On deviation matrices for birth-death processes

G.M. Koole[*] F.M. Spieksma[†]

To appear in *Probability in the Engineering and Informational Sciences*, 2001/2002

Abstract

We study deviation matrices of birth-death processes. This is relevant to the control of multi-dimensional queueing systems. We give an algorithm for computing deviation matrices for birth-death processes. As an application, we compute them explicitly for the $\mathsf{M/M/s/N}$- and $\mathsf{M/M/s/\infty}$-queues.

1 Introduction

In principle, Markov decision theory can be used to find optimal policies in controlled queueing networks. Standard methods are value iteration and policy iteration (see, e.g., [8]). To execute these algorithms one needs to store in computer memory at least one vector of size the state space. This is of course infeasible for models with an unbounded state space. But even if we bound the state space in an appropriate way, then often its size prohibits us from storing this vector. This is even more so for high-dimensional (queueing) models: depending on the number of states per component, in practice, models with more than say 4 state components can not be solved anymore. This phenomenon is called the *curse of dimensionality*. It calls for approximation methods. A successful method is *one-step improvement*.

One-step improvement is based on the policy iteration method. This method repeats the following two steps (see [8] for terminology):
1) For a given *value function* compute the minimising action in each state;
2) Compute the *value function* for this new policy.

It can be shown that this gives a sequence of policies for which every policy is better (i.e., has lower average costs) than the previous one. For one-step improvement, we assume that the value function has been determined for some fixed policy. By applying Step 1 once (only in states in which we are interested) we know to have a better policy. This policy is used as an approximation for the optimal policy. In general the value of this policy cannot be computed, let alone the optimal policy. Therefore it is hard to assess the quality of the one-step improved policy. But for low-dimensional cases the method has been tested, and has been shown to give surprisingly good results ([7, 9, 6]). The crucial step is to compute the value for some fixed policy (preferably one for which we hope that one-step improvement gives good results). Both [7] and [9] can be seen as models consisting of parallel queues with a dependency created by the control.

[*]Postal address: Division of Mathematics and Computer Science, Vrije Universiteit, De Boelelaan 1081a, 1081HV Amsterdam, the Netherlands. Email-address: koole@cs.vu.nl

[†]Postal address: Mathematical Institute, University of Leiden, P.O.Box 9512, 2300RA Leiden, the Netherlands. Email-address: spieksma@math.leidenuniv.nl

For certain classes of policies the queues behave independently. (The model of [6] is a priority queue, for which there is always a dependency.) If the queues are independent, then it can be seen that the value function is simply the sum of the values of the individual queues, thereby reducing the N-dimensional problem to N 1-dimensional problems. These are of course easy to solve, either numerically, or by using a closed formula. In [7, 9] closed formulas are computed for value functions of certain queues. These formulas depend not only on the type of queue and its parameter values, but also on the cost structure.

The *deviation matrix* of a Markov chain allows us to compute the value independent of the cost structure. Indeed, the deviation matrix D of a Markov chain is independent of the cost structure; for a cost vector c the bias vector v (for more details: see Section 3) is simply given by $v = Dc$. This means that by calculating the deviation matrix of a Markov chain we can simply compute (by a single summation) the bias vector for any cost structure that is a function of the state. This included cost functions such as queue length, idleness, but also the blocking probability in finite buffer systems.

In this paper we first introduce notation and derive some general results for the deviation matrix in Section 2. In Section 3 we show the relation with Markov decision chains through the Poisson equation. Finally in Section 4 we derive an algorithm for computing the deviation matrix for a special class of birth-death processes, including the $\mathsf{M/M/s/N}$- and $\mathsf{M/M/s/\infty}$-queues. In the same section we also derive closed-form expressions for these queues.

2 Model and basic formula for the deviation matrix

Let $\{\xi_t\}_t$ be an irreducible, aperiodic and time homogeneous Markov chain on a countable state space $\boldsymbol{S}$, which is positive recurrent. Thus it has a stationary distribution that we denote by $\pi = \{\pi_x\}_{x\in\boldsymbol{S}}$. The corresponding stationary matrix $\Pi : \boldsymbol{S} \to \boldsymbol{S}$ is the matrix with all rows equal to π.

Let us write $P^{(n)}$, $n = 0, 1, 2, \ldots$, for the n-step transition probability matrix of our Markov chain, i.e.

$$p_{xy}^{(n)} = \mathsf{P}\{\xi_n = y \,|\, \xi_0 = x\},$$

and we set $P^{(0)}$ equal to the identity matrix $\mathbf{I}$. The deviation matrix D is defined by

$$D_{xy} = \lim_{\alpha\uparrow 1} \sum_{n=0}^{\infty} \left(p_{xy}^{(n)} - \pi_y \right) \alpha^n,$$

provided this limit exists. Note that the fundamental matrix $Z = (\mathbf{I} - P + \Pi)^{-1}$ exists with D and it relates to it in the following way:

$$Z = D + \Pi.$$

When the state space is finite, the deviation and fundamental matrices always exist and they can be expressed in terms of stationary probabilities and first passage times (cf. [4]). We will introduce these quantities first and then recall the formula for the deviation matrix for the finite state case.

Let $z \in \boldsymbol{S}$ be a given state. Then the taboo transition probability matrix ${}_zP$ with taboo state z is defined by

$${}_zp_{xy} = \begin{cases} p_{xy}, & y \neq z, \\ 0, & y = z. \end{cases}$$

Write ${}_zm_{xy} = \sum_{n=0}^{\infty} {}_zp_{xy}^{(n)}$. This has the well-known interpretation of being the expected number of visits of state y, given the initial state x, before returning to z. Similarly, set $T_z = \inf\{t >$

$0 \mid \xi_t = z, \xi_1, \ldots, \xi_{t-1} \neq z\}$ and let $\tau_{xz} = \mathsf{E}\{T_z \mid \xi_0 = x\}$ the first passage time of z when starting in x. A straightforward computation yields

$$\sum_y {}_z m_{xy} = \tau_{xz}.$$

These quantities can be solved using systems of linear equations. In particular, letting δ_{xy} denote the Kronecker delta

$${}_z m_{xy} = \delta_{xy} + \sum_a {}_z p_{xa} \, {}_z m_{ay} \tag{2.1}$$

and taking the summation over y,

$$\tau_{xz} = 1 + \sum_a {}_z p_{xa} \tau_{az}. \tag{2.2}$$

Note that $\pi_x = \pi_z \cdot {}_z m_{zx}$, and $\pi_z = 1/\tau_{zz}$.

Next we summarise some known results on deviation matrices for finite state chains (cf. also [4] Theorem 4.4.7).

Theorem 2.1 *Let $\boldsymbol{S}$ be finite. The deviation matrix D is the unique solution to the following set of equations*

$$D = \mathbf{I} - \Pi + PD \tag{2.3}$$

$$\Pi D = D\Pi = 0, \quad PD = DP. \tag{2.4}$$

Moreover,

$$D_{xy} = \pi_y \Big(\sum_{v \neq y} \pi_v \tau_{vy} - \mathbf{1}_{\{x \neq y\}} \tau_{xy} \Big) \tag{2.5}$$

$$= D_{yy} - \mathbf{1}_{\{x \neq y\}} \pi_y \tau_{xy}. \tag{2.6}$$

Formula (2.6) can be heuristically explained as follows. We can say that D_{xy} counts the number of visits of y starting from x compared to starting in a stationary situation. The first visit to y occurs on average only after τ_{xy} time periods, while from a stationary initial state π_y visits are counted each time unit. Thus the difference in visits between starting in x or in stationarity from time 0 to the epoch just before y is reached is $-\pi_y \tau_{xy}$. The remaining difference starting from the moment y is reached is D_{yy}, which indeed includes the first visit after (on average) τ_{xy} time.

The theorem has two immediate interesting consequences: the first one is that $D_{yy} > 0$. The second one is that $D\Pi = 0$ implies

$$0 = \sum_y D_{xy} = \sum_y D_{yy} - \sum_{y \neq x} \pi_y \tau_{xy}. \tag{2.7}$$

In turn this implies that $\sum_{y \neq x} \pi_y \tau_{xy}$ does not depend on x! This statement can be proved to hold for the countable state space case. Unfortunately, the expressions involved may not be finite, so that changing the order of subtraction on summation in (2.7) may not be allowed. This is the case in the infinite buffer example that we will discuss later on.

For completeness, we will show independence of $\sum_{y \neq x} \pi_y \tau_{xy}$ on x. This is equivalent to showing that $\sum_y \pi_y \tau_{xy}$ is independent of x. Indeed, the term corresponding to y equals $\pi_y \tau_{yy} = 1$.

Lemma 2.1 *We have that* $\sum_y \pi_y \tau_{xy}$ *is independent of the initial state* x*, i.e., there exists a constant* $1 < c \leq \infty$ *such that*

$$\sum_y \pi_y \tau_{xy} = c, \quad \textit{for all } x \in \boldsymbol{S}.$$

As a consequence

$$\sum_{x,y} \pi_y \pi_x \tau_{xy} = c,$$

as well.

Proof. Denote $f(x) = \sum_y \pi_y \tau_{xy}$: this may be infinite. We will first argue that the $f(x)$ are either all finite or all infinite.

Assume that $f(x) < \infty$ for some state x. Choose any state $l \neq x$, some (finite) path from x to l and denote the probability of this path by q. Then $\tau_{ly} \leq \tau_{xy}/q$ and so $f(l) \leq f(x)/q < \infty$.

So me may assume that $f(x) < \infty$ for all x. By (2.2)

$$\begin{aligned} f(x) &= \sum_y \pi_y \big(1 + \sum_{l \neq y} p_{xl} \tau_{ly}\big) \\ &= 1 + \sum_l p_{xl} \sum_{y \neq l} \pi_y \tau_{ly} \\ &= 1 + \sum_l p_{xl} \big(f(l) - 1\big). \end{aligned}$$

Bu subtracting 1 from the right hand side, and writing $g(x) = f(x) - 1$, we get

$$g(x) = f(x) - 1 = \sum_l p_{xl} g(l),$$

in matrix notation

$$g = Pg.$$

Interating this, yields

$$g = P^{(n)} g,$$

and taking the limit on both sides and using Fatou's lemma we get

$$g \geq \Pi g.$$

Suppose that $g \not\equiv \Pi g$. Then there must be at least one state x for which $g(x) > (\Pi g)(x)$. Since

$$g(l) \geq \sum_y \pi_y g(y),$$

for all states l, we obtain by multiplying by π_l and taking the summation over l

$$\sum_l \pi_l g(l) > \sum_l \pi_l \sum_y \pi_y g(y) = \sum_y \pi_y g(y),$$

a contradiction. Thus $g(x) = \sum_y \pi_y g(y)$ for *all* states x. This proves the lemma. QED

The next theorem shows essentially that formula (2.6) holds, whenever the deviation matrix exists.

Theorem 2.2 *Suppose that there exists a unique solution D to (2.3) and (2.4). Then (2.5) holds.*

Proof. First we show that the matrix A with entries

$$a_{xy} = D_{yy} - \mathbf{1}_{\{x\neq y\}}\pi_y\tau_{xy}$$

solves (2.3). Indeed,

$$\begin{aligned}\delta_{xy} - \pi_y + \sum_k p_{yk}a_{ky} &= \delta_{xy} - \pi_y + D_{yy} - \sum_{k\neq y} p_{yk}\pi_y\tau_{ky} \\ &= D_{yy} + \delta_{xy} - \pi_y\bigl(1 - \sum_{k\neq y} p_{yk}\tau_{ky}\bigr) \\ &= D_{yy} + \delta_{xy} - \pi\tau_{xy} = a_{xy}.\end{aligned}$$

Next we show that this must be the only solution. We have that

$$D - A = P\bigl(D - A\bigr).$$

This difference matrix has all diagonal elements equal to 0. Also

$$\bigl(D - A\bigr)^+ \le P\bigl(D - A\bigr)^+.$$

Iterating this, yields

$$\bigl(D - A\bigr)^+ \le P^{(t)}\bigl(D - A\bigr)^+.$$

Taking the limsup for $t \to \infty$ and using Fatou's lemma gives

$$\bigl(D - A\bigr)^+ \le \Pi\bigl(D - A\bigr)^+.$$

Consequently, $\bigl(D-A\bigr)^+$ has bounded columns. In the same manner, one can show that $\bigl(D-A\bigr)^-$ has bounded columns. Using the fact that $D - A = P^{(t)}\bigl(D - A\bigr)$ and dominated convergence, shows that

$$D - A = \Pi\bigl(D - A\bigr).$$

Thus $D - A$ has constant columns. Since it has zero diagonal elements, the whole matrix must be identically 0. As a consequence, $D = A$.

This shows the validity of formula (2.6), that is $D_{xy} = D_{yy} - \mathbf{1}_{\{x\neq y\}}\pi_y\tau_{xy}$. Using that $\Pi D = 0$, we obtain $D_{yy} = \pi_y \sum_{v\neq y} \pi_v\tau_{vy}$. QED

The natural question arises whether the reverse implication holds. That is, provided the right-hand side of (2.6) is finite, does it yield a unique solution of (2.3) and (2.4)? It yields a solution of (2.3) indeed. However, it is not clear whether (2.4) holds without any further conditions.

A possible counterexample could be an ergodic, embedded $\mathsf{M}/\mathsf{GI}/1$-queue with a suitable service time distribution. If the service time distribution has a finite first moment, but an infinite second one, then the stationary distribution has an infinite first moment (see for instance Chapter 14.4 [3]). By homogeneity properties and the fact that downward jumps have size at most 1, one can show that $\tau_{xy} = c(x - y)$, $x > y$, for some constant c. Indeed, applying Theorem 2.2 yields that a unique solution to (2.3) and (2.4) cannot exist. In that case $\sum_{v>y} \pi_v\tau_{vy} = \sum_{v>y} \pi_v c(v-y)$ is necessarily finite. This contradicts the fact that the stationary distribution does not have a finite first moment.

Requiring the service time distribution to have at best a finite $(2+\epsilon)$-th moment, for some sufficiently small $\epsilon > 0$, yields finite expressions in the right-hand side of (2.6). Defining the matrix A with entries a_{xy} through (2.6), it easily follows that (2.3) is satisfied. However, it is not clear whether ΠA converges. We believe that not.

To guarantee the existence of unique solutions to (2.3) and (2.4), one can use the following theorem from [2]. The contractive Lyapunov function condition used there, is satisfied by our examples studied later on.

Theorem 2.3 *Consider an aperiodic, irreducible and time homogeneous Markov chain ξ_t on a countable state space $\boldsymbol{S}$. Suppose that ξ_t satisfies the following contractive Lyapunov function criterion: there exists a state $z \in \boldsymbol{S}$, a positive function $f : \boldsymbol{S} \to \mathbf{R}$ with $\inf_x f(x) > 0$, and a positive constant α such that ${}_zPf \leq \exp^{-\alpha} f$. Then the Markov chain is positive recurrent, in particular it is f-exponentially ergodic (see [1]). Then the deviation matrix D is the unique f-bounded solution to the following set of equations and the following formula holds:*

$$D = \Big(\mathbf{I} - \Pi\Big) \sum_{n=0}^{\infty} {}_zP^{(n)} \Big(\mathbf{I} - \Pi\Big). \tag{2.8}$$

3 The Poisson equation

The deviation matrix plays an important role in Markov decision chains. In a Markov decision chain, the transition matrix P depends on the policy $f : \boldsymbol{S} \to A$, with A the set of actions. Therefore we often write $P(f)$ instead of P. There are also immediate costs $c = \{c(x)\}_{x \in \boldsymbol{S}}$.

An intermediate step in many algorithms (e.g., policy iteration, or the approximation algorithm described in the Introduction) is solving the Poisson equation

$$c - g = v - P(f)v$$

for a given policy f. Under the conditions of Theorem 2.3 the Poisson equation has a unique solution upto a constant, namely $v = Dc$, and $g = \Pi c$ (for $P = P(f)$).

We make some observations on computational issues. Choose any set $A \subset \boldsymbol{S}$ and suppose that $\sum_{y \in A} D_{yy} c(y)$ converges. This is true for any finite set. Then by Theorem 2.2, v_A given by

$$v_A(x) = v(x) - \sum_{y \in A} D_{yy} c(y) = - \sum_{y \in A, \neq x} \pi_y \tau_{xy} c(y) + \sum_{y \notin A} \Big(D_{yy} - \mathbf{1}_{\{y \neq x\}} \pi_y \tau_{xy} \Big) c(y)$$

is a solution as well. When the state space is finite, we can always take $A = \boldsymbol{S}$. Suppose the costs have equal sign, or can be made to have equal sign by adding a constant. The latter is the case for instance when the costs have a monotone structure. Then it follows immediately that the $v_{\boldsymbol{S}}(x)$ have equal sign. This allows for numerically stable algorithms for computing a solution of the Poisson equation.

4 The deviation matrix for the M/M/s/N- and M/M/s/∞-queues

By virtue of Theorem 2.2 we need to compute hitting times and stationary probabilities in order to explicitly calculate the deviation matrix. We will do so through formulae for ${}_ym_{xv}$.

We will first compute these quantities for the time discretised approximation of a birth-death processes on the non-negative integers with the following boundedness conditions on the jump rates. For λ_x and μ_x the birth and death rates in state x respectively, such that $\mu_0 = 0$, assume that

$$0 < \liminf_{x\to\infty} \frac{\lambda_x}{\mu_x} \leq \limsup_{x\to\infty} \frac{\lambda_x}{\mu_x} < 1$$
$$0 < \inf_x \big(\lambda_x + \mu_x\big) \leq \sup_x \big(\lambda_x + \mu_x\big) < \infty.$$

Let $N = \inf\{x \,|\, \lambda_x = 0\}$, where N may be infinite if all birth rates are positive. After suitable renormalisation, we obtain an approximating Markov chain on $\{0, \ldots, N\}$ with the following transition probabilities

$$p_{xy} = \begin{cases} \lambda_x, & y = x+1, x < N \\ \mu_x, & y = x-1, x > 0 \\ 1 - \sum_{s\neq x} p_{xs}, & y = x. \end{cases}$$

This Markov chain trivially satisfies the conditions of Theorem 2.3, whenever N is finite; it also satisfies these conditions when N is infinite. This can be shown by constructing a suitable Lyapunov function.

Lemma 4.1 *Choose state* $z > 0$ *with* $\sup_{x\geq z} \lambda_x/\mu_x < 1$*. Next we determine a number of positive constants satisfying the following conditions:*

i) *choose* δ *with* $\exp\{\delta\} \leq \inf_{x\geq z} \mu_x/\lambda_x$*;*

ii) *let* γ *and* $c < 1$ *be such that* $\exp\{-\gamma\} \leq \inf_{x\leq z} \mu_x/\lambda$ *and* $(1{+}c)\big(1{+}\big(\exp\{-\gamma\}/4\big)^z\big) \leq \exp\{\delta\}$*;*

iii) *finally, let* α *satisfy* $\sup_x \big(1{-}\exp\{-\alpha\}\big)/\lambda_x \leq c{\cdot}\big(\exp\{-\gamma\}/4\big)^z$ *and* $\lambda_0\big(1{+}\big(\exp\{-\gamma\}/4\big)\big) \leq \exp\{-\alpha\}$*.*

Let

$$\beta_x = \begin{cases} \big(\exp\{-\gamma\}/4\big)^x, & x \leq z \\ \beta_z, & x > z. \end{cases}$$

Then the function f *defined recursively by* $f(0) = 1$ *and* $f(x) = (1 + \beta_x) f(x-1)$*,* $x > 0$*, is a Lyapunov function that satisfies the conditions of Theorem 2.3 with taboo state* 0 *and contraction factor* $\exp\{-\alpha\}$*.*

Proof. We need to check that $\sum_{y\neq 0} {}_0p_{xy} f(y) \leq \exp\{-\alpha\} f(x)$. For $x = 0$ this reduces to checking

$$\lambda_0\big(1 + \beta_1\big) \leq \exp\{-\alpha\}.$$

This follows immediately from (iii). For $x > 0$ we have got to check that

$$\mu_x\Big(\frac{f(x)}{1+\beta_x}\Big) + (1 - \lambda_x - \mu_x) f(x) + \lambda_x f(x)\big(1 + \beta_{x+1}\big) \leq \exp\{-\alpha\} f(x),$$

or

$$\beta_{x+1} + \frac{1 - \exp\{-\alpha\}}{\lambda_x} \leq \frac{\mu_x}{\lambda_x}\frac{\beta_x}{1+\beta_x}. \tag{4.1}$$

For $x < z$, this follows from the fact that

$$2\beta_{x+1} = 2\Big(\frac{\exp\{-\gamma\}}{4}\Big)^{x+1} \leq 2\frac{\mu_x}{4\lambda_x}\beta_x \leq \frac{\mu_x}{\lambda_x}\frac{\beta_x}{1+\beta_x},$$

since $1+\beta_x<2$ and $(1-\exp\{-\alpha\})/\lambda_x \le c\cdot\beta_z \le \beta_{x+1}$ by (iii). For $x \ge z$, (4.1) reduces to

$$\beta_z + \frac{1-\exp\{-\alpha\}}{\lambda_x} \le \frac{\mu_x}{\lambda_x}\frac{\beta_z}{1+\beta_z}.$$

This is implied by

$$\beta_z(1+c) \le \exp\{\delta\}\frac{\beta_z}{1+\beta_z},$$

or, dividing both sides by β_z and multiplying them by $1+\beta_z$, by

$$(1+c)(1+\beta_z) \le \exp\{\delta\}.$$

The latter is true by (ii). QED

We would like to point out that one can take any state for the taboo state. In that case it suffices to change the function f defined in the lemma, in state 1.

The stationary distribution expressed in terms of π_y is known to be given by

$$\pi_x = \begin{cases} \pi_y \dfrac{\lambda_{x-1}\cdots\lambda_y}{\mu_x\cdots\mu_{y+1}}, & x>y \\ \pi_y \dfrac{\mu_y\cdots\mu_{x+1}}{\lambda_{y-1}\cdots\lambda_x}, & x<y. \end{cases}$$

For computing the ${}_y m_{xv}$ we have to do some work. Note that for $x \le v < y$ and $x \ge v > y$ we have ${}_y m_{xv} = {}_y m_{vv}$. Indeed, to reach y from such x, we must reach v first and from then on the number of visits to state v is exactly the same as if we had started in v.

Similarly, for $x>y>v$ and $x<y<v$ we have ${}_y m_{xv}=0$. Further, for $v=y$

$$ {}_y m_{xy} = \begin{cases} 1, & x=y \\ 0, & x\ne y. \end{cases} \tag{4.2}$$

Finally, since $\pi_v = {}_y m_{yv}\pi_y$, we have for $x=y$

$$ {}_y m_{yv} = \begin{cases} \cdot\dfrac{\lambda_{v-1}\cdots\lambda_y}{\mu_v\cdots\mu_{y+1}}, & v>y \\ \dfrac{\mu_y\cdots\mu_{v+1}}{\lambda_{y-1}\cdots\lambda_v}, & v<y. \end{cases}$$

The only cases left to consider are the cases $v \le x < y$ and $y < x \le v$.

Let $y<x<v$. Then by (2.1)

$$ {}_y m_{yv} = \lambda_y\, {}_y m_{y+1\,v}.$$

For $x=y+1$ we get, writing $\Delta_y m_{xv} = {}_y m_{x+1\,v} - {}_y m_{xv}$,

$$\Delta_y m_{y+1\,v} = \frac{\mu_{y+1}}{\lambda_{y+1}}\, {}_y m_{y+1\,v},$$

and for $x>y+1$

$$ {}_y m_{xv} = (1-\lambda_x-\mu_x)\,{}_y m_{xv} + \lambda_x\cdot {}_y m_{x+1\,v} + \mu_x\cdot {}_y m_{x-1\,v},$$

so that

$$\Delta_y m_{xv} \quad = \quad \frac{\mu_x}{\lambda_x}\Delta_y m_{x-1\,v}$$

$$\begin{aligned}
&= \frac{\mu_x \cdots \mu_{y+1}}{\lambda_x \cdots \lambda_{y+1}} {}_y m_{y+1\,v} \\
&= \frac{\mu_x \cdots \mu_{y+1}}{\lambda_x \cdots \lambda_y} {}_y m_{y\,v} \\
&= \frac{\mu_x \cdots \mu_{y+1}}{\lambda_x \cdots \lambda_y} \frac{\lambda_{v-1} \cdots \lambda_y}{\mu_v \cdots \mu_{y+1}} = \frac{\lambda_{v-1} \cdots \lambda_{x+1}}{\mu_v \cdots \mu_{x+1}}.
\end{aligned}$$

As a consequence, for $x, v > y$

$$\begin{aligned}
{}_y m_{xv} &= {}_y m_{y+1v} + \sum_{l=y+1}^{x-1} \Delta_y m_{lv} \\
&= \sum_{l=y+1}^{\min\{x,v\}} \frac{\lambda_l \cdots \lambda_{v-1}}{\mu_l \cdots \mu_v} = \sum_{l=y+1}^{\min\{x,v\}} \frac{\pi_v}{\lambda_{l-1}\pi_{l-1}} = \sum_{l=y+1}^{\min\{x,v\}} \frac{\pi_v}{\mu_l \pi_l}.
\end{aligned} \tag{4.3}$$

By the same reasoning we find for $x, v < y$

$${}_y m_{xv} = \sum_{l=\max\{x,v\}}^{y-1} \frac{\mu_{v+1} \cdots \mu_l}{\lambda_v \cdots \lambda_l l} = \sum_{l=\max\{x,v\}}^{y-1} \frac{\pi_v}{\lambda_l \pi_l}. \tag{4.4}$$

For the first passage times, we thus find for $x > y$

$$\begin{aligned}
\tau_{xy} &= \sum_{v>y} {}_y m_{xv} \\
&= \sum_{v=y+1}^{x} {}_y m_{vv} + \sum_{v \geq x+1} {}_y m_{xv} \\
&= \sum_{v=y+1}^{x} \sum_{l=y+1}^{v} \frac{\lambda_l \cdots \lambda_{v-1}}{\mu_l \cdots \mu_v} + \sum_{v \geq x+1} \sum_{l=y+1}^{x} \frac{\lambda_l \cdots \lambda_{v-1}}{\mu_l \cdots \mu_v} \\
&= \sum_{l=y+1}^{x} \sum_{v \geq l} \frac{\lambda_l \cdots \lambda_{v-1}}{\mu_l \cdots \mu_v} = \sum_{l=y+1}^{x} \sum_{v \geq l} \frac{\pi_v}{\mu_l \pi_l},
\end{aligned} \tag{4.5}$$

and similarly for $x < y$

$$\tau_{xy} = \sum_{l=x}^{y-1} \sum_{v \leq l} \frac{\mu_{v+1} \cdots \mu_l}{\lambda_v \cdots \lambda_l} = \sum_{l=x}^{y-1} \sum_{v \leq l} \frac{\pi_v}{\lambda_l \pi_l}. \tag{4.6}$$

Next we will calculate D_{yy} in terms of the stationary probabilities:

$$\begin{aligned}
D_{yy} &= \pi_y \sum_{v \neq y} \pi_v \tau_{vy} \\
&= \pi_y \sum_{v>y} \pi_v \sum_{l=y+1}^{v} \sum_{r \geq l} \frac{\pi_r}{\mu_l \pi_l} + \pi_y \sum_{v<y} \pi_v \sum_{l=v}^{y-1} \sum_{r \leq l} \frac{\pi_r}{\lambda_l \pi_l} \\
&= \pi_y \sum_{l>y} \Big(\sum_{v \geq l} \pi_v \Big)^2 \frac{1}{\mu_l \pi_l} + \pi_y \sum_{l<y} \Big(\sum_{v \leq l} \pi_v \Big)^2 \frac{1}{\lambda_l \pi_l}.
\end{aligned} \tag{4.7}$$

By expression (4.7)

$$\frac{D_{y+1\,y+1}}{\pi_{y+1}} - \frac{D_{yy}}{\pi_y} = -\Big(\sum_{v \geq y+1} \pi_v \Big)^2 \frac{1}{\mu_{y+1}\pi_{y+1}} + \Big(\sum_{v \leq y} \pi_v \Big)^2 \frac{1}{\lambda_y \pi_y}$$

$$
\begin{aligned}
&= \frac{1}{\lambda_y \pi_y}\Big\{ -\Big(\sum_{v\geq y+1} \pi_v\Big)^2 + \Big(\sum_{v\leq y}\pi_v\Big)^2\Big\} \\
&= \frac{1}{\lambda_y \pi_y}\Big\{ \sum_{v\leq y}\pi_v - \sum_{v\geq y+1}\pi_v\Big\}.
\end{aligned}
$$

Therefore, writing

$$
v_{y+1} = \sum_{v\leq y}\pi_v - \sum_{v\geq y+1}\pi_v, \tag{4.8}
$$

we find that

$$
D_{y+1\,y+1} = \frac{\lambda_y}{\mu_{y+1}} D_{yy} + \frac{1}{\mu_{y+1}} v_{y+1} \tag{4.9}
$$

$$
D_{y-1\,y-1} = \frac{\mu_y}{\lambda_{y-1}} D_{yy} - \frac{1}{\lambda_{y-1}} v_y. \tag{4.10}
$$

This suggests the following algorithm for computing the deviation matrix.

Algorithm for computing *D*

Step 1 Choose a reference state s and compute D_{ss}.

Step 2 For $y = s, s-1, \ldots$ do: compute v_y and set $D_{y-1\,y-1} = \frac{\mu_y}{\lambda_{y-1}} D_{yy} - \frac{1}{\lambda_{y-1}} v_y$.

Step 3 For $y = s, s+1, \ldots$ do: compute v_{y+1} and set $D_{y+1,y+1} = \frac{\lambda_y}{\mu_{y+1}} D_{yy} + \frac{1}{\mu_{y+1}} v_{y+1}$.

Step 4 For any y and x compute D_{xy} for $x \neq s$ using formulae (2.6) and (4.5) or (4.6).

N-state truncations.
Suppose that the stationary distribution for the infinite system has exponentially decaying tails. Let us truncate the state space by throwing away all states larger than N and sending the disappearing mass flowing from state N back to itself. Then the entries of the deviation matrix for the truncated system and the corresponding entries for the original system differ by a factor that decays exponentially quickly with N. This follows easily from the above algorithm and formula (2.5).

Let us next introduce the time-discretised approximations of the M/M/s/N- and M/M/s/∞-queues and calculate their deviation matrices.

M/M/s/N-queue.
This is a system with s servers, and a buffer (waiting room) of size $N-s$ for some integer $N \geq s$. This means that the total amount of jobs in the system can be at most N.

The time discretised approximation has the following service rates

$$
\mu_x = \begin{cases} x\mu, & x \leq s \\ s\mu, & x > s. \end{cases}
$$

A suitable reference state is the state s which is the boundary of the set of states where the service rates are non-homogeneous and the one where they are homogeneous.

The stationary distribution is given by

$$
\pi_x = \begin{cases} \pi_s \dfrac{s!}{x!}\Big(\dfrac{\mu}{\lambda}\Big)^{s-x} = \pi_s \dfrac{s!}{x! s^{s-x}} \rho^{x-s}, & x < s \\ \pi_s \Big(\dfrac{\lambda}{s\mu}\Big)^{x-s} = \pi_s \rho^{x-s}, & x > s, \end{cases} \tag{4.11}
$$

where $\rho = \lambda/s\mu$ and where by normalisation

$$\pi_s = \frac{1}{\sum_{l=1}^{s} \rho^{-l} \frac{(s)!}{(s-l)!s^l} + \sum_{l=0}^{N-s} \rho^l} = \frac{1-\rho}{\sum_{l=1}^{s} \rho^{-l} \frac{s!l}{(s-l)!s^{l+1}} - \rho^{N+1-s}}.$$

Let us first calculate D_{ss} and v_y in terms of π_s.

$$\begin{aligned} \frac{D_{ss}}{\pi_s} &= \sum_{l>s} \Big(\sum_{v \geq l} \pi_v\Big)^2 \frac{1}{s\mu\pi_l} + \sum_{l<s} \Big(\sum_{v \leq l} \pi_v\Big)^2 \frac{1}{\lambda \pi_l} \\ &= \frac{\pi_s}{s\mu} \sum_{l>s} \rho^{l-s} \Big(\sum_{v=0}^{N-l} \rho^v\Big)^2 + \frac{s!\rho^{-s}\pi_s}{s^s \lambda} \sum_{l<s} \frac{l!}{\rho^l s^l} \Big(\sum_{v \leq l} \frac{\rho^v s^v}{v!}\Big)^2. \end{aligned} \tag{4.12}$$

The first term in (4.12) is simple to further comprise:

$$\begin{aligned} \frac{\pi_s}{s\mu} \sum_{l>s} \rho^{l-s} \Big(\sum_{v=0}^{N-l} \rho^v\Big)^2 &= \frac{\pi_s}{s\mu(1-\rho)^2} \sum_{l>s} \rho^{l-s} \Big(1 - 2\rho^{N-l+1} + \rho^{2N-2l+2}\Big) \\ &= \frac{\pi_s}{s\mu(1-\rho)^3} \Big(\rho - \rho^{N+1-s} - 2(N-s)(1-\rho)\rho^{N+1-s} + \rho^{N-s+2} \\ &\qquad -\rho^{2N-2s+2}\Big) \\ &= \frac{\pi_s}{s\mu(1-\rho)^3} \Big(\rho - (1-\rho)(2N-2s+1)\rho^{N+1-s} - \rho^{2N-2s+2}\Big). \end{aligned}$$

As a consequence,

$$\begin{aligned} D_{ss} &= \frac{\pi_s^2}{s\mu(1-\rho)^3} \Big(\rho - (1-\rho)(2N-2s+1)\rho^{N+1-s} - \rho^{2N-2s+2}\Big) + \\ &\qquad \frac{s!\rho^{-s}\pi_s^2}{s^s\lambda} \sum_{l<s} \frac{l!}{\rho^l s^l} \Big(\sum_{v \leq l} \frac{\rho^v s^v}{v!}\Big)^2. \end{aligned} \tag{4.13}$$

Calculation of v_y yields

$$v_y = \begin{cases} 1 - 2\sum_{v \geq y} \pi_v = 1 - 2\pi_s \dfrac{\rho^{y-s} - \rho^{N+1-s}}{1-\rho}, & y \geq s \\ 2\sum_{v \leq y-1} \pi_v - 1 = 2\pi_s \sum_{v=0}^{y-1} \dfrac{s!}{v! s^{s-v}} \rho^{v-s} - 1 & y < s. \end{cases} \tag{4.14}$$

Next, we will express D_{yy} in terms of D_{ss} and π_s. Using (4.9) and the expression for v_{s+1}, it follows that

$$D_{s+1\,s+1} = \rho D_{ss} + \frac{1}{s\mu} - \frac{2\pi_s}{s\mu} \frac{\rho - \rho^{N+1-s}}{1-\rho}.$$

Consequently,

$$\begin{aligned} D_{s+2\,s+2} &= \rho D_{s+1\,s+1} + \frac{1}{s\mu} - \frac{2\pi_s}{s\mu} \frac{\rho^2 - \rho^{N+1-s}}{1-\rho} \\ &= \rho^2 D_{ss} + \frac{1}{s\mu} \frac{1-\rho^2}{1-\rho} - \frac{4\pi_s}{s\mu(1-\rho)} \cdot \rho^2 + \frac{2\pi_s}{s\mu(1-\rho)} \cdot \frac{1-\rho^2}{1-\rho} \rho^{N+1-s}. \end{aligned}$$

Iterating this yields

$$D_{s+k\,s+k} = \rho^k D_{ss} + \frac{1-\rho^k}{s\mu(1-\rho)} - \frac{2k\pi_s}{s\mu(1-\rho)} \cdot \rho^k + \frac{2\pi_s}{s\mu(1-\rho)} \cdot \frac{1-\rho^k}{1-\rho}\rho^{N+1-s}.$$

Next, we obtain $D_{s-1\,s-1}$ from (4.10) and the expression for v_s:

$$D_{s-1\,s-1} = \rho^{-1} D_{ss} - \frac{2\pi_s}{\lambda}\sum_{r=0}^{s-1} \frac{s!}{r!s^{s-r}}\rho^{r-s} + \frac{1}{\lambda}.$$

Iterating this, easily yields

$$\begin{aligned} D_{s-k\,s-k} &= \frac{s!}{(s-k)!s^k}\rho^{-k} D_{ss} - \frac{2\pi_s}{\lambda}\sum_{l=1}^{k} \frac{(s-l)!}{(s-k)!s^{k-l}}\rho^{l-k}\sum_{r=0}^{s-l}\frac{s!}{r!s^{s-r}}\rho^{r-s} \\ &\quad + \frac{1}{\lambda}\sum_{l=1}^{k}\frac{(s-l)!}{(s-k)!s^{k-l}}\rho^{l-k}. \end{aligned}$$

We will finally calculate D_{xy} for $x \neq y$, and express these in terms of D_{yy}. We need to consider a number of different cases.

$\boldsymbol{s \leq x}$

i)$\boldsymbol{y > x}$**.** We have

$$\begin{aligned} D_{xy} &= D_{yy} - \pi_y \tau_{xy} \\ &= D_{yy} - \frac{\pi_s \rho^{y-s}}{\lambda}\sum_{l=x}^{y-1}\Big(\frac{1-\sum_{v>l}\pi_v}{\pi_l}\Big) \\ &= D_{yy} - \frac{\pi_s \rho^{y-s}}{\lambda}\Big(\frac{\rho^s}{\pi_s}\sum_{l=x}^{y-1}\rho^{-l} - \sum_{l=x}^{y-1}\sum_{v>l}\rho^{v-l}\Big) \\ &= D_{yy} - \frac{\pi_s \rho^{y-s}}{\lambda(1-\rho)}\Big(\frac{\rho^{s+1-y}-\rho^{s+1-x}}{\pi_s} - \sum_{l=x}^{y-1}\big(\rho - \rho^{N-l+1}\big)\Big) \\ &= D_{yy} - \frac{\rho-\rho^{y+1-x}}{\lambda(1-\rho)} + \frac{\pi_s\rho^{y+1-s}(y-x)}{\lambda(1-\rho)} - \frac{\pi_s}{\lambda(1-\rho)^2}(\rho^{N-s+2} - \rho^{N+y-x-s+2}) \\ &= D_{yy} - \frac{1-\rho^{y-x}}{s\mu(1-\rho)} + \frac{\pi_s\rho^{y-s}(y-x)}{s\mu(1-\rho)} - \frac{\pi_s(\rho^{N+1-s} - \rho^{N+1+y-x-s})}{s\mu(1-\rho)^2}. \end{aligned}$$

ii)$\boldsymbol{s \leq y < x}$**.** In this case

$$\begin{aligned} D_{xy} &= D_{yy} - \pi_y \tau_{xy} \\ &= D_{yy} - \frac{\pi_s\rho^{y-s}}{s\mu}\sum_{l=y+1}^{x}\sum_{v\geq l}\rho^{v-l} \\ &= D_{yy} - \frac{\pi_s\rho^{y-s}}{s\mu(1-\rho)}\sum_{l=y+1}^{x}\big(1-\rho^{N+1-l}\big) \\ &= D_{yy} + \frac{\pi_s\rho^{y-s}(y-x)}{s\mu(1-\rho)} - \frac{\pi_s(\rho^{N+1-s} - \rho^{N+1+y-x-s})}{s\mu(1-\rho)^2}. \end{aligned}$$

iii) $\boldsymbol{y < s}$**.** Then

$$\begin{aligned}
D_{xy} &= D_{yy} - \pi_y \tau_{xy} \\
&= D_{yy} - \pi_y \sum_{l=y+1}^{s} \frac{1 - \sum_{v<l} \pi_v}{l\mu\pi_l} - \pi_y \sum_{l=s+1}^{x} \sum_{v \geq l} \frac{\pi_v}{s\mu\pi_l} \\
&= D_{yy} - \frac{\rho^y s^y}{y!\mu} \sum_{l=y+1}^{s} \frac{(l-1)!}{\rho^l s^l} + \frac{\pi_s s! \rho^{y-s}}{y! s^{s-y}\mu} \sum_{l=y+1}^{s} \sum_{v<l} \frac{(l-1)!\rho^{v-l}}{v! s^{l-v}} - \frac{\pi_s s! \rho^{y-s}}{y! s^{s+1-y}\mu} \frac{x-s}{1-\rho} + \\
&\quad + \frac{\pi_s s! \rho^{y-s}}{y! s^{s+1-y}\mu} \frac{\rho^{N+1-x} - \rho^{N+1-s}}{(1-\rho)^2} \\
&= D_{yy} + \frac{\pi_s s! \rho^{y-s}(s-x)}{y! s^{s-y} s\mu(1-\rho)} - \frac{\pi_s s! (\rho^{N+1+y-2s} - \rho^{N+1+y-x-s})}{y! s^{s-y} s\mu(1-\rho)^2} + \\
&\quad - \frac{\rho^y s^y}{y!\mu} \sum_{l=y+1}^{s} \frac{(l-1)!}{\rho^l s^l} + \frac{\pi_s s! \rho^{y-s}}{y! s^{s-y}\mu} \sum_{l=y+1}^{s} \sum_{v<l} \frac{(l-1)!\rho^{v-l}}{v! s^{l-v}}.
\end{aligned}$$

$\boldsymbol{s > x}$

i) $\boldsymbol{y \geq s}$**.** Now we have

$$\begin{aligned}
D_{xy} &= D_{yy} - \pi_y \tau_{xy} \\
&= D_{yy} - \pi_y \sum_{l=x}^{s-1} \sum_{v \leq l} \frac{\pi_v}{\lambda\pi_l} - \pi_y \sum_{l=s}^{y-1} \frac{1 - \sum_{v>l} \pi_v}{\lambda\pi_l} \\
&= D_{yy} - \frac{\pi_s \rho^{y-s}}{\lambda} \sum_{l=x}^{s-1} \sum_{v \leq l} \frac{l!\rho^{v-l}}{v! s^{l-v}} - \frac{\rho}{\lambda(1-\rho)}(1 - \rho^{y-s}) + \frac{\pi_s \rho^{y-s+1}(y-s)}{\lambda(1-\rho)} + \\
&\quad - \frac{\pi_s}{\lambda(1-\rho)^2} \left(\rho^{N+2-s} - \rho^{N+2+y-2s} \right) \\
&= D_{yy} - \frac{1 - \rho^{y-s}}{s\mu(1-\rho)} + \frac{\pi_s \rho^{y-s}(y-s)}{s\mu(1-\rho)} - \frac{\pi_s (\rho^{N+1-s} - \rho^{N+1+y-2s})}{s\mu(1-\rho)^2} \\
&\quad - \frac{\pi_s \rho^{y-s}}{\mu} \sum_{l=x+1}^{s} \sum_{v<l} \frac{(l-1)!\rho^{v-l}}{v! s^{l-v}}.
\end{aligned}$$

ii) $\boldsymbol{s > y > x}$**.** Now

$$\begin{aligned}
D_{xy} &= D_{yy} - \pi_y \tau_{xy} \\
&= D_{yy} - \pi_y \sum_{l=x}^{y-1} \sum_{v \leq l} \frac{\pi_v}{\lambda\pi_l} \\
&= D_{yy} - \frac{\pi_s s! \rho^{y-s}}{y! s^{s-y}\lambda} \sum_{l=x}^{y-1} \sum_{v \leq l} \frac{l!}{v! s^{l-v}} \rho^{v-l} \\
&= D_{yy} - \frac{\pi_s s! \rho^{y-s}}{y! s^{s-y}\mu} \sum_{l=x+1}^{y} \sum_{v<l} \frac{(l-1)!}{v! s^{l-v}} \rho^{v-l}.
\end{aligned}$$

iii) $\boldsymbol{s > x > y}$**.** Finally,

$$\begin{aligned} D_{xy} &= D_{yy} - \pi_y \tau_{xy} \\ &= D_{yy} - \pi_y \sum_{l=y+1}^{x} \frac{1 - \sum_{v<l} \pi_v}{l\mu\pi_l} \\ &= D_{yy} - \frac{\rho^y s^y}{y!\mu} \sum_{l=y+1}^{x} \frac{(l-1)!}{\rho^l s^l} + \frac{\pi_s s! \rho^{y-s}}{y! s^{s-y} \mu} \sum_{l=y+1}^{x} \sum_{v<l} \frac{(l-1)! \rho^{v-l}}{v! s^{l-v}}. \end{aligned}$$

Let us summarise these expressions. For any two numbers x and y we use the notation $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$.

Theorem 4.1 *The deviation matrix of the* M/M/s/N*-queue has the following entries:*

$$\begin{aligned} D_{ss} &= \frac{\pi_s^2}{s\mu(1-\rho)^3}\Big(\rho - (1-\rho)(2N-2s+1)\rho^{N+1-s} - \rho^{2N-2s+2}\Big) \\ &\quad + \frac{s!\rho^{-s}\pi_s^2}{s^s\lambda} \sum_{l<s} \frac{l!}{\rho^l s^l} \Big(\sum_{v \le l} \frac{\rho^v s^v}{v!}\Big)^2 \end{aligned}$$

and for $(x, y) \neq (s, s)$

$$\begin{aligned} D_{xy} &= \frac{\rho^{y-s} s!}{(y\wedge s)! s^{s-y\wedge s}} D_{ss} + \frac{\rho^{(y\vee s - x\vee s)\vee 0} - \rho^{y\vee s - s}}{s\mu(1-\rho)} \\ &\quad + \frac{\pi_s s! \rho^{y-s}}{(y\wedge s)! s^{s-y\wedge s} s\mu(1-\rho)} \big(2s - x\vee s - y \vee s\big) \\ &\quad + \frac{\pi_s s! \rho^{y-s}}{(y\wedge s)! s^{s-y\wedge s} s\mu(1-\rho)^2} \Big(\rho^{N+1-s} + \rho^{N+1+y-x\vee s - s} - \rho^{N+1+y-2s} - \rho^{N+1+y\vee s - 2s}\Big) \\ &\quad + \frac{\rho^y s^y}{y!\mu} \Big(\sum_{l=y\wedge s+1}^{s} \frac{(l-1)!}{\rho^l s^l} - \sum_{l=y\wedge s+1}^{x\wedge s} \frac{(l-1)!}{\rho^l s^l} \Big) \\ &\quad - \frac{\pi_s s! \rho^{y-s}}{(y\wedge s)! s^{s-y\wedge s}\mu} \Big(\sum_{l=y\wedge s+1}^{s} \sum_{v<l} \frac{(l-1)!\rho^{v-l}}{v! s^{l-v}} + \sum_{l=x\wedge s+1}^{s} \sum_{v<l} \frac{(l-1)!\rho^{v-l}}{v! s^{l-v}} \Big). \end{aligned}$$

For $s = 1$ the expressions become much simpler since the inhomogeneous part of the state space disappears (cf. also [5]). The stationary probabilities are given by

$$\pi_x = \rho^x \frac{1-\rho}{1-\rho^{N+1}}.$$

Then

$$v_x = 1 - 2\frac{\rho^x - \rho^{N+1}}{1-\rho}, \quad x > 0.$$

Note that we only need v_x, $x > 0$, for calculating the diagonal of D.

First computing the diagonal element corresponding to $s = 1$, yields

$$D_{11} = \frac{1}{(1-\rho^{N+1})^2(1-\rho)\mu}\Big(1 - 3\rho + 3\rho^2 - (1-\rho)\rho^{N+2}(2N-1) - \rho^{2N+2}\Big).$$

Plugging this into the expressions for D_{yy} gives

$$D_{yy} = \frac{1}{(1-\rho^{N+1})^2(1-\rho)\mu}\Big(1 - (2y+1)\rho^y(1-\rho) + (2y-1-2N)\rho^{N+y+1}(1-\rho) - \rho^{2N+2}\Big).$$

Checking the off-diagonal elements, we finally find the following general expression for the entries of the deviation matrix (cf. [5] where the same formula has been derived in a slightly different format).

Corollary 4.1 (of Theorem 4.1) *In case $s = 1$, the deviation matrix has entries*

$$\begin{aligned} D_{xy} &= \frac{1}{\mu(1-\rho)(1-\rho^{N+1})^2}\Big(\rho^{(y-x)\vee 0}(1-\rho^{N+1})^2 - (y+x-1)\rho^y(1-\rho)(1-\rho^{N+1}) \\ &\quad +\rho^{N+y-x+1} - \rho^{2N+y-x+2} - 2\rho^y(1-\rho) + \rho^{N+1} - \rho^{2N+2} \\ &\quad -2N\rho^{N+y+1}(1-\rho)\Big). \end{aligned}$$

M/M/s/∞-queue.

This queue has exactly the same transition mechanism as the M/M/s/N-queue, only the buffer size is infinitely large, i.e., $N = \infty$. Note that the infinite buffer system is only positive recurrent, when $\lambda < s\mu$. In that case, it satisfies the conditions of Lemma 4.1. Consequently, Theorem 2.3 applies, and so the deviation matrix exists and satisfies the conditions of Theorem refhmm.

The formulae derived for the general one buffer network evidence, that the stationary probabilities, first passage times and thus the deviation matrix for the infinite buffer case, can be obtained from the corresponding expressions for the finite buffer case by taking the limit $N \to \infty$.

Thus the stationary probabilities satisfy (4.11) where in this case

$$\pi_s = \frac{1-\rho}{\sum_{l=1}^{s} \rho^{-l}\frac{s!l}{(s-l)!s^{l+1}}}.$$

For the diagonal entries of the deviation matrix, taking the limit $N \to \infty$ in the expressions in Theorem 4.1 gives the following result.

Theorem 4.2

$$D_{ss} = \frac{\pi_s^2\rho}{s\mu(1-\rho)^3} + \frac{s!\rho^{-s}\pi_s^2}{s^s\lambda}\sum_{l<s}\frac{l!}{\rho^l s^l}\Big(\sum_{v\le l}\frac{\rho^v s^v}{v!}\Big)^2$$

and for $(x, y) \neq (s, s)$

$$\begin{aligned} D_{xy} &= \frac{\rho^{y-s}s!}{(y\wedge s)!s^{s-y\wedge s}}D_{ss} + \frac{\rho^{(y\vee s - x\vee s)\vee 0} - \rho^{y\vee s - s}}{s\mu(1-\rho)} \\ &\quad + \frac{\pi_s s!\rho^{y-s}}{(y\wedge s)!s^{s-y\wedge s}s\mu(1-\rho)}\big(2s - x\vee s - y\vee s\big) \\ &\quad + \frac{\rho^y s^y}{y!\mu}\Big(\sum_{l=y\wedge s+1}^{s}\frac{(l-1)!}{\rho^l s^l} - \sum_{l=y\wedge s+1}^{x\wedge s}\frac{(l-1)!}{\rho^l s^l}\Big) \\ &\quad - \frac{\pi_s s!\rho^{y-s}}{(y\wedge s)!s^{s-y\wedge s}\mu}\Big(\sum_{l=y\wedge s+1}^{s}\sum_{v<l}\frac{(l-1)!\rho^{v-l}}{v!s^{l-v}} + \sum_{l=x\wedge s+1}^{s}\sum_{v<l}\frac{(l-1)!\rho^{v-l}}{v!s^{l-v}}\Big). \end{aligned}$$

The formulae for D_{yy} in the finite and infinite buffer systems have the same format whenever $y < s$, and for D_{xy}, $x \neq y$, whenever $x, y < s$. They only differ through D_{ss} and π_s.

Next we consider the single server case $s = 1$. The stationary probabilities are given by

$$\pi_x = \rho^x(1-\rho).$$

A general formula for the deviation matrix entries is obtained by passing to the limit $N \to \infty$ in the Corollary to Theorem 4.1.

Corollary 4.2 (of Theorem 4.2)

$$D_{xy} = \frac{1}{\mu(1-\rho)}\Big(\rho^{(y-x)\wedge 0} - (y+x+1)\rho^y(1-\rho)\Big).$$

References

[1] A. Hordijk and F.M. Spieksma (1992), On ergodicity and recurrence properties of a Markov chain with an application to an open Jackson network. *Adv. Appl. Prob.* **24**, 343–376.

[2] A. Hordijk and F.M. Spieksma (1994), A new formula for the Deviation matrix. In *Probability, Statistics and Optimisation*, 497–507. J. Wiley.

[3] S. Karlin (1966), *A first Course in Stochastic Processes.* Academic Press, London.

[4] J.G. Kemeny and J.L. Snell (1960), *Finite Markov Chains.* Van Nostrand, New York.

[5] G.M. Koole (1998), The deviation matrix of the $M/M/1/\infty$ and $M/M/1/N$ queue, with applications to controlled queueing models. In *Proceedings of the 37th IEEE CDC*, 55–59, Tampa.

[6] G.M. Koole and P. Nain (2000), On the value function of a priority queue with an application to a controlled polling model. *Queueing Systems*, **34**, 199–214.

[7] T.J. Ott and K.R. Krishnan (1992), Separable routing: A scheme for state-dependent routing of circuit switched telephone traffic. *Annals of Operations Research*, **35**, 43–68.

[8] M.L. Puterman (1994), *Markov Decision Processes.* Wiley, New York.

[9] S.A.E. Sassen, H.C. Tijms, and R.D. Nobel (1997), A heuristic rule for routing customers to parallel servers. *Statistica Neerlandica*, **51**, 107–121.