_0, Z_0), then, owing to PASTA (Poisson process see time average) property (cf. e.g. [3]), (X_k, Y_k, Z_k) has the same law as (X_0, Y_0, Z_0).

Let $V_k(x, y, z)$ be the value function (from $\mathbb{R}^{l+m+n} \to \mathbb{R}$) defined as the expected costs incurred after k jumps of the uniformization process:

$$V_0(x, y, z) = C(z), \tag{1}$$

$$V_{k+1}(x, y, z) = \sum_{(x',y',z') \in \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3} \delta_{(x,y,z),(x',y',z')} V_k(x', y', z'), \qquad k \geq 0, \tag{2}$$

where $C(\cdot)$ (from $\mathbb{R}^n \to \mathbb{R}$) is the direct cost, and δ is the transition rate. We assume that $\sum_{(x',y',z')} \delta_{(x,y,z),(x',y',z')} = 1$ for each (x, y, z). This can be done by adding a term $\delta_{(x,y,z),(x,y,z)}$ if necessary, because we assumed that $\sum_{(x',y',z')} \delta_{(x,y,z),(x',y',z')} \leq 1$.

Note that in the above dynamic programming formulation, there is no immediate cost, but a "terminal" cost function C. One can easily see that this approach is more general than those with immediate costs independent of the age (the number of jumps) or with discounted costs. Indeed, a value function associated with such costs can be obtained by appropriately summing our value functions associated with terminal costs.

Unlike the usual dynamic programming problems, we have no control decisions here. We shall use the value function to compare state variables of the MC.

Let $X_0 = x$ and $Z_0 = z$, and denote by $P_{x,z}$ the conditional probability distribution and $E_{x,z}$ the conditional expectation. Let $V_k(x, \cdot, z)$ be the conditional expected cost after k jumps, i.e.

$$V_k(x, \cdot, z) = E_{x,z} V_k(x, Y_0, z) = \sum_{y \in \mathcal{K}_2} P_{x,z}(Y_0 = y) V_k(x, y, z).$$

2.3 Comparison of Value Functions and Random Variables

Let there be another MC (X'_t, Y'_t, Z'_t) on $\mathcal{K}'_1 \times \mathcal{K}'_2 \times \mathcal{K}'_3 \subseteq \mathbb{R}^{l'} \times \mathbb{R}^{m'} \times \mathbb{R}^n$ with initial state (X'_0, Y'_0, Z'_0) and bounded transition rates, where $l' \geq 0$, $m' \geq 0$ and $n \geq 1$. Note that l' and m' can be different from l and m.

Let $X'_0 = x'$ and $Z'_0 = z'$, and denote by $P'_{x',z'}$ the conditional probability distribution and $E'_{x',z'}$ the conditional expectation. Let $V'_k(x', \cdot, z')$ be the conditional expected cost after k jumps in the same uniformization process, i.e.

$$V'_k(x',\cdot,z') = E'_{x',z'}V'_k(x',Y'_0,z') = \sum_{y'\in\mathcal{K}'_2} P'_{x',z'}(Y'_0 = y')V'_k(x',y',z').$$

Theorem 2.3 *Let the initial states (x,z) and (x',z') be fixed. Let $Z_{t|x,z}$ and $Z'_{t|x',z'}$ be the state variables given these initial states. Let $\mathcal{C}_\mathcal{L}$ be a class of functions from $I\!R^n$ to $I\!R$. If for all direct cost function $C \in \mathcal{C}_\mathcal{L}$ and for all $k \geq 0$,*

$$V_k(x,\cdot,z) \leq V'_k(x',\cdot,z'),$$

then $Z_{t|x,z}$ is smaller than $Z'_{t|x',z'}$ in the sense of $\leq_\mathcal{L}$, i.e.

$$Z_{t|x,z} \leq_\mathcal{L} Z'_{t|x',z'}.$$

Proof. By the dynamic programming formulation, $V_k(x,\cdot,z) = E_{x,z}C(Z_k)$ and $V'_k(x',\cdot,z') = E'_{x',z'}C(Z'_k)$. Let t be fixed, and p_k be the probability of having k points in the uniformization process in the interval $(0,t)$. (Thus $p_k = \frac{t^k}{k!}\mathrm{e}^{-t}$, because we assumed that $\sum_{(x',y',z')}\delta_{(x,y,z),(x',y',z')} = 1$ for each (x,y,z).) Then,

$$E_{x,z}C(Z_t) = \sum_{k=0}^{\infty} p_k E_{x,z}C(Z_k).$$

Thus, for all $C \in \mathcal{C}_\mathcal{L}$,

$$E_{x,z}C(Z_t) = \sum_{k=0}^{\infty} p_k E_{x,z}C(Z_k) \leq \sum_{k=0}^{\infty} p_k E'_{x',z'}C(Z'_k) = E'_{x',z'}C(Z'_t),$$

so that $Z_{t|x,z} \leq_\mathcal{L} Z'_{t|x',z'}$. ∎

When $\mathcal{C}_\mathcal{L}$ is the class of increasing functions or the class of increasing and convex functions, we obtain (strong) stochastic order $\leq_{\mathrm{st}}$ and increasing convex order $\leq_{\mathrm{icx}}$. Now, using the convergence properties of Lemmas 2.1 and 2.2, we obtain

Corollary 2.4 *Let the initial states (x,z) and (x',z') be fixed. Let $Z_{t|x,z}$ and $Z'_{t|x',z'}$ be the state variables given these initial states. Assume that $Z_{t|x,z}$ and $Z'_{t|x',z'}$ converge in distribution to Z_∞ and Z'_∞. If for all increasing cost function $C : I\!R \to I\!R$ and for all $k \geq 0$,*

$$V_k(x,\cdot,z) \leq V'_k(x',\cdot,z'),$$

then Z_∞ *is stochastically smaller than* Z'_∞*, i.e.*

$$Z_\infty \leq_{\text{st}} Z'_\infty.$$

Corollary 2.5 *Let the initial states* (x, z) *and* (x', z') *be fixed. Let* $Z_{t|x,z}$ *and* $Z'_{t|x',z'}$ *be the state variables given these initial states. Assume that* $Z_{t|x,z}$ *and* $Z'_{t|x',z'}$ *converge in distribution and in expectation to* Z_∞ *and* Z'_∞*. If for all increasing and convex cost function* $C : I\!R \to I\!R$ *and for all* $k \geq 0$*,*

$$V_k(x, \cdot, z) \leq V'_k(x', \cdot, z'),$$

then Z_∞ *is smaller than* Z'_∞ *in the increasing convex ordering sense, i.e.*

$$Z_\infty \leq_{\text{icx}} Z'_\infty.$$

3 Queue Lengths in the Infinite-Capacity Queueing System

3.1 The Traffic Model and the Queueing System

The infinite-capacity queueing system under consideration is fed by background traffic and foreground traffic. The background traffic is modeled by an arbitrary MAP, which is a continuous-time MC with transition rates α_{xy} for the transition from state x to y. Arrivals occur only at transition epochs. When state changes from x to y, an arrival occurs with probability β_{xy}. Note that such a model is more general than MMPP and is dense in the class of arbitrary arrival processes (see [2]).

The foreground traffic is the superposition of N homogeneous and stochastically independent on-off sources. A source goes from state "off" to state "on" with rate p, and from state "on" to state "off" with rate q. When a source is in the "on" state, arrivals occur according to a Poisson process with parameter λ. Otherwise, in the "off" state a source generates no arrivals.

All arrived customers are queued in an infinite-capacity buffer and are served in FCFS (First Come First Serve) order. The service times are exponentially distributed with parameter μ_i when there are i customers in the queue. In other words we are assuming that the customer service requirements are independently and identically distributed with an exponential distribution of parameter 1, and that the server serves at speed μ_i when there are i customers in the queue.

We derive our main results for the case that the service rate μ_i is increasing, concave and upper bounded by μ. Such an assumption holds in the case of multiple servers, but not for the infinite-server queue where μ_i is unbounded. For each result however we mention explicitly the conditions.

We assume that $\mu_0 = 0$, and, without loss of generality, that $N(\lambda + p + q) + \mu + \sum_y \alpha_{xy} = 1$ for all x, i.e., the system is normalized. By rescaling time this can be done without loss of generality.

Note that our traffic model can also deal with two-state MMPP sources as the foreground traffic. In such a case, a source generates arrivals at rates λ_0 and λ_1 according to the state it is in. Assume $\lambda_0 \leq \lambda_1$. Then it is easy to see that this two-state source is the superposition of a Poisson source with rate λ_0 and an on-off source with rate $\lambda_1 - \lambda_0$ in the "on" state. This Poisson source can be incorporated in the background traffic so that the foreground traffic consists only of on-off sources.

3.2 Dynamic Programming Equations and Monotonicity and Convexity Properties

It is easy to see that the queueing system can be described by the continuous-time MC with state variable (X_t, S_t, Q_t), where X_t is the state of the MAP, S_t is the number of active (or "on") sources and Q_t the number of customers in the system.

Denote by V_k^N the value function after k jumps in the uniformization process, defined by the following recursive equations.

$$V_0^N(x,n,i) = C(i), \tag{3}$$

$$V_{k+1}^N(x,n,i) = n\lambda V_k^N(x,n,i+1) + (N-n)\lambda V_k^N(x,n,i) \tag{4}$$

$$+ nqV_k^N(x,n-1,i) + (N-n)qV_k^N(x,n,i) \tag{5}$$

$$+ (N-n)pV_k^N(x,n+1,i) + npV_k^N(x,n,i) \tag{6}$$

$$+ \mu_i V_k^N(x,n,i-1) + (\mu - \mu_i)V_k^N(x,n,i) \tag{7}$$

$$+ \sum_y \alpha_{xy}\Big(\beta_{xy}V_k^N(y,n,i+1) + (1-\beta_{xy})V_k^N(y,n,i)\Big), \qquad k \geq 0. \tag{8}$$

In the above, function C represents the direct costs. Note that there are only terminal costs, thus $V_k^N(x,n,i)$ represents the expected costs after k steps, if the initial state is (x,n,i).

In what follows, we call the terms on the right hand side of (4) the λ-terms, those in (5) the q-terms, those in (6) the p-terms, those in (7) the μ-terms, and finally, those in (8) the MAP-terms.

We show first the convexity and the supermodularity of the value function.

Lemma 3.1 *If $C(i)$ is increasing and convex in i and μ_i is concave in i, then $V_k^N(x,n,i)$ is increasing and convex in i for all k, x and n, i.e.,*

$$V_k^N(x,n,i) \leq V_k^N(x,n,i+1) \qquad i \geq 0, \quad k \geq 0, \quad 0 \leq n \leq N, \tag{9}$$

$$2V_k^N(x,n,i+1) \leq V_k^N(x,n,i) + V_k^N(x,n,i+2), \qquad i \geq 0, \quad k \geq 0, \quad 0 \leq n \leq N. \tag{10}$$

Proof. We use induction to prove (9) and (10). As $C(i)$ is increasing and convex in i, the result holds for V_0^N. Assuming that V_k^N is increasing and convex in i, we show (9) and (10) hold for $k+1$.

According to (4–8), for the proof of the increasingness of V_{k+1}^N, i.e., $V_{k+1}^N(x,n,i) \leq V_{k+1}^N(x,n,i+1)$, it suffices to consider the μ-terms and to show, cf. (4–8),

$$\mu_i V_k^N(x,n,i-1) + (\mu - \mu_i) V_k^N(x,n,i) \leq \mu_{i+1} V_k^N(x,n,i) + (\mu - \mu_{i+1}) V_k^N(x,n,i+1),$$

which follows from the inductive assumption, i.e., $V_k^N(y,n,i) \leq V_k^N(y,n,i+1)$ and $V_k^N(y,n,i-1) - V_k^N(y,n,i) \leq 0$. Thus $V_{k+1}^N(x,n,i)$ is increasing in i.

Consider now the convexity. We see from (4–8) that V_{k+1}^N is a convex combination of V_k^N in various states. Thus, we only need to show the convexity of the different terms in i. For example, the convexity of the λ-term follows simply from inductive assumption of the convexity, i.e., $2V_k^N(x,n,i+1) \leq V_k^N(x,n,i) + V_k^N(x,n,i+2)$ and $2V_k^N(x,n,i+2) \leq V_k^N(x,n,i+1) + V_k^N(x,n,i+3)$.

The only terms that need investigation are the μ-terms. If we sum the following three inequalities we get exactly its convexity relation:

$$\begin{aligned} 2\mu_i V_k^N(x,n,i) &\leq \mu_i V_k^N(x,n,i-1) + \mu_i V_k^N(x,n,i+1) \\ 2(\mu - \mu_{i+2}) V_k^N(x,n,i+1) &\leq (\mu - \mu_{i+2}) V_k^N(x,n,i) + (\mu - \mu_{i+2}) V_k^N(x,n,i+2) \\ (2\mu_{i+1} - \mu_i - \mu_{i+2}) V_k^N(x,n,i) &\leq (2\mu_{i+1} - \mu_i - \mu_{i+2}) V_k^N(x,n,i+1) \end{aligned}$$

The last inequality comes from the increasingness of V_k^N and the concavity of μ_i in i. ∎

Note that as far as the convexity of the value function is concerned (Lemma 3.1), the proof does not need to consider on-off sources as they can be incorporated in the MAP background traffic.

Lemma 3.2 *If $V_k^N(x,n,i)$ is convex in i for all k and n, then $V_k^N(x,n,i)$ is supermodular in n and i, i.e.,*

$$V_k^N(x,n,i) + V_k^N(x,n+1,i+1) \geq V_k^N(x,n+1,i) + V_k^N(x,n,i+1) \tag{11}$$

Proof. We use induction again. For $k = 0$, (11) trivially holds (both sides are equal to $C(i) + C(i+1)$). Assume it is true for some $k \geq 0$. Consider $k+1$. As in the previous lemma, we consider all terms one by one. For the λ-terms, we need to show the following:

$$\begin{aligned} & nV_k^N(x,n,i+1) + (N-n)V_k^N(x,n,i) \\ & \quad +(n+1)V_k^N(x,n+1,i+2) + (N-n-1)V_k^N(x,n+1,i+1) \\ \geq & \quad (n+1)V_k^N(x,n+1,i+1) + (N-n-1)V_k^N(x,n+1,i) \\ & \quad +nV_k^N(x,n,i+2) + (N-n)V_k^N(x,n,i+1) \end{aligned}$$

Due to the inductive assumption, the above inequality holds if the following inequality

$$V_k^N(x,n,i) + V_k^N(x,n+1,i+2) \geq V_k^N(x,n,i+1) + V_k^N(x,n+1,i+1)$$

is valid. One can easily see this relation is a combination of supermodularity and convexity (cf. Lemma 3.1):

$$\begin{aligned} V_k^N(x,n,i) + V_k^N(x,n+1,i+1) &\geq V_k^N(x,n,i+1) + V_k^N(x,n+1,i) \\ V_k^N(x,n+1,i+2) - V_k^N(x,n+1,i+1) &\geq V_k^N(x,n+1,i+1) - V_k^N(x,n+1,i) \end{aligned}$$

Hence the result.

For the p-terms and q-terms, the relations can be shown in an analogous way. Now consider the μ-terms. We need to show

$$\begin{aligned} & \mu_i V_k^N(x,n,i-1) + (\mu-\mu_i)V_k^N(x,n,i) + \mu_{i+1}V_k^N(x,n+1,i) + (\mu-\mu_{i+1})V_k^N(x,n+1,i+1) \\ \geq & \quad \mu_i V_k^N(x,n+1,i-1) + (\mu-\mu_i)V_k^N(x,n+1,i) + \mu_{i+1}V_k^N(x,n,i) + (\mu-\mu_{i+1})V_k^N(x,n,i+1) \end{aligned}$$

which follows from the inductive assumption.

The MAP-terms follow easily too. The proof is thus completed. ∎

Theorem 3.3 *If $C(i)$ is increasing and convex in i and μ_i is concave in i, then for all k, x and n, $V_k^N(x,n,i)$ is increasing and convex in i, and for all k and x, $V_k^N(x,n,i)$ is supermodular in n and i.*

3.3 Monotonicity in the Number of On-Off Sources

We compare two queueing systems having the same background traffic. The first one has N on-off sources as foreground traffic. The second one has a single on-off source as foreground traffic and keeps the same total traffic intensity. This on-off source in the second queueing system has the same transition rates as any of the on-off sources in the first system. The only difference is its arrival rate being $N\lambda$ when it is in the "on" state. Thus, this second system can be seen as the same as the first one, except that the N sources are "coupled" in the sense that they change state all at the same time.

In order to make the comparison possible we use the same uniformization parameter. Thus, for the system with N sources, we refer to the equations (4–8). For the single source system, the value function V^1 is formulated as follows.

$$
\begin{aligned}
V_0^1(x,n,i) &= C(i), \qquad n=0,1. && (12)\\
V_{k+1}^1(x,n,i) &= nN\lambda V_k^1(x,n,i+1) + (1-n)N\lambda V_k^1(x,n,i) && (13)\\
&\quad + nqV_k^1(x,n-1,i) + (N-n)qV_k^1(x,n,i) && (14)\\
&\quad + (1-n)pV_k^1(x,n+1,i) + (N-n+1)pV_k^1(x,n,i) && (15)\\
&\quad + \mu_i V_k^1(x,n,i-1) + (\mu-\mu_i)V_k^1(x,n,i) && (16)\\
&\quad + \sum_y \alpha_{xy}\Big(\beta_{xy}V_k^1(y,n,i+1) + (1-\beta_{xy})V_k^1(y,n,i)\Big), \quad n=0,1, \quad k\geq 0. && (17)
\end{aligned}
$$

Thus the transitions of source 2 up to N are replaced by dummy transitions. We have the following comparison lemma. Note that the coefficients are proportional to the stationary probabilities of the on-off sources. It will be used with $j=0$ in the main theorem.

Lemma 3.4 *If $V_k^N(x,n,i)$ is convex in i for all x, k and n, then*

$$
\begin{aligned}
&N(p+q)^{N-1}qV_k^1(x,0,i) + N(p+q)^{N-1}pV_k^1(x,1,i+j)\\
&\quad \geq \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n}\Big((N-n)V_k^N(x,n,i) + nV_k^N(x,n,i+j)\Big) \qquad (18)
\end{aligned}
$$

for all x, k, i and $j\geq 0$, where V^1 is the value function of the model with a single source and arrival rate $N\lambda$.

Proof. We use induction to show (18).

For $k = 0$, it is easily seen that both sides are equal to $N(p+q)^{N-1}qC(i)+N(p+q)^{N-1}pC(i+j)$. Assume (18) holds for k. Consider $k+1$.

With respect to the λ-terms, we need to show

$$\begin{aligned} & N^2(p+q)^{N-1}qV_k^1(x,0,i) + N^2(p+q)^{N-1}pV_k^1(x,1,i+j+1) \\ \geq \; & \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \Big((N-n)^2 V_k^N(x,n,i) + n(N-n)V_k^N(x,n,i+1) \\ & + n(N-n)V_k^N(x,n,i+j) + n^2 V_k^N(x,n,i+j+1) \Big) \end{aligned} \tag{19}$$

Using the inductive assumption, we have

$$\begin{aligned} & N^2(p+q)^{N-1}qV_k^1(x,0,i) + N^2(p+q)^{N-1}pV_k^1(x,1,i+j+1) \\ \geq \; & \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \Big(N(N-n)V_k^N(x,n,i) + nNV_k^N(x,n,i+j+1) \Big), \end{aligned}$$

and by convexity,

$$\begin{aligned} & (N-n)NV_k^N(x,n,i) + nNV_k^N(x,n,i+j+1) \geq (N-n)^2 V_k^N(x,n,i) \\ & +(N-n)nV_k^N(x,n,i+1) + n(N-n)V_k^N(x,n,i+j) + n^2 V_k^N(x,n,i+j+1). \end{aligned}$$

Relation (19) thus holds.

The q-terms and p-terms are obtained by summing the following three inequalities:

$$N(p+q)^{N-1}q^2V_k^1(x,0,i) + N(p+q)^{N-1}pqV_k^1(x,1,i) \geq q \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} NV_k^N(x,n,i),$$

$$N(p+q)^{N-1}pqV_k^1(x,0,i+j) + N(p+q)^{N-1}p^2V_k^1(x,1,i+j) \geq p \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} NV_k^N(x,n,i+j),$$

and

$$\begin{aligned} & N(N-1)(p+q)^N qV_k^1(x,0,i) + N(N-1)(p+q)^N pV_k^1(x,1,i+j) \\ \geq \; & (N-1)(p+q) \sum_{n=0}^{N} \binom{N}{n} p^n q^{N-n} \Big((N-n)V_k^N(x,n,i) + nV_k^N(x,n,i+j) \Big), \end{aligned}$$

which are all valid due to the inductive assumption.

Now consider the μ-terms. We need to show that

$$
\begin{aligned}
& N(p+q)^{N-1}q\Big(\mu_i V_k^1(x,0,i-1)+(\mu-\mu_i)V_k^1(x,0,i)\Big) \\
& \quad +N(p+q)^{N-1}p\Big(\mu_{i+j}V_k^1(x,1,i+j-1)+(\mu-\mu_{i+j})V_k^1(x,1,i+j)\Big) \\
\geq & \sum_{n=0}^{N}\tbinom{N}{n}p^n q^{N-n}\Big((N-n)\mu_i V_k^N(x,n,i-1)+(N-n)(\mu-\mu_i)V_k^N(x,n,i) \\
& \quad +n\mu_{i+j}V_k^N(x,n,i+j-1)+n(\mu-\mu_{i+j})V_k^N(x,n,i+j)\Big) \qquad (20)
\end{aligned}
$$

If $j=0$ then (20) trivially follows from the inductive assumption. Assume that $j \geq 1$. Let $\underline{\mu}=\min\{\mu_i,\mu_{i+j}\}$ and $\Delta=|\mu_i-\mu_{i+j}|$. If $\mu_i \leq \mu_{i+j}$, then relation (20) is the sum of the following three inequalities

$$
\begin{aligned}
& \underline{\mu}N(p+q)^{N-1}\Big(qV_k^1(x,0,i-1)+pV_k^1(x,1,i+j-1)\Big) \\
\geq & \ \underline{\mu}\sum_{n=0}^{N}\tbinom{N}{n}p^n q^{N-n}\Big((N-n)V_k^N(x,n,i-1)+nV_k^N(x,n,i+j-1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
& (\mu-\underline{\mu}-\Delta)N(p+q)^{N-1}\Big(qV_k^1(x,0,i)+pV_k^1(x,1,i+j)\Big) \\
\geq & \ (\mu-\underline{\mu}-\Delta)\sum_{n=0}^{N}\tbinom{N}{n}p^n q^{N-n}\Big((N-n)V_k^N(x,n,i)+nV_k^N(x,n,i+j)\Big)
\end{aligned}
$$

$$
\begin{aligned}
& \Delta N(p+q)^{N-1}\Big(qV_k^1(x,0,i)+pV_k^1(x,1,i+j-1)\Big) \\
\geq & \ \Delta\sum_{n=0}^{N}\tbinom{N}{n}p^n q^{N-n}\Big((N-n)V_k^N(x,n,i)+nV_k^N(x,n,i+j-1)\Big)
\end{aligned}
$$

which are all true due to the inductive assumption. If, however $\mu_i \geq \mu_{i+j}$, then relation (20) is still the sum of the above three inequalities with the last one replaced by

$$
\begin{aligned}
& \Delta N(p+q)^{N-1}\Big(qV_k^1(x,0,i-1)+pV_k^1(x,1,i+j)\Big) \\
\geq & \ \Delta\sum_{n=0}^{N}\tbinom{N}{n}p^n q^{N-n}\Big((N-n)V_k^N(x,n,i-1)+nV_k^N(x,n,i+j)\Big)
\end{aligned}
$$

so that relation (20) is valid.

Finally the MAP-terms follow trivially. ∎

We are now in a position to prove the main result of this subsection. Let Q_t^N (respectively Q_t^1) denote the queue length at time t in the queueing system with N homogeneous on-off sources (respectively one on-off source). The single on-off source has N times the arrival rate of any of the N homogeneous on-off sources in the other system.

Theorem 3.5 *Consider two infinite-capacity buffer queueing systems with the same concave and bounded service rate μ_i and the same MAP background traffic, but with different number of homogeneous on-off sources. Assume that at time 0, all the on-off sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Then,*

$$Q_t^N \leq_{\mathrm{icx}} Q_t^1, \qquad \forall t \geq 0.$$

If these queue lengths converge in distribution and in expectation to Q_∞^N and Q_∞^1, then,

$$Q_\infty^N \leq_{\mathrm{icx}} Q_\infty^1.$$

Proof. Denote by $Q_t^N(x,i)$ (respectively $Q_t^1(x,i)$) the queue length at time t in the queueing system with N homogeneous on-off sources (respectively one on-off source) when the initial state of MAP is x and the initial queue length i. Then, by taking $j = 0$ in Lemma 3.4 we obtain

$$\frac{q}{p+q} V_k^1(x,0,i) + \frac{p}{p+q} V_k^1(x,1,i) \geq \sum_{n=0}^{N} \binom{N}{n} \frac{p^n q^{N-n}}{(p+q)^N} V_k^N(x,n,i),$$

where the coefficients are the stationary probabilities of the on-off sources. Using further Theorems 2.3 and 3.3, we obtain

$$Q_t^N(x,i) \leq_{\mathrm{icx}} Q_t^1(x,i), \qquad \forall t \geq 0.$$

Unconditioning with respect to x and i and using the assumption that the MAP and the initial queue length have the same distribution in both systems, we obtain

$$Q_t^N \leq_{\mathrm{icx}} Q_t^1, \qquad \forall t \geq 0.$$

Using further Corollary 2.4 we obtain $Q_\infty^N \leq_{\mathrm{icx}} Q_\infty^1$. ∎

As on-off sources are special cases of MAP sources, we can use an inductive argument to extend the above results to the comparison between N sources and MN sources by applying N times the comparison between M sources and single source.

Theorem 3.6 *Consider two infinite-capacity buffer queueing systems with the same concave and bounded service rate μ_i and the same MAP background traffic. The first one has N homogeneous on-off sources as the foreground traffic whereas the second one has MN homogeneous on-off sources with the same transition rates and the same total intensity. Assume that at time 0, all the on-off sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Then,*

$$Q_t^{MN} \leq_{\text{icx}} Q_t^N, \qquad \forall t \geq 0.$$

If these queue lengths converge in distribution and in expectation to Q_∞^{MN} and Q_∞^N, then,

$$Q_\infty^{MN} \leq_{\text{icx}} Q_\infty^N.$$

Note that the weak convergence assumption made in Theorems 3.5 and 3.6 are verified in multi-server queueing systems, see [3].

Theorem 3.6 allows us to compare N on-off sources with MN on-off sources. We conjecture that a more general monotonicity result holds, namely, that the queue length is decreasing (in the sense of increasing convex ordering) in the number of on-off sources.

3.4 Monotonicity in the Transition Rates

We now consider the effects of changing the transition rates and the arrival rates. We first compare two systems, both with a single source as the foreground traffic, which differ only in the rates with which the source changes state. The first system, with value function V, has transition rates p and q (as before), and the other, with value function V', has transition rates δp and δq, $0 < \delta < 1$. We assume also that the direct cost functions are identical in both systems. The value function V' is defined under the same uniformization parameter as V^N:

$$\begin{aligned}
V_0'(x,n,i) &= C(i), \\
V_{k+1}'(x,n,i) &= n\lambda V_k'(x,n,i+1) + (N-n)\lambda V_k'(x,n,i) \\
&\quad + n\delta q V_k'(x,n-1,i) + (Nq - n\delta q)V_k'(x,n,i) \\
&\quad + (N-n)\delta p V_k'(x,n+1,i) + (N(1-\delta)p + n\delta p)V_k'(x,n,i) \\
&\quad + \mu_i V_k'(x,n,i-1) + (\mu - \mu_i)V_k'(x,n,i) \\
&\quad + \sum_y \alpha_{xy}\Big(\beta_{xy}V_k'(y,n,i+1) + (1-\beta_{xy})V_k'(y,n,i)\Big), \qquad k \geq 0.
\end{aligned}$$

Lemma 3.7 *Let $N = 1$. If $V_k(x, n, i)$ is supermodular in n and i for all x and k, then*

$$qV'_k(x, 0, i) + pV'_k(x, 1, i + j) \geq qV_k(x, 0, i) + pV_k(x, 1, i + j) \tag{21}$$

holds for all x, k, i and $j \geq 0$.

Proof. As in the previous proofs, we use induction. The induction basis is trivial. So assume the relation holds for k. Consider $k+1$. The only terms of that need investigation are those related to the state changes of the on-off process. We need to show:

$$\delta pqV'_k(x, 1, i) + q(q + (1 - \delta)p)V'_k(x, 0, i) + \delta pqV'_k(x, 0, i + j) + p(p + (1 - \delta)q)V'_k(x, 1, i + j)$$

$$\geq pqV_k(x, 1, i) + q^2V_k(x, 0, i) + pqV_k(x, 0, i + j) + p^2V_k(x, 1, i + j). \tag{22}$$

This is the sum of the following three inequalities:

$$\delta q^2V'_k(x, 0, i) + \delta pqV'_k(x, 1, i) \geq \delta q^2V_k(x, 0, i) + \delta pqV_k(x, 1, i)$$

$$\delta pqV'_k(x, 0, i + j) + \delta p^2V'_k(x, 1, i + j) \geq \delta pqV_k(x, 0, i + j) + \delta p^2V_k(x, 1, i + j)$$

$$\begin{aligned} &(1 - \delta)(p + q)qV'_k(x, 0, i) + (1 - \delta)(p + q)pV'_k(x, 1, i + j) \\ \geq\ & (1 - \delta)\Big(pqV_k(x, 1, i) + q^2V_k(x, 0, i) + pqV_k(x, 0, i + j) + p^2V_k(x, 1, i + j)\Big) \end{aligned} \tag{23}$$

The first two follows directly from the inductive assumption. For the last inequality, note that

$$\begin{aligned} &(1 - \delta)(p + q)qV'_k(x, 0, i) + (1 - \delta)(p + q)pV'_k(x, 1, i + j) \\ \geq\ & (1 - \delta)(p + q)qV_k(x, 0, i) + (1 - \delta)(p + q)pV_k(x, 1, i + j). \end{aligned}$$

Applying the supermodularity (for j times) gives

$$(1 - \delta)pqV_k(x, 0, i) + (1 - \delta)pqV_k(x, 1, i + j) \geq (1 - \delta)pqV_k(x, 0, i + j) + (1 - \delta)pqV_k(x, 1, i)$$

Thus, (23) is valid, so is (22). Hence the result. ∎

The above lemma allows us to compare two systems, both with $N \geq 1$ homogeneous on-off sources as foreground traffic, which differ only in the rates with which the sources change state. The first system has transition rates p and q, and the other one has transition rates δp and δq, $0 < \delta < 1$. Let Q_t^N and $Q_t^{N,\delta}$ denote the queue lengths at time t in the two queueing systems.

Theorem 3.8 *Consider two infinite-capacity buffer queueing systems with the same concave and bounded service rate* μ_i *and the same MAP background traffic and the same number of homogeneous on-off sources. The transition rates of the on-off sources in the second system is* δ *times those of the first system, where* $0 < \delta < 1$*. Assume that at time 0, all the on-off sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Then,*

$$Q_t^N \leq_{\text{icx}} Q_t^{N,\delta}, \qquad \forall t \geq 0.$$

If these queue lengths converge in distribution and in expectation to Q_∞^N *and* $Q_\infty^{N,\delta}$*, then,*

$$Q_\infty^N \leq_{\text{icx}} Q_\infty^{N,\delta}.$$

Proof. Denote by $Q_t^N(x,i)$ and $Q_t^{N,\delta}(x,i)$ the queue lengths at time t in the two queueing systems when the initial state of the MAP is x and initial queue length is i. Consider the case $N = 1$. By Lemma 3.7 (taking $j = 0$) together with Theorems 2.3 and 3.3,

$$Q_t^1(x,i) \leq_{\text{icx}} Q_t^{1,\delta}(x,i), \qquad \forall t \geq 0.$$

Unconditioning with respect to x and i and using the assumption that the MAP and the initial queue length have the same distribution in both systems, we obtain

$$Q_t^1 \leq_{\text{icx}} Q_t^{1,\delta}, \qquad \forall t \geq 0.$$

Using further Corollary 2.4 we obtain $Q_\infty^N \leq_{\text{icx}} Q_\infty^{1,\delta}$.

The general case of N is easily shown by an inductive argument. Indeed, one by one we can put the on-off sources into the background traffic and use the inductive assumption on n sources to derive the comparison on $n+1$ sources. The detailed proof is left to the interested reader. ∎

3.5 Convexity in the Arrival Rates

Last, we investigate the effect of changing the arrival rates in a two-state MMPP source which generates arrivals at rates λ_0 and λ_1 according to the state it is in (0 or 1). Assume $\lambda_0 \leq \lambda_1$. We shall analyze the effect on the queue length when we increase λ_0 and decrease λ_1 in such a way that the weighted sum $q\lambda_0 + p\lambda_1$ is kept unchanged.

As we mentioned previously, the two-state source is the superposition of a Poisson source with rate λ_0 and an on-off source with rate $\lambda_1 - \lambda_0$ in the "on" state. Thus, when $\lambda_0 = \lambda_1$, the two-state source is reduced to the Poisson process.

We shall compare this (foreground) source to the on-off source with rate λ, with value function given by (4–8). In order for the two models to have the same total arrival density, we assume that

$$p\lambda = q\lambda_0 + p\lambda_1.$$

In the comparison we shall use the same uniformization parameter so that we assume without loss of generality that that $N(\lambda + \lambda_0 + \lambda_1 + p + q) + \mu + \sum_y \alpha_{xy} = 1$ for all x.

Let V be the value function associated with the on-off source model:

$$\begin{aligned}
V_0(x,n,i) &= C(i),\\
V_{k+1}(x,n,i) &= n\lambda V_k(x,n,i+1) + (N-n)\lambda V_k(x,n,i) + N(\lambda_0+\lambda_1)V_k(x,n,i)\\
&\quad + nqV_k(x,n-1,i) + (N-n)qV_k(x,n,i)\\
&\quad + (N-n)pV_k(x,n+1,i) + npV_k(x,n,i)\\
&\quad + \mu_i V_k(x,n,i-1) + (\mu-\mu_i)V_k(x,n,i)\\
&\quad + \sum_y \alpha_{xy}\Big(\beta_{xy}V_k(y,n,i+1) + (1-\beta_{xy})V_k(y,n,i)\Big), \qquad k \geq 0.
\end{aligned}$$

Let V'' be the value function associated with the two-state source model:

$$\begin{aligned}
V''_0(x,n,i) &= C(i),\\
V''_{k+1}(x,n,i) &= n\lambda_1 V''_k(x,n,i+1) + (N-n)\lambda_1 V''_k(x,n,i)\\
&\quad + (N-n)\lambda_0 V''_k(x,n,i+1) + n\lambda_0 V''_k(x,n,i) + N\lambda V''_k(x,n,i)\\
&\quad + nqV''_k(x,n-1,i) + (N-n)qV''_k(x,n,i)\\
&\quad + (N-n)pV''_k(x,n+1,i) + npV''_k(x,n,i)\\
&\quad + \mu_i V''_k(x,n,i-1) + (\mu-\mu_i)V''_k(x,n,i)\\
&\quad + \sum_y \alpha_{xy}\Big(\beta_{xy}V''_k(y,n,i+1) + (1-\beta_{xy})V''_k(y,n,i)\Big), \qquad k \geq 0.
\end{aligned}$$

Lemma 3.9 *Assume $N = 1$. If $V_k(x,n,i)$ is supermodular in n and i for all x and k, then*

$$qV_k(x,0,i) + pV_k(x,1,i+j) \geq qV''_k(x,0,i) + pV''_k(x,1,i+j) \tag{24}$$

for all x, k, i and $j \geq 0$.

Proof. We use induction and will only consider λ-terms, the other terms being simple. The induction basis trivially holds. Assume (24) holds for some $k \geq 0$. Then, for $k+1$, we need to show

$$\begin{aligned} & 2q\lambda V_k(x,0,i) + p\lambda V_k(x,1,i+j) + p\lambda V_k(x,1,i+j+1) \\ \geq \; & q(2\lambda-\lambda_0)V_k''(x,0,i) + q\lambda_0 V_k''(x,0,i+1) + p(2\lambda-\lambda_1)V_k''(x,1,i+j) + p\lambda_1 V_k''(x,1,i+j+1) \end{aligned} \tag{25}$$

By induction, we have

$$\begin{aligned} \lambda_0\left(qV_k(x,0,i+1) + pV_k(x,1,i+j+1)\right) &\geq \lambda_0\left(qV_k''(x,0,i+1) + pV_k''(x,1,i+j+1)\right) \\ (2\lambda-\lambda_1)\left(qV_k(x,0,i) + pV_k(x,1,i+j)\right) &\geq (2\lambda-\lambda_1)\left(qV_k''(x,0,i) + pV_k''(x,1,i+j)\right) \\ (\lambda_1-\lambda_0)\left(qV_k(x,0,i) + pV_k(x,1,i+j+1)\right) &\geq (\lambda_1-\lambda_0)\left(qV_k''(x,0,i) + pV_k''(x,1,i+j+1)\right) \end{aligned}$$

Thus, (25) holds if

$$\begin{aligned} & 2q\lambda V_k(x,0,i) + p\lambda V_k(x,1,i+j) + p\lambda V_k(x,1,i+j+1) \\ \geq \; & q(2\lambda-\lambda_0)V_k(x,0,i) + q\lambda_0 V_k(x,0,i+1) + p(2\lambda-\lambda_1)V_k(x,1,i+j) + p\lambda_1 V_k(x,1,i+j+1) \end{aligned}$$

which, by noting that $p(\lambda-\lambda_1) = q\lambda_0$, is valid due to the supermodularity of V. Hence the result. ∎

The above lemma allows us to establish the "convexity" in the arrival rates of the two-state MMPPs. Consider two systems with the same MAP background traffic. Both systems have N homogeneous two-state sources as foreground traffic, which have the same transition rates (p and q) but differ in the arrival rates. The first system has rates $\lambda_0 \leq \lambda_1$, the other has rates $\tilde{\lambda}_0 \leq \tilde{\lambda}_1$. They have the same density:

$$q\lambda_0 + p\lambda_1 = q\tilde{\lambda}_0 + p\tilde{\lambda}_1.$$

Let Q_t^N and $\widetilde{Q}_t^N$ denote the queue length at time t in the two queueing system.

Theorem 3.10 *Consider two infinite-capacity buffer queueing systems with the same concave and bounded service rate μ_i and the same MAP background traffic and the same number of homogeneous two-state MMPP sources with the same transition rates p and q. Assume that at time 0, all the two-state MMPP sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. If the arrival rates of two-state sources in the two systems are such that*

$$q\lambda_0 + p\lambda_1 = q\tilde{\lambda}_0 + p\tilde{\lambda}_1,$$

and $\tilde{\lambda}_0 < \lambda_0$ (so that $\tilde{\lambda}_1 > \lambda_1$), then

$$Q_t^N \leq_{\text{icx}} \widetilde{Q}_t^N, \qquad \forall t \geq 0.$$

If these queue lengths converge in distribution and in expectation to Q_∞^N *and* $\widetilde{Q}_\infty^N$*, then,*

$$Q_\infty^N \le_{\rm icx} \widetilde{Q}_\infty^N.$$

Proof. Denote by $Q_t^N(x,i)$ and $\widetilde{Q}_t^N(x,i))$ the queue lengths at time t in the two queueing systems when the initial state of MAP is x and initial queue length i. We shall only consider the case $N = 1$. The general case of N is easily shown by an inductive argument, as we mentioned in the proof of Theorem 3.8.

In comparing the two systems, we decompose the two-state MMPP in the first system into a Poisson source with rate $\tilde{\lambda}_0$ and a two-state MMPP source with rates $\lambda_0 - \tilde{\lambda}_0$ and $\lambda_1 - \tilde{\lambda}_0$. We also decompose the two-state MMPP in the second system into a Poisson source with rate $\tilde{\lambda}_0$ and an on-off source with rate $\tilde{\lambda}_1 - \tilde{\lambda}_0$. Thus, by Lemma 3.9 (taking $j = 0$) together with Theorems 2.3 and 3.3,

$$Q_t^1(x,i) \le_{\rm icx} \widetilde{Q}_t^1(x,i), \qquad \forall t \ge 0.$$

Unconditioning with respect to x and i and using the assumption that the MAP and the initial queue length have the same distribution in both systems, we obtain

$$Q_t^1 \le_{\rm icx} \widetilde{Q}_t^1, \qquad \forall t \ge 0.$$

Using further Corollary 2.4 we obtain $Q_\infty^1 \le_{\rm icx} \widetilde{Q}_\infty^1$. ∎

To conclude this section, we note that the last three theorems of this section, namely, Theorems 3.6, 3.8 and 3.10, all imply that a Poisson source with rate $\frac{p}{p+q}\lambda$ is smoother than N on-off sources. Indeed, by letting the number of sources go to infinity, or by letting the transition rates go to infinity, or by letting the low-rate go to the high-rate in the two-state MMPP model, one obtains Poisson arrivals.

Corollary 3.11 *Consider two infinite-capacity buffer queueing systems with the same concave and bounded service rate* μ_i *and the same MAP background traffic. The first system has* N *on-off sources with transition rates* p *and* q *and arrival rate* λ*. The second system has a Poisson source with rate* $\frac{Np}{p+q}\lambda$*. Let* Q_t^P *be the queue length in the system with Poisson foreground traffic. Assume that at time 0, all the on-off sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Then*

$$Q_t^P \le_{\rm icx} Q_t^N, \qquad \forall t \ge 0.$$

If these queue lengths converge in distribution and in expectation to Q^P_∞ and Q^N_∞, then,

$$Q^P_\infty \leq_{\rm icx} Q^N_\infty.$$

Remark. It is easy to show that $V_k^N(x,n,i) \leq V_k^N(x,n{+}1,i)$, if $n < N$, and therefore $V_k^N(x,\cdot,i) \leq V_k^N(x,N,i)$, following the notation of Subsection 2.3. Therefore corollary 3.11 is also valid if all on-off processes are initially in the on-state.

4 Losses in the Finite-Capacity Queueing System

In this section we consider the case of a finite-capacity buffer. We investigate the same model of input traffic, i.e., MAP background traffic and foreground traffic composed of N homogeneous on-off sources, with transition rates p and q and arrival rate λ (in the "on" state). Arriving customers which find the buffer full are lost. We analyze the loss process, i.e. the number of lost customers by time t.

4.1 Dynamic Programming Formulation and Monotonicity and Convexity

Let B be the buffer size and μ_i be the service rate when there are i customers in the queue, with $\mu_0 = 0$. Let $\mu := \max_{0\leq i\leq B}\ \mu_i$. We shall assume that μ_i is increasing and concave, which implies that $\mu = \mu_B$. Note that such an assumption is verified for multi-server queueing systems, and also Erlang blocking model fall within our framework.

We could take the same Markov chain as in the previous section, i.e. (X_t, S_t, Q_t) and take costs equal to 1 for each customer that is rejected due to space limitations. This however would only allow us to compute the expected number of rejected customers by time t.

In order to analyze the loss process, we add a counter to the state description which counts the number of rejected customers. Thus, we consider the continuous-time MC (X_t, S_t, Q_t, L_t), where X_t is the state of the MAP, S_t is the number of active (or "on") sources, Q_t the number of customers at time t, and L_t is the number of total losses by time t.

We look again at terminal costs, allowing us to establish the stochastic comparison results. Denote by V_k^N the value function after k jumps in the uniformization process, defined by the following recursive equations.

$$
\begin{aligned}
V_0^N(x,n,i,m) &= C(m), && (26)\\
V_{k+1}^N(x,n,i,m) &= \begin{cases} n\lambda V_k^N(x,n,i+1,m) + (N-n)\lambda V_k^N(x,n,i,m), & i < B,\\ n\lambda V_k^N(x,n,i,m+1) + (N-n)\lambda V_k^N(x,n,i,m), & i = B \end{cases} && (27)\\
&\quad + nqV_k^N(x,n-1,i,m) + (N-n)qV_k^N(x,n,i,m) && (28)\\
&\quad + (N-n)pV_k^N(x,n+1,i,m) + npV_k^N(x,n,i,m) && (29)\\
&\quad + \mu_i V_k^N(x,n,i-1,m) + (\mu-\mu_i)V_k^N(x,n,i,m) && (30)\\
&\quad + \sum_y \alpha_{xy}\Big(\beta_{xy}V_k^N(y,n,i+1,m) + (1-\beta_{xy})V_k^N(y,n,i,m)\Big),\ k \geq 0. && (31)
\end{aligned}
$$

In the above, the direct cost function C takes into account only the number of losses. Thus $V_k^N(x,n,i,m)$ represents the expectation of the costs after k steps, if the initial state is (x,n,i,m). When C is the identity function, $V_k^N(x,n,i,0)$ is the expected number of losses after k steps.

Before dealing with the comparisons between different systems, we prove some monotonicity and convexity properties of the value function.

Lemma 4.1 *If μ_i is increasing in i and C is an increasing function, then for all $k \geq 0$ and all x, n, i and m,*

$$
\begin{aligned}
V_k^N(x,n,i,m+1) &\geq V_k^N(x,n,i,m), && (32)\\
V_k^N(x,n,i,m+1) &\geq V_k^N(x,n,i+1,m). && (33)
\end{aligned}
$$

Proof. The proof of inequality (32) is simple and is omitted. For (33), the only interesting terms that need investigation are the μ-terms. As $\mu_i \leq \mu_{i+1}$, we need to show:

$$\mu_i V_k^N(x,n,i-1,m+1)+(\mu-\mu_i)V_k^N(x,n,i,m+1) \geq \mu_{i+1}V_k^N(x,n,i,m)+(\mu-\mu_{i+1})V_k^N(x,n,i+1,m),$$

which, by noting that $\mu - \mu_i = \mu - \mu_{i+1} + \mu_{i+1} - \mu_i$ and $\mu_{i+1} = \mu_i + \mu_{i+1} - \mu_i$, follows from the inductive assumption and (32). ∎

We also need the supermodularity in i and m.

Lemma 4.2 *If μ_i is increasing in i and C is an increasing function, then $V_k^N(x,n,i,m)$ is supermodular in i and m, i.e., for all $k \geq 0$ and all x, n, m and $i \leq B-1$,*

$$V_k^N(x,n,i+1,m+1) + V_k^N(x,n,i,m) \geq V_k^N(x,n,i,m+1) + V_k^N(x,n,i+1,m).$$

Proof. The proof is simple and is left to the interested reader. ∎

Lemma 4.3 *If μ_i is increasing and concave in i and C is an increasing function, then $V_k^N(x,n,i,m)$ is convex in i, i.e., for all $k \geq 0$ and all x, n, m and $i \leq B-2$,*

$$2V_k^N(x,n,i+1,m) \leq V_k^N(x,n,i,m) + V_k^N(x,n,i+2,m).$$

Proof. The proof is analogous to that of Lemma 3.1. We shall only deal with the case $i = B-2$ for the λ-terms. We need to show

$$\begin{aligned} & 2nV_k^N(x,n,i+2,m) + 2(N-n)V_k^N(x,n,i+1,m) \\ \leq \quad & nV_k^N(x,n,i+1,m) + (N-n)V_k^N(x,n,i,m) \\ & +nV_k^N(x,n,i+2,m+1) + (N-n)V_k^N(x,n,i+2,m). \end{aligned}$$

As by inductive assumption,

$$2(N-n)V_k^N(x,n,i+1,m) \leq (N-n)V_k^N(x,n,i,m) + (N-n)V_k^N(x,n,i+2,m),$$

it suffices to show

$$2V_k^N(x,n,i+2,m) \leq V_k^N(x,n,i+1,m) + V_k^N(x,n,i+2,m+1),$$

which follows from the monotonicity (cf. Lemma 4.1)

$$V_k^N(x,n,i+2,m) \leq V_k^N(x,n,i+1,m+1)$$

and the supermodularity (cf. Lemma 4.2)

$$V_k^N(x,n,i+1,m+1) + V_k^N(x,n,i+2,m) \leq V_k^N(x,n,i+1,m) + V_k^N(x,n,i+2,m+1).$$

This completes the proof. ∎

4.2 Stochastic Comparison of Loss Processes

We shall use the same scenarios of comparison as in the previous section. More specifically, we shall investigate the effect of multiplying the number of foreground sources, multiplying the transition rates of the sources, and of scattering the arrival rates. The main results are summerized in the following three theorems.

Theorem 4.4 *Consider two finite-capacity buffer queueing systems with the same increasing and concave service rate μ_i and the same MAP background traffic. The first one has N homogeneous on-off sources as foreground traffic whereas the second one has MN homogeneous on-off sources with the same transition rates and the same total intensity. Denote by L_t^N and L_t^{MN} the numbers of losses by time t in these systems. Assume that at time 0, all the on-off sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Then,*

$$L_t^{MN} \leq_{\mathrm{st}} L_t^N, \qquad \forall t \geq 0.$$

Theorem 4.5 *Consider two finite-capacity buffer queueing systems with the same increasing and concave service rate μ_i, the same MAP background traffic, and the same number of homogeneous on-off sources. The transition rates of the on-off sources in the second system is δ times those of the first system, where $0 < \delta < 1$. Let L_t^N and $L_t^{N,\delta}$ denote the number of losses by time t in the two queueing systems. Assume that at time 0, all the on-off sources are in steady state and that the MAP and the initial queue length have the same distribution in both systems. Then,*

$$L_t^N \leq_{\mathrm{st}} L_t^{N,\delta}, \qquad \forall t \geq 0.$$

Theorem 4.6 *Consider two finite-capacity buffer queueing systems with the same increasing and concave service rate μ_i, the same MAP background traffic, and the same number of homogeneous two-state MMPP sources with the same transition rates p and q. Assume that at time 0, all the two-state MMPP sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Let L_t^N and $\widetilde{L}_t^N$ denote the number of losses by time t in the two queueing system. If the arrival rates of two-state sources in the two systems are such that*

$$q\lambda_0 + p\lambda_1 = q\tilde{\lambda}_0 + p\tilde{\lambda}_1,$$

and $\tilde{\lambda}_0 < \lambda_0$ (so that $\tilde{\lambda}_1 > \lambda_0$), then

$$L_t^N \leq_{\mathrm{st}} \widetilde{L}_t^N, \qquad \forall t \geq 0.$$

As a corollary of any of the above theorems, we obtain:

Corollary 4.7 *Consider two finite-capacity buffer queueing systems with the same increasing and concave service rate μ_i and the same MAP background traffic. The first system has N on-off sources with transition rates p and q and arrival rate λ. The second system has a Poisson source with rate*

$\frac{Np}{p+q}\lambda$. *Let* L_t^P *be the number of losses by time* t *in the system with Poisson foreground traffic. Assume that at time 0, all the on-off sources are in steady state and the MAP and the initial queue length have the same distribution in both systems. Then*

$$L_t^P \leq_{\text{st}} L_t^N, \qquad \forall t \geq 0.$$

The proofs of the above theorems are analogous to those in the previous section. For illustration purpose, we sketch the proof of Theorem 4.4. The other proofs are omitted.

Proof of Theorem 4.4. We first compare N sources against single source, with value function V^1 defined under the same uniformization parameter:

$$\begin{aligned}
V_0^1(x,n,i,m) &= C(m), \qquad n = 0,1. \\
V_{k+1}^1(x,n,i,m) &= \begin{cases} nN\lambda V_k^1(x,n,i+1,m) + (1-n)N\lambda V_k^1(x,n,i,m), & i < B, \\ nN\lambda V_k^1(x,n,i,m+1) + (1-n)N\lambda V_k^1(x,n,i,m), & i = B \end{cases} \\
&\quad + nqV_k^1(x,n-1,i,m) + (N-n)qV_k^1(x,n,i,m) \\
&\quad + (1-n)pV_k^1(x,n+1,i,m) + (N-n+1)pV_k^1(x,n,i,m) \\
&\quad + \mu_i V_k^1(x,n,i-1,m) + (\mu-\mu_i)V_k^1(x,n,i,m) \\
&\quad + \sum_y \alpha_{xy}\Big(\beta_{xy}V_k^1(y,n,i+1,m) + (1-\beta_{xy})V_k^1(y,n,i,m)\Big), \quad n = 0,1,\ k \geq 0.
\end{aligned}$$

We can show

Lemma 4.8 *If* μ_i *is increasing in* i *and* C *is an increasing function, then*

$$\begin{aligned}
&N(p+q)^{N-1}qV_k^1(x,0,i,m) + N(p+q)^{N-1}pV_k^1(x,1,i+j,m) \\
&\geq \sum_{n=0}^{N} \tbinom{N}{n} p^n q^{N-n}\Big((N-n)V_k^N(x,n,i,m) + nV_k^N(x,n,i+j,m)\Big) \qquad (34)
\end{aligned}$$

for all x, k, i *and* $j \geq 0$ *such that* $i + j \leq B$.

Indeed, under the above assumption, $V_k^N(x,n,i,m)$ is convex in i for all x, k and n. Thus, one can use induction and the arguments of Lemma 3.4 to show (34). The proof of λ-terms is different when $i+j = B$. Using $V_k^N(x,n,i,m+1) \geq V_k^N(x,n,i+1,m)$ readily gives the required inequality. The MAP-terms are also different, but follow easily.

It then follows from the lemma (taking $j = 0$) together with Theorems 2.3 and 3.3,

$$L_t^N(x,i) \leq_{\text{st}} L_t^1(x,i), \qquad \forall t \geq 0.$$

Note that the direct cost C is an increasing function. Unconditioning with respect to x and i and using the assumption that the MAP and the initial queue length have the same distribution in both systems, we obtain

$$L_t^N \leq_{\text{st}} L_t^1, \qquad \forall t \geq 0.$$

Replacing N by M yields

$$L_t^M \leq_{\text{st}} L_t^1, \qquad \forall t \geq 0.$$

Applying N times the above inequality and using an inductive argument allow us to conclude

$$L_t^{MN} \leq_{\text{st}} L_t^N, \qquad \forall t \geq 0.$$

∎

Note that, as for the infinite buffer system, we can assume in Corollary 4.7 that all the on-off processes are initially in the on-state.

5 Concluding Remarks

In this paper we have analyzed the multi-server queueing model with MAP background traffic and a foreground traffic composed of N homogeneous two-state MMPP sources. We have investigated the impact of multiplying the number of foreground sources, of multiplying the transition rates of the sources, and of scattering the arrival rates on the queue length and on the loss (in the case of finite-capacity buffer). We have obtained stochastic comparison results in terms of the increasing and convex order for the queue length and in terms of the stochastic order for the losses.

These results have been derived under the assumption that the service rate μ_i depends only on the queue length. It is simple to extend it to the case when μ_i depends also on the state of the MAP.

We conjecture that the monotonicity (of queue length with respect to $\leq_{\text{icx}}$ order and of loss process with respect to $\leq_{\text{st}}$ order) in the number of on-off sources holds, i.e., one can compare N sources with $N+1$ sources having the same arrival density.

It will be interesting to analyze the effect of changing above mentioned parameters in the framework of networks. For example, one expects that the similar monotonicity and convexity holds for a tandem queueing system.

The method we used is a new approach to derive stochastic comparison results. It is based on the dynamic programming recursive equations, where it is assumed that the on-off sources are in steady state. Such an approach could possibly be applied to other discrete-time and continuous-time Markov chains.

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