Stochastic scheduling with event-based dynamic programming

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Abstract

In this paper we apply a new framework for the study of monotonicity in queueing systems to stochastic scheduling models. This allows us a unified treatment of many different models, among which are multiple and single server models (with and without feedback), discrete and continuous time models, models with controlled and uncontrolled arrivals, etc.

1 Introduction

Structural results for optimal policies for queuing models are usually derived in the following way. After having formulated the dynamic programming (dp) value function for a particular model, it is shown inductively that this value function satisfies certain properties, from which the results are derived. This value function represents either a discrete time model, or a continuous time model (through the well-known uniformization technique, first applied in Lippman [17]). In this paper we use *event-based dynamic programming*. Event-based dp deals with event operators, which can be seen as building blocks of the value function. Typically we associate an operator with each basic event in the system, such as an arrival at a queue, a service completion, etc. Event-based dp focuses on the underlying properties of the value and cost functions, and allows us to study many models at the same time. It is explained in the next section.

Event-based sp can be used to study all kinds of monotonicity results; in this paper we apply it to stochastic scheduling problems. These problems are characterized by multiple customers classes having different service time distributions. There are one or more servers that need to be assigned optimally to the customer classes. Typical objectives include the weighted average number of customers at the queues, or the fraction of time that the system is empty. We consider finite and infinite source models, models with controlled and uncontrolled arrivals, single and multi-server models, and models in which feedback is possible, i.e., in which customers can join other queues after being served.

First we consider single and multiple server models without feedback. Results from Chang et al. [7], Hordijk & Koole [8], and Koole & Vrijenhoek [16] are generalized.

After that we study a single server model with feedback to the other queues. The continuous time version has already been studied in Nain [18] and Koole [10], the discrete time version is the subject of Weishaupt [23]. Their results are slightly generalized; in the cited papers only feedback to queues with a lower priority were allowed, we allow also feedback to the next higher queue. This model has the following interesting application. One of the problems of dynamic programming is the difficulty of working with general distributions. A possible solution is the use of phase-type distributions, as in Koole [10]. DFR and IFR distributions are characterized in terms of phase-type distributions, and using the feedback result limiting results for G/DFR/1 and G/IFR/1 queues are derived. The single class results can also be found in Righter & Shanthikumar [19].

Event-based dp is based on ideas developed in Altman & Koole [1] and first formulated in its present form in Koole [13]. In [13] second-order properties related to convexity are studied, for one and two-dimensional systems. Convexity for more than two dimensions is the subject of Koole [14]. The current study is about first-order properties.

A preliminary version of this paper has appeared as [12].

2 Dynamic Programming

2.1 Event-based dynamic programming

In this section we formulate the dynamic programming value function in general terms and prove some theorems which form the basis of our method.

We take $x \in X = \mathbb{N}_0^{m+1}$ to be our state space. Define operators $T_0, \dots, T_{k-1}$, as follows:

$$T_i(f_1, \dots, f_{l_i})(x) = \min_{a \in A_i(x)} \Big\{ c_i(x,a) + \sum_{j=1}^{l_i} \sum_{y \in X} p_i^j(x,a,y) f_j(y) \Big\},$$

with $f_1, \dots, f_{l_i} : X \to \mathbb{R}$. $A_i(x)$ is called the action set, $c_i(x,a)$ the direct costs and $p_i^j(x,a,y)$ the transition probabilities. We often take $l_i = 1$, as we will see in the next sections. An important exception however is the uniformization operator for which we need $l_i > 1$. If $l_i = 1$ then T_i is the standard dp operator, given that $p^1(x,a,y) \ge 0$ for all x, a, y, and that $\sum_y p^1(x,a,y) = 1$ (or α, the discount factor) for each pair x and a. (In this case we omit the superscript of p.) In applications we choose the event operators as simple as possible, by associating one with every possible event in the system.

The value function V_{n+1} is constructed from V_n and the operators T_i as follows. Assume that V_0 is given. Define, for $n = 0, 1, \ldots$, $V_n^{(0)}, \ldots, V_n^{(k)}$ by taking $V_n^{(k)} = V_n$, for $j = 0, \ldots, k-1$ $V_n^{(j)} = T_j(V_n^{(k_1)}, \ldots, V_n^{(k_{l_j})})$, for some $j < k_1, \ldots, k_{l_j} \leq k$ (where the assumption that $k_1, \ldots, k_{l_s} > j$ is made to avoid circularity), and $V_{n+1} = V_n^{(0)}$.

Although this definition is notationally quite burdensome, the intuition is simple: each step of the dp consists of the parallel and/or consecutive execution of several events. If $l_i > 1$ for some i, then also the determination of which events are to be executed depends on the state, the realization, or the action. The central ideas are summarized in the following (trivial) theorem.

Theorem 2.1 *Let $\mathcal{F}$ be some class of functions from $\mathbb{N}_0^m$ to $\mathbb{R}$, and $V_0 \in \mathcal{F}$. If, for all i, for $f_1, \ldots, f_{l_i} \in \mathcal{F}$ holds that $T_i(f_1, \ldots, f_{l_i}) \in \mathcal{F}$ then $V_n \in \mathcal{F}$ for all n.*

In what follows we consider special event operators and show that $T_i f \in \mathcal{F}$ for all i. This proves that the value function $V_n \in \mathcal{F}$ for all models that can be constructed with the T_i. We choose $\mathcal{F}$ such that certain structural properties of the optimal control policies can be derived from it.

On the other hand, it is possible to show (see [13]) that V_n as defined can be rewritten in the standard MDP formulation, given by

$$W_{n+1}(x) = \min_{a \in A(x)} \Big\{ k(x, a) + \sum_y q(x, a, y) W_n(y) \Big\}.$$

This allows us to use techniques and results from the theory of MDP's.

Finally let us consider optimality criteria. Normally we assume that all $p_i^j(x, a, y) \geq 0$ and that for all i, x and a $\sum_{y,j} p_i^j(x, a, y) = 1$: then V_n represents the total minimal n-stage costs, and under certain conditions the policy minimizing V_n as $n \to \infty$ is average optimal. These conditions however are non-trivial and should be checked for each model separately, unless the state space is finite. If we make the exception that $\sum_{y,j} p_0^j(x, a, y) = \alpha$, then V_n converges (again, under certain conditions) to the minimal discounted costs. Our focus in this paper is on the properties of the value function, not on the existence of limiting policies. In the case of a finite state space existence is guaranteed for all models. For discounting results from Schäl [20] often provide the necessary existence result; for average costs some useful conditions are summarized in Cavazos-Cadena & Sennott [6].

Other optimality criteria are also possible. For discrete-time models there are no problems to expect when minimizing total finite-stage costs. Other choices are also possible. Take for example $V_0 = C$ and no further costs in the definition of V_n, $n > 0$. Then V_n gives the minimal expected final costs after n time. Also for continuous time models we can consider the costs at say T. This can simply be done by conditioning on the number of jumps of the uniformization process (see Ch. 5 of Koole [11] or Koole & Liu [15]).

2.2 The value function

In this subsection we present the operators.

Notationally we make use of the following conventions: e_i denotes the ith unity vector, $1 \leq i \leq m$, while e_0 is the 0 vector, each vector (in)equality is taken componentwise, $I\{\dots\}$ is the indicator function, $x^+ = x$ if $x \geq 0$, 0 otherwise, increasing and decreasing are used in the non-strict sense.

The operators related to the service process are:

- $T_{SS}f(x) = \min_{0\leq i\leq m}\{\mu(i)f(x-e_i) + (1-\mu(i))f(x)\}$, where the minimization ranges over those i with $x_i > 0$. This models a single server that services m parallel queues. Idleness, action 0, is always allowed.
- $T_{MS}f(x) = \min_{i_1,\dots,i_s}\{\sum_{k=1}^{s}(\mu(i_k)f(x-e_{i_k})+(1-\mu(i_k))f(x))\}$ where $\sum_k I\{i_k = j\} \geq x_j$, i.e., no more servers can work on a queue than that there are customers in that queue. This models s parallel servers.
- $T_{SSFB}f(x) = \min_{1\leq i\leq m}\{\sum_{k=0}^{m}\mu(i,k)f(x-e_i+e_k)\}$ if $x \neq 0$, $f(x)$ if $x = 0$. Action i models again serving queue i. This operator allows feedback to other queues. Note that we do not allow idleness if there are customers available. With $\mu(i,k)$ we denote the probability that a customer in queue i which is being served moves to queue k. Queue 0 means leaving the system. (Recall that e_0 was the 0 vector.) We assume that $\sum_{k=0}^{m}\mu(i,k) = 1$ for all i.

Operators related to arrivals are:

- $T_{A(i)}f(x) = f(x+e_i)$, $1 \leq i \leq m$. This operator models an arrival at queue i.
- $T_{FS}f(x) = \sum_i \lambda(i)f^i(x)$ with $\sum_i \lambda(i) = 1$ and $f^i(x) = f(x+e_i)$ if $x_i = 0$, and $f(x)$ if $x_i = 1$. This models a finite source queue: with probability $\lambda(i)$ the single class i customer in the system arrives, assuming that it is not yet there. Thus we assume that there is at most one customer in the system for each class, restricting the state space to $X = \mathbb{N}_0 \times \{0,1\}^m$. This does not restrict generality, and it simplifies the notation. Of course T_{FS} cannot be used together with $T_{A(i)}$.

The direct costs and the discounting are represented by the following operators:

- $T_{costs}f(x) = C(x) + \alpha f(x)$. Here α is the discount factor, thus we often take $\alpha \in (0,1]$, with $\alpha = 1$ representing total costs, but we only need $\alpha \geq 0$. We write sometimes $T_{costs}(C,f)$ to indicate that the conditions on f must also hold for C.

Remember that we had an $m+1$ dimensional state space. Component 1 up to m are used for the queues, as we saw in the operators above, the 0th state component will be used for the environment. This environment allows us to model general arrival streams, server vacations, etc. For examples see the following subsection.

- $T_{env(0)}(f_1,\dots,f_l)(x) = \sum_{y\in\mathbb{N}_0}\lambda(x_i,y)\sum_{j=1}^{l}q^j(x_i,y)f_j(x^*)$, where x^* is equal to x with the 0th component replaced by y. This operator models Markov Arrival Processes, which are discussed in the next section.

A special case of $T_{env(0)}$ is T_{unif}, the uniformization operator, given by

- $T_{unif}(f_1,\dots,f_l)(x) = \sum_j p(j)f_j(x)$ with $p(j) > 0$ for all j. This is a convex combination of the f_j. The value function of a continuous time model typically has this form, due to the uniformization. This technique was first introduced in Lippman [17], and further developed in Serfozo [22]. It is the basis of the analysis of most continuous-time Markovian models.

An interesting extension of $T_{env(0)}$ is

- $T_{Cenv(0)}(f_1,\dots,f_l)(x) = \min_a\{\sum_{y\in\mathbb{N}_0}\lambda(x_i,a,y)\sum_{j=1}^{l}q^j(x_i,a,y)f_j(x^*)\}$. This operator allows for control in the environment. We will not go into details here, but we refer to the discussion of MDAP's (Markov *Decision* Arrival Processes) in [8].

2.3 Examples of value functions

Here we give some examples of value functions. A simple model with arrivals and service independent of some environment state (i.e., Poisson arrivals and constant service rates) has value function

$$V_{n+1} = T_{costs}(C, T_{unif}[T_{A(1)}V_n, \ldots, T_{A(m)}V_n, T_{SS}V_n]).$$

We could let the arrivals (and also the server completion times) depend on some environment state. This can be done with a Markov Arrival Process, which has the property that the class of all MAP's is dense in the class of all arrival processes (Asmussen & Koole [2]). An MAP consists of a Markov process on the environment states (with transition rates $\lambda(x, y)$), and event probabilities: if the environment moves from x to y then with probability $q^j(x, y)$ an event of type j (which can be an arrival in a certain class, or a possible service completion) occurs. Uniformization of such an MAP leads in a continuous time setting to a value function of the form

$$V_{n+1} = T_{costs}(C, T_{env(0)}[T_{A(1)}V_n, \ldots, T_{A(m)}V_n, T_{SS}V_n]),$$

where the MDAP is modeled by the operator $T_{env(0)}$. This is the model of Buyukkoc et al. [5], for which they show the optimality of the μc rule. If we replace T_{SS} by T_{SSFB} then we get the model of Section 3 of Nain [18]. If we take T_{MS} then we find the model of Chang et al. [7].

Also discrete time models can be modeled with event-based dynamic programming. The crucial property of discrete time models is that events occur after each other. Choices can be made here; we give below the value function of the model of Weishaupt [23]. It is given by:

$$V_{n+1} = T_{SSFB}(T_{costs}[C, T_{unif}(T^{b(1,1)}_{A(1)} \cdots T^{b(1,m)}_{A(m)}V_n, \ldots, T^{b(l,1)}_{A(1)} \cdots T^{b(l,m)}_{A(m)}V_n)]).$$

The uniformization operator represents the arrivals. Thus with probability $p(j)$ a batch of customers arrives, with $b(j, i)$ arrivals in queue i (a superscript b means the b-fold application of the operator).

3 Models without Feedback

In this section we study models with service operators T_{SS} or T_{MS}. We define the class of functions $\mathcal{F}$ as follows. $f \in \mathcal{F}$ if the following two inequalities hold:

$$\mu(i)f(x - e_i) + (1 - \mu(i))f(x) \le \mu(j)f(x - e_j) + (1 - \mu(j))f(x) \quad (1)$$

for all $x \in X$ such that $x_i > 0$ and $x_j > 0$, $1 \le i < j \le m$, and

$$f(x) \le f(x + e_i) \quad (2)$$

for all $x, x + e_i \in X$ and $1 \leq i \leq m$.

The first inequality has a simple interpretation. The terms involved can be found in T_{SS}. Thus if $f \in \mathcal{F}$, then the minimizing action for $T_{SS}f$ consists of serving the customer of the lowest class number available, which is called the Smallest Index Policy (SIP) in [8]. The second inequality shows that idling is not optimal. It is also readily seen that for T_{MS} the servers should be assigned to the group of s servers with the lowest indices. The following lemma is the basis for our results, it shows under what conditions $V_n \in \mathcal{F}$.

Lemma 3.1 *The following hold:*

$$f \in \mathcal{F} \Longrightarrow T_{SS}f \in \mathcal{F},\ T_{A(i)}f \in \mathcal{F},$$

$$f \in \mathcal{F},\ \mu(1) \leq \cdots \leq \mu(m) \Longrightarrow T_{MS}f \in \mathcal{F},$$

$$f \in \mathcal{F},\ \lambda(1) \leq \cdots \leq \lambda(m) \Longrightarrow T_{FS}f \in \mathcal{F},$$

$$C, f \in \mathcal{F} \Longrightarrow T_{costs}(C, f) \in \mathcal{F},$$

$$f_1, \ldots, f_l \in \mathcal{F} \Longrightarrow T_{env(0)}(f_1, \ldots, f_l) \in \mathcal{F},$$

$$f_1, \ldots, f_l \in \mathcal{F},\ \mu(1) \leq \cdots \leq \mu(m) \Longrightarrow T_{Cenv(0)}(f_1, \ldots, f_l) \in \mathcal{F}.$$

Proof We start with the proof for $T_{A(k)}$. We have to show

$$\mu(i)T_{A(k)}f(x - e_i) + (1 - \mu(i))T_{A(k)}f(x) \leq \mu(j)T_{A(k)}f(x - e_j) + (1 - \mu(j))T_{A(k)}f(x)$$

and

$$T_{A(k)}f(x) \leq T_{A(k)}f(x + e_i).$$

This follows immediately from (1) and (2), in state $x + e_k$ instead of x. The proof for T_{SS} is more complicated. Consider first (1). We have to distinguish between the minimizing actions in the different state. Let i^* be the minimizing action of T_{SS} in x. First note that $i^* \neq j$, because for all $f \in \mathcal{F}$ the SIP minimizes T_{SS}. If $i^* \neq i$ or $i^* = i$ and $x_i > 1$, then i^* is also the minimizer in $x - e_i$ and $x - e_j$ (because of the interpretation of $\mathcal{F}$ for T_{SS}), and T_{SS} applied to (1) is thus equivalent to (1) in state $x - e_{i^*}$. The remaining case is $i^* = i$ and $x_i = 1$. Then i^* is not allowed in $x - e_i$, but still optimal in $x - e_j$. In this case we have (introduce the notation $\bar{\mu}(k) = 1 - \mu(k)$)

$$\mu(i)T_{SS}f(x - e_i) + \bar{\mu}(i)T_{SS}f(x) \leq$$

$$\mu(i)\mu(j)f(x - e_i - e_j) + \mu(i)\bar{\mu}(j)f(x - e_i) + \bar{\mu}(i)\mu(j)f(x - e_j) + \bar{\mu}(i)\bar{\mu}(j)f(x) =$$

$$\mu(j)T_{SS}f(x - e_j) + \bar{\mu}(j)T_{SS}f(x).$$

The proof of (2) for T_{SS} is similar.

Next we consider T_{MS}. The proof is similar to that for T_{SS}, but more complicated. In fact, it has already been given in [8], where a continuous time model is considered that has T_{MS} as part of the value function. The proof for T_{MS} starts on top of p. 992.

Consider T_{FS}. If we apply T_{FS} to (1), then we get

$$\sum_k \lambda(k)\mu(i)f^k(x-e_i) + \sum_k \lambda(k)\bar{\mu}(i)f^k(x) \le \sum_k \lambda(k)\mu(j)f^k(x-e_j) + \sum_k \lambda(k)\bar{\mu}(j)f^k(x).$$

This holds by induction if

$$\lambda(i)\mu(i)f^i(x-e_i) + \lambda(i)\bar{\mu}(i)f^i(x) + \lambda(j)\mu(i)f^j(x-e_i) + \lambda(j)\bar{\mu}(i)f^j(x) \le$$
$$\lambda(i)\mu(j)f^i(x-e_j) + \lambda(i)\bar{\mu}(j)f^i(x) + \lambda(j)\mu(j)f^j(x-e_j) + \lambda(j)\bar{\mu}(j)f^j(x).$$

This is equivalent to

$$\lambda(i)f(x) + \lambda(j)\mu(i)f(x-e_i) + \lambda(j)\bar{\mu}(i)f(x) \le \lambda(j)f(x) + \lambda(i)\mu(j)f(x-e_j) + \lambda(i)\bar{\mu}(j)f(x),$$

which holds by (1) and (2) if $\lambda(i) \le \lambda(j)$.

Proving that $T_C f \in \mathcal{F}$ is trivial, because $\mathcal{F}$ is closed under convex combinations. The same holds for $T_{env(0)}$.

Finally consider $T_{Cenv(0)}$. Write $h_a(x) = \sum_{y\in\mathbb{N}_0} \lambda(x_i,a,y)\sum_{j=1}^l q^j(x_i,a,y)f_j(x^*)$, then $T_{Cenv(0)}(f_1,\dots,f_l)(x) = \min_a\{h_a(x)\}$. Because $\mu(i) \le \mu(j)$ (1) is equivalent to

$$\mu(i)f(x-e_i) + (\mu(j)-\mu(i))f(x) \le \mu(j)f(x-e_j).$$

Suppose that a^* is the minimizing action of $T_{Cenv(0)}$ in $x-e_j$. Then

$$\mu(i)T_{Cenv(0)}f(x-e_i) + (\mu(j)-\mu(i))T_{Cenv(0)}f(x) \le \mu(i)h_{a^*}(x-e_i) + (\mu(j)-\mu(i))h_{a^*}(x) \le$$
$$\mu(j)h_{a^*}(x-e_j) = \mu(j)T_{Cenv(0)}f(x-e_j),$$

the first inequality by the suboptimality of a^* in x and $x-e_i$, the second inequality because $f \in \mathcal{F}$. □

Our results are summarized in the following theorem.

Theorem 3.2 *For value functions consisting of the operators $T_{A(1)},\dots,T_{A(m)}$ or T_{FS} (with $\lambda(1) \le \cdots \le \lambda(m)$), T_{SS}, T_{MS} (with $\mu(1) \le \cdots \le \mu(m)$), T_{costs}, $T_{env(0)}$ and/or $T_{Cenv(0)}$ (with $\lambda(1) \le \cdots \le \lambda(m)$), the SIP is optimal if $C \in \mathcal{F}$.*

An extensive study of allowable cost functions can be found in [8]. One of the results is that for C of the form $C(x) = \sum_i c(i)x_i$, $C \in \mathcal{F}$ is equivalent to $c_i \ge 0$ and $\mu(1)c(1) \ge \cdots \ge \mu(m)c(m)$. Thus the SIP serves in decreasing order of $\mu(i)c(i)$. This is called the μc rule. If $\mu(1) \le \cdots \le \mu(m)$ then the SIP serves the customers with the least expected processing times, which is called LEPT.

Another interesting cost function is $C(x) = I\{|x| > 0\} \in \mathcal{F}$. This shows that the fraction of time that the system is empty is maximized by the SIP. If $\mu(1) \ge \cdots \ge \mu(m)$ then also $I\{|x| > s\} \in \mathcal{F}$ for all s. For the right choice of value function (with V_0 as only costs, see Section 2.1) this leads to the conclusion that $\mathbb{P}(|X_T| > s) = \mathbb{E}I\{|X_T| > s\}$ is minimized by SIP, with X_T the random variable denoting the state at T. We can study the makespan, i.e., the time for the system to empty, by changing $T_{A(i)}$ to $f(x+e_i)$ if $x \ne 0$ and $f(x)$ if $x = 0$. This corresponds to keeping the system empty as soon as it is empty. It is easily seen that all results remain valid. See also Section 4 of [8].

In the following corollary we summarize the results for the main models.

Corollary 3.3 *In the model with:*
independent arrivals and a single server the μc *rule is optimal;*
controlled arrivals and multiple servers the μc *rule is optimal if it coincides with LEPT;*
a finite source and a single server the μc *rule is optimal if* $\lambda(1) \leq \cdots \leq \lambda(m)$*;*
a finite source and multiple servers the μc *rule is optimal if it coincides with LEPT and if* $\lambda(1) \leq \cdots \leq \lambda(m)$*.*

The result for the first model can also be found in Buyukkoc et al. [5] or Baras et al. [3]. The second result can be found in Hordijk & Koole [8], the result for uncontrolled arrivals can also be found in Chang et al. [7]. The third result is that of Koole & Vrijenhoek [16], the fourth is a generalization of it.

4 Single-Server Model with Feedback

We continue with the inequality that we consider for the operator T_{SSFB}. We define the class of functions $\mathcal{G}$ as follows:

$$f \in \mathcal{G} \Longleftrightarrow \sum_k \mu(i,k) f(x - e_i + e_k) \leq \sum_k \mu(j,k) f(x - e_j + e_k), \tag{3}$$

for all x such that $x_i > 0$, $x_j > 0$, and $1 \leq i < j \leq m$.

The terms in the inequality can be found in T_{SSFB}. Thus if $f \in \mathcal{G}$, then the minimizing action for $T_{SSFB} f$ consists again of serving the customer of the lowest class number available, the SIP. The following lemma is the basis for our results, it shows under what conditions $V_n \in \mathcal{G}$.

Lemma 4.1 *If, in the definition of* T_{SSFB}*,* μ *is such that, for* $1 \leq i \leq m$*,* $\mu(i,k) = 0$ *for* $0 < k < i-1$*, then the following hold:*

$$f, C \in \mathcal{G} \Longrightarrow T_{A(i)} f \in \mathcal{G},\ T_{SSFB} f \in \mathcal{G},\ T_{costs}(C, f) \in \mathcal{G},\ T_{env(0)} f \in \mathcal{G}.$$

Proof The proof for all operators except T_{SSFB} is equal to that of Lemma 3.1. Consider T_{SSFB}. We have to show, given that $f \in \mathcal{G}$, that

$$\sum_k \mu(i,k) T_{SSFB} f(x - e_i + e_k) \leq \sum_k \mu(j,k) T_{SSFB} f(x - e_j + e_k). \tag{4}$$

Consider the minimizing actions at the right hand side. Because k with $\mu(j,k) > 0$ have $k \geq j-1$, and $i < j$, we have that $i \leq k$. Thus, for each of the states $x - e_j + e_k$ there are two possibilities: either action i is optimal in each state (note that $x_i > 0$), or there is a $l < i$ with $x_l > 0$ optimal. In the latter case (4) is equal to

$$\sum_n \mu(l,n) \sum_k \mu(i,k) f(x - e_i + e_k - e_l + e_n) \leq \sum_n \mu(l,n) \sum_k \mu(j,k) f(x - e_j + e_k - e_l + e_n),$$

which holds by applying (3) in each of the states $x - e_l + e_n$. If action i is optimal, then

$$\sum_k \mu(i,k) T_{SSFB} f(x - e_i + e_k) \leq \sum_n \mu(j,n) \sum_k \mu(i,k) f(x - e_i + e_k - e_j + e_n) =$$

$$\sum_n \mu(j,n) T_{SSFB} f(x - e_j + e_n).$$

This completes the proof. □

Our result can be summarized as follows.

Theorem 4.2 *For value functions consisting of the operators $T_{A(1)}, \ldots, T_{A(m)}$, T_{SSFB}, T_{costs} and/or $T_{env(0)}$, the SIP is optimal under the following conditions:*
(i) Feedback in T_{SSFB} to queues with a lower index number should be restricted to the neighboring queue;
(ii) The direct costs C should be such that $C \in \mathcal{G}$.

Let us compare this result with those obtained in the literature. As we saw already in the previous section we can deal at the same time with the continuous time model of Nain [18] (in [10] an equivalent result is obtained) and the discrete time model of Weishaupt [23]. Compared to their results there is a second difference: we allow not only feedback to higher indexed queues, but also to the next lower queue. This small difference will allow us in the next section to extend substantially the limiting results of Koole [10] for G/IFR/1 and G/DFR/1 queues.

Finally let us look at some special cases of the theorem. First assume that only $\mu(i,0)$ and $\mu(i,i)$ are non-zero, and that C has the following special form: $C(x) = \sum_i c_i x_i$. Then $C \in \mathcal{G}$ is equivalent to:

$$\mu(i,0)C(x - e_i) + (1 - \mu(i,0))C(x) \leq \mu(j,0)C(x - e_j) + (1 - \mu(j,0))C(x),$$

which is, due to the special form of C, equivalent to $\mu(i,0)c_i \geq \mu(j,0)c_j$. Thus we find again the well known μc rule.

Now consider cost functions which are only functions of the total number of customers in the system. Obvious functions are $C(x) = |x|$ or $-|x|$, corresponding respectively to minimizing and maximizing the number of customers in the system, but also $C(x) = I\{|x| > s\}$ and $I\{|x| < s\}$ are of interest: they correspond to the probability that there are more or less than s customers in the system. Thus for increasing cost functions (such as $|x|$ and $I\{|x| > s\}$) $C \in \mathcal{G}$ is equivalent to $\mu(i,0)$ decreasing in i, and vice versa. We will use this in the next section.

5 Phase-Type Distributions and Limit Results

Based on results in Schassberger [21] it is shown in [10] that any distribution function F can be approximated with distributions F_n of the form

$$F_n(x) = \sum_{k=1}^{\infty} \beta_k E_n^k(x),$$

with $\beta_1 = F(1/n)$ and $\beta_k = F(k/n) - F((k-1)/n)$ for $k > 1$. E_n^k is the density function of a gamma distributed r.v. with k phases and intensity n.

Thus β_k is the probability of exactly k phases. Another way to construct this r.v. is as follows. Define

$$\alpha_m = \begin{cases} \frac{\beta_m}{1-\sum_{k=1}^{m-1}\beta_k} & \text{if } \sum_{k=1}^{m-1}\beta_k < 1, \\ 1 & \text{if } \sum_{k=1}^{m-1}\beta_k = 1. \end{cases}$$

Let the r.v. G_n be the time until absorption of the following Markov process. The initial state is 1. The process rests in each state k an E_n^1 (exponentially) distributed amount of time, after that absorption takes place with probability α_n, or the system moves to state $k+1$ with probability $1-\alpha_n$. This is a special case of a Cox distribution. Because $\beta_m = (1-\alpha_1)\cdots(1-\alpha_{m-1})\alpha_m$ we see that $F_n \overset{\mathrm{d}}{=} G_n$.

Now we discuss DFR and IFR distributions, and their relation to phase-type distributions of the form G_n. We use the following definition of Barlow & Prochan [4], which is only in terms of $\bar{F}(t) = 1 - F(t)$ (thus the failure rate itself does not need to exist).

Definition 5.1 *(DFR and IFR) A non-negative distribution function is:*
DFR if $\bar{F}(t+s)/\bar{F}(t)$ is increasing in $t \geq 0$ with $\bar{F}(t) > 0$, for each $s \geq 0$;
IFR if $\bar{F}(t+s)/\bar{F}(t)$ is decreasing in $-\infty < t < \infty$ with $\bar{F}(t) > 0$, for each $s \geq 0$.

Examples of IFR (DFR) distributions are distribution with a non-decreasing (non-increasing) failure rate (defined by $f(t)/1-F(t)$, with f the density), assuming that it exist for all t. But also $F(t) = I\{t \geq x\}$, the deterministic distribution, is IFR, although its failure rate does not exist.

Now we can formulate the result on phase-type distributions:

Theorem 5.2 *If F is DFR (IFR) then α_m is decreasing (increasing) in m, for all n.*

The proof of this theorem can be found in [10]. Hordijk & Ridder [9] proved the DFR part of this theorem for general Cox distributions.

A disadvantage of this method is that we need an infinite number of states. We can make the state space finite by changing the approximation into

$$F_n(x) = \sum_{k=1}^{n^2} \beta_k E_n^k(x) + (1 - \sum_{k=1}^{n^2} \beta_k) \sum_{k=1}^{\infty} (1-\beta_{n^2})^{k-1} \beta_{n^2} E_n^{n^2+k}(x).$$

It is easily checked that the approximation result also holds for this F_n, and that α_k becomes constant from $k = n^2$ on, so that we only need a finite number of states for the representation with the α's.

Now assume that we have a single class of customers, and identify the queues of theorem 4.2 with the stages of the phase-type distribution. We let customers arrive in queue 1, and we take $\mu(i,0) = \alpha_i$ and $\mu(i,i+1) = 1-\alpha_i$, $i < m$, and for $m = n^2$ we take $\mu(i,i) = 1-\alpha_i$, using the finite state approximation given above. At the end of the last section it was shown that if $C(x) = I\{|x| > s\}$, then $\mu(i,0) = \alpha_i$ should be decreasing in i, i.e., F_n is

DFR. Thus to minimize the number of customers, all having the same DFR distribution, the customers with the least attained service times (LAST) should be served. Similarly, to maximize the number of customers when they have IFR distributions LAST should also be used. Note that LAST, in the limit, leads to processor sharing between all the customers which have the same minimal attained service time.

Another possibility is arrivals at queue $m = n^2$, with $\mu(i, 0) = \alpha_{n^2+1-i}$ and $\mu(i, i-1) = 1 - \alpha_{n^2+1-i}$, $i > 1$, and $\mu(1, 1) = 1 - \alpha_{n^2}$. The optimal policy serves the customer which has the most attained service time (MAST). Note that this is equivalent to non-preemptive service.

All these results hold for all F_n, and therefore also for their limit F. This gives the following corollary. These results can also be found in Righter & Shanthikumar [19].

Corollary 5.3 *The number of customers at any time T in a G/G/1 queue is:*
minimized (maximized) by LAST in case of DFR (IFR) service times;
minimized (maximized) by MAST in case of IFR (DFR) service times.

Generalizations can be obtained to different customer classes having different service times. As long as feedback occurs to lower numbered queues no problems are to be expected. This is the case for DFR (IFR) distributions with positive (negative) holding times; see [10] for details. For the other two cases the situation is less clear. As different queues can belong to different customer classes, we have no guarantee that a change in phase means a transition to the next higher queue. For each case separately the conditions should be checked.

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