Comparison and majorization of queueing systems with on-off sources

Ger Koole
Vrije Universiteit
De Boelelaan 1081a
1081 HV Amsterdam
The Netherlands

koole@cs.vu.nl

Zhen Liu
INRIA
B.P. 93
06902 Sophia Antipolis
France

liu@sophia.inria.fr

Don Towsley
Dept. of Computer Science
University of Massachusetts
Amherst, MA 01003
U.S.A.

towsley@cs.umass.edu

May 16, 2001

Abstract

We investigate the smoothing effect of superposing on-off sources. We consider a queueing system where the input traffic consists of background traffic and foreground traffic modeled by $N \geq 1$ on-off sources. The queueing system has an increasing and concave service rate, which includes as a special case multi-server queueing systems. When the on and off durations are exponentially distributed, we show that the queue length increases in the increasing convex order sense when the vector of arrival rates of the on-off sources increases in the majorization sense. As a consequence, the queue length is monotone in the number of homogeneous on-off sources in the increasing convex order sense, provided that the total average intensity of the on-off sources is constant.

Keywords: On-off source, Markov Modulated Poisson Process, Markov Arrival Process, stochastic comparison, dynamic programming.

1 Introduction

One of the most widely used traffic models in the performance analyses of communication networks is the so-called on-off source. This is due in part to its small number of parameters and to its accurate characterization of networked audio and video traffic. This model is also of interest because of observations and the belief that on-off sources are the most resource consuming traffic, see Bean [5] and Boyer et al. [6].

We investigate the smoothing effect of feeding a superposition of on-off sources into a server.More precisely, we consider a the following queueing model. The input traffic consists of background traffic and foreground traffic modeled by $N \geq 1$ on-off sources. The server is characterized by an increasing and concave service rate, which includes as a particular case multi-server queueing systems.

There is an extensive literature on the comparison of queues. The reader is referred to the books of Ross [19], Stoyan [21], Baccelli and Brémaud [2], and Shaked and Shanthikumar [20]. In particular, there exist results on the comparison of queueing systems with Doubly Stochastic Poisson (DSP) processes, see Ross [18], Rolski [16, 17], Svoronos and Green [22], and Chang et al. [7, 8, 9]. The results of [8, 9] are applicable to the case of a single on-off source (or a single two-state Markov Modulated Poisson Process (MMPP) source), but are not applicable to multiple on-off sources. The reader is referred to these papers for further references on that matter.

The results most closely related to our model are those of Koole and Liu [14] and of Bäuerle [3, 4]. Koole and Liu [14] considered the Markovian model where the background traffic is modeled by a Markov Arrival Process (MAP), and foreground traffic modeled by $N \geq 1$ homogeneous on-off sources whose on and off durations are exponentially distributed. They showed that the queue length in the infinite-capacity buffer system (respectively the number of losses in finite-capacity buffer system) is larger in the increasing convex order sense (respectively the strong stochastic order sense) than the queue length (respectively number of losses) of the queueing system with the same background traffic and MN homogeneous on-off sources of the same total intensity as the foreground traffic, where M is an arbitrary integer. As a consequence, the queue length and the loss with foreground traffic consisting of multiple homogeneous on-off sources is upper bounded by that with a single on-off source and lower bounded by a Poisson source, where the comparison is in the increasing convex sense (respectively the strong stochastic sense). They also compared $N \geq 1$ homogeneous arbitrary two-state Markov Modulated Poisson Process (MMPP) sources and proved the monotonicity of the queue length in the transition rates and its convexity in the arrival rates.

Parallel to the work of Koole and Liu [14], Bäuerle [3, 4] obtained similar results for a general fluid model. More precisely, Bäuerle showed that for a fluid single-server queueing system, the stationary workload is smaller in the increasing convex ordering sense with N homogeneous fluid sources, each with rate process $A(t)/N$, than with M homogeneous fluid sources, each with rate process $A(t)/M$, where $M < N$. When the fluid queueing system has finite-capacity buffer, the loss is smaller in the increasing convex ordering sense with N homogeneous fluid sources than with M homogeneous fluid sources, where $M < N$.These results were extended to a tandem network in [11].

In this paper, we extend the results of Koole and Liu [14]. We show that the queue length increases in the increasing convex order sense when the vector of arrival rates of the on-off sources increases in the majorization sense. As a consequence, the queue length is monotone in the number of homogeneous on-off sources in the increasing convex order sense, provided the total intensity of the foreground traffic is constant.

The paper is organized as follows. In Section 2 we present some notions and preliminary results of stochastic orders and techniques of stochastic comparison of Markov chains. In Section 3 we present the monotonicity results for the Markovian model. Finally in Section 4 we point out extensions of the results and further research directions. The lengthy dynamic programming proofs can be found in the Appendices.

2 Preliminaries on Majorization and Dynamic Programming

Throughout the paper, the notions of increasing, decreasing, convex and concave are used in nonstrict sense.

2.1 Majorization

Let $\boldsymbol{x}, \boldsymbol{y} \in I\!R^K$, be two K-dimensional real-valued vectors. We introduce the notation $x_{[k]}$ to denote the *k-th largest element* in vector $\boldsymbol{x}$ and define the following ordering (see [15]).

Vector $\boldsymbol{y}$ is said to majorize vector $\boldsymbol{x}$ (written $\boldsymbol{x} \prec \boldsymbol{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, \cdots, K-1, \tag{1}$$

$$\sum_{i=1}^{K} x_i \quad = \quad \sum_{i=1}^{K} y_i \quad . \tag{2}$$

A weaker ordering can be defined by replacing the equality in (2) by an inequality. This implies, $\sum_{i=1}^{K} x_i \leq \sum_{i=1}^{K} y_i$, $k = 1, \cdots, K$. In this case, vector $\boldsymbol{y}$ is said to *weakly submajorize* vector $\boldsymbol{x}$ (or, weakly majorize $\boldsymbol{x}$ from below) written $\boldsymbol{x} \prec_w \boldsymbol{y}$.

2.2 Markov Chains and Dynamic Programming Formulation

Consider a continuous-time Markov Chain (MC) (X_t, Y_t, Z_t) where $X_t \in \mathcal{K}_1 \subseteq I\!R^l$, $Y_t \in \mathcal{K}_2 \subseteq I\!R^m$ and $Z_t \in \mathcal{K}_3 \subseteq I\!R^n$, with initial state (X_0, Y_0, Z_0) and bounded transition rates. Here $l \geq 0$, $m \geq 0$ and $n \geq 1$. Without loss of generality we can assume that the sum of the transition rates is upper bounded by 1 in each state. Thus, we can apply the technique of *uniformization* to obtain a discrete-time MC whose possible transition times are generated by some Poisson process representing the uniformization. See e.g. Çinlar [10], pp. 236–237.

Let (X_k, Y_k, Z_k) be the state of the MC after k jumps. If the MC has stationary initial state (X_0, Y_0, Z_0), then, owing to PASTA (Poisson process see time average) property (cf. e.g. [2]), (X_k, Y_k, Z_k) has the same law as (X_0, Y_0, Z_0).

Let $V_k(x, y, z)$ be the value function (from $I\!R^{l+m+n} \to I\!R$) defined as the expected cost incurred after k jumps of the uniformization process:

$$V_0(x, y, z) \quad = \quad C(z), \tag{3}$$

$$V_{k+1}(x, y, z) \quad = \quad \sum_{(x', y', z') \in \mathcal{K}_1 \times \mathcal{K}_2 \times \mathcal{K}_3} \delta_{(x,y,z),(x',y',z')} \; V_k(x', y', z'), \qquad k \geq 0, \tag{4}$$

where $C(\cdot)$ (from $I\!R^n \to I\!R$) is the direct cost, and δ is the transition rate. We assume that $\sum_{(x',y',z')} \delta_{(x,y,z),(x',y',z')} = 1$ for each (x, y, z). This can be done by adding a term $\delta_{(x,y,z),(x,y,z)}$ if necessary, because we assumed that $\sum_{(x',y',z')} \delta_{(x,y,z),(x',y',z')} \leq 1$.

Note that in the above dynamic programming formulation, there is no immediate cost, but a "terminal" cost function C. One can easily see that this approach is more general than those with immediate costs independent of the age (the number of jumps) or with discounted costs. Indeed, a value function associated with such costs can be obtained by appropriately summing our value functions associated with terminal costs. (See Koole [12], Chapter 5, for an extensive discussion of the use of terminal costs in dynamic programming equations.)

Unlike the usual dynamic programming problems, we have no control decisions here. We shall use the value function to compare state variables of the MC.

Denote by $P_{x,z}$ the conditional probability distribution and $E_{x,z}$ the conditional expectation, given $X_0 = x$ and $Z_0 = z$. Let $V_k(x, \cdot, z)$ be the conditional expected cost after k jumps, i.e.

$$V_k(x, \cdot, z) = E_{x,z} V_k(x, Y_0, z) = \sum_{y \in \mathcal{K}_2} P_{x,z}(Y_0 = y) V_k(x, y, z).$$

Let there be another MC (X'_t, Y'_t, Z'_t) on $\mathcal{K}'_1 \times \mathcal{K}'_2 \times \mathcal{K}'_3 \subseteq I\!R^{l'} \times I\!R^{m'} \times I\!R^n$ with initial state (X'_0, Y'_0, Z'_0) and bounded transition rates, where $l' \geq 0$, $m' \geq 0$ and $n \geq 1$. Note that l' and m' can be different from l and m.

Let $X'_0 = x'$ and $Z'_0 = z'$, and denote by $P'_{x',z'}$ the conditional probability distribution and $E'_{x',z'}$ the conditional expectation. Let $V'_k(x', \cdot, z')$ be the conditional expected cost after k jumps in the same uniformization process, i.e.

$$V'_k(x', \cdot, z') = E'_{x',z'} V'_k(x', Y'_0, z') = \sum_{y' \in \mathcal{K}'_2} P'_{x',z'}(Y'_0 = y') V'_k(x', y', z').$$

The following results were obtained by Koole and Liu [14].

Theorem 2.1 *Let the initial states (x, z) and (x', z') be fixed. Let $Z_{t|x,z}$ and $Z'_{t|x',z'}$ be the state variables given these initial states. Let $\mathcal{C}_{\mathcal{L}}$ be a class of functions from $I\!R^n$ to $I\!R$. If for all direct cost function $C \in \mathcal{C}_{\mathcal{L}}$ and for all $k \geq 0$,*

$$V_k(x, \cdot, z) \leq V'_k(x', \cdot, z'),$$

then $Z_{t|x,z}$ is smaller than $Z'_{t|x',z'}$ in the sense of $\leq_{\mathcal{L}}$, i.e.

$$Z_{t|x,z} \leq_{\mathcal{L}} Z'_{t|x',z'}.$$

When $\mathcal{C}_{\mathcal{L}}$ is the class of increasing functions or the class of increasing and convex functions, we obtain (strong) stochastic order $\leq_{\text{st}}$ and increasing convex order $\leq_{\text{icx}}$. Now, using the convergence properties such orders (cf. [21]), we have (cf. [14])

Corollary 2.2 *Let the initial states (x, z) and (x', z') be fixed. Let $Z_{t|x,z}$ and $Z'_{t|x',z'}$ be the state variables given these initial states. Assume that $Z_{t|x,z}$ and $Z'_{t|x',z'}$ converge in distribution to Z_∞ and Z'_∞. If for all increasing cost function $C : I\!R \to I\!R$ and for all $k \geq 0$,*

$$V_k(x, \cdot, z) \leq V'_k(x', \cdot, z'),$$

then Z_∞ is stochastically smaller than Z'_∞, i.e.

$$Z_\infty \leq_{\mathrm{st}} Z'_\infty.$$

If $Z_{t|x,z}$ and $Z'_{t|x',z'}$ converge in distribution and in expectation to Z_∞ and Z'_∞, and if for all increasing and convex cost function $C : I\!R \to I\!R$ and for all $k \geq 0$,

$$V_k(x, \cdot, z) \leq V'_k(x', \cdot, z'),$$

then Z_∞ is smaller than Z'_∞ in the increasing convex ordering sense, i.e.

$$Z_\infty \leq_{\mathrm{icx}} Z'_\infty.$$

3 Superposition of On-Off Sources

3.1 Queueing model and traffic sources

We consider a queueing system fed by background traffic and foreground traffic. The background traffic is modeled by an arbitrary Markov Arrival Process (MAP), which is described by a continuous-time Markov Chain with transition rates α_{xy} for the transition from state x to y. Arrivals occur only at transition epochs. When the state changes from x to y, an arrival occurs with probability β_{xy}. Note that such a model is more general than a Markov Modulated Poisson Process (MMPP) and is dense in the class of arbitrary arrival processes (see [1]).

The foreground traffic is the superposition of N stochastically independent on-off sources. Source n goes from state "off" to state "on" with rate $\delta_n p$, and from state "on" to state "off" with rate $\delta_n q$. When source n is in the "on" state, arrivals occur according to a Poisson process with parameter λ_n. Otherwise, in the "off" state a source generates no arrivals. Note that the average arrival rate of source n is equal to $p\lambda_n/(p+q)$, and is thus independent of δ_n. The coefficient δ_n is related to the burstiness of the on-off source: the smaller, the burstier. Note that (the superposition of) on-off sources can also be modelled as a MAP.

The queueing system has an increasing and concave service rate, which includes as a particular case multi-server queueing systems.

All arrived customers are queued in an infinite-capacity buffer and are served in FCFS (First Come First Serve) order. The service times are exponentially distributed with parameter μ_i when there are i customers in the queue. In other words we are assuming that the customer service requirements are independently and identically distributed with an exponential distribution of parameter 1, and that the server serves at speed μ_i when there are i customers in the queue.

We derive our main results for the case that the service rate μ_i is increasing, concave and upper bounded by μ. Such an assumption holds in the case of multiple servers, but not for the infinite-server queue where μ_i is unbounded. For each result however we mention explicitly the conditions.

We assume that $\mu_0 = 0$, and, without loss of generality, that $N(\lambda+p+q)+\mu+\sum_y \alpha_{xy} = 1$ for all x, i.e., the system is normalized. By rescaling time this can be done without loss of generality.

3.2 Dynamic Programming Equation

It is easy to see that the queueing system can be described by the continuous-time MC with state variable (X_t, S_t, Q_t), where X_t is the state of the MAP, $S_t \in \{0,1\}^N$ is the state of the sources (with 1 representing "on") and Q_t the number of customers in the system.

Denote by V_k the value function after k jumps in the uniformization process, defined by the following recursive equations. The state of the on-off processes is written as $u \in \{0,1\}^N$. Define: $u^{-n} = \max\{u - e_n, (0,\dots,0)\}$, $u^{+n} = \min\{u + e_n, (1,\dots,1)\}$, where the maximization and the minimization are taken componentwise.

$$V_0(x,u,i) = C(i), \tag{5}$$

$$V_{k+1}(x,u,i) = \sum_{n=1}^{N} \Big[\lambda_n u_n V_k(x,u,i+1) + \lambda_n(1-u_n)V_k(x,u,i)\Big] \tag{6}$$

$$+ \sum_{n=1}^{N} \Big[\delta_n q V_k(x,u^{-n},i) + \delta_n p V_k(x,u^{+n},i)\Big] \tag{7}$$

$$+ \mu_i V_k(x,u,i-1) + (\mu - \mu_i) V_k(x,u,i) \tag{8}$$

$$+ \sum_y \alpha_{xy} \Big[\beta_{xy} V_k(y,u,i+1) + (1-\beta_{xy}) V_k(y,u,i) \Big], \qquad k \geq 0. \tag{9}$$

In the above, the function C represents the direct costs. Note that there are only terminal costs, thus $V_k(x,u,i)$ represents the expected cost after k steps, if the initial state is (x,u,i).

In what follows, we call the terms on the right hand side of (6) the λ-terms, those in (7) the pq-terms, those in (8) the μ-terms, and finally, those in (9) the MAP-terms.

The cost function V_k has the following properties, the proof of which is provided in Appendix A.

Lemma 3.1 *If $C(i)$ is increasing and convex in i, and if μ_i is concave in i, then $V_k(x,u,i)$ is increasing and convex in i for all k, x and u, i.e.,*

$$V_k(x,u,i) \leq V_k(x,u,i+1) \qquad i \geq 0, \quad k \geq 0 \tag{10}$$

$$2V_k(x,u,i+1) \leq V_k(x,u,i) + V_k(x,u,i+2) \qquad i \geq 0, \quad k \geq 0 \tag{11}$$

3.3 Dynamic Programming Results For Two Sources

Consider first the case $N = 2$. As usual, we start by deriving some properties of V; then we compare two different systems. It will simplify notation if we write:

$$W_k(x,i,j,m,l) = q^2 V_k(x,(0,0),i) + pqV_k(x,(1,0),j) + pqV_k(x,(0,1),m) + p^2 V_k(x,(1,1),l). \tag{12}$$

Lemma 3.2 *Assume that $C(i)$ is increasing and convex in i, and that μ_i is increasing and concave in i. Assume further that $\lambda_1 \leq \lambda_2$ and that $\delta_1 = \delta_2$ or $\delta_1 \geq \delta_2$ and $p = q$. Then for all $i,j,m,l \geq 0$ and x we have*

$$W_k(x,i,i+j+1,i+j+m,i+j+m+l+1) \leq W_k(x,i,i+j,i+j+m+1,i+j+m+l+1). \tag{13}$$

The proof is provided in Appendix B.

The above lemma will allow us to compare two different systems, both with two on-off sources as foreground traffic. Let the second system have value function V' and W', and all parameters equal except for the arrival rates of the two on-off sources which are λ'_1 and λ'_2.

Lemma 3.3 *Assume that $C(i)$ is increasing and convex in i, and that μ_i is increasing and concave in i. Assume that $\lambda_1 \le \lambda_2$, $\lambda_1' \le \lambda_2'$, $\lambda \prec \lambda'$, and that $\delta_1 = \delta_2$ or $\delta_1 \ge \delta_2$ and $p = q$. For all $i, j, m, l \ge 0$ and x we have*

$$W_k(x, i, i+j, i+j+m, i+j+m+l) \le W_k'(x, i, i+j, i+j+m, i+j+m+l). \qquad (14)$$

The proof is again based on induction. It is provided in Appendix C.

3.4 The Comparison Result

We now compare two different systems, both with N on-off sources as foreground traffic, that differ in their arrival rate parameters. Our main result follows from Lemma 3.3.

Theorem 3.4 *Assume that μ_i is increasing and concave in i. Assume also that $\lambda \prec \lambda'$, and that δ_n are identical for all $n = 1, 2, \ldots, N$, or $\lambda \prec \lambda'$, λ_n and λ_n' are non-decreasing and δ_n is non-increasing in n. Then for all t*

$$Q_t \le_{\text{icx}} Q_t'.$$

Furthermore, if Q_t and Q_t' converge in distribution and in mean, then

$$Q_\infty \le_{\text{icx}} Q_\infty'.$$

Proof. We consider only the case where $\lambda \prec \lambda'$ and δ_n are identical for all $n = 1, 2, \ldots, N$. The case that $\lambda \prec \lambda'$, and λ_n and λ_n' are non-decreasing and δ_n is non-increasing in n can be treated in a similar way.

It follows from the majorization relation $\lambda \prec \lambda'$ that, cf. [15, Lemma B.1, page 21], there is a finite sequence of vectors $\{\lambda^{(j)}\}_{j=1}^k$ such that $\lambda = \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(k)} = \lambda'$ and that for each $j = 0, 1, \cdots, k-1$, $\lambda^{(j)}$ and $\lambda^{(j+1)}$ differ only in two components.

Thus, we only need to consider two systems that differ only in two on-off sources, the other $N-2$ on-off sources are identical. Let the second system have value function V' and W', and all parameters equal except for the arrival rates of the two on-off sources which are λ_1' and λ_2'. By putting the $N-2$ identical on-off sources into background traffic, we obtain from Lemma 3.3 (with $j = k = l = 0$ to each environment state) that

$$W_k(x, i, i+j+1, i+j+m, i+j+m+l+1) \le W_k(x, i, i+j, i+j+m+1, i+j+m+l+1),$$

so that

$$\sum_{u\in\{0,1\}^N} p^{\sum_n u_n} q^{N-\sum_n u_n} V_k(x,u,i) \le \sum_{u\in\{0,1\}^N} p^{\sum_n u_n} q^{N-\sum_n u_n} V'_k(x,u,i).$$

Applying now Theorem 2.1 and Corollary 2.2 readily yields the desired result. ∎

As a corollary we can compare a system with N homogeneous sources and rates λ/N with a system with $M < N$ sources and rates λ/M. It suffices to consider the second system with N sources with the first M sources having rates λ/M in on-state, and the last $N-M$ sources having zero rate in on-state. Thus,

Corollary 3.5 *Consider a system with N homogeneous sources and rates λ/N and a system with $M < N$ sources and rates λ/M. Let Q_t^N and Q_t^M be their queue lengths at time t. Then for all t*

$$Q_t^N \le_{\text{icx}} Q_t^M.$$

Furthermore, if Q_t^N and Q_t^M converge in distribution and in mean, then

$$Q_\infty^N \le_{\text{icx}} Q_\infty^M.$$

4 Concluding Remarks

In this paper we have considered the smoothing effect when superposing on-off sources in a multiserver queueing system with an increasing and concave service rate. We have shown that the queue length increases in the increasing convex order sense when the vector of arrival rates of the on-off sources increases in the majorization sense. As a consequence, the queue length is monotone in the number of homogeneous on-off sources in the increasing convex order sense, provided the total intensity of the foreground traffic is constant.

These results have been derived under the assumption that the service rate μ_i depends only on the queue length. It is simple to extend it to the case when μ_i depends also on the state of the background traffic.

It will be interesting to analyze the smoothing effect of superposing on-off sources in the framework of networks. As a first result in this direction, Koole [13] generalizes the results of Koole and Liu [14] to tandem systems.

A Proof of Lemma 3.1

We use induction to prove (10) and (11). As $C(i)$ is increasing and convex in i, the result holds for V_0. Assuming that V_k is increasing and convex in i, we show (10) holds for $k+1$.

According to (6–9), for the proof of the increasingness of V_{k+1}, i.e., $V_{k+1}(x,u,i) \leq V_{k+1}(x,u,i+1)$, it suffices to consider the μ-terms and to show, cf. (6–9),

$$\mu_i V_k(x,u,i-1) + (\mu-\mu_i)V_k^N(x,u,i) \leq \mu_{i+1}V_k(x,u,i) + (\mu-\mu_{i+1})V_k(x,u,i+1),$$

which follows from the fact that $\mu_i \leq \mu$ and from the inductive assumption, i.e., $V_k(x,u,i) \leq V_k(x,u,i+1)$ and $V_k(x,u,i-1) - V_k(x,u,i) \leq 0$. Thus $V_{k+1}(x,n,i)$ is increasing in i.

Consider now the convexity. We see from (6–9) that V_{k+1} is a convex combination of V_k in various states. Thus, we only need to show the convexity of the different terms in i. For example, the convexity of the λ-term follows simply from inductive assumption of the convexity, i.e., $2V_k(x,u,i+1) \leq V_k(x,u,i) + V_k(x,u,i+2)$ and $2V_k(x,u,i+2) \leq V_k(x,u,i+1) + V_k(x,u,i+3)$.

The only terms that need investigation are the μ-terms. If we sum the following three inequalities we get exactly its convexity relation:

$$\begin{aligned} 2\mu_i V_k(x,u,i) &\leq \mu_i V_k(x,u,i-1) + \mu_i V_k(x,u,i+1) \\ 2(\mu-\mu_{i+2})V_k(x,u,i+1) &\leq (\mu-\mu_{i+2})V_k(x,u,i) + (\mu-\mu_{i+2})V_k(x,u,i+2) \\ (2\mu_{i+1}-\mu_i-\mu_{i+2})V_k(x,u,i) &\leq (2\mu_{i+1}-\mu_i-\mu_{i+2})V_k(x,u,i+1) \end{aligned}$$

The last inequality comes from the increasingness of V_k and the concavity of μ_i in i. ∎

B Proof of Lemma 3.2

Note that, in view of (12), the relation (13) is equivalent to the following inequality

$$V_k(x,(1,0),i+j+1)+V_k(x,(0,1),i+j+m) \leq V_k(x,(1,0),i+j)+V_k(x,(0,1),i+j+m+1) \quad (15)$$

We first consider the case $\delta_1 = \delta_2$.

We prove (15) by induction. The induction basis is valid under the assumption that $C(i)$ is convex in i. Assume that (15) holds for some $k \geq 0$. Consider $k+1$.

$$
\begin{aligned}
V_{k+1}(x,(1,0),s) &= \lambda_1 V_k(x,(1,0),s+1) + \lambda_2 V_k(x,(1,0),s) \\
&+ \delta_1 q V_k(x,(0,0),s) + \delta_1 p V_k(x,(1,0),s) \\
&+ \delta_2 q V_k(x,(1,0),s) + \delta_2 p V_k(x,(1,1),s) \\
&+ \mu_s q V_k(x,(1,0),s-1) + (\mu-\mu_s) V_k(x,(1,0),s) \\
&+ \sum_y \alpha_{xy} \Big[\beta_{xy} V_k(y,(1,0),s) + (1-\beta_{xy}) V_k(y,(1,0),s)\Big]
\end{aligned}
$$

$$
\begin{aligned}
V_{k+1}(x,(0,1),s) &= \lambda_1 V_k(x,(0,1),s) + \lambda_2 V_k(x,(0,1),s+1) \\
&+ \delta_1 q V_k(x,(0,1),s) + \delta_1 p V_k(x,(1,1),s) \\
&+ \delta_2 q V_k(x,(0,0),s) + \delta_2 p V_k(x,(0,1),s) \\
&+ \mu_s q V_k(x,(0,1),s-1) + (\mu-\mu_s) V_k(x,(0,1),s) \\
&+ \sum_y \alpha_{xy} \Big[\beta_{xy} V_k(y,(0,1),s) + (1-\beta_{xy}) V_k(y,(0,1),s)\Big]
\end{aligned}
$$

Consider first the λ-terms. We need to show

$$
\begin{aligned}
&\lambda_1 V_k(x,(1,0),s+2) + \lambda_2 V_k(x,(1,0),s+1) \\
&+\lambda_1 V_k(x,(0,1),s+m) + \lambda_2 V_k(x,(0,1),s+m+1) \\
&\quad\leq \lambda_1 V_k(x,(1,0),s+1) + \lambda_2 V_k(x,(1,0),s) \\
&\qquad +\lambda_1 V_k(x,(0,1),s+m+1) + \lambda_2 V_k(x,(0,1),s+m+2)
\end{aligned}
$$

It is easy to see that the above inequality is the summation of the following three equalities:

$$
\begin{aligned}
&\lambda_1 \{V_k(x,(1,0),s+2) + V_k(x,(0,1),s+m+1)\} \\
&\quad\leq \lambda_1 \{V_k(x,(1,0),s+1) + V_k(x,(0,1),s+m+2)\} \\
&\lambda_1 \{V_k(x,(1,0),s+1) + V_k(x,(0,1),s+m+1)\} \\
&\quad\leq \lambda_1 \{V_k(x,(1,0),s) + V_k(x,(0,1),s+m+2)\} \\
&(\lambda_2-\lambda_1) \{V_k(x,(1,0),s+2) + V_k(x,(0,1),s+m+1)\} \\
&\quad\leq (\lambda_2-\lambda_1) \{V_k(x,(1,0),s+1) + V_k(x,(0,1),s+m+2)\}
\end{aligned}
$$

which hold owing to the inductive assumption.

Consider now the pq-terms, We need to show

$$
\begin{aligned}
&q\{V_k(x,(0,0),s+1)+V_k(x,(1,0),s+1)+V_k(x,(0,1),s+m)+V_k(x,(0,0),s+m)\} \\
&+p\{V_k(x,(1,0),s+1)+V_k(x,(1,1),s+1)+V_k(x,(1,1),s+m)+V_k(x,(0,1),s+m)\} \\
&\quad\leq\ q\{V_k(x,(0,0),s)+V_k(x,(1,0),s)+V_k(x,(0,1),s+m+1)+V_k(x,(0,0),s+m+1)\} \\
&\qquad +p\{V_k(x,(1,0),s)+V_k(x,(1,1),s)+V_k(x,(1,1),s+m+1)+V_k(x,(0,1),s+m+1)\}
\end{aligned}
$$

which follows from the inductive assumption and the convexity of V_k:

$$V_k(x,u,s+1)+V_k(x,u,s+m)\leq V_k(x,u,s)+V_k(x,u,s+m+1).$$

For the μ-terms, we need to show

$$
\begin{aligned}
&\mu_{s+1}V_k(x,(1,0),s+1)+(\mu-\mu_{s+1})V_k(x,(1,0),s+1) \\
&+\mu_{s+m}V_k(x,(0,1),s+m-1)+(\mu-\mu_{s+m})V_k(x,(0,1),s+m) \\
&\quad\leq\ \mu_sV_k(x,(1,0),s-1)+(\mu-\mu_s)V_k(x,(1,0),s) \\
&\qquad+\mu_{s+m+1}V_k(x,(0,1),s+m)+(\mu-\mu_{s+m+1})V_k(x,(0,1),s+m+1) \qquad (16)
\end{aligned}
$$

Using the increasingness of V_k and the concavity of μ_i, we have

$$
\begin{aligned}
&(\mu_{s+m}-\mu_s-(\mu_{s+m+1}-\mu_{s+1}))V_k(x,(0,1),s+m-1) \\
&\quad\leq\ (\mu_{s+m}-\mu_s-(\mu_{s+m+1}-\mu_{s+1}))V_k(x,(0,1),s+m).
\end{aligned}
$$

Owing to the increasingness of μ_i and to the inductive assumption we have

$$
\begin{aligned}
&(\mu_{s+m+1}-\mu_{s+1})\{V_k(x,(0,1),s+m-1)+V_k(x,(1,0),s+1)\} \\
&\quad\leq\ (\mu_{s+m+1}-\mu_{s+1})\{V_k(x,(0,1),s+m)+V_k(x,(1,0),s)\}.
\end{aligned}
$$

By summing the last two inequalities we obtain

$$
\begin{aligned}
&(\mu_{s+m}-\mu_s)V_k(x,(0,1),s+m-1)+(\mu_{s+m+1}-\mu_{s+1})V_k(x,(1,0),s+1) \\
&\quad\leq\ (\mu_{s+m}-\mu_s)V_k(x,(0,1),s+m)(\mu_{s+m+1}-\mu_{s+1})V_k(x,(1,0),s),
\end{aligned}
$$

which, in addition to the following two inequalities resulting from the inductive assumption

$$\mu_s\{V_k(x,(1,0),s+1)+V_k(x,(0,1),s+m-1)\}\leq\mu_s\{V_k(x,(1,0),s)+V_k(x,(0,1),s+m)\},$$

$$
\begin{aligned}
&(\mu-\mu_{s+m+1})\{V_k(x,(1,0),s+1)+V_k(x,(0,1),s+m)\} \\
&\quad\leq\ (\mu-\mu_{s+m+1})\{V_k(x,(1,0),s)+V_k(x,(0,1),s+m+1)\}
\end{aligned}
$$

imply relation (16).

Last, the MAP-terms are trial.

If, instead of $\delta_1 = \delta_2$, we have $\delta_1 \geq \delta_2$ and $p = q$, then, in addition to (15), we need to show

$$V_k(x,(0,0),i{+}j{+}1){+}V_k(x,(1,1),i{+}j{+}m) \leq V_k(x,(0,0),i{+}j){+}V_k(x,(1,1),i{+}j{+}m{+}1) \quad (17)$$

The proofs of (15) and (17) are identical except for the pq-terms which require to show (for (15))

$$\begin{aligned}
&\delta_1 V_k(x,(0,0),s+1) + \delta_2 V_k(x,(1,0),s+1) + \delta_1 V_k(x,(0,1),s+m) \\
&+\delta_2 V_k(x,(0,0),s+m) + \delta_1 V_k(x,(1,0),s+1) + \delta_2 V_k(x,(1,1),s+1) \\
&+\delta_1 V_k(x,(1,1),s+m) + \delta_2 V_k(x,(0,1),s+m) \\
&\quad \leq \quad \delta_1 V_k(x,(0,0),s) + \delta_2 V_k(x,(1,0),s) + \delta_1 V_k(x,(0,1),s+m+1) \\
&\qquad\quad +\delta_2 V_k(x,(0,0),s+m+1) + \delta_1 V_k(x,(1,0),s) + \delta_2 V_k(x,(1,1),s) \\
&\qquad\quad +\delta_1 V_k(x,(1,1),s+m+1) + \delta_2 V_k(x,(0,1),s+m+1)
\end{aligned}$$

and for (17)

$$\begin{aligned}
&\delta_1 V_k(x,(0,0),s+1) + \delta_2 V_k(x,(0,0),s+1) + \delta_1 V_k(x,(0,1),s+m) \\
&+\delta_2 V_k(x,(1,0),s+m) + \delta_1 V_k(x,(1,0),s+1) + \delta_2 V_k(x,(0,1),s+1) \\
&+\delta_1 V_k(x,(1,1),s+m) + \delta_2 V_k(x,(1,1),s+m) \\
&\quad \leq \quad \delta_1 V_k(x,(0,0),s) + \delta_2 V_k(x,(0,0),s) + \delta_1 V_k(x,(0,1),s+m+1) \\
&\qquad\quad +\delta_2 V_k(x,(1,0),s+m+1) + \delta_1 V_k(x,(1,0),s) + \delta_2 V_k(x,(0,1),s) \\
&\qquad\quad +\delta_1 V_k(x,(1,1),s+m+1) + \delta_2 V_k(x,(1,1),s+m+1)
\end{aligned}$$

which both follow from the inductive assumptions and the convexity of V_k

$$V_k(x,u,s+1) + V_k(x,u,s+m) \leq V_k(x,u,s) + V_k(x,u,s+m+1).$$

∎

C Proof of Lemma 3.3

We use again induction to show (14). The case of $k = 0$ is trivial. Assume that (14) holds for some $k \geq 0$. Then

$$
\begin{aligned}
& W_{k+1}(x, i, i+j, i+j+m, i+j+m+l) \\
= \quad & \lambda_1 \Big\{ q^2 V_k(x,(0,0),i) + pqV_k(x,(1,0),i+j+1) \\
& \quad +pqV_k(x,(0,1),i+j+m) + p^2 V_k(x,(1,1),i+j+m+l) \Big\} \\
& + \lambda_2 \Big\{ q^2 V_k(x,(0,0),i) + pqV_k(x,(1,0),i+j) \\
& \quad +pqV_k(x,(0,1),i+j+m+1) + p^2 V_k(x,(1,1),i+j+m+l) \Big\} \\
& + \delta_1 \Big\{ q^3 V_k(x,(0,0),i) + pq^2 V_k(x,(1,0),i) + pq^2 V_k(x,(0,0),i+j) \\
& \quad + p^2 q V_k(x,(1,0),i+j) + pq^2 V_k(x,(0,1),i+j+m) + p^2 q V_k(x,(1,1),i+j+m) \\
& \quad +p^2 q V_k(x,(0,1),i+j+m+l) + p^3 V_k(x,(1,1),i+j+m+l) \Big\} \\
& + \delta_2 \Big\{ q^3 V_k(x,(0,0),i) + pq^2 V_k(x,(0,1),i) + pq^2 V_k(x,(1,0),i+j) \\
& \quad + p^2 q V_k(x,(1,1),i+j) + pq^2 V_k(x,(0,0),i+j+m) + p^2 q V_k(x,(0,1),i+j+m) \\
& \quad +p^2 q V_k(x,(1,0),i+j+m+l) + p^3 V_k(x,(1,1),i+j+m+l) \Big\} \\
& + q^2 \mu_i V_k(x,(0,0),i-1) + q^2(\mu - \mu_i) V_k(x,(0,0),i) \\
& + pq\mu_{i+j} V_k(x,(1,0),i+j-1) + pq(\mu - \mu_{i+j}) V_k(x,(1,0),i+j) \\
& + pq\mu_{i+j+m} V_k(x,(0,1),i+j+m-1) + pq(\mu - \mu_{i+j+m}) V_k(x,(0,1),i+j+m) \\
& + p^2 \mu_{i+j+m+l} V_k(x,(1,1),i+j+m+l-1) \\
& + p^2(\mu - \mu_{i+j+m+l}) V_k(x,(1,1),i+j+m+l) \\
& + \sum_y \alpha_{xy} \Big[\beta_{xy} W_k(x, i+1, i+j+1, i+j+m+1, i+j+m+l+1) \\
& \quad + (1-\beta_{xy}) W_k(x, i, i+j, i+j+m, i+j+m+l) \Big] \\
= \quad & \lambda_1' W_k(x, i, i+j+1, i+j+m, i+j+m+l+1) \\
& \quad + (\lambda_1 - \lambda_1') pq \left\{ V_k(x,(1,0),i+j+1) + pqV_k(x,(0,1),i+j+m) \right\} \\
& + \lambda_2' W_k(x, i, i+j, i+j+m+1, i+j+m+l+1) \\
& \quad - (\lambda_2' - \lambda_2) pq \left\{ V_k(x,(1,0),i+j) + V_k(x,(0,1),i+j+m+1) \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \delta_1 q W_k(x, i, i, i+j+m, i+j+m) \\
&+ \delta_1 p W_k(x, i+j, i+j, i+j+m+l, i+j+m+l) \\
&+ \delta_2 q W_k(x, i, i, i+j, i+j) \\
&+ \delta_1 p W_k(x, i+j+m, i+j+m, i+j+m+l, i+j+m+l) \\
&+ \mu_i W_k(x, (i-1)^+, (i+j-1)^+, (i+j+m-1)^+, (i+j+m+l-1)^+) \\
&+ (\mu_{i+j} - \mu_i) W_k(x, i, (i+j-1)^+, (i+j+m-1)^+, (i+j+m+l-1)^+) \\
&+ (\mu_{i+j+m} - \mu_{i+j}) W_k(x, i, i+j, (i+j+m-1)^+, (i+j+m+l-1)^+) \\
&+ (\mu_{i+j+m+l} - \mu_{i+j+m}) W_k(x, i, i+j, i+j+m, (i+j+m+l-1)^+) \\
&+ (\mu - \mu_{i+j+m+l}) W_k(x, i, i+j, i+j+m, i+j+m+l) \\
&+ \sum_y \alpha_{xy} \Big[\beta_{xy} W_k(x, i+1, i+j+1, i+j+m+1, i+j+m+l+1) \\
&\quad + (1-\beta_{xy}) W_k(x, i, i+j, i+j+m, i+j+m+l) \Big]
\end{aligned}
$$

Thus, by the inductive assumption and Lemma 3.2 we have (note that $\lambda_1 - \lambda_1' = \lambda_2' - \lambda_2$),

$$
W_{k+1}(x, i, i+j, i+j+m, i+j+m+l) \le W_{k+1}'(x, i, i+j, i+j+m, i+j+m+l).
$$

Thus by induction (14) holds for all $k \ge 0$. ∎

References

[1] S. Asmussen and G. M. Koole. Marked point processes as limits of Markovian arrival streams. *Journal of Applied Probability*, 30:365–372, 1993.

[2] F. Baccelli, and P. Brémaud. *Elements of Queueing Theory.* Springer Verlag (Applications of Mathematics Series), 1994.

[3] N. Bäuerle, "The advantage of small machines in a stochastic fluid production process", *Mathematical Methods of Operations Research*, 47:83–97, 1998.

[4] N. Bäuerle, "How to improve the performance of ATM multiplexers, *Operations Research Letters*, 24:81–89, 1999.

[5] N. G. Bean. Robust connection acceptance for ATM networks with incomplete source information. *Annals of Operations Research*, 48:357–379, 1994.

[6] P.E. Boyer, F.M. Guillemin, M.J. Servel, and J.-P. Coudreuse. Spacing cells protects and enhances utilization of ATM network links. *IEEE Network*, pages 38–49, 1992.

[7] C. S. Chang, and M. Pinedo. Bounds and inequalities for single server loss systems. *Queueing Systems*, 6:425–436, 1990.

[8] C. S. Chang, X. Chao, and M. Pinedo. Monotonicity results for queues with doubly stochastic Poisson arrivals: Ross's conjecture. *Advances in Applied Probability*, 23:210–228, 1991.

[9] C. S. Chang, X. Chao, M. Pinedo and J. G. Shanthikumar. Stochastic convexity for multidimensional processes and its applications. *IEEE Transactions on Automatic Control*, 36:1347–1355, 1991.

[10] E. Çinlar. *Introduction to Stochastic processes*. Prentice-Hall, 1975.

[11] A. Hordijk, Z. Liu, D. Towsley. "Smoothing Effect of the Superposition of Homogeneous Sources in Tandem Networks," to appear in *J. Appl. Prob.*, Fall 2000.

[12] G.M. Koole. *Stochastic Scheduling and Dynamic Programming*. CWI, Amsterdam, 1995. CWI Tract 113.

[13] G.M. Koole. Convexity in tandem queues. Technical Report WS-516, Vrije Universiteit Amsterdam, 1999. Electronically available as www.cs.vu.nl/~koole/papers/WS516.ps.

[14] G.M. Koole and Z. Liu. Stochastic bounds for queueing systems with multiple on-off sources. *Probability in the Engineering and Informational Sciences*, Vol. 12, pp. 25-48, Jan. 1998.

[15] A. W. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Applications*. Academic Press, New York, 1979.

[16] T. Rolski. Queues with non-stationary input stream: Ross's conjecture. *Advances in Applied Probability*, 13:603–618, 1981.

[17] T. Rolski. "Upper bounds for single server queues with Doubly stochastic Poisson arrivals", *Mathematics of Operations Research*, 11:442–450, 1986.

[18] S. M. Ross. Average delay in queues with non-stationary Poisson arrivals. *Journal of Applied Probability*, 15:602–609, 1978.

[19] S. M. Ross. *Stochastic Processes*. Wiley, 1983.

[20] M. Shaked, J. G. Shanthikumar. *Stochastic Orders and Their Applications*. Academic Press, Boston, 1994.

[21] D. Stoyan. *Comparison Methods for Queues and Other Stochastic Models*. English translation (D.J. Daley editor), Wiley, New York, 1983.

[22] A. Svoronos, L. Green. A convexity result for single server exponential loss systems with nonstationary arrivals. *Journal of Applied Probability*, 25:224–227, 1988.